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One-dimensional compressible heat-conducting gas with temperature-dependent viscosity

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We consider the one-dimensional compressible Navier–Stokes system for a viscous and heat-conducting ideal polytropic gas when the viscosity μ and the heat conductivity κ depend on the specific volume v and the temperature θ and are both proportional to $h(v)\theta^\alpha$ for certain non-degenerate smooth function h. We prove the existence and uniqueness of a global-in-time non-vacuum solution to its Cauchy problem under certain assumptions on the parameter α and initial data, which imply that the initial data can be large if $|\alpha|$ is sufficiently small. Such a result appears to be the first global existence result for general adiabatic exponent and large initial data when the viscosity coefficient depends on both the density and the temperature.

Keywords: Compressible Navier–Stokes system; temperature-dependent viscosity; ideal polytropic gas; global solutions; large initial data; general adiabatic exponent.

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1. Introduction

The one-dimensional motion of a compressible viscous and heat-conducting fluid can be formulated in the Lagrangian coordinates as

$$\begin{cases} v_t - u_x = 0, \\ u_t + P_x = \left[\dfrac{\mu u_x}{v}\right]_x, \\ \left[e + \dfrac{u^2}{2}\right]_t + (uP)_x = \left[\dfrac{\kappa\theta_x}{v} + \dfrac{\mu u u_x}{v}\right]_x. \end{cases} \tag{1.1}$$

Here $t > 0$ is the time variable, $x \in \mathbb{R}$ is the Lagrangian spatial variable, and the primary-dependent variables are the specific volume v, the fluid velocity u, and the temperature θ. The pressure P, the specific internal energy e, and the transport coefficients μ (viscosity) and κ (heat conductivity) are prescribed through constitutive relations as functions of the specific volume v and the temperature θ. The thermodynamic variables v, P, e, and θ are related through Gibbs equation $\mathrm{d}e = \theta \mathrm{d}s - P\mathrm{d}v$ with s being the specific entropy.

This paper concerns the construction of globally smooth non-vacuum solutions to the Cauchy problem of (1.1) for an ideal polytropic gas, which is identified by the constitutive relations

$$P = \frac{\theta}{v} = v^{-\gamma}\exp\left(\frac{s}{c_v}\right), \quad e = c_v\theta, \tag{1.2}$$

with prescribed initial data

$$(v(t,x), u(t,x), \theta(t,x))|_{t=0} = (v_0(x), u_0(x), \theta_0(x)) \quad \text{for } x \in \mathbb{R}. \tag{1.3}$$

Here $c_v = 1/(\gamma - 1)$ is the specific heat at constant volume with $\gamma > 1$ being the adiabatic exponent and some gas constants involved have been normalized to be unity without loss of generality. It is assumed that the initial data (v_0, u_0, θ_0) satisfy the far-field condition

$$\lim_{x\to\pm\infty}(v_0(x), u_0(x), \theta_0(x)) = (1, 0, 1). \tag{1.4}$$

We are interested in the case when the transport coefficients μ and κ, especially the viscosity μ, depend on both the specific volume v and the temperature θ. Recall that the study on such a dependence is motivated by the following three observations:

(i) For certain class of solid-like materials considered in Refs. 6 and 7, both the viscosity coefficient μ and the heat conductivity coefficient κ may depend on the density and/or temperature.

(ii) Experimental results in Ref. 36 show that the transport coefficients μ and κ vary in terms of temperature and density for gases at very high temperature and density.

(iii) If the compressible Navier–Stokes equations (1.1) are derived from the Boltzmann equation with slab symmetry for the monatomic gas by using the Chapman–Enskog expansion, the constitutive relations between thermodynamic variables satisfy (1.2) and the transport coefficients μ and κ depend only on the temperature. Moreover, the functional dependence is the same for both coefficients (see Refs. 4 and 32). In particular, if the intermolecule potential varies as r^{-a} with r being the molecule distance, then

$$\mu = \tilde{\mu}\theta^{\alpha}, \quad \kappa = \tilde{\kappa}\theta^{\alpha}, \tag{1.5}$$

where $\tilde{\mu}$, $\tilde{\kappa}$, and $\alpha = \frac{a+4}{2a} > \frac{1}{2}$ are positive constants.

The crucial step to construct the global solutions of the compressible Navier–Stokes equations (1.1) with large initial data is to obtain the positive upper and lower bounds of the specific volume v and the temperature θ, which has been shown in Ref. 18 for small and sufficiently smooth data. When the viscosity and the heat conductivity coefficients are positive constants, Kazhikhov *et al.*[2,20,21] succeeded in deriving a representation for specific volume v by employing the special structure of ideal polytropic gases (1.1) and (1.2). By means of the representation for v and the maximum principle, the positive upper and lower bounds of v and θ as well as the existence and uniqueness of globally smooth solutions have been obtained in Refs. 2, 20 and 21 for (1.1) and (1.2) with arbitrarily large initial data. See also Refs. 1, 13, 14, 15, 17 and 37 for related studies. In all of these works no vacuum nor concentration of mass occur in a finite time.

We note that this argument can be applied to the case when the viscosity μ is a constant and the heat conductivity κ is some function of temperature θ (see Refs. 12, 30 and 34). But this methodology seems not valid if the viscosity μ is a non-constant function of v and θ. For the case when the viscosity μ is a function of the specific volume v alone, as observed by Kanel′[16] for the isentropic flow, the identity

$$\left[\frac{\mu(v)v_x}{v}\right]_t = u_t + P_x \tag{1.6}$$

holds even for general gases. By employing this identity, one can deduce global solvability results on the compressible Navier–Stokes equations (1.1) with large data for certain types of density-dependent viscosity, and density and temperature-dependent heat conductivity. See Refs. 5–7, 19, 31, and references therein for some representative works in this direction.

When the viscosity μ depends on the temperature θ and the specific volume v, the identity corresponding to (1.6) becomes

$$\left[\frac{\mu(v,\theta)v_x}{v}\right]_t = u_t + P_x + \frac{\mu_\theta(v,\theta)}{v}\left(\theta_t v_x - u_x\theta_x\right) \tag{1.7}$$

with $\mu_\theta(v,\theta) := \partial\mu(v,\theta)/\partial\theta$. The temperature dependence of the viscosity μ has a strong influence on the solution and leads to difficulty in mathematical analysis for global solvability with large data. As pointed out in Ref. 12, such a dependence has turned out to be especially problematic and challenging. One of the main difficulties in analysis arises from the last term in (1.7), which is a highly nonlinear term.

A possible way to go on is to use some "smallness mechanism" induced by the structure of Eqs. (1.1) to control the last term in (1.7) suitably. A recent progress along this way is a Nishida–Smøller-type global solvability result with large data obtained in Ref. 25 for the Cauchy problem (1.1)–(1.4) when the viscosity μ and the heat conductivity κ are both functions of the temperature. The main observation in Ref. 25 is that for ideal polytropic gases (1.1) and (1.2), the temperature θ satisfies

$$\theta = v^{1-\gamma}\mathrm{e}^{(\gamma-1)s} \quad \text{and} \quad \frac{\theta_t}{\gamma-1} + \frac{\theta u_x}{v} = \frac{\mu u_x^2}{v} + \left[\frac{\kappa\theta_x}{v}\right]_x,$$

from which one can deduce that $\|(\theta-1,\theta_t,\theta_x)\|_{L^\infty([0,T]\times\mathbb{R})}$ can be small under the condition that the adiabatic exponent γ is close to 1. Thus one can perform the desired energy-type *a priori* estimates as in Refs. 8, 17, 28 and 29 based on the *a priori* assumption

$$\frac{1}{2} \le \theta(t,x) \le 2 \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R}. \tag{1.8}$$

It is to close the *a priori* assumption (1.8) on $\theta(t,x)$ that one needs to impose that the initial data satisfies a Nishida–Smøller-type condition, that is,

$$(\gamma-1)\times C\left(\|(v_0-1,u_0,s_0-1)\|_{H^3(\mathbb{R})}, \inf_{x\in\mathbb{R}} v_0(x)\right) \le 1$$

for some $(\gamma-1)$-independent smooth function C.

The result obtained in Ref. 25 shows that $\|(v_0-1,u_0,s_0-1)\|_{H^3(\mathbb{R})}$ can be large. However, the oscillation of the temperature, $\|\theta-1\|_{L^\infty([0,T]\times\mathbb{R})}$, is not arbitrarily large but has to be small. Thus a natural question is: *Whether can we obtain a global solvability result for the Cauchy problem* (1.1)–(1.4) *with large initial data and general adiabatic exponent* γ *for a class of temperature and density-dependent viscosity coefficient* μ *or not*?

The main goal of this paper is devoted to the above problem and our motivation is essentially the same as that of Ref. 25 mentioned above, that is *to use some "smallness mechanism" induced by the structure of Eqs.* (1.1) *to control the last term in* (1.7) *suitably.* We note that $\gamma-1$ cannot be assumed to be small for the case with general adiabatic exponent. In this paper, we will try to use the smallness of $|\mu_\theta(v,\theta)|$ to control the possible growth of the solutions of the Cauchy problem (1.1)–(1.4) caused by the last term in (1.7). Motivated by such an idea, we assume throughout the rest of this paper that the viscosity μ and the heat conductivity κ are smooth functions of the temperature θ and the specific volume v, which are given by

$$\mu = \tilde{\mu}h(v)\theta^\alpha, \quad \kappa = \tilde{\kappa}h(v)\theta^\alpha, \tag{1.9}$$

where $\tilde{\mu}$ and $\tilde{\kappa}$ are positive constants, and there exist positive constants C, ℓ_1, and ℓ_2 such that

$$Ch(v) \geq v^{\ell_1} + v^{-\ell_2}, \quad h'(v)^2 v \leq Ch(v)^3 \quad \text{for all } v \in (0, \infty). \tag{1.10}$$

We expect to obtain a global solvability result to the Cauchy problem (1.1)–(1.4) with large data and transport coefficients (1.9) and (1.10) for general adiabatic exponent γ provided that $|\alpha|$ is sufficiently small.

The very reason why we choose μ and κ as in (1.9) is that the transport coefficients (1.9) and (1.10) with $\ell_1 = \ell_2 = 0$ can include (1.5) as a special example. Moreover, the special form (1.9) of the viscosity μ with $h(v)$ satisfying (1.10) is essential in our argument and the role is two-fold:

(i) First, we will employ the smallness of resulting factor $|\alpha|$ to control the last term in (1.7).
(ii) Second, the assumption (1.10) imposed on $h(v)$ will be used to yield some estimates on the lower and upper bounds for the specific volume $v(t, x)$ in terms of $\|\theta\|_{L^\infty([0,T]\times\mathbb{R})}$.

As for the heat conductivity κ, the choice as in (1.9) is not so crucial and can be replaced by some more general function of v and θ which satisfies certain conditions in terms of the parameters ℓ_1, ℓ_2, and α. Such a generalization is straightforward and hence we will focus on the case when κ is given by (1.9) and (1.10) for simplicity of presentation.

We introduce

$$H(w) := \sup_{w \leq \sigma \leq w^{-1}} |(h(\sigma), h'(\sigma), h''(\sigma), h'''(\sigma))| \quad \text{for } w > 0, \tag{1.11}$$

and state our main result as follows.

Theorem 1.1. *Assume that the viscosity μ and the heat conductivity κ satisfy* (1.9) *and* (1.10) *for some $\ell_1 \geq 1$ and $\ell_2 \geq 1$. Let the initial data (v_0, u_0, θ_0) satisfy that*

$$(v_0 - 1, u_0, \theta_0 - 1) \in H^3(\mathbb{R}), \qquad \|(v_0 - 1, u_0, \theta_0 - 1)\|_{H^3(\mathbb{R})} \leq \Pi_0, \tag{1.12}$$

$$V_0 \leq v_0(x) \leq V_0^{-1}, \qquad \theta_0(x) \geq V_0 \quad \textit{for all } x \in \mathbb{R}, \tag{1.13}$$

where Π_0 and V_0 are positive constants. Then there exists $\epsilon_0 > 0$, which depends only on Π_0, V_0, and $H(C_0)$ with positive constant C_0 depending only on Π_0, V_0, and $H(V_0)$, such that the Cauchy problem (1.1)–(1.4) *with $|\alpha| \leq \epsilon_0$ admits a unique solution $(v(t, x), u(t, x), \theta(t, x))$ satisfying*

$$(v - 1, u, \theta - 1) \in C([0, \infty), H^3(\mathbb{R})), \tag{1.14}$$

$$v_x \in L^2(0, \infty; H^2(\mathbb{R})), \quad (u_x, \theta_x) \in L^2(0, \infty; H^3(\mathbb{R})), \tag{1.15}$$

and

$$\inf_{(t,x)\in[0,\infty)\times\mathbb{R}} \{v(t, x), \theta(t, x)\} > 0, \qquad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} \{v(t, x), \theta(t, x)\} < +\infty. \tag{1.16}$$

Furthermore, the solution (v, u, θ) *converges to* $(1, 0, 1)$ *uniformly as time tends to infinity*:

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |(v(t,x) - 1, u(t,x), \theta(t,x) - 1)| = 0. \tag{1.17}$$

Remark 1.1. We deduce from (1.12), (1.13) and (1.16) that no vacuum will be developed if the initial data do not contain a vacuum. It follows from (1.14) and (1.15) and Sobolev's imbedding theorem that the unique solution constructed in Theorem 1.1 is a globally smooth non-vacuum solution with large initial data. Moreover, this result in Lagrangian coordinates can easily be converted to equivalent statement for the corresponding problem in Eulerian coordinates.

Remark 1.2. As far as we are aware, for ideal polytropic gases with general adiabatic exponent γ, Theorem 1.1 is the first result on the global well-posedness of smooth non-vacuum solutions to the compressible Navier–Stokes equations (1.1) and (1.2) with temperature-dependent viscosity and large initial data.

Remark 1.3. The assumption we imposed on the parameters ℓ_1 and ℓ_2 in Theorem 1.1 is just for illustrating our main idea to deduce the desired result and is far from being optimal. In fact, our approach can be applied to prove a similar global solvability result when the parameters ℓ_1 and ℓ_2 satisfy

$$\ell_1 > 0, \quad \ell_2 > 0 \quad \text{and} \quad \frac{18}{1+2\ell_1}\max\{0, 1-\ell_1\} + \frac{19}{2\ell_2}\max\{0, 1-\ell_2\} < 4.$$

Unfortunately, our result cannot cover the model satisfying (1.5) since the parameters ℓ_1 and ℓ_2 are assumed to be positive. The extension of our result to the case with $\ell_1 = \ell_2 = 0$ is an open problem for future research.

Remark 1.4. The existence of global strong solutions to the one-dimensional compressible Navier–Stokes equations for isentropic flows has been established in Ref. 27 with the viscosity μ given by (1.9) and (1.10) for $\ell_1 = 0$, $0 \le \ell_2 < \frac{1}{2}$, and $\alpha = 0$, and also in Ref. 11 for the shallow water system, where the viscosity μ satisfies (1.9) with $h(v) = v^{-1}$ and $\alpha = 0$. Note that our derivation of the uniform bounds on $v(t,x)$ and $\theta(t,x)$ relies heavily on the assumption that the initial data is sufficiently smooth. It is an interesting and difficult problem to extend the results in Refs. 11 and 27 to the non-isentropic case with transport coefficients satisfying (1.9) for nonzero α.

Now we outline the main ideas to deduce our main result Theorem 1.1. As pointed out before, the key point for the global solvability result with large data is to deduce the desired positive lower and upper bounds on the specific volume $v(t,x)$ and the temperature $\theta(t,x)$ uniformly in space x as in Refs. 20, 21, and 31. Since we are trying to use the smallness of $|\alpha|$ to control the possible growth of the solutions caused by the last term in (1.7), the amplitude of $|\alpha|$ should be determined by pointwise bounds for the specific volume $v(t,x)$ and the temperature $\theta(t,x)$. The main point in our analysis is to determine the positive parameter ϵ_0 (namely, the upper bound of $|\alpha|$) in Theorem 1.1 in terms of the initial data, such that the whole analysis can be carried out for $|\alpha| \leq \epsilon_0$. To guarantee the existence of such an ϵ_0 (i.e. to insure that the parameter α does not vanish) when we extend the local solutions step-by-step to the global ones, we have to obtain the lower and upper bounds for $v(t,x)$ and $\theta(t,x)$ uniformly in time t and space x. It is worth noting that, even for the Cauchy problem (1.1)–(1.4) with constant transport coefficients, such uniform bounds on $\theta(t,x)$ are obtained only very recently by Li and Liang,[23] although the corresponding global solvability result was addressed by Kazhikhov[20] a long time ago. The starting points of the argument in Ref. 23 are the following:

(i) the global existence result obtained in Ref. 20;
(ii) the uniform positive lower and upper bounds on $v(t,x)$ obtained in Refs. 14 and 15 by using a decent localized version of the expression for $v(t,x)$.

Based on these two points, Li and Liang further deduce the uniform positive lower and upper bounds on the temperature $\theta(t,x)$ in Ref. 23 through a time-asymptotically nonlinear stability analysis. However, the approach in Refs. 14 and 15 cannot be applied to the case when the viscosity μ is a non-constant function of v and θ. To overcome such a difficulty, we employ the argument developed by Kanel′[16,25] to prove that the specific volume $v(t,x)$ can be bounded in terms of the upper bound of the temperature $\theta(t,x)$. Then we combine the local-in-time lower bound on the temperature $\theta(t,x)$ induced by the maximum principle and a well-designed continuation argument to obtain the positive lower and upper bounds of the temperature $\theta(t,x)$ uniformly in time and space as well as the global existence of smooth solutions. Such a continuation argument is of some interest itself and can be used to study some other problems, such as nonlinear stability of the non-degenerate stationary solutions to the outflow problem of the compressible Navier–Stokes equations (1.1) and (1.2) with large initial perturbation and general adiabatic exponent γ in Ref. 33.

Before concluding this section, let us point out that our result shows that no vacuum, mass or heat concentration will be developed in any finite time, although the motion of the flow has large oscillations. For the corresponding results on the compressible Navier–Stokes equations with large data and vacuum, we refer to Refs. 3, 10, 22, 24, 35, and the references therein.

The layout of the rest of this paper is organized as follows. In Sec. 2.1, we deduce the estimate for $\left\|\frac{\mu v_x}{v}(t)\right\|$ under some *a priori* assumptions as in Lemma 2.2, and by applying the argument developed by Kanel′, we prove in Lemma 2.3 that the bounds of the specific volume $v(t,x)$ can be controlled in terms of the upper bound of the temperature $\theta(t,x)$. In Sec. 2.2, we estimate the $H^1(\mathbb{R})$-norm of the temperature $\theta(t,x)$ and obtain the upper and lower bound on the temperature $\theta(t,x)$. The estimates on second-order and third-order derivatives of the solution $(v(t,x), u(t,x), \theta(t,x))$ will be deduced in Secs. 2.3 and 2.4, respectively. Finally, in Sec. 3, by combining the *a priori* estimates and a well-designed continuation argument, we derive the positive lower and upper bounds of the temperature $\theta(t,x)$ and the specific volume $v(t,x)$ uniformly in time and space and extend the local solution step-by-step to the global one.

Notations. Throughout this paper, $L^q(\mathbb{R})$ $(1 \le q \le \infty)$ stands for the usual Lebesgue space on $\mathbb{R}$ with norm $\|\cdot\|_{L^q}$ and $H^k(\mathbb{R})$ $(k \in \mathbb{N})$ the usual Sobolev space in the L^2-sense with norm $\|\cdot\|_k$. We introduce $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}$ for notational simplicity. We denote by $C(I; H^p)$ the space of continuous functions on the interval I with values in $H^p(\mathbb{R})$ and $L^2(I; H^p)$ the space of L^2-functions on I with values in $H^p(\mathbb{R})$. We introduce $A \lesssim B$ (or $B \gtrsim A$) if $A \le CB$ holds uniformly for some constant C depending solely on Π_0, V_0, and $H(V_0)$, where Π_0, V_0, and H are given by (1.11), (1.12) and (1.13).

2. *A Priori* Estimates

We define, for constants N, m_i, s, and t $(i = 1, 2,\ t \ge s)$, the set

$$\begin{aligned} X(s,t;m_1,m_2,N) := \{(v,u,\theta) : {} & (v-1,u,\theta-1) \in C([s,t];H^3), \\ & v_x \in L^2(s,t;H^2), (u_x,\theta_x) \in L^2(s,t;H^3), \\ & \mathcal{E}(s,t) \le N^2, v(\tau,x) \ge m_1, \theta(\tau,x) \ge m_2, \forall(\tau,x) \in [s,t]\times\mathbb{R}\}, \end{aligned}$$

where

$$\mathcal{E}(s,t) := \sup_{\tau\in[s,t]} \|(v-1,u,\theta-1)(\tau)\|_3^2 + \int_s^t \left[\|v_x(\tau)\|_2^2 + \|(u_x,\theta_x)(\tau)\|_3^2\right]\mathrm{d}\tau.$$

The main purpose of this section is to derive certain *a priori* estimates on the solution $(v,u,\theta) \in X(0,T;m_1,m_2,N)$ to the Cauchy problem (1.1)–(1.4) with constitutive relations (1.9) and (1.10) for $T > 0$ and $0 < m_i \le 1 \le N < +\infty$ $(i = 1, 2)$. It follows from the Sobolev's inequality that

$$m_1 \le v(t,x) \le 4N, \quad m_2 \le \theta(t,x) \le 4N \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R}. \tag{2.1}$$

To make the presentation clearly, we divide this section into the following four parts, where we use $\|\!|\cdot|\!\| := \|\cdot\|_{L^\infty([0,T]\times\mathbb{R})}$ for notational simplicity.

2.1. *Pointwise bounds on specific volume*

In this part, we will deduce the lower and upper bounds on the specific volume $v(t,x)$ in terms of $\|\!|\theta|\!\|$. To this end, we first have the basic energy estimate.

Lemma 2.1. *Assume that the conditions listed in Theorem* 1.1 *hold. Then*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} \eta(v,u,\theta)(t,x)\mathrm{d}x + \int_0^T \int_{\mathbb{R}} \left[\frac{\mu u_x^2}{v\theta} + \frac{\kappa\theta_x^2}{v\theta^2}\right] \lesssim 1, \tag{2.2}$$

where

$$\eta(v,u,\theta) := \phi(v) + \frac{1}{2}u^2 + c_v\phi(\theta), \tag{2.3}$$

$$\phi(z) := z - \ln z - 1. \tag{2.4}$$

Proof. In light of (1.1), we deduce

$$c_v\theta_t + \frac{\theta u_x}{v} = \left[\frac{\kappa\theta_x}{v}\right]_x + \frac{\mu u_x^2}{v}. \tag{2.5}$$

Multiplying $(1.1)_1$ (the first equation of (1.1)), $(1.1)_2$, and (2.5) by $(1 - v^{-1})$, u, and $(1 - \theta^{-1})$, respectively, we find

$$\eta(v,u,\theta)_t + \frac{\mu u_x^2}{v\theta} + \frac{\kappa\theta_x^2}{v\theta^2} = \left[\frac{\mu u u_x}{v} + \left(1 - \frac{1}{\theta}\right)\frac{\kappa\theta_x}{v} + \left(1 - \frac{\theta}{v}\right)u\right]_x.$$

Integrate the above identity over $[0,T] \times \mathbb{R}$ to have

$$\int_{\mathbb{R}} \eta(v,u,\theta)(t,x)\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \left[\frac{\mu u_x^2}{v\theta} + \frac{\kappa\theta_x^2}{v\theta^2}\right] = \int_{\mathbb{R}} \eta(v_0,u_0,\theta_0)(x)\mathrm{d}x. \tag{2.6}$$

It follows from the identity $\phi(z) = \int_0^1\int_0^1 \theta_1\phi''(1+\theta_1\theta_2(z-1))\mathrm{d}\theta_2\mathrm{d}\theta_1(z-1)^2$ that

$$(z+1)^{-2}(z-1)^2 \lesssim \phi(z) \lesssim (z^{-1}+1)^2(z-1)^2. \tag{2.7}$$

Applying the last inequality to $\phi(v_0)$ and $\phi(\theta_0)$, we obtain

$$\eta(v_0,u_0,\theta_0)(x) \lesssim 1.$$

Plug this last inequality into (2.6) to derive (2.2). The proof of this lemma is completed. □

Our analysis will rely on the following lemma.

Lemma 2.2. *Suppose that the conditions listed in Theorem* 1.1 *hold. Then there is a constant* $0 < \epsilon_1 \le 1$, *depending only on* Π_0, V_0, *and* $H(V_0)$, *such that if*

$$m_2^{-|\alpha|} \le 2, \quad N^{|\alpha|} \le 2, \quad \Xi(m_1,m_2,N)|\alpha| \le \epsilon_1, \tag{2.8}$$

where

$$\Xi(m_1,m_2,N) := \left[m_1^{-1} + m_2^{-1} + N + \sup_{m_1\le\sigma\le 4N} h(\sigma) + 1\right]^{80},$$

then

$$\sup_{t\in[0,T]}\left\|\frac{\mu v_x}{v}(t)\right\|^2 + \int_0^T\int_{\mathbb{R}} \frac{\mu\theta v_x^2}{v^3} \lesssim 1 + \|\theta\|. \tag{2.9}$$

Proof. According to the chain rule, we have

$$\begin{aligned}
\left(\frac{\mu v_x}{v}\right)_t &= \mu\left(\frac{v_t}{v}\right)_x + \frac{v_x}{v}(\mu_v v_t + \mu_\theta\theta_t) \\
&= \left(\frac{\mu v_t}{v}\right)_x - \frac{v_t}{v}\mu_x + \frac{v_x}{v}(\mu_v v_t + \mu_\theta\theta_t) \\
&= \left(\frac{\mu v_t}{v}\right)_x + \frac{\mu_\theta}{v}(v_x\theta_t - \theta_x v_t),
\end{aligned}$$

which combined with (1.1) implies

$$\left(\frac{\mu v_x}{v}\right)_t = u_t + \left(\frac{\theta}{v}\right)_x + \frac{\mu_\theta}{v}(v_x\theta_t - \theta_x u_x). \tag{2.10}$$

Multiply (2.10) by $\mu v_x/v$ to deduce

$$\begin{aligned}\left[\frac{1}{2}\left(\frac{\mu v_x}{v}\right)^2\right]_t &+ \frac{\mu\theta v_x^2}{v^3} + \left(\frac{\mu u u_x}{v}\right)_x - \left(\frac{\mu v_x}{v}u\right)_t \\ &= \frac{\mu u_x^2}{v} + \frac{\mu v_x\theta_x}{v^2} + \frac{\mu_\theta}{v^2}(\mu v_x - uv)(v_x\theta_t - \theta_x u_x),\end{aligned}$$

and hence

$$\begin{aligned}\left\|\frac{\mu v_x}{v}(t)\right\|^2 + \int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3} &\lesssim 1 + \left|\int_{\mathbb{R}}\frac{\mu v_x}{v}u\mathrm{d}x\right| + \int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v} \\ &+ \left|\int_0^t\int_{\mathbb{R}}\frac{\mu v_x\theta_x}{v^2}\right| + \left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v^2}(\mu v_x - uv)(v_x\theta_t - \theta_x u_x)\right|.\end{aligned}$$

We deduce from Cauchy's inequality and (2.2) that

$$\begin{aligned}\left\|\frac{\mu v_x}{v}(t)\right\|^2 + \int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3} &\lesssim 1 + \int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v} \\ &+ \int_0^t\int_{\mathbb{R}}\frac{\mu\theta_x^2}{v\theta} + \left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v^2}(\mu v_x - uv)(v_x\theta_t - \theta_x u_x)\right|.\end{aligned} \tag{2.11}$$

We first estimate the last term in (2.11). It follows from (2.5) that

$$\theta_t = \frac{1}{c_v}\left[\frac{\kappa_v\theta_x}{v} - \frac{\kappa\theta_x}{v^2}\right]v_x + \frac{1}{c_v}\left[\frac{\kappa_\theta\theta_x^2}{v} + \frac{\kappa\theta_{xx}}{v} + \frac{\mu u_x^2}{v} - \frac{\theta u_x}{v}\right], \tag{2.12}$$

which yields

$$(\mu v_x - uv)(v_x\theta_t - \theta_x u_x) = uv\theta_x u_x + \mathcal{R}_1 v_x + \mathcal{R}_2 v_x^2 \tag{2.13}$$

with

$$\begin{aligned}\mathcal{R}_1 &:= -\mu\theta_x u_x - \frac{u}{c_v}\left(\kappa_\theta\theta_x^2 + \kappa\theta_{xx} + \mu u_x^2 - \theta u_x\right), \\ \mathcal{R}_2 &:= \mu\theta_t - \frac{u}{c_v}\left[\kappa_v\theta_x - \frac{\kappa\theta_x}{v}\right].\end{aligned}$$

Plug (2.13) into (2.11) to obtain

$$\begin{aligned}\left\|\frac{\mu v_x}{v}(t)\right\|^2 + \int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3} &\lesssim 1 + \int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v} + \int_0^t\int_{\mathbb{R}}\frac{\mu\theta_x^2}{v\theta} \\ &+ \left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v}u\theta_x u_x\right| + \left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v^2}v_x\mathcal{R}_1\right| + \left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v^2}v_x^2\mathcal{R}_2\right|.\end{aligned} \tag{2.14}$$

Next we estimate the terms on the right-hand side of (2.14). In view of (2.2), we have

$$\left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v}u\theta_x u_x\right| \le \left\|\frac{\mu_\theta\sqrt{\theta}\theta}{\sqrt{\kappa\mu}}u\right\|\int_0^t\left\|\frac{\sqrt{\kappa}\theta_x}{\sqrt{v}\theta}\right\|\left\|\frac{\sqrt{\mu}u_x}{\sqrt{v\theta}}\right\| \lesssim \left\|\frac{\mu_\theta\sqrt{\theta}\theta}{\sqrt{\kappa\mu}}u\right\|.$$

We deduce from the identity $\mu_\theta = \alpha\mu/\theta$ and (2.1) that

$$\left\|\frac{\mu_\theta\sqrt{\theta}\theta}{\sqrt{\kappa\mu}}u\right\| \lesssim \left\|\alpha\sqrt{\theta}u\right\| \lesssim |\alpha|N^{\frac{1}{2}}\sup_{t\in[0,T]}\|u(t)\|_1 \lesssim |\alpha|N^{\frac{3}{2}}.$$

Hence

$$\left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v}u\theta_x u_x\right| \lesssim |\alpha|N^{\frac{3}{2}}. \tag{2.15}$$

Apply Cauchy's inequality to get

$$\left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v^2}v_x\mathcal{R}_1\right| \le \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3} + C(\epsilon)\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta^2}{\mu\theta v}\mathcal{R}_1^2. \tag{2.16}$$

Since

$$\mathcal{R}_1^2 \lesssim \mu^2\theta_x^2u_x^2 + u^2\kappa_\theta^2\theta_x^4 + u^2\kappa^2\theta_{xx}^2 + \mu^2u^2u_x^4 + u^2\theta^2u_x^2,$$

we have from (2.2) that

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta^2\mathcal{R}_1^2}{\mu\theta v} &\lesssim \int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v\theta^2}\frac{\mu_\theta^2\theta}{\mu\kappa}(\mu^2u_x^2 + u^2\kappa_\theta^2\theta_x^2) + \int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}\frac{\mu_\theta^2\kappa}{\mu\theta}u^2\\ &\quad + \int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v\theta}\frac{\mu_\theta^2}{\mu^2}(u^2\theta^2 + \mu^2u^2u_x^2)\\ &\lesssim \left\|\frac{\mu_\theta^2\theta}{\mu\kappa}(\mu^2u_x^2 + u^2\kappa_\theta^2\theta_x^2)\right\| + \left\|\frac{\mu_\theta^2\kappa}{\mu\theta}u^2\right\|\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}\\ &\quad + \left\|\frac{\mu_\theta^2}{\mu^2}(u^2\theta^2 + \mu^2u^2u_x^2)\right\|.\end{aligned} \tag{2.17}$$

The *a priori* assumption (2.1) implies

$$\left\|\theta^\alpha + \theta^{-\alpha}\right\| \le m_2^{-|\alpha|} + (4N)^{|\alpha|}. \tag{2.18}$$

Then we have from (2.8), (2.18) and Sobolev's inequality that

$$\begin{aligned}&\left\|\frac{\mu_\theta^2\theta}{\mu\kappa}\left(\mu^2u_x^2 + u^2\kappa_\theta^2\theta_x^2\right)\right\| + \left\|\frac{\mu_\theta^2\kappa}{\mu\theta}u^2\right\|\\ &\quad + \left\|\frac{\mu_\theta^2}{\mu^2}(u^2\theta^2 + \mu^2u^2u_x^2)\right\| \lesssim \alpha^2\Xi(m_1, m_2, N)^{\frac{1}{8}}.\end{aligned} \tag{2.19}$$

Combine the estimates (2.16)–(2.19) to derive

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v^2}v_x\mathcal{R}_1\right|\\ &\quad \le \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3} + C(\epsilon)\alpha^2\Xi(m_1, m_2, N)^{\frac{1}{8}}\left[1 + \int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}\right].\end{aligned} \tag{2.20}$$

For the last term on the right-hand side of (2.14), we have

$$\left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v^2}v_x^2\mathcal{R}_2\right|\le\left\|\frac{\mu_\theta v}{\mu\theta}\mathcal{R}_2\right\|\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}. \tag{2.21}$$

It follows from (2.8) and (2.12) that

$$|\theta_t|\le\Xi(m_1,m_2,N)^{\frac{1}{8}}\left(u_x^2+\theta_x^2+|\theta_x v_x|+|\theta_{xx}|+|u_x|\right) \tag{2.22}$$

and

$$|\mathcal{R}_2|\le\Xi(m_1,m_2,N)^{\frac{1}{8}}\left(u_x^2+\theta_x^2+|\theta_x v_x|+|\theta_{xx}|+|u_x|+|u\theta_x|\right).$$

Hence

$$\left\|\frac{\mu_\theta v}{\mu\theta}\mathcal{R}_2\right\|\le|\alpha|\Xi(m_1,m_2,N)^{\frac{1}{4}}.$$

We plug this last estimate into (2.21) to find that

$$\left|\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta}{v^2}v_x^2\mathcal{R}_2\right|\le|\alpha|\Xi(m_1,m_2,N)^{\frac{1}{4}}\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}. \tag{2.23}$$

Plugging (2.15), (2.20), and (2.23) into (2.14) and choosing $\epsilon>0$ sufficiently small, we derive

$$\begin{aligned}\left\|\frac{\mu v_x}{v}(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}\lesssim 1&+\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}+\int_0^t\int_{\mathbb{R}}\frac{\mu\theta_x^2}{v\theta}\\&+\alpha^2\Xi(m_1,m_2,N)^{\frac{1}{8}}\left[1+\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}\right]\\&+|\alpha|\Xi(m_1,m_2,N)^{\frac{1}{4}}\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}.\end{aligned} \tag{2.24}$$

If the parameter ϵ_1 in (2.8) is chosen to be suitably small, then we obtain

$$\left\|\frac{\mu v_x}{v}(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}\lesssim 1+\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}+\int_0^t\int_{\mathbb{R}}\frac{\mu\theta_x^2}{v\theta}+|\alpha|\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}. \tag{2.25}$$

We next estimate the last term in (2.25). To this end, we multiply (2.12) by θ_{xx} to get

$$\begin{aligned}&\left(\frac{c_v}{2}\theta_x^2\right)_t-(c_v\theta_x\theta_t)_x+\frac{\kappa\theta_{xx}^2}{v}\\&\quad=\theta_{xx}\left[\frac{\theta u_x}{v}-\frac{\kappa_v v_x\theta_x}{v}+\frac{\kappa\theta_x v_x}{v^2}-\frac{\kappa_\theta\theta_x^2}{v}-\frac{\mu u_x^2}{v}\right].\end{aligned}$$

Integrating this last identity over $[0,t]\times\mathbb{R}$ and employing Cauchy's inequality give us

$$\begin{aligned}\|\theta_x(t)\|^2+\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}&\lesssim 1+\int_0^t\int_{\mathbb{R}}\left[\frac{\theta^2u_x^2}{v\kappa}+\frac{\kappa_v^2v_x^2\theta_x^2}{v\kappa}+\frac{\kappa\theta_x^2v_x^2}{v^3}+\frac{\kappa_\theta^2\theta_x^4}{v\kappa}+\frac{\mu^2u_x^4}{v\kappa}\right]\\&\lesssim 1+\left\|\frac{v\theta}{\mu}\left(\frac{\theta^2}{v\kappa}+\frac{\mu^2u_x^2}{v\kappa}\right)\right\|\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v\theta}\\&\quad+\left\|\frac{v\theta^2}{\kappa}\left(\frac{\kappa_v^2v_x^2}{v\kappa}+\frac{\kappa v_x^2}{v^3}+\frac{\kappa_\theta^2\theta_x^2}{v\kappa}\right)\right\|\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v\theta^2}. \qquad (2.26)\end{aligned}$$

It follows from (2.8) that

$$\left\|\frac{v\theta}{\mu}\left(\frac{\theta^2}{v\kappa}+\frac{\mu^2u_x^2}{v\kappa}\right)\right\|+\left\|\frac{v\theta^2}{\kappa}\left(\frac{\kappa_v^2v_x^2}{v\kappa}+\frac{\kappa v_x^2}{v^3}+\frac{\kappa_\theta^2\theta_x^2}{v\kappa}\right)\right\|\lesssim\Xi(m_1,m_2,N)^{\frac{1}{8}}.$$

Insert the above inequality and (2.2) into (2.26) to derive

$$\|\theta_x(t)\|^2+\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}\lesssim 1+\Xi(m_1,m_2,N)^{\frac{1}{8}}. \qquad (2.27)$$

Combining (2.25) and (2.27), we obtain

$$\begin{aligned}&\left\|\frac{\mu v_x}{v}(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}\\&\quad\lesssim 1+\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}+\int_0^t\int_{\mathbb{R}}\frac{\mu\theta_x^2}{v\theta}+|\alpha|\Xi(m_1,m_2,N)^{\frac{1}{8}}. \qquad (2.28)\end{aligned}$$

Under the assumptions (2.8), we take $\epsilon_1>0$ small enough to infer

$$\left\|\frac{\mu v_x}{v}(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}\lesssim 1+\int_0^t\int_{\mathbb{R}}\left[\frac{\mu u_x^2}{v}+\frac{\mu\theta_x^2}{v\theta}\right]. \qquad (2.29)$$

The estimate (2.2) implies

$$\int_0^t\int_{\mathbb{R}}\left[\frac{\mu u_x^2}{v}+\frac{\mu\theta_x^2}{v\theta}\right]\le\|\theta\|\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v\theta}+\left\|\frac{\mu\theta}{\kappa}\right\|\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v\theta^2}\lesssim\|\theta\|. \qquad (2.30)$$

Plugging (2.30) into (2.29) yields (2.9). This completes the proof of this lemma. □

In the next lemma, we apply the technique developed by Kanel′[16,25] to estimate the upper and lower bounds for the specific volume $v(t,x)$ in terms of $\|\theta\|$.

Lemma 2.3. *Assume that the conditions listed in Lemma* 2.2 *hold. Then*

$$\|v\|\lesssim 1+\|\theta\|^{\frac{1}{1+2\ell_1}},\quad \left\|v^{-1}\right\|\lesssim 1+\|\theta\|^{\frac{1}{2\ell_2}}. \qquad (2.31)$$

Proof. Define

$$\Phi(v):=\int_1^v\frac{\sqrt{\phi(z)}}{z}h(z)\mathrm{d}z.$$

We infer from (1.10) that for suitably large constant C and $v \geq C$,

$$\Phi(v) \gtrsim \int_C^v \frac{\sqrt{\phi(z)}}{z} z^{\ell_1} \mathrm{d}z \gtrsim \int_C^v z^{-\frac{1}{2}+\ell_1} \mathrm{d}z.$$

Hence

$$v^{\frac{1}{2}+\ell_1} \lesssim 1 + |\Phi(v)| \quad \text{for all } v \in (0, \infty).$$

Similarly, it follows that

$$v^{-\ell_2} \lesssim 1 + |\Phi(v)| \quad \text{for all } v \in (0, \infty).$$

Thus, we have

$$\|v\|^{\frac{1}{2}+\ell_1} + \left\|v^{-1}\right\|^{\ell_2} \lesssim 1 + \sup_{(t,x)\in[0,T]\times\mathbb{R}} |\Phi(v(t,x))|. \tag{2.32}$$

On the other hand, the *a priori* assumption $v - 1 \in C([0,T]; H^3)$ implies

$$\begin{aligned} |\Phi(v)(t,x)| &= \left| \int_x^\infty \frac{\partial}{\partial x} \Phi(v(t,y)) \mathrm{d}y \right| \\ &\leq \int_{\mathbb{R}} \sqrt{\phi(v(t,y))} \left| \left(\frac{h(v)}{v} v_x \right)(t,y) \right| \mathrm{d}y \\ &\leq \left\| \sqrt{\phi(v(t))} \right\| \left\| \left(\frac{h(v) v_x}{v} \right)(t) \right\|, \end{aligned}$$

which combined with Lemmas 2.1–2.2 and the conditions (2.8) yields

$$|\Phi(v)(t,x)| \lesssim \left\| \left(\theta^{-\alpha} \frac{\mu v_x}{v} \right)(t) \right\| \lesssim 1 + \|\theta\|^{\frac{1}{2}}. \tag{2.33}$$

Combine (2.32) and (2.33) to deduce (2.31). The proof is complete. □

2.2. *Pointwise bounds on temperature*

This part is devoted to obtaining pointwise upper and lower bounds of the temperature $\theta(t,x)$ as well as the estimates on H_x^1-norm of $(v(t,x) - 1, u(t,x), \theta(t,x) - 1)$. We first consider the estimate on the $L_x^2(\mathbb{R})$-norm of $(\theta(t,x) - 1)$ in the following lemma.

Lemma 2.4. *Assume that the conditions listed in Lemma* 2.2 *hold. Then*

$$\sup_{t\in[0,T]} \left[\|(\theta - 1)(t)\|^2 + \|u(t)\|_{L^4}^4 \right] + \int_0^T \int_{\mathbb{R}} \left[\frac{\kappa \theta_x^2}{v} + \theta \frac{\mu u_x^2}{v} + \frac{\mu u^2 u_x^2}{v} \right] \lesssim 1. \tag{2.34}$$

Proof. For each $t \geq 0$ and $a > 1$, define

$$\Omega_a(t) := \{x \in \mathbb{R} : \theta(t,x) > a\}.$$

Multiply (2.5) by $(\theta-2)_+ := \max\{\theta-2,0\}$, and integrate the resulting identity over $[0,t]\times\mathbb{R}$ to find

$$\begin{aligned}\frac{c_v}{2}\int_{\mathbb{R}}(\theta-2)_+^2\mathrm{d}x - \frac{c_v}{2}\int_{\mathbb{R}}(\theta_0-2)_+^2\mathrm{d}x + \int_0^t\int_{\Omega_2(\tau)}\frac{\kappa\theta_x^2}{v}\\ = -\int_0^t\int_{\mathbb{R}}\frac{\theta u_x}{v}(\theta-2)_+ + \int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}(\theta-2)_+.\end{aligned} \tag{2.35}$$

To estimate the last term in this last identity, we multiply $(1.1)_2$ by $2u(\theta-2)_+$ and then integrate the resulting identity over $[0,t]\times\mathbb{R}$ to infer

$$\begin{aligned}\int_{\mathbb{R}}u^2(\theta-2)_+\mathrm{d}x - \int_{\mathbb{R}}u_0^2(\theta_0-2)_+\mathrm{d}x + 2\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}(\theta-2)_+\\ = 2\int_0^t\int_{\mathbb{R}}\frac{\theta}{v}u_x(\theta-2)_+ + \int_0^t\int_{\Omega_2(\tau)}\left[2\frac{\theta}{v}u\theta_x - 2\frac{\mu u u_x}{v}\theta_x + u^2\theta_t\right].\end{aligned} \tag{2.36}$$

Combine (2.35) and (2.36) to get

$$\begin{aligned}\int_{\mathbb{R}}\left[\frac{c_v}{2}(\theta-2)_+^2 + u^2(\theta-2)_+\right]\mathrm{d}x + \int_0^t\int_{\Omega_2(\tau)}\left[\frac{\kappa\theta_x^2}{v} + \frac{\mu u_x^2}{v}(\theta-2)_+\right]\\ = \int_{\mathbb{R}}\left[\frac{c_v}{2}(\theta_0-2)_+^2 + u_0^2(\theta_0-2)_+\right]\mathrm{d}x + \sum_{p=1}^{5}\mathcal{J}_p,\end{aligned} \tag{2.37}$$

where each term $\mathcal{J}_p$ in the decomposition will be defined below. First, we consider the term

$$\mathcal{J}_1 := \int_0^t\int_{\mathbb{R}}\frac{\theta}{v}(\theta-2)_+u_x.$$

We deduce from the condition (1.10) that

$$\|h(v)^{-1}v^{-1}\| + \|h(v)^{-1}v\| \lesssim 1, \tag{2.38}$$

which along with (2.8) implies

$$\|\mu^{-1}v^{-1}\| + \|\mu^{-1}v\| \lesssim 1. \tag{2.39}$$

It follows from Cauchy's inequality and (2.39) that

$$\begin{aligned}|\mathcal{J}_1| &\le \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}(\theta-2)_+ + C(\epsilon)\|\mu^{-1}v^{-1}\|\int_0^t\int_{\Omega_2(\tau)}\theta^2(\theta-2)_+\\ &\le \epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}(\theta-2)_+ + C(\epsilon)\int_0^t\sup_{x\in\mathbb{R}}\left(\theta-\frac{3}{2}\right)_+^2\mathrm{d}\tau.\end{aligned} \tag{2.40}$$

Here we have used

$$\int_{\Omega_2(\tau)}\theta\mathrm{d}x \lesssim \int_{\mathbb{R}}\phi(\theta)\mathrm{d}x \lesssim 1. \tag{2.41}$$

For the terms

$$\mathcal{J}_2 := 2\int_0^t \int_{\Omega_2(\tau)} \frac{\theta}{v} u\theta_x \quad \text{and} \quad \mathcal{J}_3 := -2\int_0^t \int_{\Omega_2(\tau)} \frac{\mu u u_x}{v}\theta_x,$$

we derive from Cauchy's inequality, (2.2), and (2.39) that

$$\begin{aligned}|\mathcal{J}_2| &\le \epsilon \int_0^t \int_{\Omega_2(\tau)} \frac{\kappa\theta_x^2}{v} + C(\epsilon)\|\kappa^{-1}v^{-1}\| \int_0^t \int_{\Omega_2(\tau)} \theta^2 u^2 \\ &\le \epsilon \int_0^t \int_{\Omega_2(\tau)} \frac{\kappa\theta_x^2}{v} + C(\epsilon)\int_0^t \sup_{x\in\mathbb{R}} \left(\theta - \frac{3}{2}\right)_+^2 \mathrm{d}\tau, \end{aligned} \tag{2.42}$$

and

$$|\mathcal{J}_3| \le \epsilon \int_0^t \int_{\Omega_2(\tau)} \frac{\kappa\theta_x^2}{v} + C(\epsilon)\int_0^t \int_{\Omega_2(\tau)} \frac{\mu u^2 u_x^2}{v}. \tag{2.43}$$

For the term

$$\mathcal{J}_4 := \frac{1}{c_v}\int_0^t \int_{\Omega_2(\tau)} u^2 \left[\mu\frac{u_x^2}{v} - \frac{\theta}{v}u_x\right],$$

similar to the estimate for $\mathcal{J}_2$, we have

$$\begin{aligned}|\mathcal{J}_4| &\lesssim \int_0^t \int_{\Omega_2(\tau)} \frac{\mu u^2 u_x^2}{v} + \|\mu^{-1}v^{-1}\| \int_0^t \int_{\Omega_2(\tau)} \theta^2 u^2 \\ &\lesssim \int_0^t \int_{\Omega_2(\tau)} \frac{\mu u^2 u_x^2}{v} + \int_0^t \sup_{x\in\mathbb{R}} \left(\theta - \frac{3}{2}\right)_+^2 \mathrm{d}\tau. \end{aligned} \tag{2.44}$$

For the last term

$$\mathcal{J}_5 := \int_0^t \int_{\Omega_2(\tau)} \frac{1}{c_v} u^2 \left(\frac{\kappa\theta_x}{v}\right)_x,$$

we apply Lebesgue's dominated convergence theorem to find

$$\begin{aligned}\mathcal{J}_5 &= \int_0^t \int_{\mathbb{R}} \lim_{\nu\to 0^+} \varphi_\nu(\theta)\frac{u^2}{c_v}\left(\frac{\kappa\theta_x}{v}\right)_x \\ &= \frac{1}{c_v}\lim_{\nu\to 0^+} \int_0^t \int_{\mathbb{R}} \left[-2\varphi_\nu(\theta)uu_x\frac{\kappa\theta_x}{v} - \varphi_\nu'(\theta)u^2\frac{\kappa\theta_x^2}{v}\right],\end{aligned}$$

where φ_ν is defined by

$$\varphi_\nu(\theta) := \begin{cases} 1, & \theta - 2 \ge \nu, \\ (\theta-2)/\nu, & 0 \le \theta - 2 < \nu, \\ 0, & \theta - 2 < 0. \end{cases}$$

Hence

$$\begin{aligned}
\mathcal{J}_5 &\le -\frac{2}{c_v}\lim_{\nu\to 0^+}\int_0^t\int_{\mathbb{R}}\varphi_\nu(\theta)uu_x\frac{\kappa\theta_x}{v} \\
&\le -\frac{2}{c_v}\int_0^t\int_{\Omega_2(\tau)}\frac{\kappa uu_x\theta_x}{v} \\
&\le \epsilon\int_0^t\int_{\Omega_2(\tau)}\frac{\kappa\theta_x^2}{v}+C(\epsilon)\int_0^t\int_{\Omega_2(\tau)}\frac{\mu u^2u_x^2}{v}.
\end{aligned}\tag{2.45}$$

Plug the estimates (2.40)–(2.45) into (2.37) to infer

$$\begin{aligned}
&\int_{\mathbb{R}}(\theta-2)_+^2\mathrm{d}x+\int_0^t\int_{\Omega_2(\tau)}\left[\frac{\kappa\theta_x^2}{v}+\frac{\mu u_x^2}{v}(\theta-2)_+\right] \\
&\quad\lesssim 1+\int_0^t\sup_{x\in\mathbb{R}}\left(\theta-\frac{3}{2}\right)_+^2\mathrm{d}\tau+\int_0^t\int_{\Omega_2(\tau)}\frac{\mu u^2u_x^2}{v}.
\end{aligned}\tag{2.46}$$

It follows from (2.2) that

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v} &= \int_0^t\left[\int_{\Omega_2(\tau)}+\int_{\mathbb{R}\backslash\Omega_2(\tau)}\right]\frac{\kappa\theta_x^2}{v} \\
&\le \int_0^t\int_{\Omega_2(\tau)}\frac{\kappa\theta_x^2}{v}+C\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v\theta^2} \\
&\lesssim 1+\int_0^t\int_{\Omega_2(\tau)}\frac{\kappa\theta_x^2}{v},
\end{aligned}\tag{2.47}$$

and

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}}\theta\frac{\mu u_x^2}{v} &= \int_0^t\left[\int_{\Omega_3(\tau)}+\int_{\mathbb{R}\backslash\Omega_3(\tau)}\right]\theta\frac{\mu u_x^2}{v} \\
&\lesssim \int_0^t\int_{\Omega_2(\tau)}\frac{\mu u_x^2}{v}(\theta-2)_++\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v\theta} \\
&\lesssim 1+\int_0^t\int_{\Omega_2(\tau)}\frac{\mu u_x^2}{v}(\theta-2)_+.
\end{aligned}\tag{2.48}$$

Insert (2.47) and (2.48) into (2.46) to discover

$$\begin{aligned}
&\int_{\mathbb{R}}(\theta-2)_+^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\left[\frac{\kappa\theta_x^2}{v}+\theta\frac{\mu u_x^2}{v}\right] \\
&\quad\lesssim 1+\int_0^t\sup_{x\in\mathbb{R}}\left(\theta-\frac{3}{2}\right)_+^2\mathrm{d}\tau+\int_0^t\int_{\Omega_2(\tau)}\frac{\mu u^2u_x^2}{v}.
\end{aligned}\tag{2.49}$$

In order to estimate the last term in (2.49), we multiply $(1.1)_2$ by u^3 to have

$$\left(\frac{1}{4}u^4\right)_t+\left[u^3\left(\frac{\theta}{v}-1\right)-u^3\frac{\mu u_x}{v}\right]_x=3u^2u_x\left[\frac{\theta}{v}-1-\frac{\mu u_x}{v}\right].$$

Integrate the above identity over $[0,t]\times\mathbb{R}$ to obtain

$$\int_{\mathbb{R}} u^4 \mathrm{d}x + \int_0^t \int_{\mathbb{R}} \frac{\mu u^2 u_x^2}{v} \lesssim 1 + \sum_{p=1}^{4} \mathcal{I}_p, \tag{2.50}$$

where each term $\mathcal{I}_p$ in the decomposition will be defined and estimated as follows. First we consider the term

$$\mathcal{I}_1 := \int_0^t \int_{\Omega_2(\tau)} u^2 u_x \frac{\theta - 1}{v}.$$

Applying Cauchy's inequality, (2.2), and (2.39), we get

$$\begin{aligned} |\mathcal{I}_1| &\le \nu \int_0^t \int_{\mathbb{R}} \frac{\mu u^2 u_x^2}{v} + C(\nu)\|\!|\mu^{-1}v^{-1}|\!\| \int_0^t \int_{\Omega_2(\tau)} (\theta-1)^2 u^2 \\ &\le \nu \int_0^t \int_{\mathbb{R}} \frac{\mu u^2 u_x^2}{v} + C(\nu) \int_0^t \sup_{x\in\mathbb{R}} \left(\theta - \frac{3}{2}\right)_+^2 \mathrm{d}\tau. \end{aligned} \tag{2.51}$$

For the terms

$$\mathcal{I}_2 := \int_0^t \int_{\mathbb{R}\setminus\Omega_2(\tau)} u^2 u_x \frac{\theta-1}{v} \quad \text{and} \quad \mathcal{I}_3 := \int_0^t \int_{v\le 2} u^2 u_x \frac{1-v}{v},$$

we have from (2.7) and (2.2) that

$$\int_{\mathbb{R}\setminus\Omega_2(\tau)} (\theta-1)^2 \mathrm{d}x + \int_{v(\tau,x)\le 2} (v-1)^2 \mathrm{d}x \lesssim \int_{\mathbb{R}} \eta(v,u,\theta)\mathrm{d}x \lesssim 1.$$

In view of Hölder's inequality and (2.2), we deduce

$$\mathcal{I}_2 + \mathcal{I}_3 \lesssim \int_0^t \|u\|_{L^\infty}^2 \left\|\frac{u_x}{v}\right\| \lesssim \int_0^t \|u_x\| \left\|\frac{u_x}{v}\right\| \lesssim \int_0^t \int_{\mathbb{R}} u_x^2 + \int_0^t \int_{\mathbb{R}} \frac{u_x^2}{v^2}.$$

Applying Cauchy's inequality again, we infer from (2.2) and (2.39) that

$$\begin{aligned} \mathcal{I}_2 + \mathcal{I}_3 &\lesssim \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\theta u_x^2}{v} + C(\epsilon) \int_0^t \int_{\mathbb{R}} \left[\frac{v u_x^2}{\mu\theta} + \frac{u_x^2}{v^3 \mu\theta}\right] \\ &\lesssim \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\theta u_x^2}{v} + C(\epsilon)\|\!|\mu^{-2}v^2 + \mu^{-2}v^{-2}|\!\| \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v\theta} \\ &\lesssim \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\theta u_x^2}{v} + C(\epsilon). \end{aligned} \tag{2.52}$$

Let us now consider the term

$$\mathcal{I}_4 := \int_0^t \int_{v\ge 2} u^2 u_x \frac{1-v}{v}.$$

In view of (2.7), we obtain that $\left|\frac{1-v}{v}\right| \lesssim \sqrt{\phi(v)}$ for all $v \ge 2$, which combined with (2.2) implies

$$\mathcal{I}_4 \lesssim \int_0^t \int_{v\ge 2} u^2 |u_x| \sqrt{\phi(v)} \lesssim \int_0^t \|u\|_{L^\infty}^2 \|u_x\| \|\sqrt{\phi(v)}\| \lesssim \int_0^t \|u_x\|^2.$$

Hence we have from (2.2) and (2.39) that

$$\mathcal{I}_4 \lesssim \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\theta u_x^2}{v} + C(\epsilon)\left\|\mu^{-2}v^2\right\| \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v\theta} \lesssim \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\theta u_x^2}{v} + C(\epsilon). \tag{2.53}$$

Insert (2.51)–(2.53) into (2.50) and let $\nu > 0$ suitably small to derive

$$\int_{\mathbb{R}} u^4 \mathrm{d}x + \int_0^t \int_{\mathbb{R}} \frac{\mu u^2 u_x^2}{v} \lesssim C(\epsilon) + \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\theta u_x^2}{v} + \int_0^t \sup_{x\in\mathbb{R}} \left(\theta - \frac{3}{2}\right)_+^2 \mathrm{d}\tau. \tag{2.54}$$

Combining (2.49) and (2.54), we take $\epsilon > 0$ small enough to get

$$\begin{aligned} &\int_{\mathbb{R}} \left[(\theta-2)_+^2 + u^4\right] \mathrm{d}x + \int_0^t \int_{\mathbb{R}} \left[\frac{\kappa\theta_x^2}{v} + \theta\frac{\mu u_x^2}{v} + \frac{\mu u^2 u_x^2}{v}\right] \\ &\quad \lesssim 1 + \int_0^t \sup_{x\in\mathbb{R}} \left(\theta - \frac{3}{2}\right)_+^2 \mathrm{d}\tau. \end{aligned} \tag{2.55}$$

In light of the fundamental theorem of calculus and (2.41), we infer

$$\begin{aligned} \int_0^t \sup_{x\in\mathbb{R}} \left(\theta - \frac{3}{2}\right)_+^2 \mathrm{d}\tau &\lesssim \int_0^t \int_{\Omega_{3/2}(\tau)} \frac{\theta_x^2}{\theta} \\ &\lesssim \delta \int_0^t \int_{\mathbb{R}} \frac{\kappa\theta_x^2}{v} + C(\delta) \int_0^t \int_{\mathbb{R}} \frac{v\theta_x^2}{\kappa\theta^2} \\ &\lesssim \delta \int_0^t \int_{\mathbb{R}} \frac{\kappa\theta_x^2}{v} + C(\delta) \int_0^t \int_{\mathbb{R}} \frac{\kappa\theta_x^2}{v\theta^2} \left\|\frac{v^2}{\kappa^2}\right\| \\ &\lesssim \delta \int_0^t \int_{\mathbb{R}} \frac{\kappa\theta_x^2}{v} + C(\delta). \end{aligned} \tag{2.56}$$

If we plug this last inequality into (2.55) and choose $\delta > 0$ sufficiently small, then we can deduce (2.34) and hence finish the proof of the lemma. □

The next lemma concerns the estimate for the first-order derivative with respect to x of $v(t,x)$.

Lemma 2.5. *Assume that the conditions listed in Lemma* 2.2 *hold. Then*

$$\sup_{t\in[0,T]} \left\|\frac{\mu v_x}{v}(t)\right\|^2 + \int_0^T \int_{\mathbb{R}} \frac{\mu\theta v_x^2}{v^3} \lesssim 1. \tag{2.57}$$

Proof. Applying Cauchy's inequality, we deduce from (2.29), (2.2) and (2.34) that

$$\left\|\frac{\mu v_x}{v}(t)\right\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\mu\theta v_x^2}{v^3} \lesssim 1 + \int_0^t \int_{\mathbb{R}} \left[\frac{\mu\theta u_x^2}{v} + \frac{\mu u_x^2}{v\theta} + \frac{\mu\theta_x^2}{v} + \frac{\mu\theta_x^2}{v\theta^2}\right] \lesssim 1.$$

The proof of the lemma is complete. □

For the estimate on the first-order derivative of $u(t,x)$, we have

Lemma 2.6. *Assume that the conditions listed in Lemma* 2.2 *hold. Then*

$$\sup_{t\in[0,T]}\|u_x(t)\|^2+\int_0^T\int_{\mathbb{R}}\frac{\mu u_{xx}^2}{v}\lesssim 1+|\!|\!|\theta|\!|\!|. \tag{2.58}$$

Proof. Multiply $(1.1)_2$ by u_{xx} to get

$$\left(\frac{1}{2}u_x^2\right)_t-(u_xu_t)_x-\left(\frac{\theta}{v}\right)_x u_{xx}=-\frac{\mu u_{xx}^2}{v}+\frac{\mu u_{xx}v_xu_x}{v^2}-\frac{\mu_xu_xu_{xx}}{v}.$$

We integrate the above identity over $[0,t]\times\mathbb{R}$ and apply Cauchy's inequality to have

$$\|u_x(t)\|^2+\int_0^t\int_{\mathbb{R}}\frac{\mu u_{xx}^2}{v}\lesssim 1+\left|\!\left|\!\left|\frac{1}{\mu\kappa}\right|\!\right|\!\right|\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v}+\left|\!\left|\!\left|\frac{\theta}{\mu^2}\right|\!\right|\!\right|\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}$$
$$+\left|\!\left|\!\left|\frac{1}{\mu v}\right|\!\right|\!\right|\int_0^t\|u_x\|_{L^\infty}^2\left\|\frac{\mu v_x}{v}\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\mu_x^2u_x^2}{\mu v}.$$

In view of (1.10) and (2.8), we obtain

$$\left|\!\left|\!\left|\frac{1}{\mu\kappa}\right|\!\right|\!\right|+\left|\!\left|\!\left|\frac{1}{\mu v}\right|\!\right|\!\right|\lesssim 1,$$

which combined with (2.57) and (2.34) yields

$$\|u_x(t)\|^2+\int_0^t\int_{\mathbb{R}}\frac{\mu u_{xx}^2}{v}\lesssim 1+|\!|\!|\theta|\!|\!|+\int_0^t\|u_x\|_{L^\infty}^2+\int_0^t\int_{\mathbb{R}}\frac{\mu_x^2u_x^2}{\mu v}. \tag{2.59}$$

We have from (2.39) that

$$\begin{aligned}\int_0^t\|u_x\|_{L^\infty}^2&\lesssim\int_0^t\|u_x\|\|u_{xx}\|\\&\lesssim\epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu u_{xx}^2}{v}+C(\epsilon))\left|\!\left|\!\left|\frac{v}{\mu}\right|\!\right|\!\right|^2\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}\\&\lesssim\epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu u_{xx}^2}{v}+C(\epsilon).\end{aligned} \tag{2.60}$$

For the last term on the right-hand side of (2.59), we use (2.8), (1.10), (2.2), (2.57), and (2.60) to discover

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\frac{\mu_x^2u_x^2}{\mu v}&\le\int_0^t\int_{\mathbb{R}}\frac{\mu_v^2v_x^2u_x^2}{\mu v}+\int_0^t\int_{\mathbb{R}}\frac{\mu_\theta^2\theta_x^2u_x^2}{\mu v}\\&\le\left|\!\left|\!\left|\frac{\mu_v^2v}{\mu^3}\right|\!\right|\!\right|\int_0^t\|u_x\|_{L^\infty}^2\left\|\frac{\mu v_x}{v}\right\|^2+\left|\!\left|\!\left|\frac{\mu_\theta^2\theta^2u_x^2}{\mu}\right|\!\right|\!\right|\int_0^t\int_{\mathbb{R}}\frac{\mu\theta_x^2}{v\theta^2}\\&\le\left|\!\left|\!\left|\frac{h'(v)^2v}{h(v)^3}\right|\!\right|\!\right|\int_0^t\|u_x\|_{L^\infty}^2+\alpha^2N^2\int_0^t\int_{\mathbb{R}}\frac{\mu\theta_x^2}{v\theta^2}\\&\lesssim\epsilon\int_0^t\int_{\mathbb{R}}\frac{\mu u_{xx}^2}{v}+C(\epsilon)+1.\end{aligned} \tag{2.61}$$

Plug (2.60) and (2.61) into (2.59) and choose ϵ small enough to derive (2.58). □

We now turn to deduce an upper bound on the temperature $\theta(t,x)$.

Lemma 2.7. *Assume that the conditions listed in Lemma* 2.2 *hold. Then there exist positive constants* C_i $(i=1,2,3)$, *which depend only on* Π_0, V_0, *and* $H(V_0)$, *such that for all* $(t,x)\in[0,T]\times\mathbb{R}$,

$$\theta(t,x)\le C_1, \tag{2.62}$$

$$C_2\le v(t,x)\le C_2^{-1}, \tag{2.63}$$

$$|(v-1,u,\theta-1)(t)\|_1^2+\int_0^t\left[\left\|\sqrt{\theta}v_x(s)\right\|^2+\|(\theta_x,u_x)(s)\|_1^2\right]\mathrm{d}s\le C_3^2. \tag{2.64}$$

Proof. Multiply (2.5) by θ_{xx} and integrate the resulting identity to find

$$\begin{aligned}&\frac{c_v}{2}\|\theta_x(t)\|^2-\frac{c_v}{2}\|\theta_{0x}\|^2+\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}\\&\quad=\int_0^t\int_{\mathbb{R}}\left[\frac{\kappa v_x\theta_x}{v^2}\theta_{xx}+\frac{\theta}{v}u_x\theta_{xx}-\frac{\mu u_x^2}{v}\theta_{xx}-\frac{\kappa_x\theta_x}{v}\theta_{xx}\right].\end{aligned} \tag{2.65}$$

Next we estimate each term in (2.65). First,

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}\frac{\kappa v_x\theta_x}{v^2}\theta_{xx}\right|&\lesssim\int_0^t\|\theta_x\|_{L^\infty}\left\|\sqrt{\frac{\kappa}{v}}\theta_{xx}\right\|\left\|\sqrt{\frac{\kappa}{v}}\frac{v_x}{v}\right\|\\&\lesssim\left\|\frac{1}{\sqrt{\kappa v}}\right\|\int_0^t\|\theta_x\|^{\frac12}\|\theta_{xx}\|^{\frac12}\left\|\sqrt{\frac{\kappa}{v}}\theta_{xx}\right\|\left\|\frac{\kappa v_x}{v}\right\|\\&\lesssim\left\|\frac{v}{\kappa}\right\|^{\frac14}\left\|\frac{1}{\kappa v}\right\|^{\frac12}\int_0^t\|\theta_x\|^{\frac12}\left\|\sqrt{\frac{\kappa}{v}}\theta_{xx}\right\|^{\frac32}\left\|\frac{\kappa v_x}{v}\right\|.\end{aligned} \tag{2.66}$$

In light of Young's inequality, we combine (2.66), (2.39), (2.34), and (2.57) to get

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}\frac{\kappa v_x\theta_x}{v^2}\theta_{xx}\right|&\lesssim\epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}+C(\epsilon)\int_0^t\|\theta_x\|^2\left\|\frac{\kappa v_x}{v}\right\|^4\\&\lesssim\epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}+C(\epsilon)\sup_{[0,T]}\left\|\frac{\mu v_x}{v}\right\|^4\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v}\\&\lesssim\epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}+C(\epsilon),\end{aligned} \tag{2.67}$$

and

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}\frac{\theta}{v}u_x\theta_{xx}\right|&\lesssim\|\theta\|\int_0^t\left\|\sqrt{\frac{\kappa}{v}}\theta_{xx}\right\|\left\|\frac{1}{\sqrt{\kappa v}}u_x\right\|\\&\lesssim\epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}+C(\epsilon)\|\theta\|^2\left\|\frac{1}{\kappa\mu}\right\|\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}\\&\lesssim\epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v}+C(\epsilon)\|\theta\|^2.\end{aligned} \tag{2.68}$$

Using Hölder's inequality, we have from (2.39) that

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}\theta_{xx}\right| &\lesssim \int_0^t \|u_x\|_{L^\infty}\left\|\sqrt{\frac{\mu}{v}}u_x\right\|\left\|\sqrt{\frac{\mu}{v}}\theta_{xx}\right\| \\ &\lesssim \int_0^t \|u_x\|^{\frac{1}{2}}\|u_{xx}\|^{\frac{1}{2}}\left\|\sqrt{\frac{\mu}{v}}u_x\right\|\left\|\sqrt{\frac{\mu}{v}}\theta_{xx}\right\| \\ &\lesssim \sup_{[0,T]}\|u_x\|\int_0^t\left\|\sqrt{\frac{\mu}{v}}u_{xx}\right\|^{\frac{1}{2}}\left\|\sqrt{\frac{\mu}{v}}u_x\right\|^{\frac{1}{2}}\left\|\sqrt{\frac{\mu}{v}}\theta_{xx}\right\|,\end{aligned}$$

which combined with (2.2), (2.34), and (2.58) implies

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}\theta_{xx}\right| &\lesssim \epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v} + C(\epsilon)\sup_{[0,T]}\|u_x\|^2\int_0^t\int_{\mathbb{R}}\left[\frac{\mu u_x^2}{v}+\frac{\mu u_{xx}^2}{v}\right] \\ &\lesssim \epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v} + C(\epsilon)(1+\|\theta\|)^2. \end{aligned}\tag{2.69}$$

For the last term on the right-hand side of (2.65), we have

$$\left|\int_0^t\int_{\mathbb{R}}\frac{\kappa_x\theta_x}{v}\theta_{xx}\right| \le \left|\int_0^t\int_{\mathbb{R}}\frac{\kappa_\theta\theta_x^2}{v}\theta_{xx}\right| + \left|\int_0^t\int_{\mathbb{R}}\frac{\kappa_v v_x\theta_x}{v}\theta_{xx}\right|.$$

We obtain from (2.2) and (2.8) that

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}\frac{\kappa_\theta\theta_x^2}{v}\theta_{xx}\right| &\lesssim \epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v} + C(\epsilon)\alpha^2\|\theta_x\|^2\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v\theta^2} \\ &\lesssim \epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v} + C(\epsilon). \end{aligned}\tag{2.70}$$

In view of (1.10), (2.38), (2.57), and (2.34), we infer

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}\frac{\kappa_v v_x\theta_x}{v}\theta_{xx}\right| &\lesssim \left\|\frac{\kappa_v}{\mu}\sqrt{\frac{v}{\kappa}}\right\|\int_0^t\|\theta_x\|_{L^\infty}\left\|\sqrt{\frac{\kappa}{v}}\theta_{xx}\right\|\left\|\frac{\mu v_x}{v}\right\| \\ &\lesssim \int_0^t\left\|\sqrt{\frac{\kappa}{v}}\theta_{xx}\right\|^{\frac{3}{2}}\left\|\sqrt{\frac{\kappa}{v}}\theta_x\right\|^{\frac{1}{2}}\left\|\frac{\mu v_x}{v}\right\| \\ &\lesssim \epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v} + C(\epsilon)\int_0^t\left\|\sqrt{\frac{\kappa}{v}}\theta_x\right\|^2\left\|\frac{\mu v_x}{v}\right\|^4 \\ &\lesssim \epsilon\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v} + C(\epsilon). \end{aligned}\tag{2.71}$$

We plug (2.67)–(2.71) into (2.65), and take $\epsilon > 0$ suitably small to derive

$$\|\theta_x(t)\|^2 + \int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_{xx}^2}{v} \lesssim 1 + \|\theta\|^2. \tag{2.72}$$

Combining (2.34) and (2.72) gives

$$\|\theta\|^2 = \sup_{t\in[0,T]}\|\theta(t)\|_{L^\infty}^2 \le \sup_{t\in[0,T]}\|\theta(t)\|\|\theta_x(t)\| \lesssim 1 + \|\theta\|.$$

Apply Cauchy's inequality to the last inequality to obtain (2.62). We then derive (2.63) by plugging (2.62) into (2.31).

Insert (2.62) into (2.58) and (2.72) to give

$$\|(u_x, \theta_x)(t)\|^2 + \int_0^t \int_{\mathbb{R}} \left[\frac{\mu u_{xx}^2}{v} + \frac{\kappa \theta_{xx}^2}{v} \right] \lesssim 1. \tag{2.73}$$

In view of (2.63) and (2.8), we can obtain (2.64) from (2.2), (2.34), (2.57), and (2.73). The proof of the lemma is finished. □

We present a local-in-time lower bound for the temperature $\theta(t,x)$ in the following lemma.

Lemma 2.8. *Assume that the conditions listed in Lemma* 2.2 *hold. Then there exist positive constant* C_4 *depending only on* Π_0, V_0, *and* $H(V_0)$ *such that*

$$\inf_{\mathbb{R}} \theta(t,\cdot) \geq \frac{\inf_{\mathbb{R}} \theta(s,\cdot)}{C_4 \inf_{\mathbb{R}} \theta(s,\cdot)(t-s) + 1} \quad \text{for all } 0 \leq s \leq t \leq T. \tag{2.74}$$

Proof. Multiply (2.5) by θ^{-2} to have

$$c_v \left(\frac{1}{\theta}\right)_t = \left[\frac{\kappa}{v}\left(\frac{1}{\theta}\right)_x\right]_x - \frac{2\theta\kappa}{v}\left|\left(\frac{1}{\theta}\right)_x\right|^2 - \frac{\mu}{v\theta^2}\left[u_x - \frac{\theta}{2\mu}\right]^2 + \frac{1}{4\mu v}.$$

In view of (2.8) and (2.63), we deduce that

$$c_v \left(\frac{1}{\theta}\right)_t \leq \left[\frac{\kappa}{v}\left(\frac{1}{\theta}\right)_x\right]_x + C_4,$$

for some positive constant C_4, depending only on Π_0, V_0, and $H(V_0)$.

Let $s \in [0,T]$ be fixed and define

$$H(t,x) := \frac{1}{\theta(t,x)} - C_4(t-s).$$

Then we derive that H satisfies

$$\begin{cases} c_v H_t \leq \left(\dfrac{\kappa}{v} H_x\right)_x & \text{for } (t,x) \in (s,T] \times \mathbb{R}, \\ H(s,x) = \dfrac{1}{\theta(s,x)} \leq \dfrac{1}{\inf_{\mathbb{R}} \theta(s,\cdot)} & \text{for } x \in \mathbb{R}. \end{cases}$$

Employing the maximum principle (see Ref. 9), we infer that

$$H(t,x) \leq \frac{1}{\inf_{\mathbb{R}} \theta(s,\cdot)} \quad \text{for all } (t,x) \in [s,T] \times \mathbb{R},$$

which implies (2.74). The proof is complete. □

2.3. *Estimates of second-order derivatives*

In Secs. 2.3 and 2.4, to simplify the presentation, we introduce $A \lesssim_h B$ if $A \leq C_h B$ holds uniformly for some constant C_h, depending only on Π_0, V_0, and $H(C_2)$ with C_2 given in Lemma 2.7. The letter $C(m_2)$ will be employed to denote some positive constant which depends only on m_2, Π_0, V_0, and $H(C_2)$. We note from (1.11) and (2.63) that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |(h(v(t,x)), h'(v(t,x)), h''(v(t,x)), h'''(v(t,x)))| \leq H(C_2). \tag{2.75}$$

We estimate the second-order derivatives of $(u(t,x), \theta(t,x))$ in the next lemma.

Lemma 2.9. *Assume that the conditions listed in Lemma* 2.2 *hold. Then*

$$\sup_{t\in[0,T]} \|(u_{xx}, \theta_{xx})(t)\|^2 + \int_0^T \|(u_{xxx}, \theta_{xxx})(t)\|^2 \mathrm{d}t$$
$$\lesssim_h C(m_2) + \int_0^T \|v_{xx}(t)\|^2 \mathrm{d}t + \sup_{t\in[0,T]} \|v_{xx}(t)\|^2. \tag{2.76}$$

Proof. The proof is divided into the following steps:

Step 1. Differentiating $(1.1)_2$ with respect to x, and multiplying the resulting identity by u_{xxx} give

$$\left[\frac{1}{2}u_{xx}^2\right]_t - [u_{xt}u_{xx}]_x + \frac{\mu u_{xxx}^2}{v} = P_{xx}u_{xxx} + \left[\frac{\mu u_{xxx}}{v} - \left(\frac{\mu u_x}{v}\right)_{xx}\right] u_{xxx}.$$

Integrate the above identity over $[0,t] \times \mathbb{R}$, and use (2.63), (2.8), and Cauchy's inequality to obtain

$$\|u_{xx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} u_{xxx}^2 \lesssim 1 + \int_0^t \int_{\mathbb{R}} |P_{xx}|^2 + \int_0^t \int_{\mathbb{R}} \left|\frac{\mu u_{xxx}}{v} - \left(\frac{\mu u_x}{v}\right)_{xx}\right|^2. \tag{2.77}$$

We next make the estimates for the terms on the right-hand side of (2.77). In light of (2.63), we deduce for general smooth function $f(v)$ that

$$\begin{cases} |f(v)_x| \lesssim |f'(v)||v_x|, \\ |f(v)_{xx}| \lesssim |(f', f'')(v)|(|v_{xx}| + v_x^2), \\ |f(v)_{xxx}| \lesssim |(f', f'', f''')(v)|(|v_{xxx}| + |v_{xx}|v_x^2 + |v_x|^3). \end{cases} \tag{2.78}$$

Hence by using (2.62) and

$$P_{xx} = \frac{\theta_{xx}}{v} + 2\theta_x \left(\frac{1}{v}\right)_x + \theta \left(\frac{1}{v}\right)_{xx}, \tag{2.79}$$

we infer

$$|P_{xx}|^2 \lesssim |(v_{xx}, \theta_{xx})|^2 + |(v_x, \theta_x)|^4.$$

From (2.64) and (2.1), we have

$$\int_0^T \|v_x\|^2 \le C(m_2),$$

which combined with (2.64) implies

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} |(v_x,u_x,\theta_x)|^4 &\lesssim \int_0^t \|(v_{xx},u_{xx},\theta_{xx})\|\|(v_x,u_x,\theta_x)\|^3\\ &\lesssim \sup_{[0,T]}\|(v_x,u_x,\theta_x)\|^2\int_0^t \|(v_x,u_x,\theta_x)\|_1^2\\ &\lesssim C(m_2)+\int_0^t \|v_{xx}\|^2. \end{aligned}\tag{2.80}$$

Consequently, we have

$$\int_0^t\int_{\mathbb{R}} |P_{xx}|^2 \lesssim \int_0^t \|v_{xx}\|^2 + C(m_2). \tag{2.81}$$

To estimate the last term in (2.77), we first make some estimate of θ^α. It follows from (2.8) that

$$\begin{cases}|(\theta^\alpha)_x| \lesssim |\theta_x|,\\ |(\theta^\alpha)_{xx}| \lesssim |\theta_{xx}|+\theta_x^2,\\ |(\theta^\alpha)_{xxx}| \lesssim |\theta_{xxx}|+|\theta_{xx}|\theta_x^2+|\theta_x|^3,\end{cases}\tag{2.82}$$

which combined with (2.78) yields

$$\begin{cases}|(f(v)\theta^\alpha)_x| \lesssim |(f,f')(v)||(v_x,\theta_x)|,\\ |(f(v)\theta^\alpha)_{xx}| \lesssim |(f,f',f'')(v)|[|(v_{xx},\theta_{xx})|+|(v_x,\theta_x)|^2],\\ |(f(v)\theta^\alpha)_{xxx}| \lesssim |(f,f',f'',f''')(v)|[|(v_{xxx},\theta_{xxx})|\\ \qquad\qquad +|(v_{xx},\theta_{xx})||(v_x,\theta_x)|^2+|(v_x,\theta_x)|^3].\end{cases}\tag{2.83}$$

Taking $f(v)=h(v)/v$, we can combine the identity

$$\left(\frac{\mu u_x}{v}\right)_{xx} = \left(\frac{\mu}{v}\right)_{xx} u_x + 2\left(\frac{\mu}{v}\right)_x u_{xx} + \frac{\mu}{v}u_{xxx}$$

and (2.75) to conclude

$$\left|\left(\frac{\mu u_x}{v}\right)_{xx} - \frac{\mu u_{xxx}}{v}\right| \lesssim_h |(v_x,u_x,\theta_x)|^3 + |v_{xx}||u_x| + |(u_{xx},\theta_{xx})||(v_x,u_x,\theta_x)|.$$

From this estimate, we derive

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\left|\frac{\mu u_{xxx}}{v} - \left(\frac{\mu u_x}{v}\right)_{xx}\right|^2 &\lesssim_h \int_0^t\int_{\mathbb{R}} |(v_x,u_x,\theta_x)|^6 + \int_0^t\int_{\mathbb{R}} v_{xx}^2u_x^2\\ &\quad + \int_0^t\int_{\mathbb{R}} |(u_{xx},\theta_{xx})|^2|(v_x,u_x,\theta_x)|^2.\end{aligned}\tag{2.84}$$

Employ Sobolev's inequality and (2.64) to get

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}|(v_x,u_x,\theta_x)|^6 &\lesssim \int_0^t \|(v_x,u_x,\theta_x)\|_{L^\infty}^4\|(v_x,u_x,\theta_x)\|^2\\ &\lesssim \int_0^t \|(v_{xx},u_{xx},\theta_{xx})\|^2\|(v_x,u_x,\theta_x)\|^4\\ &\lesssim 1+\int_0^t\|v_{xx}\|^2, \end{aligned}\tag{2.85}$$

$$\int_0^t\int_{\mathbb{R}} v_{xx}^2u_x^2 \le \sup_{[0,t]}\|v_{xx}\|^2\int_0^t\|u_x\|_1^2 \le \sup_{[0,t]}\|v_{xx}\|^2, \tag{2.86}$$

and

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}|(u_{xx},\theta_{xx})|^2|(v_x,u_x,\theta_x)|^2 &\lesssim \int_0^t \|(u_{xx},\theta_{xx})\|_{L^\infty}^2\|(v_x,u_x,\theta_x)\|^2\\ &\lesssim \int_0^t \|(u_{xx},\theta_{xx})\|\|(u_{xxx},\theta_{xxx})\|\\ &\lesssim C(\delta)+\delta\int_0^t\|(u_{xxx},\theta_{xxx})\|^2. \end{aligned}\tag{2.87}$$

We plug (2.81) and (2.84) into (2.77), and use (2.85)–(2.87) to have

$$\begin{aligned}&\|u_{xx}(t)\|^2+\int_0^t\int_{\mathbb{R}}u_{xxx}^2\\ &\quad\lesssim_h C(m_2)+\int_0^t\|v_{xx}\|^2+\sup_{[0,t]}\|v_{xx}\|^2+\delta\int_0^t\|(u_{xxx},\theta_{xxx})\|^2. \end{aligned}\tag{2.88}$$

Step 2. Next, we differentiate (2.5) with respect to x and multiply the result by θ_{xxx} to find

$$\begin{aligned}&\left[\frac{c_v}{2}\theta_{xx}^2\right]_t-[c_v\theta_{xt}\theta_{xx}]_x+\frac{\kappa\theta_{xxx}^2}{v}\\ &\quad=(Pu_x)_x\theta_{xxx}-\left(\frac{\mu u_x^2}{v}\right)_x\theta_{xxx}+\left[\frac{\kappa\theta_{xxx}}{v}-\left(\frac{\kappa\theta_x}{v}\right)_{xx}\right]\theta_{xxx}.\end{aligned}$$

Integrating this last identity over $[0,t]\times\mathbb{R}$, we obtain from Cauchy's inequality, (2.63) and (2.8) that

$$\begin{aligned}\|\theta_{xx}(t)\|^2+\int_0^t\int_{\mathbb{R}}\theta_{xxx}^2 &\lesssim 1+\int_0^t\int_{\mathbb{R}}|(Pu_x)_x|^2+\int_0^t\int_{\mathbb{R}}\left|\left(\frac{\mu u_x^2}{v}\right)_x\right|^2\\ &\quad+\int_0^t\int_{\mathbb{R}}\left|\frac{\kappa\theta_{xxx}}{v}-\left(\frac{\kappa\theta_x}{v}\right)_{xx}\right|^2. \end{aligned}\tag{2.89}$$

We estimate the terms on the right-hand side of (2.89) below. First it follows from (2.62) and (2.63) that

$$|(Pu_x)_x|\lesssim|P_xu_x|+|Pu_{xx}|\lesssim|(v_x,u_x,\theta_x)|^2+|u_{xx}|,$$

which along with (2.64) and (2.80) implies

$$\int_0^t \int_{\mathbb{R}} |(Pu_x)_x|^2 \lesssim \int_0^t \int_{\mathbb{R}} [|(v_x, u_x, \theta_x)|^4 + u_{xx}^2] \lesssim C(m_2) + \int_0^t \|v_{xx}\|^2. \tag{2.90}$$

We deduce from (2.83) with $f(v) = h(v)/v$ that

$$\left| \left(\frac{\mu u_x^2}{v} \right)_x \right| = \left| \left(\frac{\mu}{v} \right)_x u_x^2 + \frac{2\mu u_x u_{xx}}{v} \right| \lesssim_h |(v_x, u_x, \theta_x)|^3 + |u_x u_{xx}|.$$

In light of (2.85) and (2.87), we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \left| \left(\frac{\mu u_x^2}{v} \right)_x \right|^2 &\lesssim_h \int_0^t \int_{\mathbb{R}} [|(v_x, u_x, \theta_x)|^6 + u_x^2 u_{xx}^2] \\ &\lesssim_h C(\delta) + \int_0^t \|v_{xx}\|^2 + \delta \int_0^t \|(u_{xxx}, \theta_{xxx})\|^2. \end{aligned} \tag{2.91}$$

For the last term in (2.89), we deduce by applying the argument in Step 1 that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \left| \frac{\kappa \theta_{xxx}}{v} - \left(\frac{\kappa \theta_x}{v} \right)_{xx} \right|^2 \\ &\quad \lesssim_h C(\delta) + \int_0^t \|v_{xx}\|^2 + \sup_{[0,t]} \|v_{xx}\|^2 + \delta \int_0^t \|(u_{xxx}, \theta_{xxx})\|^2. \end{aligned} \tag{2.92}$$

Plug (2.90)–(2.92) into (2.89) to get

$$\begin{aligned} &\|\theta_{xx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta_{xxx}^2 \\ &\quad \lesssim_h C(m_2) + \int_0^t \|v_{xx}\|^2 + \sup_{[0,t]} \|v_{xx}\|^2 + \delta \int_0^t \|(u_{xxx}, \theta_{xxx})\|^2. \end{aligned} \tag{2.93}$$

Combining (2.88) and (2.93), we take δ small enough to prove (2.76). This completes the proof. □

We next obtain a m_2-dependent bound for the second-order derivatives with respect to x of the solution $(v(t,x), u(t,x), \theta(t,x))$.

Lemma 2.10. *Assume that the conditions listed in Lemma* 2.2 *hold. Then*

$$\sup_{t\in[0,T]} \|(v_{xx}, u_{xx}, \theta_{xx})(t)\|^2 + \int_0^T \|(v_{xx}, u_{xxx}, \theta_{xxx})(t)\|^2 \mathrm{d}t \le C(m_2). \tag{2.94}$$

Proof. Differentiate (2.10) with respect to x and multiply the result by $\left(\frac{\mu v_x}{v}\right)_x$ to find

$$\begin{aligned} &\left[\frac{1}{2} \left(\frac{\mu v_x}{v} \right)_x^2 \right]_t - \left[u_x \left(\frac{\mu v_x}{v} \right)_x \right]_t + \left[u_x \left(\frac{\mu v_x}{v} \right)_t \right]_x \\ &\quad = u_{xx} \left(\frac{\mu v_x}{v} \right)_t + \left(\frac{\theta}{v} \right)_{xx} \left(\frac{\mu v_x}{v} \right)_x + \left(\frac{\mu v_x}{v} \right)_x \left[\frac{\mu_\theta}{v} (v_x \theta_t - \theta_x u_x) \right]_x. \end{aligned}$$

We integrate the above identity over $[0,t]\times\mathbb{R}$ and use Cauchy's inequality to derive

$$\left\|\left(\frac{\mu v_x}{v}\right)_x(t)\right\|^2 \lesssim_h 1+\int_0^t\int_{\mathbb{R}} u_{xx}\left(\frac{\mu v_x}{v}\right)_t+\int_0^t\int_{\mathbb{R}}\left(\frac{\theta}{v}\right)_{xx}\left(\frac{\mu v_x}{v}\right)_x + \int_0^t\int_{\mathbb{R}}\left(\frac{\mu v_x}{v}\right)_x\left[\frac{\mu_\theta}{v}(v_x\theta_t-\theta_x u_x)\right]_x. \tag{2.95}$$

It follows from (2.22), (2.8), and

$$\left(\frac{\mu v_x}{v}\right)_t=\frac{v\mu_v-\mu}{v^2}v_x u_x+\frac{\mu_\theta\theta_t v_x}{v}+\frac{\mu u_{xx}}{v}$$

that

$$\left|\left(\frac{\mu v_x}{v}\right)_t\right| \lesssim_h |v_x u_x|+Nm_2^{-1}|\alpha||v_x|+|u_{xx}|.$$

We then deduce from Cauchy's inequality and (2.64) that

$$\int_0^t\int_{\mathbb{R}} u_{xx}\left(\frac{\mu v_x}{v}\right)_t \lesssim C(m_2). \tag{2.96}$$

In view of (2.79) and

$$\left(\frac{\mu v_x}{v}\right)_x=\frac{\mu}{v}v_{xx}+\left(\frac{\mu}{v}\right)_x v_x, \tag{2.97}$$

we have

$$\int_0^t\int_{\mathbb{R}}\left(\frac{\theta}{v}\right)_{xx}\left(\frac{\mu v_x}{v}\right)_x \le -\int_0^t\int_{\mathbb{R}}\frac{\theta\mu v_{xx}^2}{2v^3}+C(m_2)\int_0^t\int_{\mathbb{R}}[\theta_{xx}^2+|(\theta_x,v_x)|^4]. \tag{2.98}$$

Apply Sobolev's inequality to get

$$\int_0^t\int_{\mathbb{R}}|(\theta_x,v_x)|^4 \lesssim \int_0^t\|(\theta_x,v_x)\|^2\|(\theta_x,v_x)\|_1^2 \lesssim C(m_2)+\int_0^t\|(\theta_x,v_x)\|^2\|v_{xx}\|^2.$$

Inserting the last inequality and (2.64) into (2.98), we infer

$$\int_0^t\int_{\mathbb{R}}\left(\frac{\theta}{v}\right)_{xx}\left(\frac{\mu v_x}{v}\right)_x \le -\int_0^t\int_{\mathbb{R}}\frac{\theta\mu v_{xx}^2}{2v^3}+C(m_2)+C(m_2)\int_0^t\|(\theta_x,v_x)\|^2\|v_{xx}\|^2. \tag{2.99}$$

For the last term in (2.95), we use (2.97),

$$\left[\frac{\mu_\theta}{v}(v_x\theta_t-\theta_x u_x)\right]_x=\frac{\mu_\theta}{v}(v_{xx}\theta_t+v_x\theta_{xt}-\theta_{xx}u_x-\theta_x u_{xx}) +\left[\frac{\mu_{\theta\theta}\theta_x+\mu_{\theta v}v_x}{v}-\frac{\mu_\theta v_x}{v^2}\right](v_x\theta_t-\theta_x u_x),$$

and (2.8) to derive

$$\left|\left(\frac{\mu v_x}{v}\right)_x\right| \lesssim_h |v_{xx}| + |(v_x,\theta_x)|^2,$$

$$\left|\left[\frac{\mu_\theta}{v}(v_x\theta_t - \theta_x u_x)\right]_x\right| \lesssim_h |\alpha|(|\theta_t||v_{xx}| + |\theta_{xt}||v_x| + |\theta_{xx}||u_x| + |\theta_x||u_{xx}|) + |\alpha||(v_x,\theta_x)|(|v_x\theta_t| + |\theta_x u_x|).$$

It follows from the identity

$$c_v\theta_{tx} = \left(\frac{\mu u_x^2}{v}\right)_x + \left(\frac{\kappa\theta_x}{v}\right)_{xx} - \left(\frac{\theta}{v}u_x\right)_x$$

and, (2.62) and (2.63) that

$$|\theta_{tx}| \le |\theta_{xxx}| + |(v_x,\theta_x)||\theta_{xx}| + (1+|u_x|)|u_{xx}| + |\theta_x||v_{xx}| + (1+|(u_x,\theta_x)|)|(v_x,u_x,\theta_x)|^2.$$

Hence applying Cauchy's inequality yields

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\left|\left(\frac{\mu v_x}{v}\right)_x\left[\frac{\mu_\theta}{v}(v_x\theta_t-\theta_x u_x)\right]_x\right| \\
&\lesssim_h (\epsilon + C(\epsilon)\alpha^2 \|(\theta_t,\theta_x v_x)\|^2)\int_0^t\int_{\mathbb{R}} v_{xx}^2 + C(\epsilon)\alpha^2\,\|v_x\|^2\int_0^t\int_{\mathbb{R}}\theta_{xxx}^2 \\
&\quad + C(\epsilon)\int_0^t\int_{\mathbb{R}}\alpha^2|(v_x,\theta_x)|^2|(v_x\theta_t,\theta_x u_x)|^2 + C(\epsilon)\int_0^t\int_{\mathbb{R}}|(v_x,\theta_x)|^4 \\
&\quad + C(\epsilon)\alpha^2\int_0^t\int_{\mathbb{R}}[|(v_x,u_x,\theta_x)|^4\theta_{xx}^2 + (1+|(u_x,\theta_x)|^2)|(v_x,u_x,\theta_x)|^6] \\
&\quad + C(\epsilon)\alpha^2\int_0^t\int_{\mathbb{R}}[(1+|u_x|^2)u_{xx}^2v_x^2 + \theta_x^2u_{xx}^2] \\
&\lesssim_h \epsilon\int_0^t\int_{\mathbb{R}} v_{xx}^2 + C(\epsilon,m_2).
\end{aligned} \tag{2.100}$$

Here we have used (2.8) and

$$\int_0^t\int_{\mathbb{R}}\theta_{xxx}^2 \lesssim N^2.$$

Plug (2.96), (2.99), and (2.100) into (2.95) to deduce

$$\|v_{xx}(t)\|^2 + \int_0^t \|v_{xx}\|^2 \lesssim_h C(m_2) + \int_0^t \|(\theta_x,v_x)\|^2\|v_{xx}\|^2.$$

We apply Gronwall's inequality to the above estimate to obtain

$$\|v_{xx}(t)\|^2 + \int_0^t \|v_{xx}\|^2 \lesssim C(m_2),$$

which combined with (2.76) implies (2.94). The proof is complete. ∎

2.4. *Estimates of third-order derivatives*

Estimates on the third-order derivatives of $(v(t,x),u(t,x),\theta(t,x))$ with respect to x will be proved in this subsection. The notation $A \lesssim_h B$ is employed to denote that $A \le C_h B$ holds uniformly for some constant C_h, depending only on Π_0, V_0, and $H(C_2)$ with C_2 given in Lemma 2.7. And we denote by $C(m_2)$ some positive constant which depends only on m_2, Π_0, V_0, and $H(C_2)$.

We first give an estimate on the third-order derivatives of u and θ.

Lemma 2.11. *Assume that the conditions listed in Lemma* 2.2 *hold. Then*

$$\sup_{t\in[0,T]} \|(u_{xxx},\theta_{xxx})(t)\|^2 + \int_0^T \|(u_{xxxx},\theta_{xxxx})(t)\|^2 \mathrm{d}t \lesssim_h C(m_2) + \int_0^T \|v_{xxx}(t)\|^2 \mathrm{d}t + \sup_{t\in[0,T]} \|v_{xxx}(t)\|^2. \tag{2.101}$$

Proof. The proof is divided into the following steps:

Step 1. Differentiating $(1.1)_2$ with respect to x twice and multiplying the resulting identity by u_{xxxx} yield

$$\frac{1}{2}(u_{xxx}^2)_t - (u_{xxt}u_{xxx})_x + \frac{\mu u_{xxxx}^2}{v} = P_{xxx}u_{xxxx} + \left[\frac{\mu u_{xxxx}}{v} - \left(\frac{\mu u_x}{v}\right)_{xxx}\right] u_{xxxx}.$$

Integrate the above identity over $[0,t]\times\mathbb{R}$ to have

$$\|u_{xxx}(t)\|^2 + \int_0^t\int_{\mathbb{R}} u_{xxxx}^2 \lesssim 1 + \int_0^t\int_{\mathbb{R}} |P_{xxx}|^2 + \int_0^t\int_{\mathbb{R}} \left|\frac{\mu u_{xxxx}}{v} - \left(\frac{\mu u_x}{v}\right)_{xxx}\right|^2. \tag{2.102}$$

We compute from (2.79) that

$$P_{xxx} = \frac{\theta_{xxx}}{v} - 3\frac{\theta_{xx}v_x}{v^2} - 3\frac{\theta_x v_{xx}}{v^2} + 6\frac{\theta_x v_x^2}{v^3} - \frac{\theta v_{xxx}}{v^2} + 6\frac{\theta v_x v_{xx}}{v^3} - 6\frac{\theta v_x^3}{v^4}.$$

Hence

$$|P_{xxx}|^2 \lesssim |(v_{xxx},\theta_{xxx})|^2 + |(v_x,\theta_x)|^6 + |(v_x,\theta_x)|^2|(v_{xx},\theta_{xx})|^2.$$

It follows from (2.85)–(2.87) and (2.94) that

$$\int_0^t\int_{\mathbb{R}} |P_{xxx}|^2 \lesssim C(m_2) + \int_0^t\int_{\mathbb{R}} v_{xxx}^2. \tag{2.103}$$

From (2.83) with $f(v) = \frac{h(v)}{v}$ and

$$\left(\frac{\mu u_x}{v}\right)_{xxx} = \left(\frac{\mu}{v}\right)_{xxx} u_x + 3\left(\frac{\mu}{v}\right)_{xx} u_{xx} + 3\left(\frac{\mu}{v}\right)_x u_{xxx} + \frac{\mu}{v}u_{xxxx},$$

we have

$$\left|\left(\frac{\mu u_x}{v}\right)_{xxx} - \frac{\mu u_{xxxx}}{v}\right| \lesssim_h |(v_x, u_x, \theta_x)|^4 + |(v_x, u_x, \theta_x)|^2 |(v_{xx}, u_{xx}, \theta_{xx})| + |u_x||v_{xxx}| + |(v_x, u_x, \theta_x)||(u_{xxx}, \theta_{xxx})| + |u_{xx}||(v_{xx}, \theta_{xx})|.$$

In view of (2.64) and (2.94), we deduce

$$\int_0^t \int_{\mathbb{R}} \left|\frac{\mu u_{xxxx}}{v} - \left(\frac{\mu u_x}{v}\right)_{xxx}\right|^2 \lesssim_h C(\delta, m_2) + \sup_{[0,t]} \|v_{xxx}\|^2 + \delta \int_0^t \|(u_{xxxx}, \theta_{xxxx})\|^2. \tag{2.104}$$

Plugging (2.103) and (2.104) into (2.102), we get

$$\|u_{xxx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} u_{xxxx}^2 \lesssim_h C(\delta, m_2) + \int_0^t \int_{\mathbb{R}} v_{xxx}^2 + \sup_{[0,t]} \|v_{xxx}\|^2 + \delta \int_0^t \|(u_{xxxx}, \theta_{xxxx})\|^2. \tag{2.105}$$

Step 2. We differentiate (2.5) with respect to x twice and multiply the resulting identity by θ_{xxxx} to obtain

$$\frac{c_v}{2}(\theta_{xxx})_t - c_v(\theta_{xxt}\theta_{xxx})_x + \kappa \frac{\theta_{xxxx}^2}{v} = (Pu_x)_{xx}\theta_{xxxx} + \left(\frac{\mu u_x^2}{v}\right)_{xx} \theta_{xxxx} + \left[\frac{\kappa \theta_{xxxx}}{v} - \left(\frac{\kappa \theta_x}{v}\right)_{xxx}\right] \theta_{xxxx}.$$

Integrate the above identity over $[0, t] \times \mathbb{R}$ to have

$$\|\theta_{xxx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta_{xxxx}^2 \lesssim \int_0^t \int_{\mathbb{R}} |(Pu_x)_{xx}|^2 + \int_0^t \int_{\mathbb{R}} \left|\left(\frac{\mu u_x^2}{v}\right)_{xx}\right|^2 + \int_0^t \int_{\mathbb{R}} \left|\frac{\kappa \theta_{xxxx}}{v} - \left(\frac{\kappa \theta_x}{v}\right)_{xxx}\right|^2. \tag{2.106}$$

Similar to the derivation of (2.104), we can obtain

$$\int_0^t \int_{\mathbb{R}} \left|\left(\frac{\kappa \theta_x}{v}\right)_{xxx} - \frac{\kappa \theta_{xxxx}}{v}\right|^2 \lesssim_h C(\delta, m_2) + \sup_{[0,t]} \|v_{xxx}\|^2 + \delta \int_0^t \|(u_{xxxx}, \theta_{xxxx})\|^2. \tag{2.107}$$

The identity

$$(Pu_x)_{xx} = (Pu_{xx} + P_x u_x)_x = Pu_{xxx} + 2P_x u_{xx} + P_{xx} u_x$$

implies

$$|(Pu_x)_{xx}|^2 \lesssim u_{xxx}^2 + \theta_x^2 u_{xx}^2 + v_x^2 u_{xx}^2 + u_x^2 v_{xx}^2 + u_x^2 \theta_{xx}^2 + v_x^2 u_x^2 \theta_x^2 + v_x^4 u_x^2.$$

Hence

$$\int_0^t \int_{\mathbb{R}} |(Pu_x)_{xx}|^2 \lesssim C(m_2). \tag{2.108}$$

On the other hand, from

$$\left(\frac{\mu u_x^2}{v}\right)_x = \frac{\mu_x u_x^2}{v} + \frac{2\mu u_x u_{xx}}{v} - \frac{\mu u_x^2 v_x}{v^2},$$

and

$$\begin{aligned}\left(\frac{\mu u_x^2}{v}\right)_{xx} &= \frac{\mu_{xx} u_x^2}{v} + \frac{4\mu_x u_x u_{xx}}{v} + \frac{2\mu u_{xx}^2}{v} + \frac{2\mu u_x u_{xxx}}{v}\\ &\quad - \frac{2\mu_x u_x^2 v_x}{v^2} - \frac{4\mu u_x u_{xx} v_x}{v^2} - \frac{\mu u_x^2 v_{xx}}{v^2} + \frac{2\mu u_x^2 v_x^2}{v^3},\end{aligned}$$

we have

$$\begin{aligned}\left|\left(\frac{\mu u_x^2}{v}\right)_{xx}\right| &\lesssim_h |(v_x, u_x, \theta_x)|^4 + u_{xx}^2 + |u_x u_{xxx}|\\ &\quad + |(v_x, u_x, \theta_x)|^2 |(v_{xx}, u_{xx}, \theta_{xx})|.\end{aligned}$$

Thus,

$$\int_0^t \int_{\mathbb{R}} \left|\left(\frac{\mu u_x^2}{v}\right)_{xx}\right|^2 \lesssim C(m_2). \tag{2.109}$$

Plug (2.107)–(2.109) into (2.106) to deduce

$$\begin{aligned}&\|\theta_{xxx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta_{xxxx}^2\\ &\quad \lesssim_h C(\delta, m_2) + \int_0^t \int_{\mathbb{R}} v_{xxx}^2 + \sup_{[0,t]} \|v_{xxx}\|^2 + \delta \int_0^t \|(u_{xxxx}, \theta_{xxxx})\|^2.\end{aligned} \tag{2.110}$$

Combining (2.105) and (2.110), we take δ suitably small to derive (2.101). $\square$

By using (2.8) and Gronwall's inequality, we can deduce the m_2-dependent bounds for the third-order derivatives of $(v(t,x), u(t,x), \theta(t,x))$. The proof is similar to that of Lemma 2.10 and hence we omit the details for brevity.

Lemma 2.12. *Assume that the conditions listed in Lemma* 2.2 *hold. Then for all* $t \in [0,T]$, *we have*

$$\|(v_{xxx}, u_{xxx}, \theta_{xxx})(t)\|^2 + \int_0^T \|(v_{xxx}, u_{xxxx}, \theta_{xxxx})(s)\|^2 \mathrm{d}s \le C(m_2). \tag{2.111}$$

By virtue of Lemmas 2.1–2.12, we can get the following corollary.

Corollary 2.1. *Assume that the conditions listed in Lemma* 2.2 *hold. Then there exists* $C(m_2) > 0$, *which depends only on* m_2, Π_0, V_0, *and* $H(C_2)$ *with* C_2 *being given in Lemma* 2.7, *such that for all* $t \in [0, T]$,

$$\|(v-1, u, \theta-1)(t)\|_3^2 + \int_0^T [\|v_x(s)\|_2^2 + \|(u_x, \theta_x)(s)\|_3^2] \mathrm{d}s \le C(m_2). \tag{2.112}$$

3. Proof of Theorem 1.1

In this section we will prove our main result, Theorem 1.1. For this purpose, we first present the local solvability result to the Cauchy problem (1.1)–(1.4), (1.9) and (1.10) in the following lemma, which can be proved by the standard iteration method (see Ref. 26).

Lemma 3.1. *If positive constants* M *and* λ_i $(i = 1, 2)$ *exist such that* $\|(v_0 - 1, u_0, \theta_0 - 1)\|_3 \le M$, $v_0(x) \ge \lambda_1$, *and* $\theta_0(x) \ge \lambda_2$ *for all* $x \in \mathbb{R}$, *then there exists* $T_0 = T_0(\lambda_1, \lambda_2, M) > 0$, *depending only on* λ_1, λ_2 *and* M, *such that the Cauchy problem* (1.1) *and* (1.4), (1.9) *and* (1.10) *has a unique solution* $(v, u, \theta) \in X(0, T_0; \frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, 2M)$.

We prove Theorem 1.1 in the following six steps by employing the continuation argument.

Step 1. Set $T_1 = 128C_3^4$, where C_3 is exactly the same constant as in (2.64). Recalling (1.12) and (1.13) and applying Lemma 3.1, we can find a positive constant $t_1 = \min\{T_1, T_0(V_0, V_0, \Pi_0)\}$ such that there exists a unique solution $(v, u, \theta) \in X(0, t_1; \frac{1}{2}V_0, \frac{1}{2}V_0, 2\Pi_0)$ to the Cauchy problem (1.1)–(1.4), (1.9) and (1.10).

Take $|\alpha| \le \alpha_1$, where α_1 is some positive constant such that

$$\left(\frac{1}{2}V_0\right)^{-\alpha_1} \le 2, \quad (2\Pi_0)^{\alpha_1} \le 2, \quad \Xi\left(\frac{1}{2}V_0, \frac{1}{2}V_0, 2\Pi_0\right)\alpha_1 \le \epsilon_1, \tag{3.1}$$

where the value of ϵ_1 is chosen in Lemma 2.2. Then we can apply Lemmas 2.7 and 2.8 with $T = t_1$ to deduce that for each $t \in [0, t_1]$, the local solution (v, u, θ) constructed above satisfies:

$$\theta(t, x) \ge \frac{V_0}{C_4 V_0 T_1 + 1} =: C_5 \quad \text{for all } x \in \mathbb{R}, \tag{3.2}$$

$$\theta(t, x) \le C_1, \quad C_2 \le v(t, x) \le C_2^{-1} \quad \text{for all } x \in \mathbb{R}, \tag{3.3}$$

$$\|(v-1, u, \theta-1)(t)\|_1^2 + \int_0^t \left[\left\|\sqrt{\theta} v_x(s)\right\|^2 + \|(u_x, \theta_x)(s)\|_1^2\right] \mathrm{d}s \le C_3^2. \tag{3.4}$$

Combining Corollary 2.1 and (3.2), we can find a positive constant C_6, which depends on C_5, Π_0, V_0, and $H(C_2)$, such that for each $t \in [0, t_1]$,

$$\|(v-1, u, \theta-1)(t)\|_3^2 + \int_0^t [\|v_x(s)\|_2^2 + \|(u_x, \theta_x)(s)\|_3^2] \mathrm{d}s \le C_6^2. \tag{3.5}$$

Step 2. If we take $(v(t_1,\cdot), u(t_1,\cdot), \theta(t_1,\cdot))$ as the initial data and apply Lemma 3.1 again, we can extend the local solution (v,u,θ) to the time interval $[0, t_1+t_2]$ with

$$t_2 = \min\{T_1 - t_1, T_0(C_2, C_5, C_6)\}.$$

Moreover, for all $(t,x) \in [t_1, t_1+t_2] \times \mathbb{R}$, we have

$$v(t,x) \geq \frac{1}{2}C_2, \quad \theta(t,x) \geq \frac{1}{2}C_5,$$

and

$$\|(v-1,u,\theta-1)(t)\|_3^2 + \int_{t_1}^t \Big[\|v_x(s)\|_2^2 + \|(u_x,\theta_x)(s)\|_3^2\Big]\mathrm{d}s \leq 4C_6^2, \tag{3.6}$$

which combined with (3.5) implies that for all $t \in [0, t_1+t_2]$,

$$\|(v-1,u,\theta-1)(t)\|_3^2 + \int_{0}^t \Big[\|v_x(s)\|_2^2 + \|(u_x,\theta_x)(s)\|_3^2\Big]\mathrm{d}s \leq 5C_6^2. \tag{3.7}$$

Take $|\alpha| \leq \min\{\alpha_1, \alpha_2\}$, where $\alpha_1 > 0$ is determined by (3.1) and α_2 is some positive constant satisfying

$$\left(\frac{1}{2}C_5\right)^{-\alpha_2} \leq 2, \quad \big(2\sqrt{5}C_6\big)^{\alpha_2} \leq 2, \quad \Xi\left(\frac{1}{2}C_2, \frac{1}{2}C_5, \sqrt{5}C_6\right)\alpha_2 \leq \epsilon_1, \tag{3.8}$$

where the value of ϵ_1 is chosen in Lemma 2.2. Then we can employ Lemma 2.7, Lemma 2.8, and Corollary 2.1 with $T = t_1 + t_2$ to infer that the local solution (v,u,θ) satisfies (3.2)–(3.5) for each $t \in [0, t_1+t_2]$.

Step 3. We repeat the argument in Step 2, to extend our solution (v,u,θ) to the time interval $[0, t_1+t_2+t_3]$, where

$$t_3 = \min\big\{T_1 - (t_1+t_2), T_0\big(C_2^{-1}, C_5, C_6\big)\big\}.$$

Assume that $|\alpha| \leq \min\{\alpha_1,\alpha_2\}$ with constants α_1 and α_2 satisfying (3.1) and (3.8). Continuing, after finitely many steps we construct the unique solution (v,u,θ) existing on $[0,T_1]$ and satisfying (3.2)–(3.5) for each $t \in [0,T_1]$.

Step 4. Since $T_1 = 128C_3^4$ and

$$\sup_{0\leq t\leq T_1} \|(\theta-1)(t)\|_1^2 + \int_{T_1/2}^{T_1} \|\theta_x(t)\|_1^2 \mathrm{d}t \leq C_3^2, \tag{3.9}$$

we can find a $t_0' \in [T_1/2, T_1]$ such that

$$\|\theta(t_0') - 1\| \leq C_3, \quad \|\theta_x(t_0')\| \leq \frac{1}{8}C_3^{-1}.$$

For if not, we have that $\|\theta_x(t)\| > \frac{1}{8}C_3^{-1}$ for each $t \in [T_1/2, T_1]$ and hence

$$\int_{T_1/2}^{T_1} \|\theta_x(t)\|_1^2 \mathrm{d}t > \frac{1}{2}T_1\left(\frac{1}{8}C_3^{-1}\right)^2 = C_3^2.$$

This contradicts (3.9). Then it follows from Sobolev's inequality that

$$\|(\theta-1)(t_0')\|_{L^\infty} \leq \sqrt{2}\|(\theta-1)(t_0')\|^{\frac{1}{2}}\|\theta_x(t_0')\|^{\frac{1}{2}} \leq \frac{1}{2},$$

from which we get

$$\theta(t_0',x) \geq 1 - \|(\theta-1)(t_0')\|_{L^\infty} \geq \frac{1}{2} \quad \text{for all } x \in \mathbb{R}. \tag{3.10}$$

We notice that

$$\|(v-1,u,\theta-1)(t_0')\|_3 \leq C_6, \quad v(t_0',x) \geq C_2 \quad \text{for all } x \in \mathbb{R}.$$

Now we apply Lemma 3.1 again by taking $(v(t_0',\cdot),u(t_0',\cdot),\theta(t_0',\cdot))$ as the initial data. Then we derive that the solution (v,u,θ) exists on $[t_0', t_0'+t_1']$ with $t_1' = \min\{T_1, T_0(C_2^{-1}, \frac{1}{2}, C_6)\}$, and for all $(t,x) \in [t_0', t_0'+t_1'] \times \mathbb{R}$,

$$\|(v-1,u,\theta-1)(t)\|_3^2 + \int_{t_0'}^t \Big[\|v_x(s)\|_2^2 + \|(u_x,\theta_x)(s)\|_3^2\Big]\mathrm{d}s \leq 4C_6^2$$

and

$$v(t,x) \geq \frac{1}{2}C_2, \quad \theta(t,x) \geq \frac{1}{4}.$$

Therefore, the solution (v,u,θ) satisfies (3.7) for all $t \in [0, t_0'+t_1']$.

We take $|\alpha| \leq \min\{\alpha_1,\alpha_2,\alpha_3\}$ with α_i $(i=1,2,3)$ being positive constants satisfying (3.1), (3.8) and

$$\left(\frac{1}{4}\right)^{-\alpha_3} \leq 2, \quad (2\sqrt{5}C_6)^{\alpha_3} \leq 2, \quad \Xi\left(\frac{1}{2}C_2, \frac{1}{4}, \sqrt{5}C_6\right)\alpha_3 \leq \epsilon_1, \tag{3.11}$$

where the value of ϵ_1 is chosen in Lemma 2.2. Then we can deduce from Lemmas 2.7 and 2.8 with $T = t_0' + t_1'$ that for each $t \in [t_0', t_0'+t_1']$, the local solution $(v(t,x), u(t,x), \theta(t,x))$ satisfies (3.3) and (3.4), and

$$\theta(t,x) \geq \frac{\inf_{x\in\mathbb{R}}\theta(t_0',x)}{C_4\inf_{x\in\mathbb{R}}\theta(t_0',x)T_1 + 1} \geq \frac{1}{C_4T_1+2} =: C_7 \quad \text{for all } x \in \mathbb{R}. \tag{3.12}$$

Here we have used the estimate (3.10). Combining (3.2) and (3.12) yields that for each $t \in [0, t_0'+t_1']$,

$$\theta(t,x) \geq \min\{C_5, C_7\} := C_8 \quad \text{for all } x \in \mathbb{R}. \tag{3.13}$$

We deduce from (3.13) and Corollary 2.1 that there exists some positive constant C_9, depending on C_8, Π_0, V_0, and $H(C_2)$, such that for each $t \in [0, t_0'+t_1']$,

$$\|(v-1,u,\theta-1)(t)\|_3^2 + \int_0^t \Big[\|v_x(s)\|_2^2 + \|(u_x,\theta_x)(s)\|_3^2\Big]\mathrm{d}s \leq C_9^2. \tag{3.14}$$

Step 5. Next if we take $(v(t_0'+t_1',\cdot), u(t_0'+t_1',\cdot), \theta(t_0'+t_1',\cdot))$ as the initial data, we apply Lemma 3.1 to construct the solution (v,u,θ) existing on the time interval

$[0, t_0' + t_1' + t_2']$ with

$$t_2' = \min\{T_1 - t_1', T_0(C_2, C_8, C_9)\},$$

such that for all $(t, x) \in [t_0' + t_1', t_0' + t_1' + t_2'] \times \mathbb{R}$,

$$v(t,x) \geq \frac{1}{2}C_2, \quad \theta(t,x) \geq \frac{1}{2}C_8,$$

and

$$\|(v-1, u, \theta-1)(t)\|_3^2 + \int_{t_0'+t_1'}^t \left[\|v_x(s)\|_2^2 + \|(u_x, \theta_x)(s)\|_3^2\right] \mathrm{d}s \leq 4C_9^2. \tag{3.15}$$

Combine (3.14) and (3.15) to obtain that for all $t \in [0, t_0' + t_1' + t_2']$,

$$\|(v-1, u, \theta-1)(t)\|_3^2 + \int_0^t \left[\|v_x(s)\|_2^2 + \|(u_x, \theta_x)(s)\|_3^2\right] \mathrm{d}s \leq 5C_9^2. \tag{3.16}$$

Take $0 < \alpha \leq \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where α_i $(i = 1, 2, 3)$ are positive constants satisfying (3.1), (3.8), (3.11), and

$$\left(\frac{1}{2}C_8\right)^{-\alpha_4} \leq 2, \quad \left(2\sqrt{5}C_9\right)^{\alpha_4} \leq 2, \quad \Xi\left(\frac{1}{2}C_2, \frac{1}{2}C_8, \sqrt{5}C_9\right)\alpha_4 \leq \epsilon_1, \tag{3.17}$$

where the value of ϵ_1 is chosen in Lemma 2.2. Then we infer from Lemma 2.7, Lemma 2.8 and Corollary 2.1 with $T = t_0' + t_1' + t_2'$ that the local solution $(v(t,x), u(t,x), \theta(t,x))$ satisfies (3.13) and (3.14) for each $t \in [0, t_0' + t_1' + t_2']$. By assuming $|\alpha| \leq \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, we can repeatedly apply the argument above to extend the local solution to the time interval $[0, t_0' + T_1]$. Furthermore, we deduce that (3.13) and (3.14) hold for each $t \in [0, t_0' + T_1]$. In view of $t_0' + T_1 \geq 3T_1/2$, we have shown that the Cauchy problem (1.1)–(1.4), (1.9) and (1.10) admits a unique solution $(v, u, \theta) \in X(0, \frac{3}{2}T_1; C_2, C_8, C_9)$ on the time interval $[0, \frac{3}{2}T_1]$.

Step 6. We take $|\alpha| \leq \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. As in Steps 4 and 5, we can find $t_0'' \in [t_0' + T_1/2, t_0' + T_1]$ such that the Cauchy problem (1.1)–(1.4), (1.9) and (1.10) admits a unique solution (v, u, θ) on $[0, t_0'' + T_1]$, which satisfies (3.13) and (3.14) for each $t \in [0, t_0'' + T_1]$. Since $t_0'' + T_1 \geq t_0' + 3T_1/2 \geq 2T_1$, we have extended the local solution (v, u, θ) to the time interval $[0, 2T_1]$. Repeating the above procedure, we can then extend the solution (v, u, θ) step-by-step to a global one provided that $|\alpha| \leq \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Choosing

$$\epsilon_0 = \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \tag{3.18}$$

where α_i $(i = 1, 2, 3, 4)$ are given by (3.1), (3.8), (3.11), and (3.17), we then derive that the Cauchy problem (1.1)–(1.4), (1.9) and (1.10) has a unique solution (v, u, θ) satisfying (3.3), (3.13), and (3.14) for each $t \in [0, \infty)$. Thus we have

$$\sup_{0 \leq t < \infty} \|(v-1, u, \theta-1)(t)\|_3^2 + \int_0^\infty \left[\|v_x(t)\|_2^2 + \|(u_x, \theta_x)(t)\|_3^2\right] \mathrm{d}t \leq C_9^2, \tag{3.19}$$

from which we derive that the solution $(v, u, \theta) \in X(0, \infty; C_2, C_8, C_9)$.

The large-time behavior (1.17) follows from (3.19) by using a standard argument (see Ref. 26).

Recall that ϵ_1, C_i $(i = 1, 2, 3, 4, 5, 7, 8)$ depend only on Π_0, V_0, and $H(V_0)$, while C_6 and C_9 depend only on Π_0, V_0, and $H(C_2)$. According to the definition (3.18) of ϵ_0, we can conclude the proof of Theorem 1.1.

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References

1. A. A. Amosov and A. A. Zlotnik, Solvability "in the large" of a system of equations for the one-dimensional motion of an inhomogeneous viscous heat-conducting gas, *Math. Notes* **52** (1992) 753–763.
2. S. N. Antontsev, A. V. Kazhikhov and V. N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, Studies in Mathematics and Its Applications (North-Holland, 1990).
3. D. Bresch and B. Desjardins, On the existence of global weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids, *J. Math. Pures Appl.* **87** (2007) 57–90.
4. S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases*, 3rd edn., Cambridge Mathematical Library (Cambridge Univ. Press, 1990).
5. G.-Q. Chen, D. Hoff and K. Trivisa, Global solutions of the compressible Navier–Stokes equations with large discontinuous initial data, *Comm. Partial Differential Equations* **25** (2000) 2233–2257.
6. C. M. Dafermos, Global smooth solutions to the initial-boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity, *SIAM J. Math. Anal.* **13** (1982) 397–408.
7. C. M. Dafermos and L. Hsiao, Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity, *Nonlinear Anal.* **6** (1982) 435–454.
8. R. Duan, H. Liu and H. Zhao, Nonlinear stability of rarefaction waves for the compressible Navier–Stokes equations with large initial perturbation, *Trans. Amer. Math. Soc.* **361** (2009) 453–493.
9. L. C. Evans, *Partial Differential Equations*, 2nd edn., Graduate Studies in Mathematics (Amer. Math. Soc. 2010).
10. E. Feireisl, Mathematical theory of compressible, viscous, and heat conducting fluids, *Comput. Math. Appl.* **53** (2007) 461–490.
11. B. Haspot, Existence of global strong solution for the compressible Navier–Stokes equations with degenerate viscosity coefficients in 1D (2014), arXiv: 1411.5503.

12. H. K. Jenssen and T. K. Karper, One-dimensional compressible flow with temperature dependent transport coefficients, *SIAM J. Math. Anal.* **42** (2010) 904–930.
13. S. Jiang, Large-time behavior of solutions to the equations of a viscous polytropic ideal gas, *Ann. Mat. Pura Appl.* **175** (1998) 253–275.
14. S. Jiang, Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains, *Commun. Math. Phys.* **200** (1999) 181–193.
15. S. Jiang, Remarks on the asymptotic behaviour of solutions to the compressible Navier–Stokes equations in the half-line, *Proc. Roy. Soc. Edinb. A* **132** (2002) 627–638.
16. J. I. Kanel′, A model system of equations for the one-dimensional motion of a gas, *Differential Equations* **4** (1968) 374–380.
17. S. Kawashima and T. Nishida, Global solutions to the initial value problem for the equations of one-dimensional motion of viscous polytropic gases, *J. Math. Kyoto Univ.* **21** (1981) 825–837.
18. S. Kawashima and M. Okada, Smooth global solutions for the one-dimensional equations in magnetohydrodynamics, *Proc. Japan Acad. Ser. A Math. Sci.* **58** (1982) 384–387.
19. B. Kawohl, Global existence of large solutions to initial-boundary value problems for a viscous, heat-conducting, one-dimensional real gas, *J. Differential Equations* **58** (1985) 76–103.
20. A. V. Kazhikhov, On the Cauchy problem for the equations of a viscous gas, *Siberian Math. J.* **23** (1982) 44–49.
21. A. V. Kazhikhov and V. V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.* **41** (1977) 273–282.
22. H.-L. Li, J. Li and Z. Xin, Vanishing of vacuum states and blow-up phenomena of the compressible Navier–Stokes equations, *Commun. Math. Phys.* **281** (2008) 401–444.
23. J. Li and Z. Liang, Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier–Stokes system in unbounded domains with large data, *Arch. Ration. Mech. Anal.* **220** (2016) 1195–1208.
24. T.-P. Liu, Z. Xin and T. Yang, Vacuum states for compressible flow, *Discrete Contin. Dynam. Syst.* **4** (1998) 1–32.
25. H. Liu, T. Yang, H. Zhao and Q. Zou, One-dimensional compressible Navier–Stokes equations with temperature dependent transport coefficients and large data, *SIAM J. Math. Anal.* **46** (2014) 2185–2228.
26. A. Matsumura and K. Nishihara, *Global Solutions for Nonlinear Differential Equations: Mathematical Analysis on Compressible Viscous Fluids* (Nippon Hyoron Sha, 2004).
27. A. Mellet and A. Vasseur, Existence and uniqueness of global strong solutions for one-dimensional compressible Navier–Stokes equations, *SIAM J. Math. Anal.* **39** (2008) 1344–1365.
28. K. Nishihara, T. Yang and H. Zhao, Nonlinear stability of strong rarefaction waves for compressible Navier–Stokes equations, *SIAM J. Math. Anal.* **35** (2004) 1561–1597.
29. M. Okada and S. Kawashima, On the equations of one-dimensional motion of compressible viscous fluids, *J. Math. Kyoto Univ.* **23** (1983) 55–71.
30. R. Pan and W. Zhang, Compressible Navier–Stokes equations with temperature dependent heat conductivity, *Commun. Math. Sci.* **13** (2015) 401–425.

31. Z. Tan, T. Yang, H. Zhao and Q. Zou, Global solutions to the one-dimensional compressible Navier–Stokes–Poisson equations with large data, *SIAM J. Math. Anal.* **45** (2013) 547–571.
32. W. G. Vincenti and C. H. Kruger Jr., *Introduction to Physical Gas Dynamics* (John Wiley and Sons, 1965).
33. L. Wan, T. Wang and Q. Zou, Stability of stationary solutions to the outflow problem for full compressible Navier–Stokes equations with large initial perturbation, *Nonlinearity* **29** (2016) 1329–1354.
34. T. Wang, One dimensional p-th power Newtonian fluid with temperature-dependent thermal conductivity, *Commun. Pure Appl. Anal.* **15** (2016) 477–494.
35. T. Yang and C. Zhu, Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum, *Commun. Math. Phys.* **230** (2002) 329–363.
36. Y. B. Zel'dovich and Y. P. Raizer, *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, Graduate Studies in Mathematics (Academic Press, 1967).
37. A. A. Zlotnik and A. A. Amosov, On the stability of generalized solutions of equations of one-dimensional motion of a viscous heat-conducting gas, *Siberian Math. J.* **38** (1997) 663–684.