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ONE DIMENSIONAL p-TH POWER NEWTONIAN FLUID WITH TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

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Abstract. We study the initial and initial-boundary value problems for the p-th power Newtonian fluid in one space dimension with general large initial data. The existence and uniqueness of globally smooth non-vacuum solutions are established when the thermal conductivity is some non-negative power of the temperature. Our analysis is based on some detailed estimates on the bounds of both density and temperature.

1. Introduction.

1.1. The Eulerian description.

The motion of one dimensional compressible flow of a p-th power Newtonian fluid can be described by the system

$$\begin{cases} \rho_t + (\rho u)_y = 0, \\ (\rho u)_t + (\rho u^2 + \mathcal{P})_y = (\mu u_y)_y, \\ (\rho \mathcal{E})_t + (\rho u \mathcal{E} + u\mathcal{P})_y = (\kappa \theta_y + \mu u u_y)_y, \end{cases} \tag{1}$$

where $t > 0$ is the time variable, $y \in \Omega \subset \mathbb{R}$ is the spatial variable, and the primary dependent variables are the density ρ, fluid velocity u and temperature θ. The specific total energy $\mathcal{E} = e + \frac{1}{2}|u|^2$ with e being the specific internal energy. The transport coefficients μ (viscosity) and κ (thermal conductivity) are prescribed by means of constitutive relations as functions of ρ and θ. The pressure $\mathcal{P}$ and specific internal energy e are given by

$$\mathcal{P} = \rho^p \theta, \quad e = c_v \theta \tag{2}$$

with the pressure exponent $p \geq 1$ and constant specific heat $c_v > 0$. The thermodynamic variables p, ρ and e are related by Gibbs equation $\theta \mathrm{d}s = \mathrm{d}e + \mathcal{P}\mathrm{d}v$, where $v = 1/\rho$ is the specific volume and s the specific entropy. As a function of (v, s), the internal energy e can be explicitly given by

$$e(v,s) = \begin{cases} C \exp\left(\dfrac{s}{c_v} + \dfrac{p-1}{c_v} v^{1-p}\right), & p > 1, \\ C v^{-c_v} \exp\left(\dfrac{s}{c_v}\right), & p = 1, \end{cases}$$

where C is some positive constant.

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The system (1)-(2) is supplemented with the initial conditions

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0) \qquad \text{in } \Omega, \tag{3}$$

and one type of the following far-field and boundary conditions:

$$\lim_{y\to\pm\infty}(\rho_0(y), u_0(y), \theta_0(y)) = (1, 0, 1), \qquad \text{if } \Omega = \mathbb{R}; \tag{4}$$

$$(u, \theta_y)\,|_{y=0} = 0, \quad \lim_{y\to\infty}(\rho_0(y), u_0(y), \theta_0(y)) = (1, 0, 1), \qquad \text{if } \Omega = (0, \infty); \tag{5}$$

$$(u, \theta_y)\,|_{\partial\Omega} = 0, \qquad \text{if } \Omega = (0, 1). \tag{6}$$

The initial data is assumed to satisfy certain compatibility conditions as usual.

The aim of this article is to show the existence and uniqueness of the globally smooth non-vacuum solutions to the initial value problem (1)-(4) and the initial-boundary value problems (1)-(3) and (5), and (1)-(3) and (6) for general large initial data. Our main interest lies in the case when the thermal conductivity κ depends on temperature θ in power law. This choice of thermal conductivity is motivated by the kinetic theory of gases. In fact, for ideal polytropic gases (i.e. pressure $\mathcal{P}$ and specific internal energy e satisfy (2) with $p = 1$), according to the first order approximation in the Chapman-Enskog expansion, the viscosity μ and thermal conductivity κ depend solely on the temperature (cf. [2]). If the intermolecular potential is proportional to $r^{-\alpha}$ with $\alpha > 1$ and r being the intermolecular distance, then μ and κ satisfy

$$\mu = \bar{\mu}\theta^{\frac{\alpha+4}{2\alpha}}, \quad \kappa = \bar{\kappa}\theta^{\frac{\alpha+4}{2\alpha}}, \tag{7}$$

where $\bar{\mu}$ and $\bar{\kappa}$ are positive constants. Apart from small and sufficiently smooth data [7] and Nishida-Smoller type global solvability result [12], there is no global solvability result currently available for (1) with constitutive relations (2) and (7).

Let us first recall some previous results related. For the case of ideal polytropic gases, global existence and uniqueness of smooth solutions are established in [9] for the initial-boundary value problem (1)-(3) and (6) and in [8] for the initial value problem (1)-(4) with constant transport coefficients, respectively. There are also many works on the construction of non-vacuum solutions for the ideal polytropic gases when the viscosity depends only on density and the thermal conductivity may depend on both density and temperature in various forms, cf. [5, 13, 16] and the references therein.

In our case, the form of pressure $\mathcal{P}$ and specific internal energy e in (2) can be seen as a generalization of the constitutive relations for the ideal polytropic gas. In this direction, Lewicka and Watson [11] showed exponential convergence of solutions to equilibria for initial-boundary value problems involving fixed endpoints held at a fixed temperature or insulated. Qin and Huang [15] proved the regularity and exponential stability of solutions in $H^i (i = 2, 4)$ for (1)-(3) and (6). Recently Cui and Yao [4] established the large time behavior of the global spherically or cylindrically symmetric solutions in H^1 for the p-th power Newtonian fluid in multi-dimension. For the reactive p-th power gas confined between two parallel plates, see [10] for the large-time behavior of the solutions and [14] for the global existence and exponential stability of solutions in $H^i (i = 2, 4)$.

We note that the papers [4, 10, 11, 14, 15] are all concerned with constant transport coefficients. Thus it is natural to investigate the global solvability for the p-th Newtonian fluid (1)-(2) with non-constant transport coefficients and general pressure exponent $p \geq 1$. Our study is motivated by the recent work [16] which is concentrated on the construction of globally smooth solutions to the one-dimensional compressible Navier-Stokes-Poisson equations with degenerate transport coefficients. We introduce the Lagrangian variables and reduce the initial value problem (1)-(4) and the initial-boundary value problems (1)-(3) and (5), and (1)-(3) and (6) into corresponding problems in the Lagrangian variables. The local existence and uniqueness of solutions can be proved by using the Banach theorem and the constructivity of the operator defined by the linearization of the problems on a small time-interval (cf. Antontsev et al. [1, Section 2.5]). The global solvability is showed by applying the continuation argument to extend the local solution step by step to the global one based on certain global a priori estimates of the solution. The key ingredient to the global a priori estimates with large data is to obtain the desired positive lower and upper bounds for both density and temperature as showed in [8, 9]. For technical reasons we will assume that the viscosity coefficient depends only on the density.

1.2. **Reformulation and main results.** We transform the initial value problem (1)-(4) and the initial-boundary value problems (1)-(3) and (5), and (1)-(3) and (6) into Lagrangian coordinates. Introduce the Lagrangian variables (t, x) with

$$x = \int_{y(t)}^{y} \rho(t, z)\mathrm{d}z,$$

where $y(t)$ is the particle path satisfying $y'(t) = u(t, y(t))$ and $y(0) = 0$. Using this transformation, we obtain the Lagrangian version of the system (1) as

$$\begin{cases} v_t - u_x = 0, \\ u_t + \mathcal{P}_x = \left(\mu \dfrac{u_x}{v}\right)_x, \\ e_t + \mathcal{P} u_x = \dfrac{\mu u_x^2}{v} + \left(\kappa \dfrac{\theta_x}{v}\right)_x, \end{cases} \tag{8}$$

where $v = 1/\rho$ is the specific volume, $\mathcal{P} = \theta/v^p$ $(p \geq 1)$ and $e = c_v\theta$. For the initial-boundary value problem (1)-(3) and (6) in bounded domain $(0, 1)$, we may assume $\int_0^1 \rho_0(y)dy = 1$ without loss of generality. Then the initial and boundary conditions (3)-(6) can be translated into similar conditions:

$$(v, u, \theta)|_{t=0} = (v_0, u_0, \theta_0) \qquad \text{in } \Omega, \tag{9}$$

and

$$\lim_{x\to\pm\infty} (v_0(x), u_0(x), \theta_0(x)) = (1, 0, 1), \qquad \text{if } \Omega = \mathbb{R}; \tag{10}$$

$$(u, \theta_x)|_{x=0} = 0, \quad \lim_{x\to\infty} (v_0(x), u_0(x), \theta_0(x)) = (1, 0, 1), \qquad \text{if } \Omega = (0, \infty); \tag{11}$$

$$(u, \theta_x)|_{\partial\Omega} = 0, \qquad \text{if } \Omega = (0, 1). \tag{12}$$

Our first result is on the case when the viscosity μ is a positive constant.

Theorem 1.1. *Suppose that μ is a positive constant and*

$$(v_0 - 1, u_0, \theta_0 - 1) \in H^1(\Omega), \quad \inf_{x\in\Omega} \{v_0(x), \theta_0(x)\} > 0. \tag{13}$$

If the thermal conductivity $\kappa = \kappa(\theta)$ is a smooth function on $[0, \infty)$ and satisfies

$$\lim_{\theta\to\infty} \kappa(\theta) \leq \infty \quad \text{and} \quad \inf_{\theta\geq\underline{\theta}} \kappa(\theta) > 0 \text{ for all } \underline{\theta} > 0, \tag{14}$$

then there exists a unique global-in-time solution (v, u, θ) *to the initial value problem* (8)-(10)*, the initial-boundary value problem* (8)-(9) *and* (11)*, or* (8)-(9) *and* (12) *such that*

$$\begin{aligned} &(v-1, u, \theta-1) \in C([0,T]; H^1(\Omega)), \quad (u_x, \theta_x) \in L^2(0,T; H^1(\Omega)), \\ &\inf_{(t,x)\in[0,T]\times\Omega} \{v(t,x), \theta(t,x)\} > 0 \quad \textit{for each positive constant } T. \end{aligned} \tag{15}$$

Remark 1. The case of p-th power Newtonian fluids with temperature-dependent thermal conductivity $\kappa = \theta^b$ ($b \geq 0$ is a constant) is included in the class of fluids investigated in Theorem 1.1.

The second result is concerned with the case when the transport coefficients μ and κ satisfy

$$\mu = v^{-a}, \quad \kappa = \theta^b \tag{16}$$

with some positive parameters a and b. For such a case, we have the following result.

Theorem 1.2. *Suppose that*

- (v_0, u_0, θ_0)*,* μ *and* κ *satisfy* (13) *and* (16)*;*
- a *satisfies* $\frac{p}{3} < a < \frac{1}{2}$, $1 \leq p \leq \frac{3a+1}{2}$;
- b *satisfies one of the following conditions:*
  (i) $1 \leq b < \frac{2a+p-1}{2p-1-a}$,
  (ii)

$$\begin{cases} 0 < b < 1, \ \dfrac{a+3-2a^2+6(p-1)(1-a)}{(3a-p)(1-2a)}(1-b) < 1, \\ \dfrac{2-b}{2} + \dfrac{4-2a+2a^2+(p-1)(9-11a-2p)}{2(3a-p)(1-2a)}(1-b) < 1. \end{cases} \tag{17}$$

Then there is a unique global solution (v, u, θ) *satisfying* (15) *to the initial value problem* (8)-(10)*, the initial-boundary value problem* (8)-(9) *and* (11)*, or* (8)-(9) *and* (12)*.*

Remark 2. Since the assumptions on a imply $1 < a + p < 2p - a < 2a + p$, it follows that $\frac{2a+p-1}{2p-1-a} > 1$ and hence the conditions stated in Theorem 1.2 are not vacuous.

Remark 3. The results in Theorems 1.1 and 1.2 show that neither shock waves nor vacuum and concentration will be developed in finite time. Also the above results in Lagrangian coordinates can easily be converted to equivalent statements for corresponding problems in Eulerian coordinates (cf. Chen [3]).

We outline the main ideas used to deduce our main results. Our analysis relies on the continuation argument. For general large initial data, the main difficulty is to control the possible growth of the solutions to the system (8) caused by the dependence on v or θ of the transport coefficients and the strong nonlinearities of the system itself. For the case when the viscosity μ is a positive constant while the thermal conductivity κ may be degenerate in θ, we cannot hope to derive the desired bounds on v and θ as in [8, 9]. However, motivated by the analysis in [8, 9], we can deduce an explicit formula for v an then use this formula to deduce the desired lower bound on v which enables us to get a positive lower bound for θ and an upper bound for v. The upper bound on θ follows from the argument used in

[16]. When μ is a power function of the density, we first get an lower bound on θ in terms of the bounds on v as in (27), then we control the lower and upper bounds on v in terms of $\|\theta^{1-b}\|_{L^\infty([0,T]\times\Omega)}$ as in (70) and (71) by applying Kanel's argument (cf. [6]). These estimates along with the estimate on the upper bound of θ can yield the desired lower and upper bounds of both v and θ provided that the parameters a and b satisfy certain relations as stated in Theorem 1.2.

The layout of this paper is as follows. In Section 2 we state the local existence theorem and deduce some a priori estimates while we give the proofs of Theorem 1.1 and 1.2 in Section 3 and 4, respectively.

2. A priori estimates.

To prove Theorem 1 and Theorem 2, we first define the set for which we seek the solution of the problem (8)-(10), (8)-(9) and (11), or (8)-(9) and (12) as follows

$$X(0,T;m,M) := \left\{ (v,u,\theta) \;\middle|\; \begin{array}{c} (v-1,u,\theta-1) \in C([0,T];H^1(\Omega)), \\ (u_x,\theta_x) \in L^2(0,T;L^2(\Omega)), \\ \inf\{v(t,x),\theta(t,x)\} \geq m \quad \forall\ (t,x) \in [0,T]\times\Omega, \\ \|(v-1,u,\theta-1)(t)\|_{H^1(\Omega)} \leq M \quad \forall\ t \in [0,T] \end{array} \right\}.$$

Here the viscosity coefficients μ may depend on v and the thermal conductivity κ may be a function of both v and θ.

We first present the local solvability of solutions to the problem (8)-(10), (8)-(9) and (11), or (8)-(9) and (12) in the following lemma, which can be proved by the standard iteration method (see [1, Section 2.5]).

Proposition 1. *If positive constants m and M exist such that*

$$\inf_{x\in\Omega}\{v_0(x),\theta_0(x)\} \geq m \quad \text{and} \quad \|(v_0-1,u_0,\theta_0-1)\|_{H^1(\Omega)} \leq M,$$

then there exists a constant $T_0 = T_0(m,M) > 0$ depending only on m and M such that the the problem (8)-(10), (8)-(9) and (11), or (8)-(9) and (12) has a unique smooth solution $(v,u,\theta) \in X(0,T_0;m/2,2M)$.

The rest of this section is devoted to deriving some a priori estimates on the solutions (v,u,θ) to the problem (8)-(10), (8)-(9) and (11), or (8)-(9) and (12) defined on $[0,T]\times\Omega$ for any fixed positive constant T. To prove Theorem 1.1 or Theorem 1.2, it suffices to deduce that there exist positive constants C_T and c_T, which depend only on T, $\inf_{x\in\Omega}\{v_0(x),\theta_0(x)\}$ and $\|(v_0-1,u_0,\theta_0-1)\|_{H^1(\Omega)}$, such that

$$\inf_{(t,x)\in[0,T]\times\Omega}\{v(t,x),\theta(t,x)\} \geq c_T, \qquad \sup_{t\in[0,T]}\|(v-1,u,\theta-1)(t)\|_{H^1(\Omega)} \leq C_T. \tag{18}$$

In fact, if the maximum interval of existence of the solution $(v(t,x),u(t,x),\theta(t,x))$ is supposed to be $[0,T)$ with $T\in(0,\infty)$, by setting

$$t_1 := T - \frac{1}{2}T_0(c_T,C_T)$$

and choosing $(v,u,\theta)(t_1,\cdot)$ as the initial data, we can use (18) and Proposition 1 to extend the unique solution (v,u,θ) to the time interval $[0,t_1+T_0(c_T,C_T)]$. This contradicts the assumption that $[0,T)$ is the maxinum interval of existence. Therefore, we can prove the uniqueness and existence of the global-in-time solutions by combining proposition 1 and the a priori estimates (18).

We introduce some notations used in the rest of this manuscript. We denote C the generic positive constant which may depend on T and may vary from line to line. We will use $A \lesssim B$ ($B \gtrsim A$) if $A \leq CB$ for some positive constant C. The notation $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$. Moreover, $\|\cdot\|_q$ and $⦀\cdot⦀$ stand for the standard norms of the Lebesgue spaces $L^q(\Omega)$ and $L^\infty([0,T]\times\Omega)$, respectively.

We begin with the fundamental entropy-type energy estimate for the general case $p \geq 1$.

Lemma 2.1. *We have*

$$\sup_{0\leq t\leq T}\int_\Omega \eta(v,u,\theta)\mathrm{d}x + \int_0^T\int_\Omega \left[\frac{\mu u_x^2}{v\theta} + \frac{\kappa\theta_x^2}{v\theta^2}\right] \lesssim 1, \tag{19}$$

where

$$\eta(v,u,\theta) = \psi(v) + \frac{1}{2}u^2 + c_v\phi(\theta) \tag{20}$$

with

$$\phi(z) = z - \ln z - 1 \tag{21}$$

and

$$\psi(z) = \begin{cases} \phi(z), & p = 1, \\ z - \dfrac{1}{p-1}\left(1 - z^{1-p}\right) - 1, & p > 1. \end{cases} \tag{22}$$

Proof. It follows from (8) that the temperature θ satisfies

$$c_v\theta_t + \frac{\theta u_x}{v^p} = \left(\frac{\kappa\theta_x}{v}\right)_x + \frac{\mu u_x^2}{v}. \tag{23}$$

Multiplying $(8)_1$ by $(1 - v^{-p})$, $(8)_2$ by u and (23) by $(1-\theta^{-1})$, we find

$$\eta(v,u,\theta)_t + \frac{\mu u_x^2}{v\theta} + \frac{\kappa\theta_x^2}{v\theta^2} = \left[\frac{\mu u u_x}{v} + \left(1 - \frac{1}{\theta}\right)\frac{\kappa\theta_x}{v} - u\left(\frac{\theta}{v^p} - 1\right)\right]_x.$$

We integrate this last identity over $[0,T]\times\Omega$ to deduce (19) by using the far-field and boundary conditions (10), (11) or (12). □

We have the following lemma from the estimate (19) by applying the argument in [8, 9].

Lemma 2.2. *A positive constant C_0 exists such that for all $i \in \mathbb{Z}$ and $t \in [0,T]$,*

$$C_0^{-1} \leq \int_{\Omega_i} v(t,x)\mathrm{d}x \leq C_0, \quad C_0^{-1} \leq \int_{\Omega_i} \theta(t,x)\mathrm{d}x \leq C_0, \tag{24}$$

where $\Omega_i := \Omega \cap [i, i+1]$. Moreover, for each integer i and $t \in [0,T]$, there are $a_i(t), b_i(t) \in \Omega_i$ such that

$$C_0^{-1} \leq v(t, a_i(t)) \leq C_0, \quad C_0^{-1} \leq \theta(t, b_i(t)) \leq C_0. \tag{25}$$

Proof. From Lemma 2.1, we find a positive constant e_0 such that

$$\int_{\Omega_i}\psi(v), \int_{\Omega_i}\phi(\theta) \leq e_0 \tag{26}$$

for all integer i. If we use (26) and apply Jensen's inequality to the convex functions $\psi(z)$ and $\phi(z)$, we deduce

$$\psi\left(\int_{\Omega_i} v\right), \phi\left(\int_{\Omega_i}\theta\right) \leq e_0,$$

which yields

$$\alpha_1 \le \int_{\Omega_i} v \le \alpha_2, \quad \beta_1 \le \int_{\Omega_i} \theta \le \beta_2,$$

where α_1 and α_2 are two positive roots of equation $\psi(z) = e_0$, and β_1 and β_2 are two positive roots of equation $\phi(z) = e_0$. Moreover, in virtue of the mean value theorem, for each $t \ge 0$, there are $a_i(t), b_i(t) \in \Omega_i$ such that

$$\alpha_1 \le v(t, a_i(t)) \le \alpha_2, \quad \beta_1 \le \theta(t, b_i(t)) \le \beta_2.$$

The proof of the lemma is completed. □

The next lemma is concerned with the lower bound estimate on the temperature θ.

Lemma 2.3. *For all $x \in \Omega$ and $t \in [0, T]$,*

$$\frac{1}{\theta(t,x)} \lesssim 1 + \left|\left|\left| \frac{1}{\mu v^{2p-1}} \right|\right|\right|. \tag{27}$$

Proof. Let $q > 1$. We multiply (23) by $2q\theta^{-2q-1}$ to find

$$\begin{aligned}\left[c_v\theta^{-2q}\right]_t + 2q(2q+1)\frac{\kappa\theta_x^2}{v\theta^{2q+2}} &\le -2q\left[\frac{\kappa\theta_x}{v\theta^{2q+1}}\right]_x + 2q\theta^{-2q+1}\left(\frac{u_x}{v^p\theta} - \frac{\mu u_x^2}{v\theta^2}\right)\\ &\le -2q\left[\frac{\kappa\theta_x}{v\theta^{2q+1}}\right]_x + \frac{q}{2}\theta^{-2q+1}\frac{1}{\mu v^{2p-1}}.\end{aligned}$$

Integrating this identity over Ω, we have

$$\frac{d}{dt}\left\|\theta^{-1}(t)\right\|_{2q}^{2q} \lesssim q\int_\Omega \theta^{-2q+1}\frac{1}{\mu v^{2p-1}} \lesssim q\left\|\theta^{-1}(t)\right\|_{2q}^{2q-1}\left\|\frac{1}{\mu v^{2p-1}}(t)\right\|_{2q},$$

which gives

$$\left\|\theta^{-1}(t)\right\|_{2q} \lesssim 1 + \int_0^t \left\|\frac{1}{\mu v^{2p-1}}(s)\right\|_{2q} \mathrm{d}s.$$

Letting q go to infinity, we can derive (27). This lemma follows. □

Lemma 2.4. *For each $t \in [0, T]$,*

$$\int_0^t\int_\Omega \frac{\mu u_x^2}{v} \lesssim \left[1 + \left|\left|\left|\frac{1}{\mu v^{2p-1}}\right|\right|\right|\right]\left[1 + \int_0^t \|\theta(s)\|_\infty \mathrm{d}s\right]. \tag{28}$$

Proof. Multiply $(8)_2$ by u to find

$$\left(\frac{1}{2}u^2\right)_t + \frac{\mu u_x^2}{v} = \left[\mu\frac{u u_x}{v} + \left(1 - \frac{\theta}{v^p}\right)u\right]_x + \left(\frac{\theta}{v^p} - 1\right)u_x.$$

We integrate this last identity over $[0, t] \times \Omega$ to obtain

$$\begin{aligned}&\int_0^t\int_\Omega \frac{\mu u_x^2}{v}\\ &\lesssim 1 + \left|\int_0^t\int_\Omega\left(\frac{1}{v^p} - 1\right)u_x\right| + \left|\int_0^t\int_\Omega \frac{\theta - 1}{v^p}u_x\right|\\ &\lesssim 1 + \left|\int_0^t\int_\Omega \psi(v)_t\right| + \epsilon\int_0^t\int_\Omega\frac{\mu u_x^2}{v} + C(\epsilon)\left|\left|\left|\frac{1}{\mu v^{2p-1}}\right|\right|\right|\int_0^t\int_\Omega(\theta-1)^2\\ &\lesssim 1 + \epsilon\int_0^t\int_\Omega\frac{\mu u_x^2}{v} + C(\epsilon)\left|\left|\left|\frac{1}{\mu v^{2p-1}}\right|\right|\right|\int_0^t\int_\Omega(\theta-1)^2.\end{aligned} \tag{29}$$

To complete the proof of this lemma, it remains to show

$$\int_0^t \int_\Omega (\theta-1)^2 \lesssim 1 + \int_0^t \|\theta(s)\|_\infty \mathrm{d}s.$$

For this, we notice that ϕ, which is defined by (21), satisfies

$$\begin{cases} \phi(z) & \geq \frac{1}{2}(z-1), \qquad \forall\ z \geq C, \\ \phi(z) & \geq C^{-1}(z-1)^2, \qquad \forall\ 0 < z \leq C, \end{cases}$$

for some sufficiently large constant $C > 1$. Hence we use (19) to deduce

$$\int_0^t \int_\Omega (\theta-1)^2 \lesssim \int_0^t \int_\Omega (1+|\theta-1|)\phi(\theta) \lesssim 1 + \int_0^t \|\theta(s)\|_\infty \mathrm{d}s.$$

We complete the proof of this lemma. □

We will employ Kanel's technique [6] to estimate positive lower and upper bounds on v in the proof of Theorem 1.2. For this, we first make an estimate on $\left\|\frac{\mu}{v}v_x(t)\right\|_2$ in the following lemma.

Lemma 2.5. *For each $t \in [0,T]$,*

$$\left\|\frac{\mu(v)v_x}{v}(t)\right\|_2^2 + \int_0^t \int_\Omega \frac{\mu(v)\theta v_x^2}{v^{p+2}} \lesssim 1 + \int_0^t \int_\Omega \frac{\mu(v)u_x^2}{v} + \int_0^t \int_\Omega \frac{\mu(v)\theta_x^2}{v^p\theta}. \tag{30}$$

Proof. We utilize (8) to derive

$$\left(\frac{\mu(v)}{v}v_x\right)_t = \left(\frac{\mu(v)}{v}v_t\right)_x = u_t + \frac{\theta_x}{v^p} - p\frac{\theta v_x}{v^{p+1}}.$$

Multiplying the above identity by $\frac{\mu(v)}{v}v_x$, we infer

$$\left[\frac{1}{2}\left(\frac{\mu}{v}v_x\right)^2\right]_t + p\frac{\mu(v)\theta v_x^2}{v^{p+2}} + \left[\frac{\mu u u_x}{v}\right]_x = \left[\frac{\mu v_x u}{v}\right]_t + \frac{\mu u_x^2}{v} + \frac{\mu v_x\theta_x}{v^{p+1}}.$$

Integrate this last identity over $[0,T]\times\Omega$ and use (19) to find

$$\begin{aligned} &\left\|\frac{\mu v_x}{v}(t)\right\|_2^2 + \int_0^t \int_\Omega \frac{\mu\theta v_x^2}{v^{p+2}} \\ \lesssim\ & 1 + \int_0^t \int_\Omega \frac{\mu u_x^2}{v} + \int_0^t \int_\Omega \frac{\mu|v_x\theta_x|}{v^{p+1}} \\ \lesssim\ & 1 + \int_0^t \int_\Omega \frac{\mu u_x^2}{v} + \epsilon\int_0^t \int_\Omega \frac{\mu\theta v_x^2}{v^{p+2}} + C(\epsilon)\int_0^t \int_\Omega \frac{\mu\theta_x^2}{v^p\theta}. \end{aligned} \tag{31}$$

We can achieve (30) by taking $\epsilon > 0$ suitable small. □

We next make an estimate of the upper bound on θ.

Lemma 2.6. *It holds for all $t \in [0,T]$ that*

$$\|\theta(t)\|_\infty \lesssim 1 + \int_0^t \|\theta(s)\|_\infty^2 \mathrm{d}s + \left[\left\|\frac{\mu}{v}\right\| + \left\|\frac{1}{v^{2p}}\right\|\right]\int_0^t \|u_x(s)\|_\infty^2 \mathrm{d}s. \tag{32}$$

Proof. We multiply (23) by $2q(\theta-1)^{2q-1}$ to find

$$\begin{aligned} &c_v\left[(\theta-1)^{2q}\right]_t + 2q(2q-1)\frac{\kappa\theta_x^2}{v} - \left[2q(\theta-1)^{2q-1}\frac{\kappa\theta_x}{v}\right]_x \\ &\quad = 2q(\theta-1)^{2q-1}\left(\frac{\mu u_x^2}{v} - \frac{\theta u_x}{v^p}\right). \end{aligned} \tag{33}$$

Integrating (33) over Ω, we obtain

$$\frac{d}{dt}\|(\theta-1)(t)\|_{2q}^{2q} \lesssim \|(\theta-1)(t)\|_{2q}^{2q-1}\left\|v^{-1}\mu u_x^2 + v^{-p}\theta|u_x|\right\|_{2q},$$

which implies

$$\|(\theta-1)(t)\|_{2q} \lesssim 1 + \int_0^t \left\|v^{-1}\mu u_x^2 + v^{-p}\theta|u_x|\right\|_{2q} \mathrm{d}s.$$

We then derive (32) by taking $q\to\infty$ and applying Cauchy's inequality. $\square$

Lemma 2.7. *For each $t\in[0,T]$,*

$$\begin{aligned} &\|u_x(t)\|_2^2 + \int_0^t\int_\Omega \frac{\mu u_{xx}^2}{v} \\ &\quad \lesssim 1 + \int_0^t\int_\Omega \frac{\theta_x^2}{\mu v^{2p-1}} + \int_0^t\int_\Omega \frac{\theta^2 v_x^2}{\mu v^{2p+1}} + \int_0^t\int_\Omega \frac{v}{\mu}\left(\frac{\mu}{v}\right)_x^2 u_x^2. \end{aligned} \tag{34}$$

Proof. Multiply (8) by u_{xx} to find

$$\left(\frac{1}{2}u_x^2\right)_t + \frac{\mu u_{xx}^2}{v} = [u_x u_t]_x + \mathcal{P}_x u_{xx} - \left(\frac{\mu}{v}\right)_x u_x u_{xx},$$

which along with Cauchy's inequality gives

$$\|u_x(t)\|_2^2 + \int_0^t\int_\Omega \frac{\mu u_{xx}^2}{v} \lesssim 1 + \int_0^t\int_\Omega \frac{v}{\mu}\left(\frac{\mu}{v}\right)_x^2 u_x^2 + \int_0^t\int_\Omega |\mathcal{P}_x u_{xx}|. \tag{35}$$

The last term on the right hand side of (35) is estimated as

$$\begin{aligned} \int_0^t\int_\Omega |\mathcal{P}_x u_{xx}| &\lesssim \epsilon\int_0^t\int_\Omega \frac{\mu u_{xx}^2}{v} + C(\epsilon)\int_0^t\int_\Omega \frac{v}{\mu}\mathcal{P}_x^2 \\ &\lesssim \epsilon\int_0^t\int_\Omega \frac{\mu u_{xx}^2}{v} + C(\epsilon)\int_0^t\int_\Omega \frac{v}{\mu}\left(\frac{\theta_x^2}{v^{2p}} + \frac{\theta^2 v_x^2}{v^{2p+2}}\right). \end{aligned} \tag{36}$$

We plug (36) into (35) and take $\epsilon>0$ suitable small to achieve (34). $\square$

3. **Proof of Theorem 1.1.** In this section we prove Theorem 1.1 by extending the argument in [8, 9] for constant transport coefficients to the case when μ is a positive constant and κ may depend on temperature θ.

We recall that Kazhikhov and Shelukhin in [9] and Kazhikhov in [8] have discovered a representation of specific volume for the ideal polytropic gases. Thus we first derive a similar representation of the specific volume v for the p-th Newtonian fluid (8) with pressure $\mathcal{P}=\theta/v^p$ $(p\geq 1)$, and use it to deduce the lower and upper bounds of v, which is a key point to the global solvability result.

Lemma 3.1. *Under the assumptions in Theorem 1.1, we have for all $(t,x)\in[0,T]\times\Omega$ that*

$$v(t,x)\sim 1, \quad \theta(t,x)\gtrsim 1. \tag{37}$$

Proof. Let $x \in \Omega_i := \Omega \cap [i, i+1]$ for any fixed integer i. We divide the proof into three steps.

Step 1. Since μ is assumed to be a positive constant, we integrate $(8)_2$ over $[0,t] \times [a_i(t), x]$ to infer

$$\int_{a_i(t)}^{x} (u(t,z) - u_0(z))\,\mathrm{d}z + \int_0^t \left(\frac{\theta(s,x)}{v^p(s,x)} - \frac{\theta(s,a_i(t))}{v^p(s,a_i(t))} \right) \mathrm{d}s$$
$$= \frac{\mu}{p} \ln \frac{v^p(t,x) v_0^p(a_i(t))}{v_0^p(x) v^p(t,a_i(t))},$$

which implies

$$\frac{1}{v^p(t,x)} \exp\left(\frac{p}{\mu} \int_0^t \frac{\theta(s,x)}{v^p(s,x)} \mathrm{d}s \right) = \frac{1}{B_i(t,x) Y_i(t)}, \tag{38}$$

where

$$B_i(t,x) = \frac{v_0^p(x) v^p(t,a_i(t))}{v_0^p(a_i(t))} \exp\left(\frac{p}{\mu} \int_{a_i(t)}^{x} (u(t,z) - u_0(z))\,\mathrm{d}z \right), \tag{39}$$

$$Y_i(t) = \exp\left(-\frac{p}{\mu} \int_0^t \frac{\theta(s,a_i(t))}{v^p(s,a_i(t))} \mathrm{d}s \right). \tag{40}$$

We multiply (38) by $\frac{p}{\mu}\theta(t,x)$ to find

$$\frac{\partial}{\partial t} \exp\left(\frac{p}{\mu} \int_0^t \frac{\theta(s,x)}{v^p(s,x)} \mathrm{d}s \right) = \frac{p}{\mu} \frac{\theta(t,x)}{B_i(t,x) Y_i(t)},$$

which yields

$$\exp\left(\frac{p}{\mu} \int_0^t \frac{\theta(s,x)}{v^p(s,x)} \mathrm{d}s \right) = 1 + \frac{p}{\mu} \int_0^t \frac{\theta(s,x)}{B_i(s,x) Y_i(s)} \mathrm{d}s. \tag{41}$$

Plugging (41) into (38), we deduce

$$v(t,x) = \left[B_i(t,x) Y_i(t) + \frac{p}{\mu} \int_0^t \theta(s,x) \frac{B_i(t,x) Y_i(t)}{B_i(s,x) Y_i(s)} \mathrm{d}s \right]^{\frac{1}{p}}. \tag{42}$$

Step 2. We obtain from (19) that

$$\left| \int_{a_i(t)}^{x} (u(t,z) - u_0(z))\,\mathrm{d}z \right| \le \int_{\Omega_i} \left(u^2(t,z) + u_0^2(z) \right) \mathrm{d}z \lesssim 1,$$

which combined with (25) yields

$$B_i(t,x) \sim 1. \tag{43}$$

To estimate $Y_i(t)$, we first apply Jensen's inequality to $h(z) = z^p$ $(p \ge 1)$ and use (24) to obtain

$$\int_{\Omega_i} v^p \ge \left(\int_{\Omega_i} v \right)^p \gtrsim 1,$$

which combined with (42) and (43) gives

$$\frac{1}{Y_i(t)} \lesssim \frac{1}{Y_i(t)} \int_{\Omega_i} v^p \lesssim 1 + \int_0^t \frac{1}{Y_i(s)} \int_{\Omega_i} \theta(s,z) \mathrm{d}z \mathrm{d}s \lesssim 1 + \int_0^t \frac{1}{Y_i(s)} \mathrm{d}s.$$

Applying Gronwall's inequality, we deduce

$$Y_i(t) \gtrsim 1. \tag{44}$$

We plug (43)-(44) into (42) to obtain $v(t,x) \gtrsim 1$. And we get a lower bound for the specific volume v. Then we can derive that $\theta(t,x) \gtrsim 1$ for all $(t,x) \in [0,T] \times \Omega$ by using Lemma 2.3.

Step 3. We first note from (14) that $\kappa = \kappa(\theta) \gtrsim 1$. Since $Y_i(t) \lesssim 1$, we use (42)-(44) to find

$$v(t,x) \lesssim 1 + \int_0^t \theta(s,x)\mathrm{d}s. \tag{45}$$

To control the last term on the right hand side of (45), we utilize (24) and (25) to derive

$$\begin{aligned} \theta(t,x) &\lesssim \theta(t,b_i(t)) + \left|\theta^{\frac{1}{2}}(t,x) - \theta^{\frac{1}{2}}(t,b_i(t))\right|^2 \\ &\lesssim 1 + \left[\int_{\Omega_i} \frac{|\theta_x|}{\sqrt{\theta}}(t,z)\mathrm{d}z\right]^2 \\ &\lesssim 1 + \sup_{z\in\Omega_i} v(t,z) \int_{\Omega_i} \theta \int_{\Omega_i} \frac{\kappa\theta_x^2}{v\theta^2} \\ &\lesssim 1 + \sup_{z\in\Omega_i} v(t,z) \int_{\Omega_i} \frac{\kappa\theta_x^2}{v\theta^2}. \end{aligned} \tag{46}$$

Then we derive

$$v(t,x) \lesssim 1 + \int_0^t \sup_{z\in\Omega_i} v(t,z) \int_{\Omega_i} \frac{\kappa\theta_x^2}{v\theta^2}\mathrm{d}z\mathrm{d}s. \tag{47}$$

Applying Gronwall's inequality to (46), we deduce $\sup\limits_{x\in\Omega_i} v(t,x) \lesssim 1$ from (19). This completes the proof. □

We can easily deduce the following corollary by plugging (37) into (46).

Corollary 1. *Under the assumptions in Theorem 1.1, for each $t \in [0,T]$,*

$$\int_0^t \|\theta(s)\|_\infty \mathrm{d}s \lesssim 1. \tag{48}$$

With (37) and (48) in hand, we can recheck lemmas 2.4-2.7 to obtain the following corollary.

Corollary 2. *Under the assumptions in Theorem 1.1, for each $t \in [0,T]$,*

$$\int_0^t \int_\Omega u_x^2 \lesssim 1, \tag{49}$$

$$\|v_x(t)\|_2^2 + \int_0^t \int_\Omega \theta v_x^2 \lesssim 1 + \int_0^t \int_\Omega \frac{\theta_x^2}{\theta}, \tag{50}$$

$$\|u_x(t)\|_2^2 + \int_0^t \int_\Omega u_{xx}^2 \lesssim 1 + \int_0^t \int_\Omega \theta_x^2 + \int_0^t \int_\Omega \theta^2 v_x^2 + \sup_{t\in[0,T]} \|v_x(t)\|_2^4, \tag{51}$$

$$\|\theta(t)\|_\infty \lesssim 1 + \left[\int_0^t \int_\Omega u_{xx}^2\right]^{\frac{1}{2}}. \tag{52}$$

Proof. We plug (37) and (48) into (28) to derive (49).

Plugging (37) and (49) into (30) gives (50).

Estimate (51) can be proved by applying (37), (49) and Cauchy's inequality to (34).

If we use (37) and (48) and apply Gronwall's inequality to (32), we deduce

$$\|\theta(t)\|_\infty \lesssim 1 + \int_0^t \|u_x(s)\|_\infty^2 \mathrm{d}s,$$

which implies (52) by using the Sobolev's inequality. □

We will deduce an upper bound on temperature θ in the following two lemmas.

Lemma 3.2. *If the limit of $\kappa(\theta)$ at infinity is infinity, then*

$$\theta(t,x) \lesssim 1. \tag{53}$$

Proof. Estimates (19), (37) and (50) yield

$$\|v_x(t)\|_2^2 + \int_0^t \int_\Omega \theta v_x^2 \lesssim 1 + \left\|\left\| \frac{\theta}{\kappa} \right\|\right\|. \tag{54}$$

We plug (19) and (54) into (51) to find

$$\|u_x(t)\|_2^2 + \int_0^t \int_\Omega u_{xx}^2 \lesssim 1 + \left\|\left\| \frac{\theta^2}{\kappa} \right\|\right\| + \|\theta\| \left[1 + \left\|\left\| \frac{\theta}{\kappa} \right\|\right\| \right] + \left\|\left\| \frac{\theta}{\kappa} \right\|\right\|^2,$$

which along with (52) gives

$$\|\theta\| \lesssim 1 + \left\|\left\| \frac{\theta}{\sqrt{\kappa}} \right\|\right\| + \|\theta\|^{\frac{1}{2}} \left[1 + \left\|\left\| \frac{\theta}{\kappa} \right\|\right\|^{\frac{1}{2}} \right] + \left\|\left\| \frac{\theta}{\kappa} \right\|\right\|. \tag{55}$$

Appplying Cauchy's inequality to (55), we have

$$\|\theta\| \leq K + K \left\|\left\| \frac{\theta}{\sqrt{\kappa(\theta)}} \right\|\right\| + K \left\|\left\| \frac{\theta}{\kappa(\theta)} \right\|\right\| \tag{56}$$

for some positive constant K depending solely on T.

Since the limit of $\kappa(\theta)$ at infinity is infinity, there exists a positive constant C_K such that $\kappa(\theta) \geq \max\{4K, 16K^2\}$ for each $\theta \geq C_K$. We derive from (37) and (14) that $\kappa(\theta(t,x)) \geq c$ for each $(t,x) \in [0,T] \times \Omega$ and some positive constant c. Hence, we have

$$\frac{\theta(t,x)}{\sqrt{\kappa(\theta(t,x))}} \leq \frac{C_K}{\sqrt{c}} + \frac{\theta(t,x)}{4K}, \quad \frac{\theta(t,x)}{\kappa(\theta(t,x))} \leq \frac{C_K}{c} + \frac{\theta(t,x)}{4K}. \tag{57}$$

Plugging (57) into (56) yields

$$\|\theta\| \leq K + \frac{KC_K}{\sqrt{c}} + \frac{KC_K}{c} + \frac{\|\theta\|}{2},$$

from which we complete the proof of the lemma. □

Lemma 3.3. *If the limit of $\kappa(\theta)$ at infinity is a constant, then*

$$\theta(t,x) \lesssim 1. \tag{58}$$

Proof. We divide the proof into four steps.

Step 1. Since $\lim\limits_{\theta\to\infty} \kappa(\theta) < \infty$ and $\kappa(\theta)$ is smooth on $[0,\infty)$, there is some positive constant K depending only on T such that $\kappa(\theta(t,x)) \leq K$ for each $(t,x) \in [0,T] \times \Omega$.

We set $w := \frac{1}{2}u^2 + c_v(\theta - 1)$ and deduce

$$w_t = \left[\mu \frac{w_x}{v} + (\kappa - \mu c_v) \frac{\theta_x}{v} \right]_x - (\mathcal{P}u)_x. \tag{59}$$

Multiplying (59) by w and integrating the resulting identity over $[0,t]\times\Omega$, we obtain

$$\frac{1}{2}\|w(t)\|_2^2+\int_0^t\int_\Omega\frac{\mu w_x^2}{v}\le C-\int_0^t\int_\Omega(\kappa-\mu c_v)\frac{\theta_x w_x}{v}+\int_0^t\int_\Omega \mathcal{P}uw_x.$$

Plugging $w_x=uu_x+c_v\theta_x$ into the above inequality, we have

$$\frac{1}{2}\|w(t)\|_2^2+\int_0^t\int_\Omega\frac{c_v\kappa\theta_x^2}{v}\lesssim 1+\int_0^t\int_\Omega\frac{|\theta_x uu_x|}{v}+\int_0^t\int_\Omega\frac{u^2u_x^2}{v}+\int_0^t\int_\Omega|\mathcal{P}uw_x|. \quad (60)$$

We note that $\kappa=\kappa(\theta)$ is bounded away from zero due to (37). If we use (37) and apply Cauchy's inequality to (60), we infer

$$\|w(t)\|_2^2+\int_0^t\int_\Omega\theta_x^2\lesssim 1+\int_0^t\int_\Omega\theta^2u^2+\int_0^t\int_\Omega u^2u_x^2. \quad (61)$$

Step 2. To estimate the last term on the right hand side of (61), we multiply $(8)_2$ by u^3 to find

$$\|u(t)\|_4^4+\int_0^t\int_\Omega\frac{\mu u^2u_x^2}{v}\lesssim 1+\int_0^t\int_\Omega\frac{\theta_x u^2|u_x|}{v^p},$$

which combined with (37) and Cauchy's inequality yields

$$\|u(t)\|_4^4+\int_0^t\int_\Omega u^2u_x^2\lesssim 1+\int_0^t\int_\Omega\theta^2u^2. \quad (62)$$

Plugging (62) into (61), we have

$$\|u(t)\|_4^4+\|(\theta-1)(t)\|_2^2+\int_0^t\int_\Omega\left(\theta_x^2+u^2u_x^2\right)\lesssim 1+\int_0^t\int_\Omega\theta^2u^2. \quad (63)$$

Step 3. We have from (19) and Sobolev's inequality that

$$\begin{aligned}\int_0^t\int_\Omega\theta^2u^2&\lesssim 1+\int_0^t\|(\theta-1)(s)\|_2\|\theta_x(s)\|_2\mathrm{d}s\\&\lesssim 1+\epsilon\int_0^t\int_\Omega\theta_x^2+C(\epsilon)\int_0^t\int_\Omega(\theta-1)^2.\end{aligned} \quad (64)$$

Plugging (64) into (63) and employing Gronwall's inequality, we can derive

$$\|u(t)\|_4^4+\|(\theta-1)(t)\|_2^2+\int_0^t\int_\Omega\left(\theta_x^2+u^2u_x^2\right)\lesssim 1, \quad (65)$$

which along with (64) gives

$$\int_0^t\|\theta(s)\|_\infty^2\lesssim 1. \quad (66)$$

Step 4. We use (37) and (65) to estimate the last term on the right hand side of (50) and have

$$\|v_x(t)\|_2^2+\int_0^t\int_\Omega\theta v_x^2\lesssim 1. \quad (67)$$

Plugging (65), (67) into (51), we derive

$$\|u_x(t)\|_2^2+\int_0^t\int_\Omega u_{xx}^2\lesssim 1+\int_0^t\|v_x(s)\|_2^2\|\theta(s)\|_\infty^2\mathrm{d}s\lesssim 1.$$

which along with (52) implies

$$\|\theta(t)\|_\infty\lesssim 1.$$

The proof of the lemma is completed. $\square$

With the bounds (37), (53) and (58) for both v and θ in hand, Theorem 1.1 follows from the standard continuation argument.

4. **Proof of Theorem 1.2.** In this section we will apply Kanel's technique [6] to estimate the bounds for the specific volume v in terms of the bounds of θ and then give the proof of Theorem 1.2.

Lemma 4.1. *If $\kappa = \theta^b$ for some positive constant b, then we have*

$$\int_0^t \|\theta(s)\|_\infty^b \,\mathrm{d}s \lesssim 1, \tag{68}$$

$$\int_0^t \|\theta(s)\|_\infty^{b+1} \,\mathrm{d}s \lesssim 1 + \|\!|v|\!\| . \tag{69}$$

Proof. Let $x \in \Omega_i$ for any integer i. Then it follows from (25) that

$$\theta^{\frac{b+1}{2}}(t,x) \lesssim \theta^{\frac{b+1}{2}}(t,b_i(t)) + \int_{\Omega_i} \theta^{\frac{b-1}{2}}|\theta_x|(t,z)\mathrm{d}z \lesssim 1 + \left[\int_{\Omega_i} \frac{\kappa\theta_x^2}{v\theta^2}\right]^{\frac{1}{2}} \left[\int_{\Omega_i} v\theta\right]^{\frac{1}{2}}$$

and

$$\theta^{\frac{b}{2}}(t,x) \lesssim 1 + \int_{\Omega_i} \theta^{\frac{b-2}{2}}|\theta_x|(t,z)\mathrm{d}z \lesssim 1 + \left[\int_{\Omega_i} \frac{\kappa\theta_x^2}{v\theta^2}\right]^{\frac{1}{2}} \left[\int_{\Omega_i} v\right]^{\frac{1}{2}},$$

which combined with (19) and (24) imply (68) and (69). □

We now apply Kanel's approach to deduce lower and upper bounds of v in terms of $\|\!|\theta^{1-b}|\!\|$ for the general p-th power pressure $\mathcal{P} = \theta/v^p$.

Lemma 4.2. *Under the assumptions in Theorem 1.2, it holds that*

$$\|\!|v^{-1}|\!\| \lesssim 1 + \|\!|\theta^{1-b}|\!\|^{\frac{1}{3a-p}}, \tag{70}$$

$$\|\!|v|\!\| \lesssim 1 + \|\!|\theta^{1-b}|\!\|^{\frac{2a+p-1}{(3a-p)(1-2a)}}, \tag{71}$$

$$\left\|\frac{v_x}{v^{1+a}}(t)\right\|_2^2 + \int_0^t \int_\Omega \left[\frac{u_x^2}{v^{1+a}} + \frac{\theta v_x^2}{v^{p+2+a}}\right] \lesssim 1 + \|\!|\theta^{1-b}|\!\|^{\frac{2a+p-1}{3a-p}}. \tag{72}$$

Proof. The proof is divided into three steps.

Step 1. We have from (68) that

$$\int_0^t \|\theta(s)\|_\infty \,\mathrm{d}s \lesssim \|\!|\theta^{1-b}|\!\| . \tag{73}$$

Plugging (73) into (28) gives

$$\int_0^t \int_\Omega \frac{u_x^2}{v^{1+a}} \lesssim \left[1 + \|\!|v^{1+a-2p}|\!\|\right]\left[1 + \|\!|\theta^{1-b}|\!\|\right]. \tag{74}$$

We plug (74) into (30) to find

$$\begin{aligned}
&\left\|\frac{v_x}{v^{1+a}}\right\|_2^2 + \int_0^t \int_\Omega \left[\frac{\theta v_x^2}{v^{p+2+a}} + \frac{u_x^2}{v^{1+a}}\right] \\
&\lesssim 1 + \int_0^t \int_\Omega \frac{u_x^2}{v^{1+a}} + \int_0^t \int_\Omega \frac{\kappa\theta_x^2}{v\theta^2} \left\|\frac{\mu\theta}{v^{p-1}\kappa}\right\| \\
&\lesssim \left[1 + \|\!|v^{1+a-2p}|\!\| + \|\!|v^{1-a-p}|\!\|\right]\left[1 + \|\!|\theta^{1-b}|\!\|\right] \\
&\lesssim \left[1 + \|\!|v^{-1}|\!\|^{2p-a-1}\right]\left[1 + \|\!|\theta^{1-b}|\!\|\right].
\end{aligned} \tag{75}$$

Here we have used the fact that $2p-a-1>a+p-1>0$ under the assumptions in Theorem 1.2.

Step 2. We assume $\int_0^1 v_0(z)dz=1$ without loss of generality and set

$$\Psi(v):=\int_1^v \frac{\sqrt{\psi(z)}}{z^{1+a}}\mathrm{d}z.$$

Since we can easily deduce

$$\Psi(v)\sim\begin{cases} v^{\frac{1}{2}-a}, & v\to\infty,\\ v^{-a+\frac{1-p}{2}}, & v\to 0^+,\end{cases}$$

there are positive constants C_1 and C_2 such that

$$|\Psi(v)|\geq C_1\left(v^{-a+\frac{1-p}{2}}+v^{\frac{1}{2}-a}\right)-C_2. \tag{76}$$

Moreover, we have

$$|\Psi(v)(t,x)|\lesssim\left\|\sqrt{\psi(v)}(t)\right\|_2\left\|\frac{v_x}{v^{1+a}}(t)\right\|_2. \tag{77}$$

Indeed, we derive different representations for $\Psi(v)$ in two cases in order to prove (77).

- If $\Omega=\mathbb{R}$ or $\Omega=(0,\infty)$, then we have from (10) and (11) that

$$\Psi(v)(t,x)=\int_\infty^x \Psi(v(t,z))_z\mathrm{d}z. \tag{78}$$

- If $\Omega=(0,1)$, then we integrate $(8)_1$ over $[0,t]\times\Omega$ to find

$$\int_0^1 v(t,z)\mathrm{d}z=\int_0^1 v_0(z)\mathrm{d}z=1.$$

  Therefore, $h(t)\in(0,1)$ exists for all $t\in[0,T]$ such that $v(t,h(t))=1$ and

$$\Psi(v)(t,x)=\int_{h(t)}^x \Psi(v(t,z))_z\mathrm{d}z. \tag{79}$$

We deduce (77) from the identities (78) and (79) and Hölder's inequality.

Plugging (19) into (77) and using (76), we have

$$\left\|\!\left|v^{-1}\right|\!\right\|^{a+\frac{p-1}{2}}+\|\!|v|\!\|^{\frac{1}{2}-a}\lesssim\sup_{0\leq t\leq T}\left\|\frac{v_x}{v^{1+a}}(t)\right\|_2. \tag{80}$$

Step 3. If we use (75) to control the right hand side of (80) and apply Young's inequality with $p'=\frac{2a+p-1}{2p-a-1}>1$ and $q'=\frac{2a+p-1}{3a-p}$, we deduce

$$\left\|\!\left|v^{-1}\right|\!\right\|^{a+\frac{p-1}{2}}+\|\!|v|\!\|^{\frac{1}{2}-a}\lesssim 1+\epsilon\left\|\!\left|v^{-1}\right|\!\right\|^{(p-\frac{a+1}{2})p'}+C(\epsilon)\left\|\!\left|\theta^{1-b}\right|\!\right\|^{\frac{q'}{2}},$$

which implies (70) and (71). Plugging (70) into (75) gives (72). The lemma follows. $\square$

To obtain an upper bound on θ, we have to make estimate on $\|u_x(t)\|_2$.

Lemma 4.3. *Under the assumptions in Theorem 1.2, it holds that*

$$\|u_x(t)\|_2^2+\int_0^t\int_\Omega\frac{u_{xx}^2}{v^{1+a}}\lesssim 1+\left\|\!\left|\theta^{1-b}\right|\!\right\|^{a_1}+\left\|\!\left|\theta^{2-b}\right|\!\right\|\left[1+\left\|\!\left|\theta^{1-b}\right|\!\right\|^{a_2}\right]. \tag{81}$$

with $a_1=\frac{4a-4a^2+2+(p-1)(5-4a)}{(3a-p)(1-2a)}$ *and* $a_2=\frac{(2a+p-1)(a+2-2p)}{(3a-p)(1-2a)}$.

Proof. Apply (16) to estimate (34) as

$$\|u_x(t)\|_2^2+\int_0^t\int_\Omega \frac{u_{xx}^2}{v^{1+a}} \lesssim 1+\int_0^t\int_\Omega \frac{\theta_x^2}{v^{2p-1-a}}+\int_0^t\int_\Omega \frac{\theta^2 v_x^2}{v^{2p+1-a}}+\int_0^t\int_\Omega \frac{v_x^2 u_x^2}{v^{3+a}}. \tag{82}$$

Since we can deduce $2-2p+a>3a+1-2p\geq 0$ from the assumptions on a stated in Theorem 1.2, we use (69)-(72) and (19) to estimate the terms on the right hand side of (81) as

$$\begin{aligned}\int_0^t\int_\Omega \frac{\theta_x^2}{v^{2p-1-a}} &\lesssim \|\!|v|\!\|^{2-2p+a}\,\|\!|\theta^{2-b}|\!\|\int_0^t\int_\Omega \frac{\kappa\theta_x^2}{v\theta^2}\\ &\lesssim \|\!|\theta^{2-b}|\!\|\left[1+\|\!|\theta^{1-b}|\!\|^{a_2}\right],\end{aligned} \tag{83}$$

$$\begin{aligned}\int_0^t\int_\Omega \frac{\theta^2 v_x^2}{v^{2p+1-a}} &\lesssim \|\!|v|\!\|^{3a+1-2p}\sup_{0\leq s\leq t}\left\|\frac{v_x}{v^{1+a}}(s)\right\|_2^2\|\!|\theta^{1-b}|\!\|\int_0^t\|\theta(s)\|_\infty^{b+1}\mathrm{d}s\\ &\lesssim 1+\|\!|\theta^{1-b}|\!\|^{a_3}\end{aligned} \tag{84}$$

with $a_3=\frac{7a-4a^2-1+(p-1)(2-2p-a)}{(3a-p)(1-2a)}$ and

$$\begin{aligned}\int_0^t\int_\Omega \frac{v_x^2u_x^2}{v^{3+a}} &\lesssim \|\!|v^{-1}|\!\|^{1-a}\sup_{0\leq s\leq t}\left\|\frac{v_x}{v^{1+a}}(s)\right\|_2^2\int_0^t\|u_x(s)\|_\infty^2\mathrm{d}s\\ &\lesssim \left[1+\|\!|\theta^{1-b}|\!\|^{\frac{4a-4a^2+2+(p-1)(5-4a)}{2(3a-p)(1-2a)}}\right]\left[\int_0^t\int_\Omega\frac{u_{xx}^2}{v^{1+a}}\right]^{\frac12}.\end{aligned} \tag{85}$$

Here we have used

$$\begin{aligned}\int_0^t\|u_x(s)\|_\infty^2\mathrm{d}s &\lesssim \|\!|v|\!\|^{1+a}\left[\int_0^t\int_\Omega\frac{u_x^2}{v^{1+a}}\right]^{\frac12}\left[\int_0^t\int_\Omega\frac{u_{xx}^2}{v^{1+a}}\right]^{\frac12}\\ &\lesssim \left[1+\|\!|\theta^{1-b}|\!\|^{a_4}\right]\left[\int_0^t\int_\Omega\frac{u_{xx}^2}{v^{1+a}}\right]^{\frac12}\end{aligned} \tag{86}$$

with $a_4=\frac{3(2a+p-1)}{2(3a-p)(1-2a)}$. Since $a_3<a_1$ under the assumptions in Theorem 1.2, we get (81) by plugging (83)-(85) into (82) and applying Cauchy's inequality. The lemma follows. □

Proof of Theorem 1.2. To complete the proof of Theorem 1.2, it remains to get the upper and lower bounds of both θ and v. To this end, we have from (32), (69)-(72), (81) and (86) that

$$\begin{aligned}&\|\theta(t)\|_\infty\\ &\lesssim 1+\int_0^t\|\theta(s)\|_\infty^2\,\mathrm{d}s+\left[1+\|\!|v^{-1}|\!\|^{2p}\right]\left[1+\|\!|\theta^{1-b}|\!\|^{a_4}\right]\left[\int_0^t\int_\Omega\frac{u_{xx}^2}{v^{1+a}}\right]^{\frac12}\\ &\lesssim 1+\|\!|\theta^{1-b}|\!\|^{a_5}+\left[1+\|\!|\theta^{1-b}|\!\|^{a_6}\right]\left[\int_0^t\int_\Omega\frac{u_{xx}^2}{v^{1+a}}\right]^{\frac12}\\ &\lesssim 1+\|\!|\theta^{1-b}|\!\|^{a_7}+\|\!|\theta^{2-b}|\!\|^{\frac12}\left[1+\|\!|\theta^{1-b}|\!\|^{a_8}\right]\end{aligned} \tag{87}$$

with

$$\begin{cases} a_5 = \dfrac{7a-1-6a^2+2a(p-1)}{(3a-p)(1-2a)}, \quad a_6 = \dfrac{4-2a+(p-1)(7-8a)}{2(3a-p)(1-2a)}, \\ a_7 = \max\left\{a_5, a_6+\dfrac{a_1}{2}\right\} = \dfrac{a+3-2a^2+6(p-1)(1-a)}{(3a-p)(1-2a)}, \\ a_8 = a_6 + \dfrac{a_2}{2} = \dfrac{4-2a+2a^2+(p-1)(9-11a-2p)}{2(3a-p)(1-2a)}. \end{cases}$$

We first consider the case $b \geq 1$. In this case, we have from (27) and (70) that

$$\left\|\theta^{-1}\right\| \lesssim 1 + \left\|v^{-1}\right\|^{2p-1-a} \lesssim 1 + \left\|\theta^{-1}\right\|^{\frac{(2p-1-a)(b-1)}{3a-p}},$$

which along with the condition (17) implies

$$\left\|\theta^{-1}\right\| \lesssim 1. \tag{88}$$

Moreover, we plug (88) into (70) and (71) and get $v(t,x) \sim 1$ for all $(t,x) \in [0,T]\times\Omega$. Since $1 \leq b < \frac{2a+p-1}{2p-1-a} < 2$, we obtain from (87) and (88) that

$$\|\theta\| \lesssim 1 + \left\|\theta^{2-b}\right\|^{\frac{1}{2}} \lesssim 1 + \|\theta\|^{\frac{2-b}{2}},$$

which yields $\|\theta\| \lesssim 1$ by applying Young's inequality.

For the case $b < 1$, we have from (87) that

$$\|\theta\| \lesssim 1 + \|\theta\|^{(1-b)a_7} + \|\theta\|^{\frac{2-b}{2}+(1-b)a_8},$$

which along with the conditions stated in Theorem 1.2, yields $\|\theta\| \lesssim 1$. With this upper bound on θ, the lower and upper bounds on v can be easily obtained from (70) and (71). And then we obtain the lower bound on θ from (27).

Then integrating (23) multiplied with θ_{xx} over $(0,t)\times\Omega$, we have

$$\|\theta_x(t)\|_2^2 + \int_0^t \int_\Omega \theta_{xx}^2 \lesssim 1,$$

which combined with (19), (72) and (81)

$$\sup_{0\leq t\leq T} \|(v-1,u,\theta-1)(t)\|_{H^1(\Omega)}^2 + \int_0^T \left(\|v_x(t)\|_2^2 + \|(u_x,\theta_x)(t)\|_{H^1(\Omega)}^2\right) \mathrm{d}t \lesssim 1.$$

Theorem 1.2 follows from the standard continuation argument.

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