Nonlinear Analysis: Real World Applications 39 (2018) 93–119

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Asymptotic behavior for cylindrically symmetric nonbarotropic flows in exterior domains with large data

Ling Wan[a], Tao Wang[b,*]

[a] School of Mathematics and Physics, China University of Geosciences, Wuhan, 430074, China
[b] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

A R T I C L E I N F O

Article history:
Received 30 September 2016
Accepted 13 June 2017

Keywords:
Compressible Navier–Stokes equations
Cylindrically symmetric solutions
Asymptotic behavior
Large initial data
Unbounded domains

A B S T R A C T

We study the initial–boundary value problem for the compressible Navier–Stokes equations describing the cylindrically symmetric motion of a viscous nonbarotropic fluid in the domain exterior to a ball in $\mathbb{R}^3$. The global solution is proved to exist uniquely and be asymptotically stable as time tends to infinity for large initial data. Moreover, the density and temperature are shown to be bounded from above and below uniformly in both time and space. Our analysis is based on nonlinear energy methods.

1. Introduction

The cylindrically symmetric motion of a compressible, viscous, and nonbarotropic fluid in the exterior domain $\{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| > a\}$ $(a > 0)$ is formulated by the compressible Navier–Stokes equations (*cf.* Landau and Lifshitz [1]):

$$\rho_t + \frac{(r\rho u)_r}{r} = 0, \tag{1.1a}$$

$$\rho(u_t + uu_r) - \frac{\rho v^2}{r} + P_r = \nu \left[\frac{(ru)_r}{r}\right]_r, \tag{1.1b}$$

$$\rho(v_t + uv_r) + \frac{\rho uv}{r} = \mu \left[\frac{(rv)_r}{r}\right]_r, \tag{1.1c}$$

$$\rho(w_t + uw_r) = \mu w_{rr} + \frac{\mu w_r}{r}, \tag{1.1d}$$

$$\rho(e_t + ue_r) + \frac{P(ru)_r}{r} = \frac{\kappa(r\theta_r)_r}{r} + \mathcal{Q}_c, \tag{1.1e}$$

* Corresponding author.
E-mail addresses: ling.wan@whu.edu.cn (L. Wan), tao.wang@whu.edu.cn (T. Wang).

http://dx.doi.org/10.1016/j.nonrwa.2017.06.006
1468-1218/

with $\nu = 2\mu + \lambda$ and

$$\mathcal{Q}_c = \nu \left[\frac{(ru)_r}{r}\right]^2 - \frac{4\mu u u_r}{r} + \mu\Big(v_r - \frac{v}{r}\Big)^2 + \mu w_r^2. \tag{1.2}$$

Here $t > 0$ is the time, $r > a$ is the radial variable, and the primary dependent variables are the density ρ, the velocity field $\mathbf{u} = (u, v, w)$, and the temperature θ. The components of velocity field $\mathbf{u} = (u, v, w)$ represent the radial, angular, and axial velocities, respectively. For ideal polytropic gases, the pressure P and the specific internal energy e are related with ρ and θ by equations of state:

$$P = R\rho\theta, \quad e = c_v\theta, \tag{1.3}$$

where $R > 0$ and $c_v > 0$ are, respectively, the gas constant and the specific heat at constant volume. The viscosity coefficients μ, λ, and the thermal conductivity coefficient κ are assumed to be positive constants (see Secchi [2] for the mathematical theory of compressible fluids with $\mu = 0$ and $\lambda > 0$).

In this paper, we establish the existence and large-time behavior of the global-in-time solutions to (1.1)–(1.3) in the unbounded domain (a, ∞) with large initial data. We shall consider the system (1.1)–(1.3) supplemented with the initial and boundary conditions:

$$(\rho, u, v, w, \theta)(0, r) = (\rho_0, u_0, v_0, w_0, \theta_0)(r), \qquad r \geq a, \tag{1.4}$$

$$(u, v, w, \theta_r)(t, a) = 0, \qquad t \geq 0. \tag{1.5}$$

The boundary conditions (1.5) are supposed to be compatible with the initial data (1.4).

Let us first mention some related results about the global solvability and large-time behavior for the nonbarotropic compressible Navier–Stokes equations with large data. Kazhikhov and Shelukhin [3] first proved the global existence and uniqueness of solutions to the compressible Navier–Stokes equations in one-dimensional bounded domains with arbitrarily large initial data. The results in [3] have been generalized to cover the spherically and cylindrically symmetric flows. In the case of spherical symmetry, Nikolaev [4] showed the existence of global-in-time (generalized) solutions in bounded annular domains, while Chen and Kratka [5] investigated the flows between a static solid core and a free boundary connected to a surrounding vacuum state. In the cylindrically symmetric case, Frid and Shelukhin [6] obtained the global solvability with large data in a bounded annular domain. Later, Hoff and Jenssen [7] proved global existence of spherically and cylindrically symmetric weak solutions with large discontinuous data in a ball. The argument in [3,6] can be also applied to the case of constant viscosity and temperature dependent thermal conductivity; see [8–11] among others. For the one-dimensional (*resp.* spherically or cylindrically symmetric) fluid in *bounded* domains, a global solution converges exponentially to a constant state as time tends to infinity, which has been established in [12] (*resp.* [13–15]). The boundedness of domains is essential in the aforementioned results.

For the cases in *unbounded* domains, the existence and uniqueness of global solutions was showed by Kazhikhov [16] for one-dimensional ideal polytropic gases with large initial data. The key point in [16] is to get positive upper and lower bounds on the density ρ and temperature θ uniformly in space. It is worth noting that, to derive the asymptotic behavior of the global-in-time solutions, one has to obtain the pointwise bounds on ρ and θ, independent of both space and time. In this direction, Jiang [17] first proved that the density ρ is bounded from below and above uniformly in both space and time by using a decent localized version of the expression on the specific volume. Based on the results in [16,17] and a time-asymptotically nonlinear stability analysis, Li and Liang [18] recently obtain the uniform-in-time upper and lower bounds on the temperature θ as well as the large-time behavior of global solutions with large data. The techniques in the works [16–18] can be employed for deducing the global solvability and asymptotic behavior of spherically symmetric solutions to the compressible Navier–Stokes equations with large initial data; see, for instance, [19–21]. However, there is no analogous result currently available for the cylindrically symmetric flows in unbounded domains.

Our main goal is to show the large-time behavior of global solutions to the initial–boundary value problem (1.1)–(1.5) with large initial data. For this purpose, it is convenient to transform the initial–boundary value problem (1.1)–(1.5) into that in Lagrangian coordinates. We introduce the Lagrangian coordinates (t, x) and denote $(\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\theta})(t, x) = (\rho, u, v, w, \theta)(t, r)$, where

$$r = r(t, x) = r_0(x) + \int_0^t u(s, r(s, x))\mathrm{d}s, \tag{1.6}$$

and

$$r_0(x) := h^{-1}(x), \quad h(r) := \int_a^r z\rho_0(z)\mathrm{d}z. \tag{1.7}$$

Note that the function h is invertible on $[a, \infty)$ provided that $\rho_0(z) > 0$ for each $z \in [a, \infty)$ (which will be assumed in Theorem 1.1). Due to (1.1a), (1.5), and (1.6), we see

$$\frac{\partial}{\partial t}\int_a^{r(t,x)} z\rho(t, z)\mathrm{d}z = 0.$$

Then it is easy to check

$$\int_a^{r(t,x)} z\rho(t, z)\mathrm{d}z = h(r_0(x)) = x \quad \text{and} \quad r(t, 0) = a. \tag{1.8}$$

Hence the region $\{(t, r) : t \geq 0,\ a \leq r < \infty\}$ under consideration is transformed into $\{(t, x) : t \geq 0, 0 \leq x < \infty\}$. Hereafter, we denote $(\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\theta})$ by (ρ, u, v, w, θ) for simplicity. The identities (1.6) and (1.8) imply

$$r_t(t, x) = u(t, x), \quad r_x(t, x) = r^{-1}\tau(t, x), \tag{1.9}$$

where $\tau := 1/\rho$ is the specific volume. By virtue of identities (1.9), system (1.1) can be reformulated to the compressible Navier–Stokes equations in Lagrangian coordinates:

$$\tau_t = (ru)_x, \tag{1.10a}$$

$$u_t - \frac{v^2}{r} + rP_x = \nu r\left[\frac{(ru)_x}{\tau}\right]_x, \tag{1.10b}$$

$$v_t + \frac{uv}{r} = \mu r\left[\frac{(rv)_x}{\tau}\right]_x, \tag{1.10c}$$

$$w_t = \mu r\left[\frac{rw_x}{\tau}\right]_x + \mu w_x, \tag{1.10d}$$

$$e_t + P(ru)_x = \left[\frac{\kappa r^2\theta_x}{\tau}\right]_x + \mathcal{Q}, \tag{1.10e}$$

where $t > 0$, $x \in \Omega := (0, \infty)$, $P = R\theta/\tau$, $e = c_v\theta$, and

$$\mathcal{Q} = \frac{\nu(ru)_x^2}{\tau} - 4\mu uu_x + \mu\tau\left[\frac{rv_x}{\tau} - \frac{v}{r}\right]^2 + \frac{\mu r^2 w_x^2}{\tau}. \tag{1.11}$$

The initial and boundary conditions are

$$(\tau, u, v, w, \theta)(0, x) = (\tau_0, u_0, v_0, w_0, \theta_0)(x), \quad x \in \overline{\Omega}, \tag{1.12}$$

$$(u, v, w, \theta_x)(t, 0) = 0, \quad t \geq 0. \tag{1.13}$$

These boundary conditions are supposed to be compatible with the initial data.

We now state our main results in the following theorem.

Theorem 1.1. *Assume that the initial data* $(\tau_0, u_0, v_0, w_0, \theta_0)$ *are compatible with the boundary condition* (1.13) *and satisfy*

$$\inf_{x\in\Omega}\{\tau_0(x), \theta_0(x)\} > 0, \tag{1.14}$$

$$(\tau_0 - 1, u_0, r_0v_0, w_0, \theta_0 - 1, r_0\tau_{0x}, r_0u_{0x}, r_0v_{0x}, r_0w_{0x}, r_0\theta_{0x}) \in L^2(\Omega). \tag{1.15}$$

Then

(i) *The initial–boundary value problem* (1.10)–(1.13) *has a unique solution* (τ, u, v, w, θ)*, in the sense that, for each* $T > 0$*,*

$$\begin{aligned}&(\tau - 1, u, v, w, \theta - 1) \in L^\infty(0, T; H^1(\Omega)), \quad r(\tau_x, u_x, v_x, w_x, \theta_x) \in L^\infty(0, T; L^2(\Omega)),\\ &\left(u_t, v_t, w_t, \theta_t, r^2u_{xx}, r^2v_{xx}, r^2w_{xx}, r^2\theta_{xx}\right) \in L^2([0, T] \times \Omega),\end{aligned} \tag{1.16}$$

and (τ, u, v, w, θ) *satisfies* (1.10)–(1.13) *almost everywhere in* $(0, T) \times \Omega$ *and takes on the boundary and initial conditions in the sense of traces.*

(ii) *There exists a positive constant* C*, which depends on initial data* $(\tau_0, u_0, v_0, w_0, \theta_0)$*, such that the solution* (τ, u, v, w, θ) *satisfies that for all* $(t, x) \in [0, \infty) \times \overline{\Omega}$*,*

$$C^{-1} \le \tau(t, x) \le C, \quad C^{-1} \le \theta(t, x) \le C, \tag{1.17}$$

$$\int_\Omega |(\tau - 1, u, rv, w, \theta - 1, r\tau_x, ru_x, rv_x, rw_x, r\theta_x)(t, x)|^2 \mathrm{d}x \le C, \tag{1.18}$$

$$\int_0^\infty \int_\Omega \left[|r(\tau_x, u_x, u_{xx}, v_x, v_{xx}, w_x, w_{xx}, \theta_x, \theta_{xx})|^2 + |(u_t, v_t, w_t, \theta_t)|^2\right] \le C. \tag{1.19}$$

Moreover, (τ, u, v, w, θ) *converges to* $(1, 0, 0, 0, 1)$ *uniformly as time tends to infinity:*

$$\lim_{t\to\infty} \sup_{x\in\Omega} |(\tau - 1, u, v, w, \theta - 1)(t, x)| = 0. \tag{1.20}$$

Remark 1.1. The techniques in this paper can be employed to deduce analogous results for the initial–boundary value problem (1.10)–(1.12) with the following boundary conditions

$$(u, v, w, \theta)(t, 0) = (0, 0, 0, 1), \quad t \ge 0.$$

Remark 1.2. The same conclusions as in Theorem 1.1 hold for the pth power Newtonian fluid where the pressure $P(\tau, \theta) = R\theta/\tau$ is replaced by $P(\tau, \theta) = R\theta/\tau^p$ $(p \ge 1)$. See [14] for asymptotic behavior of symmetric solutions in bounded domains and [9] for global existence of large solutions in the case of temperature-dependent thermal conductivity.

Now we make some comments on the analysis for proving Theorem 1.1. As pointed out before, to obtain the large-time behavior of global solutions, one has to deduce the pointwise bounds for $\tau(t, x)$ and $\theta(t, x)$, independent of both space x and time t. Our approach for showing the uniform-in-time boundedness of $\tau(t, x)$ and $\theta(t, x)$ is motivated by that for one-dimensional compressible Navier–Stokes equations in [17,18,22].

By making use of the special structure of ideal polytropic gas (namely (1.10) with constitutive relations (1.3)), one can deduce a localized representation formula for the specific volume $\tau(t, x)$ (see (2.23)), which is essential for establishing the uniform-in-time bounds of $\tau(t, x)$. For the one-dimensional or spherically symmetric compressible Navier–Stokes equations where the angular velocity v vanishes, $\tau(t, x)$ is shown to be uniformly bounded from above through some delicate estimates in [17,20,22]. However, in cylindrically symmetric case, we need to handle the boundedness of the new term $\int_0^t \int_y^\infty \varphi v^2/r^2$ with φ defined by (2.15). Our idea to overcome this difficulty is to deduce Lemma 2.4 by imposing that the initial data v_0 satisfies $r_0 v_0 \in L^2(\Omega)$. Then we can combine Lemma 2.4 with the estimate (2.11) to establish the boundedness of $D(t, y)$ as well as the upper bound of $\tau(t, x)$. The idea for proving the uniform-in-time upper bound of temperature $\theta(t, x)$ is essentially that for one-dimensional flows in [18], but with a fairly nontrivial modification required to deal with the additional terms caused by the presence of the angular and axial velocities v, w.

The rest of this paper is organized as follows. In Section 2, we deduce a series of the a priori estimates for the solution (τ, u, v, w, θ), which will be employed in Section 3 to prove the main theorem.

2. A priori estimates

This section is devoted to deducing the a priori estimates for the solution $(\tau, u, v, w, \theta)(t, \cdot)$ that exists in the time interval $[0, T)$. Throughout this section, we employ c and C to denote various generic positive constants which are independent of T. The notation $C(\cdot)$ is used to denote some positive constant depending only on the arguments listed in the parentheses. We also introduce $A \lesssim B$ if $A \leq CB$ holds uniformly for some constant C independent of T. The notation $B \gtrsim A$ is equivalent to $A \lesssim B$, and $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$. Here and thereafter, the gas constant R is taken to be unity without loss of generality.

2.1. Preliminaries

We start with the basic energy estimate as in the following lemma.

Lemma 2.1. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$\int_\Omega \eta(\tau, u, v, w, \theta)(t, x)\mathrm{d}x + \int_0^t \int_\Omega \left[\frac{r^2 u_x^2}{\tau\theta} + \frac{\tau u^2}{r^2\theta} + \frac{(ru)_x^2}{\tau\theta} + \frac{r^4}{\tau\theta}\left(\frac{v}{r}\right)_x^2 + \frac{r^2 w_x^2}{\tau\theta} + \frac{r^2\theta_x^2}{\tau\theta^2}\right] \lesssim 1, \tag{2.1}$$

where we define

$$\eta(\tau, u, v, w, \theta) := \phi(\tau) + \frac{u^2 + v^2 + w^2}{2} + c_v \phi(\theta), \quad \phi(z) := z - \ln z - 1. \tag{2.2}$$

Proof. Multiply (1.10a), (1.10b), (1.10c), (1.10d), (1.10e) by $(1 - \tau^{-1})$, u, v, w, $(1 - \theta^{-1})$, respectively, and use (1.9) to discover

$$\eta(\tau, u, v, w, \theta)_t + \frac{\kappa r^2\theta_x^2}{\tau\theta^2} + \frac{\mathcal{Q}}{\theta} = \mathcal{R}_x, \tag{2.3}$$

where $\mathcal{Q}$ is given by (1.11) and

$$\mathcal{R} := \frac{\nu r u (ru)_x}{\tau} + \frac{\mu r^2 (v v_x + w w_x)}{\tau} + \left[1 - \frac{1}{\theta}\right]\frac{\kappa r^2 \theta_x}{\tau} + ru\left[1 - \frac{\theta}{\tau}\right] - \mu\left(2u^2 + v^2\right).$$

Thanks to the boundary conditions (1.13), we integrate (2.3) over $[0, t] \times \Omega$ to obtain

$$\int_\Omega \eta(\tau, u, v, w, \theta)(t, x)\mathrm{d}x + \int_0^t \int_\Omega \left[\frac{\mathcal{Q}}{\theta} + \frac{\kappa r^2\theta_x^2}{\tau\theta^2}\right] = \int_\Omega \eta(\tau_0, u_0, v_0, w_0, \theta_0)\mathrm{d}x. \tag{2.4}$$

By virtue of (1.9), we infer that

$$(ru)_x = ru_x + r^{-1}\tau u, \tag{2.5}$$

$$\left(\frac{v}{r}\right)_x = \frac{\tau}{r^2}\left(\frac{rv_x}{\tau} - \frac{v}{r}\right) = \frac{\tau}{r^2}\left[\frac{(rv)_x}{\tau} - \frac{2v}{r}\right], \tag{2.6}$$

and for $\sigma > 0$,

$$\begin{aligned}\nu(ru)_x^2 - 4\mu\tau u u_x &= \nu r^2 u_x^2 + \nu\frac{\tau^2 u^2}{r^2} + 2\lambda\tau u u_x \\ &= \left[\frac{\lambda r u_x}{\sqrt{\sigma}} + \frac{\sqrt{\sigma}\tau u}{r}\right]^2 + \left[\nu - \frac{\lambda^2}{\sigma}\right] r^2 u_x^2 + (\nu - \sigma)\frac{\tau^2 u^2}{r^2} \\ &\geq \left[\nu - \frac{\lambda^2}{\sigma}\right] r^2 u_x^2 + (\nu - \sigma)\frac{\tau^2 u^2}{r^2}.\end{aligned} \tag{2.7}$$

It follows from $\mu > 0$ and $2\mu + 3\lambda \geq 0$ that

$$\nu = 2\mu + \lambda > 0, \quad \frac{\lambda^2}{\nu} < \frac{\lambda^2 + 2\mu^2 + 2\mu\lambda}{\nu} < \nu.$$

Applying (2.7) with $\sigma = (\lambda^2 + 2\mu^2 + 2\mu\lambda)/\nu$, we have from (2.5) and (2.6) that

$$\begin{aligned}\frac{\mathcal{Q}}{\theta} &\gtrsim \frac{r^2 u_x^2}{\tau\theta} + \frac{\tau u^2}{r^2\theta} + \frac{\tau}{\theta}\left[\frac{rv_x}{\tau} - \frac{v}{r}\right]^2 + \frac{r^2 w_x^2}{\tau\theta} \\ &\gtrsim \frac{r^2 u_x^2}{\tau\theta} + \frac{\tau u^2}{r^2\theta} + \frac{(ru)_x^2}{\tau\theta} + \frac{r^4}{\tau\theta}\left(\frac{v}{r}\right)_x^2 + \frac{r^2 w_x^2}{\tau\theta},\end{aligned}$$

which combined with (2.4) implies (2.1). □

Employing Jensen's inequality to the convex function ϕ, we can derive the following lemma from the estimate (2.1). Since the proof is standard we omit it for simplicity (see Kazhikhov [16]).

Lemma 2.2. *For all $(t, x, i) \in [0, T) \times \Omega \times \mathbb{N}$, there are $a_i(t), b_i(t) \in \Omega_i := [i, i+1]$ such that*

$$\int_0^x \tau(t, y)\mathrm{d}y \sim x, \quad \int_{\Omega_i} \tau(t, y)\mathrm{d}y \sim 1, \quad \tau(t, a_i(t)) \sim 1, \tag{2.8}$$

$$\int_0^x \theta(t, y)\mathrm{d}y \sim x, \quad \int_{\Omega_i} \theta(t, y)\mathrm{d}y \sim 1, \quad \theta(t, b_i(t)) \sim 1. \tag{2.9}$$

It follows from (1.8) and (1.9) that

$$r^2(t, x) = a^2 + 2\int_0^x \tau(t, y)\mathrm{d}y, \tag{2.10}$$

which combined with (2.8) yields the following lemma.

Lemma 2.3. *Assume that the conditions listed in Theorem 1.1 hold. Then*

$$r(t, x) \sim \sqrt{1 + x} \quad \text{for all } (t, x) \in [0, T) \times \overline{\Omega}. \tag{2.11}$$

The next lemma is crucial for obtaining uniform-in-time pointwise bounds of τ and a uniform-in-time upper bound of θ.

Lemma 2.4. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$\int_{\Omega} (rv)^2(t,x)\mathrm{d}x + \int_0^t \int_{\Omega} \frac{r^2(rv)_x^2}{\tau} = \int_{\Omega} (r_0 v_0)^2(x)\mathrm{d}x. \tag{2.12}$$

Proof. Due to (1.9) and (1.10c), we discover

$$(rv)_t = \mu r^2 \left[\frac{(rv)_x}{\tau}\right]_x. \tag{2.13}$$

Multiply (2.13) by $2rv$ to get

$$\left[(rv)^2\right]_t + 2\mu \frac{r^2(rv)_x^2}{\tau} = \left[2\mu r^3 v \frac{(rv)_x}{\tau} - 2\mu (rv)^2\right]_x.$$

We integrate this last identity over $[0,t] \times \Omega$ and use (1.13) to conclude (2.12). □

2.2. Pointwise bounds for the specific volume

In Lemma 2.5, we deduce that the specific volume τ is bounded from above and below.

Lemma 2.5. *Assume that the conditions listed in Theorem 1.1 hold. Then for all $(t,x) \in [0,T) \times \overline{\Omega}$,*

$$\tau(t,x) \lesssim 1, \quad \tau(t,x) \geq \left\{ \begin{array}{ll} C(t), & \text{if } T < \infty, \\ C, & \text{if } T = \infty. \end{array} \right. \tag{2.14}$$

Proof. Let $z \in \overline{\Omega}$ be arbitrary but fixed. The proof is divided into six steps.
Step 1 (Representation formula for τ). First we derive a local representation for the specific volume τ.

Let φ be defined by

$$\varphi(x) = \begin{cases} 1, & x < [z]+1, \\ [z]+2-x, & [z]+1 \leq x < [z]+2, \\ 0, & x \geq [z]+2, \end{cases} \tag{2.15}$$

where $[x]$ denotes the largest integer that is less or equal to x.

In view of (1.9), (1.10a) and (1.10b), we have

$$\left(\frac{u}{r}\right)_t + \frac{u^2 - v^2}{r^2} + \left(\frac{\theta}{\tau}\right)_x = \nu\left(\frac{\tau_x}{\tau}\right)_t = \nu(\ln\tau)_{xt}. \tag{2.16}$$

Set $y \in I := ([z]-1, [z]+1) \cap \Omega$. Multiply (2.16) by φ and integrate the resulting identity over (y, ∞) to find

$$\begin{aligned} &-\int_y^{\infty} \left(\frac{\varphi u}{r}\right)_t (t,x)\mathrm{d}x - \int_y^{\infty} \frac{\varphi(u^2 - v^2)}{r^2} \\ &= \nu(\ln\tau)_t(t,y) - \frac{\theta}{\tau}(t,y) + \int_{[z]+1}^{[z]+2} \left[\frac{\theta}{\tau} - \frac{\nu(ru)_x}{\tau}\right](t,x)\mathrm{d}x. \end{aligned} \tag{2.17}$$

We integrate (2.17) over $[0,t]$ to deduce that

$$\frac{1}{\tau(t,y)} \exp\left\{\frac{1}{\nu}\int_0^t \frac{\theta(s,y)}{\tau(s,y)}\mathrm{d}s\right\} = \frac{1}{A(t,y)B(t,y)D(t,y)Y(t)} \tag{2.18}$$

with

$$A(t,y) := \tau_0(y)\exp\left\{\frac{1}{\nu}\int_y^\infty \left[\frac{u_0}{r_0}(x) - \frac{u}{r}(t,x)\right]\varphi(x)\mathrm{d}x\right\}, \tag{2.19}$$

$$B(t,y) := \exp\left\{-\frac{1}{\nu}\int_0^t\int_y^\infty \frac{\varphi u^2}{r^2}\right\}, \tag{2.20}$$

$$D(t,y) := \exp\left\{\frac{1}{\nu}\int_0^t\int_y^\infty \frac{\varphi v^2}{r^2}\right\}, \tag{2.21}$$

$$Y(t) := \exp\left\{\frac{1}{\nu}\int_0^t\int_{[z]+1}^{[z]+2}\left[\frac{\nu(ru)_x}{\tau} - \frac{\theta}{\tau}\right]\right\}. \tag{2.22}$$

We then multiply (2.18) by $\theta(t,y)/\nu$ and integrate the resulting identity over $[0,t]$ to obtain

$$\exp\left\{\frac{1}{\nu}\int_0^t \frac{\theta(s,y)}{\tau(s,y)}\mathrm{d}s\right\} = 1 + \frac{1}{\nu}\int_0^t \frac{\theta(s,y)}{A(s,y)B(s,y)D(s,y)Y(s)}\mathrm{d}s.$$

Combining this last identity with (2.18) yields

$$\tau(t,y) = A(t,y)B(t,y)D(t,y)Y(t) + \frac{1}{\nu}\int_0^t \frac{A(t,y)B(t,y)D(t,y)Y(t)}{A(s,y)B(s,y)D(s,y)Y(s)}\theta(s,y)\mathrm{d}s \tag{2.23}$$

for $(t,y) \in [0,\infty) \times I$, where A, B, D, and Y are given by (2.19)–(2.22).

Step 2 (Boundedness of A and D). We have to make the estimates for A, B, D, and Y in order to establish the pointwise boundedness of τ. We show that both $A(t,y)$ and $D(t,y)$ are uniformly bounded from below and above in this step.

Since

$$\int_y^\infty \left[\frac{u_0}{r_0}(x) - \frac{u}{r}(t,x)\right]\varphi(x)\mathrm{d}x \lesssim 1 + \int_y^{[z]+2}\left[u^2(t,x) + u_0^2(x)\right]\mathrm{d}x \lesssim 1,$$

we get

$$A(t,y) \sim 1. \tag{2.24}$$

Due to (1.13) and (2.11), we have

$$(rv)^2(t,x) = \left[\int_0^x (rv)_x(t,x_1)\mathrm{d}x_1\right]^2 \lesssim \int_0^x \tau(t,x_1)\mathrm{d}x_1 \int_0^x \frac{r^2(rv)_x^2}{\tau}(t,x_1)\mathrm{d}x_1, \tag{2.25}$$

which combined with (2.10) implies

$$\begin{aligned}\int_y^\infty \frac{\varphi v^2}{r^2} &\lesssim \int_y^{[z]+2}\frac{(rv)^2}{r^4} \lesssim \int_y^{[z]+2}\frac{1}{r^4(t,x)}\left[\int_0^x \tau\right]\left[\int_0^x \frac{r^2(rv)_x^2}{\tau}\right]\mathrm{d}x \\ &\lesssim \int_y^{[z]+2}\left[\int_0^x \frac{r^2(rv)_x^2}{\tau}\right]\mathrm{d}x \lesssim \int_0^{[z]+2}\frac{r^2(rv)_x^2}{\tau}\mathrm{d}x.\end{aligned} \tag{2.26}$$

It then follows from (2.12) that

$$D(t,y) \sim 1. \tag{2.27}$$

Step 3 (Estimates for Y). Let $t \geq s \geq 0$. Applying Jensen's inequality to the convex function z^{-1} $(z > 0)$, we have from (2.1) and (2.8) that

$$\begin{aligned}\int_s^t \int_{[z]+1}^{[z]+2} \left[\frac{\nu(ru)_x}{\tau} - \frac{\theta}{\tau}\right] &\leq C \int_s^t \int_{[z]+1}^{[z]+2} \frac{(ru)_x^2}{\tau\theta} - \frac{1}{2}\int_s^t \int_{[z]+1}^{[z]+2} \frac{\theta}{\tau} \\ &\leq C - \frac{1}{2}\int_s^t \inf_{([z]+1,[z]+2)} \theta(\cdot,\xi)\left[\int_{[z]+1}^{[z]+2} \tau\right]^{-1} \mathrm{d}\xi \\ &\leq C - C^{-1}\int_s^t \inf_{([z]+1,[z]+2)} \theta(\cdot,\xi)\mathrm{d}\xi.\end{aligned}$$

Following the argument in [22, page 186], we have

$$-\int_s^t \inf_{([z]+1,[z]+2)} \theta(\cdot,\xi)\mathrm{d}\xi \leq C - C^{-1}(t-s),$$

and hence

$$\int_s^t \int_{[z]+1}^{[z]+2} \left[\frac{\nu(ru)_x}{\tau} - \frac{\theta}{\tau}\right] \leq C - C^{-1}(t-s) \quad \text{for } t \geq s \geq 0.$$

This last estimate gives

$$0 \leq Y(t) \leq C\mathrm{e}^{-t/C}, \quad \frac{Y(t)}{Y(s)} \leq C\mathrm{e}^{-(t-s)/C} \quad \text{for } t \geq s \geq 0. \tag{2.28}$$

Step 4 (Upper bound for τ). Noting that

$$B(t,y) \leq 1, \quad \frac{B(t,y)}{B(s,y)} \leq 1 \quad \text{for } t \geq s \geq 0, \tag{2.29}$$

we insert (2.24)–(2.28) into (2.23) to deduce

$$\tau(t,y) \lesssim 1 + \int_0^t \mathrm{e}^{-(t-s)/C}\theta(s,y)\mathrm{d}s. \tag{2.30}$$

In view of (2.9), we apply Hölder's inequality to get

$$\begin{aligned}\left|\theta^{\frac{1}{2}}(t,y) - \theta^{\frac{1}{2}}(t,b_{[z]}(t))\right| &\lesssim \int_I \theta^{-\frac{1}{2}}|\theta_x|(t,x)\mathrm{d}x \\ \lesssim \sup_I \tau^{\frac{1}{2}}(t,\cdot)\left[\int_I \frac{r^2\theta_x^2}{\tau\theta^2}\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_I \theta\mathrm{d}x\right]^{\frac{1}{2}} &\lesssim \sup_I \tau^{\frac{1}{2}}(t,\cdot)\left[\int_I \frac{r^2\theta_x^2}{\tau\theta^2}\mathrm{d}x\right]^{\frac{1}{2}},\end{aligned}$$

which combined with (2.9) implies

$$\theta(t,y) \gtrsim 1 - C\sup_I \tau(t,\cdot)\int_I \frac{r^2\theta_x^2}{\tau\theta^2}\mathrm{d}x \quad \text{for all } y \in I, \tag{2.31}$$

$$\theta(t,y) \lesssim 1 + \sup_I \tau(t,\cdot)\int_I \frac{r^2\theta_x^2}{\tau\theta^2}\mathrm{d}x \quad \text{for all } y \in I. \tag{2.32}$$

Combine (2.32) with (2.30) to get

$$\sup_I \tau(t,\cdot) \lesssim 1 + \int_0^t \sup_I \tau(s,\cdot)\int_I \frac{r^2\theta_x^2}{\tau\theta^2}(s,x)\mathrm{d}x\mathrm{d}s.$$

In light of (2.1), we apply Grönwall's inequality to derive

$$\sup_I \tau(t,\cdot) \lesssim 1 \quad \text{for all } t \in [0,\infty). \tag{2.33}$$

Step 5 (Boundedness of B). By virtue of Hölder's inequality and (1.9), we have

$$\frac{u^2}{r^2}(t,x) = \left[\int_x^\infty \left(\frac{u_x}{r} - \frac{\tau u}{r^3}\right) \mathrm{d}x\right]^2 \lesssim \int_0^\infty \frac{\tau\theta}{r^4}\mathrm{d}x \int_0^\infty \left[\frac{r^2 u_x^2}{\tau\theta} + \frac{\tau u^2}{r^2\theta}\right] \mathrm{d}x. \tag{2.34}$$

It follows from (2.33), (2.11) and (2.9) that

$$\int_0^\infty \frac{\tau\theta}{r^4}\mathrm{d}x \lesssim \sum_{j=0}^\infty \int_j^{j+1} \frac{\theta}{r^4}\mathrm{d}x \lesssim \sum_{j=0}^\infty \frac{1}{(1+j)^2}\int_j^{j+1} \theta \mathrm{d}x \lesssim 1. \tag{2.35}$$

Combine (2.34) with (2.35) and utilize (2.1) to have

$$\begin{aligned}\int_0^t \int_y^\infty \frac{\varphi u^2}{r^2} &\lesssim \int_0^t \int_y^{[z]+2} \frac{u^2}{r^2} \\ &\lesssim \int_0^t \sup_{x\in\Omega} \frac{u^2}{r^2}(s,x)\mathrm{d}s \lesssim \int_0^t \int_0^\infty \left[\frac{r^2 u_x^2}{\tau\theta} + \frac{\tau u^2}{r^2\theta}\right] \lesssim 1,\end{aligned}$$

which combined with (2.29) yields

$$B(t,y) \sim 1. \tag{2.36}$$

Step 6 (Lower bound for τ). Integrating (2.23) over I, we make use of (2.24), (2.27), (2.36) and (2.8) to obtain

$$Y^{-1}(t) \lesssim Y^{-1}(t)\int_I \tau(t,y)\mathrm{d}y \lesssim 1 + \int_0^t Y^{-1}(s)\int_I \theta(s,y)\mathrm{d}y\mathrm{d}s.$$

Apply Grönwall's inequality and (2.9) to get

$$Y^{-1}(t) \lesssim \exp\left\{C\int_0^t \int_I \theta(s,y)\mathrm{d}y\mathrm{d}s\right\} \lesssim \mathrm{e}^{Ct}. \tag{2.37}$$

We plug (2.24), (2.27), (2.36) and (2.37) into (2.23) to deduce

$$\tau(t,y) \gtrsim \mathrm{e}^{-Ct}. \tag{2.38}$$

Integrating (2.23) over I again, we obtain from (2.24)–(2.28), (2.8), and (2.9) that

$$1 \lesssim \int_I Y(t)\mathrm{d}y + \int_0^t \frac{Y(t)}{Y(s)}\int_I \theta(s,y)\mathrm{d}y\mathrm{d}s \lesssim \mathrm{e}^{-t/C} + \int_0^t \frac{Y(t)}{Y(s)}\mathrm{d}s.$$

Consequently,

$$\int_0^t \frac{Y(t)}{Y(s)}\mathrm{d}s \gtrsim 1 - C\mathrm{e}^{-t/C}. \tag{2.39}$$

Then we have from (2.23)–(2.28), (2.36), (2.31)–(2.33) and (2.39) that

$$\begin{aligned}\tau(t,y) &\gtrsim \int_0^t \frac{Y(t)}{Y(s)}\left[1 - C\int_I \frac{r^2\theta_x^2}{\tau\theta^2}\mathrm{d}x\right]\mathrm{d}s \\ &\gtrsim 1 - C\mathrm{e}^{-t/C} - C\left[\int_0^{t/2} + \int_{t/2}^t\right]\frac{Y(t)}{Y(s)}\int_I \frac{r^2\theta_x^2}{\tau\theta^2}\mathrm{d}x\mathrm{d}s \\ &\gtrsim 1 - C\mathrm{e}^{-t/C} - C\int_0^{t/2} \mathrm{e}^{-(t-s)/C}\int_I \frac{r^2\theta_x^2}{\tau\theta^2}\mathrm{d}x\mathrm{d}s - C\int_{t/2}^t \int_I \frac{r^2\theta_x^2}{\tau\theta^2}\mathrm{d}x\mathrm{d}s \\ &\gtrsim 1 - C\mathrm{e}^{-t/C} - C\mathrm{e}^{-t/(2C)} - C\int_{t/2}^t \int_I \frac{r^2\theta_x^2}{\tau\theta^2}\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{2.40}$$

If $T=\infty$, from (2.1), we have

$$\int_0^\infty \frac{r^2\theta_x^2}{\tau\theta^2} \lesssim 1,$$

which combined with (2.40) implies that there exists a positive constant T_0 such that

$$\tau(t,y) \gtrsim 1 \quad \text{for all } y\in I,\ t\ge T_0. \tag{2.41}$$

We then complete the proof of this lemma by combining (2.33), (2.38), and (2.41). □

2.3. Lower bound for the temperature

We establish a local-in-time lower bound for the temperature $\theta(t,x)$ in the following lemma.

Lemma 2.6. *Under the conditions of Theorem 1.1, there exists a positive constant C_5 such that*

$$\theta(t,x)\ge C(t) \qquad \text{for } t\ge 0. \tag{2.42}$$

Proof. By virtue of (1.9) and (1.10e), it follows that

$$c_v\theta_t - \left[\frac{\kappa r^2\theta_x}{\tau}\right]_x \ge \frac{\lambda}{\tau}\left[(ru)_x - \frac{\theta}{2\lambda}\right]^2 - \frac{\theta^2}{4\lambda\tau} + \frac{2\mu(ru)_x^2}{\tau} - 4\mu u u_x \ge -\frac{\theta^2}{4\lambda\tau}.$$

Multiply this last identity by θ^{-2} to find

$$c_v\left(\frac{1}{\theta}\right)_t + \left[\frac{\kappa r^2}{\tau}\left(\frac{1}{\theta}\right)_x\right]_x \le \frac{1}{4\lambda\tau} + \frac{2\kappa r^2\theta_x^2}{\tau\theta^3} \le C_1(t). \tag{2.43}$$

By defining $H(t,x) := \theta^{-1}(t,x) - c_v^{-1}\int_0^t C_1(s)\mathrm{d}s$, we get from (2.43) that

$$\begin{cases} c_v H_t \le \left[\dfrac{\kappa r^2}{\tau}H_x\right]_x & \text{for } (t,x)\in(0,T)\times\Omega, \\ H(0,x) = \dfrac{1}{\theta(0,x)} \le \dfrac{1}{\inf_\Omega \theta_0} & \text{for } x\in\Omega. \end{cases}$$

In view of the maximum principle (see Evans [23]), we infer that

$$H(t,x) \le \frac{1}{\inf_\Omega \theta_0} \quad \text{for all } (t,x)\in[0,T]\times\Omega,$$

which implies (2.42). The proof is completed. □

2.4. Energy estimates

We just consider the case $T=\infty$ in this subsection where a series of energy estimates for the solution (τ,u,v,w,θ) will be deduced. For the case when $T<\infty$, we can obtain similar estimates with bounds depending on T due to (2.14).

Lemma 2.7. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t\ge 0$,*

$$\int_\Omega \left[(\theta-1)^2 + u^4 + v^4 + w^4\right] + \int_0^t\int_\Omega \left[r^2\theta_x^2 + r^2 v_x^2 + r^{-2}v^2\right] \\ + \int_0^t\int_\Omega (1+\theta+u^2+v^2+w^2)\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2 w_x^2\right] \lesssim 1. \tag{2.44}$$

Proof. For $b > 1$, we denote

$$(\theta - b)_+ := \max\{\theta - b, 0\}, \quad \Omega_b(t) := \{x \in \Omega : \ \theta(t,x) > b\}.$$

To make the presentation clearly, we divide the proof of this lemma into six steps.
Step 1. We multiply (1.10e) by $(\theta-2)_+$ and integrate the resulting identity over Ω to find

$$\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\theta-2)_+^2 + \int_{\Omega_2(t)} \frac{\kappa r^2\theta_x^2}{\tau} = -\int_\Omega (\theta-2)_+ P(ru)_x + \int_\Omega (\theta-2)_+\mathcal{Q}. \tag{2.45}$$

To estimate the last term in (2.45), we note from (1.10c) and (2.6) that

$$v_t + \frac{uv}{r} = \mu r\left[\frac{r^2}{\tau}\left(\frac{v}{r}\right)_x\right]_x + 2\mu r\left(\frac{v}{r}\right)_x. \tag{2.46}$$

Multiplying (1.10b), (2.46), and (1.10d) by $2u(\theta-2)_+$, $2v(\theta-2)_+$, and $2w(\theta-2)_+$, respectively, and integrating the resulting identities over Ω imply

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\theta-2)_+\left(u^2+v^2+w^2\right) + 2\int_\Omega (\theta-2)_+\left[\frac{\nu(ru)_x^2}{\tau} + \frac{\mu r^4}{\tau}\left(\frac{v}{r}\right)_x^2 + \frac{\mu r^2 w_x^2}{\tau}\right] \\
&= 2\int_\Omega P(\theta-2)_+(ru)_x + 2\int_\Omega P\partial_x(\theta-2)_+ ru \\
&\quad - 2\int_{\Omega_2(t)} \theta_x\left[\frac{\nu ru(ru)_x}{\tau} + \frac{\mu r^3 v}{\tau}\left(\frac{v}{r}\right)_x + \frac{\mu r^2 w w_x}{\tau}\right] + \int_{\Omega_2(t)} \theta_t\left(u^2+v^2+w^2\right).
\end{aligned} \tag{2.47}$$

Combine (2.45) with (2.47) to have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{c_v}{2}(\theta-2)_+^2 + (\theta-2)_+\left(u^2+v^2+w^2\right)\right] \\
&\quad + \int_{\Omega_2(t)} \frac{\kappa r^2\theta_x^2}{\tau} + \int_\Omega (\theta-2)_+\left[\frac{\nu(ru)_x^2}{\tau} + \frac{\mu r^4}{\tau}\left(\frac{v}{r}\right)_x^2 + \frac{\mu r^2 w_x^2}{\tau}\right] = \sum_{q=1}^{6}\mathcal{I}_q,
\end{aligned} \tag{2.48}$$

where each term $\mathcal{I}_q$ in the decomposition will be defined and estimated below. First we consider the term

$$\mathcal{I}_1 := -4\mu\int_\Omega (\theta-2)_+ u u_x,$$

which is trivially estimated as

$$\forall\ \delta > 0, \quad \mathcal{I}_1 = 2\mu\int_{\Omega_2(t)} \theta_x u^2 \le \delta\int_{\Omega_2(t)} r^2\theta_x^2 + C(\delta)\int_{\Omega_2(t)} u^4. \tag{2.49}$$

Let us define

$$\mathcal{I}_2 := \int_\Omega P(ru)_x(\theta-2)_+ \quad \text{and} \quad \mathcal{I}_3 := 2\int_\Omega Pru\partial_x(\theta-2)_+.$$

Then for all $\epsilon > 0$ and $\delta > 0$, we have

$$\begin{aligned}
&\mathcal{I}_2 + \mathcal{I}_3 \\
&\quad \le \epsilon\int_{\Omega_2(t)} \theta(ru)_x^2 + C(\epsilon)\int_{\Omega_2(t)} \theta(\theta-2)_+^2 + \delta\int_{\Omega_2(t)} r^2\theta_x^2 + C(\delta)\int_{\Omega_2(t)} \theta^2 u^2 \\
&\quad \le \epsilon\int_{\Omega_2(t)} \theta(ru)_x^2 + \delta\int_{\Omega_2(t)} r^2\theta_x^2 + C(\epsilon,\delta)\sup_\Omega(\theta-3/2)_+^2(t,\cdot).
\end{aligned} \tag{2.50}$$

Here we have used (2.14) and

$$\begin{aligned}
&\int_{\Omega_2(t)} \theta(\theta-2)_+^2 + \int_{\Omega_2(t)} \theta^2\left(u^2+v^2+w^2\right)\\
&\quad\lesssim \int_{\Omega_2(t)} \theta(\theta-3/2)_+^2 + \int_{\Omega_2(t)} (\theta-3/2)_+^2\left(u^2+v^2+w^2\right)\\
&\quad\lesssim \sup_{\Omega}(\theta-3/2)_+^2(t,\cdot)\int_{\Omega}\left[\phi(\theta)+u^2+v^2+w^2\right] \lesssim \sup_{\Omega}(\theta-3/2)_+^2(t,\cdot),
\end{aligned} \tag{2.51}$$

due to (2.1). For the term

$$\mathcal{I}_4 := -2\int_{\Omega_2(t)} \theta_x\left[\frac{\nu ru(ru)_x}{\tau}+\frac{\mu r^3 v}{\tau}\left(\frac{v}{r}\right)_x+\frac{\mu r^2 w w_x}{\tau}\right],$$

we have from (2.14) that

$$\mathcal{I}_4 \le \delta\int_{\Omega_2(t)} r^2\theta_x^2 + C(\delta)\int_{\Omega_2(t)}\left[u^2(ru)_x^2 + r^4v^2\left(\frac{v}{r}\right)_x^2 + r^2w^2w_x^2\right]. \tag{2.52}$$

By virtue of (2.5), (2.14) and (2.51), we estimate $\mathcal{I}_5$ as

$$\begin{aligned}
\mathcal{I}_5 &:= c_v^{-1}\int_{\Omega_2(t)}\left[\mathcal{Q}-P(ru)_x\right]\left(u^2+v^2+w^2\right)\\
&\lesssim \int_{\Omega_2(t)}\left[(ru)_x^2+\frac{u^2}{r^2}+r^4\left(\frac{v}{r}\right)_x^2+r^2w_x^2+\theta^2\right]\left(u^2+v^2+w^2\right)\\
&\lesssim \int_{\Omega_2(t)}\left[(ru)_x^2+r^4\left(\frac{v}{r}\right)_x^2+r^2w_x^2+u^2\right]\left(u^2+v^2+w^2\right)+\sup_{\Omega}(\theta-3/2)_+^2(t,\cdot).
\end{aligned} \tag{2.53}$$

Let the approximate scheme $\varphi_\xi(\theta)$ be defined by

$$\varphi_\xi(\theta) := \begin{cases} 1, & \theta-2\ge\xi,\\ (\theta-2)/\xi, & 0\le\theta-2<\xi,\\ 0, & \theta-2<0. \end{cases}$$

For all $\delta>0$, we employ Lebesgue's dominated convergence theorem and (2.14) to derive

$$\begin{aligned}
\mathcal{I}_6 &:= c_v^{-1}\int_{\Omega_2(t)}\left(\frac{\kappa r^2\theta_x}{\tau}\right)_x\left(u^2+v^2+w^2\right)\\
&= \frac{\kappa}{c_v}\lim_{\xi\to0^+}\int_{\Omega}\varphi_\xi(\theta)\left(\frac{r^2\theta_x}{\tau}\right)_x\left(u^2+v^2+w^2\right)\\
&\le -\frac{\kappa}{c_v}\lim_{\xi\to0^+}\int_{\Omega}\varphi_\xi(\theta)\frac{r^2\theta_x}{\tau}\left(u^2+v^2+w^2\right)_x\\
&\le \delta\int_{\Omega_2(t)} r^2\theta_x^2 + C(\delta)\int_{\Omega_2(t)} r^2\left(u^2u_x^2+v^2v_x^2+w^2w_x^2\right).
\end{aligned}$$

Utilizing (2.5) and (2.6), we further get

$$\mathcal{I}_6 \le \delta\int_{\Omega_2(t)} r^2\theta_x^2 + C(\delta)\int_{\Omega_2(t)}\left[u^2(ru)_x^2+r^4v^2\left(\frac{v}{r}\right)_x^2+r^2w^2w_x^2+u^4+v^4\right]. \tag{2.54}$$

Plug (2.49), (2.50), (2.52)–(2.54) into (2.48) and choose $\delta > 0$ suitably small to have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{c_v}{2}(\theta-2)_+^2 + (\theta-2)_+\left(u^2+v^2+w^2\right)\right] \\
&\quad + c\int_{\Omega_2(t)}\left\{r^2\theta_x^2 + (\theta-2)_+\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right]\right\} \\
&\lesssim \epsilon\int_{\Omega_2(t)}\theta(ru)_x^2 + C(\epsilon)\sup_\Omega(\theta-3/2)_+^2(t,\cdot) + \int_{\Omega_2(t)}\left(u^4+v^4+w^4\right) \\
&\quad + \int_\Omega\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right]\left(u^2+v^2+w^2\right). \qquad (2.55)
\end{aligned}$$

Step 2. We note from (2.1) and (2.14) that

$$\begin{aligned}
\int_0^t\int_\Omega r^2\theta_x^2 &= \int_0^t\left[\int_{\Omega_2(s)} + \int_{\Omega\setminus\Omega_2(s)}\right] r^2\theta_x^2 \\
&\lesssim \int_0^t\int_{\Omega_2(s)} r^2\theta_x^2 + \int_0^t\int_{\Omega\setminus\Omega_2(s)}\frac{r^2\theta_x^2}{\tau\theta^2} \lesssim \int_0^t\int_{\Omega_2(s)} r^2\theta_x^2 + 1, \qquad (2.56)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t\int_\Omega(1+\theta)\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] \\
&\quad\lesssim \int_0^t\int_\Omega\left[\frac{1}{\theta}+\theta\right]\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] \\
&\quad\lesssim 1 + \int_0^t\left[\int_{\Omega_3(s)} + \int_{\Omega\setminus\Omega_3(s)}\right]\theta\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] \\
&\quad\lesssim 1 + \int_0^t\int_{\Omega_3(s)}(\theta-2)_+\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right]. \qquad (2.57)
\end{aligned}$$

Integrating (2.55) over $[0,t]$ and using (2.56)–(2.57), we take $\epsilon > 0$ suitably small to deduce

$$\begin{aligned}
&\int_\Omega(\theta-2)_+^2 + \int_0^t\int_\Omega\left\{r^2\theta_x^2 + (1+\theta)\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right]\right\} \\
&\quad\lesssim 1 + \int_0^t\sup_\Omega(\theta-3/2)_+^2(s,\cdot) + \int_0^t\int_{\Omega_2(s)}\left(u^4+v^4+w^4\right) \\
&\qquad + \int_0^t\int_\Omega\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right]\left(u^2+v^2+w^2\right). \qquad (2.58)
\end{aligned}$$

Step 3. In light of (2.1), we have

$$\int_{\Omega_2(t)}\mathrm{d}x \lesssim \int_{\Omega_2(t)}\theta\mathrm{d}x \lesssim \int_{\Omega_2(t)}\phi(\theta)\mathrm{d}x \lesssim 1, \qquad (2.59)$$

which immediately implies

$$\int_{\Omega_2(t)}\left(u^4+v^4+w^4\right) \lesssim \sup_\Omega\left(v^4+u^4+w^4\right)(t,\cdot). \qquad (2.60)$$

For $\epsilon > 0$, it follows from (1.9), (1.13) and (2.1) that

$$\begin{aligned}
v^4(t,x) &= 4\int_0^x\frac{v^3}{r}\left[(rv)_x - r^{-1}\tau v\right] \le 4\int_0^x\frac{v^3(rv)_x}{r} \\
&\lesssim \epsilon^{-1}\int_\Omega r^2(rv)_x^2 + \epsilon\int_\Omega\frac{v^6}{r^4} \lesssim \epsilon^{-1}\int_\Omega r^2(rv)_x^2 + \epsilon\sup_\Omega v^4(t,\cdot),
\end{aligned}$$

from which we get

$$\sup_{\Omega} v^4(t,\cdot) \lesssim \int_{\Omega} r^2 (rv)_x^2. \tag{2.61}$$

We get from (2.1) and (2.59) that

$$\begin{aligned}
\left(u^4+w^4\right)(t,x) &= 4\int_0^x \left(u^3u_x+w^3w_x\right)\\
&\lesssim \epsilon^{-1}\int_{\Omega}\frac{r^2u_x^2+r^2w_x^2}{\theta}+\epsilon\left[\int_{\Omega\setminus\Omega_2(t)}+\int_{\Omega_2(t)}\right]\frac{\theta}{r^2}\left(u^6+w^6\right)\\
&\lesssim \epsilon^{-1}\int_{\Omega}\frac{r^2u_x^2+r^2w_x^2}{\theta}+\epsilon\sup_{\Omega}\left(u^4+w^4\right)+\epsilon\sup_{\Omega}\left(u^6+w^6\right).
\end{aligned}$$

By virtue of this last estimate, we can deduce, for $\epsilon>0$ sufficiently small, that

$$\sup_{\Omega}\left(u^4+w^4\right)\lesssim \epsilon^{-1}\int_{\Omega}\frac{r^2u_x^2+r^2w_x^2}{\theta}+\epsilon\sup_{\Omega}\left(u^6+w^6\right). \tag{2.62}$$

It then follows from (1.9) and (2.1) that

$$\begin{aligned}
\left(u^6+w^6\right)(t,x) &= 6\int_0^x\left(u^5u_x+w^5w_x\right)\\
&\le 6\int_0^x\left[\frac{u^5}{r}(ru)_x+w^5w_x\right]\lesssim \int_{\Omega}\left[\sqrt{\epsilon}\left(u^8+w^8\right)+\frac{1}{\sqrt{\epsilon}}\left(u^2(ru)_x^2+w^2w_x^2\right)\right]\\
&\lesssim \sqrt{\epsilon}\sup_{\Omega}\left[u^2+w^2\right]^3+\frac{1}{\sqrt{\epsilon}}\int_{\Omega}\left(u^2(ru)_x^2+w^2w_x^2\right).
\end{aligned}$$

For $\epsilon>0$ sufficiently small, we infer

$$\sup_{\Omega}\left(u^6+w^6\right)\lesssim\frac{1}{\sqrt{\epsilon}}\int_{\Omega}\left(u^2(ru)_x^2+w^2w_x^2\right). \tag{2.63}$$

Plugging (2.63) into (2.62) yields

$$\sup_{\Omega}\left(u^4+w^4\right)\lesssim\epsilon^{-1}\int_{\Omega}\left[\frac{r^2u_x^2}{\theta}+\frac{r^2w_x^2}{\theta}\right]+\sqrt{\epsilon}\int_{\Omega}\left(u^2(ru)_x^2+w^2w_x^2\right). \tag{2.64}$$

Inserting (2.60) into (2.58), we utilize (2.61), (2.64), (2.1), and (2.12) to discover

$$\begin{aligned}
&\int_{\Omega}(\theta-2)_+^2+\int_0^t\int_{\Omega}\left\{r^2\theta_x^2+(1+\theta)\left[(ru)_x^2+r^4\left(\frac{v}{r}\right)_x^2+r^2w_x^2\right]\right\}\\
&\lesssim 1+\int_0^t\sup_{\Omega}(\theta-3/2)_+^2(s,\cdot)+\int_0^t\int_{\Omega}\left[(ru)_x^2+r^4\left(\frac{v}{r}\right)_x^2+r^2w_x^2\right]\left(u^2+v^2+w^2\right).
\end{aligned} \tag{2.65}$$

Step 4. We now make the estimates for the last term in (2.65). Multiply (1.10d) by w^3 and integrate the resulting identity over Ω to get

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}w^4+\int_{\Omega}\frac{3\mu}{\tau}r^2w^2w_x^2=0. \tag{2.66}$$

We then multiply (2.46) by v^3 to obtain

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}v^4+\int_{\Omega}\frac{3\mu}{\tau}r^4v^2\left(\frac{v}{r}\right)_x^2=-2\mu\int_{\Omega}rv^3\left(\frac{v}{r}\right)_x-\int_{\Omega}\frac{uv^4}{r}\\
&\lesssim\int_{\Omega}\left[\epsilon_1r^4\left(\frac{v}{r}\right)_x^2+\epsilon_1^{-1}\left(v^6+\frac{uv^4}{r}\right)\right]\lesssim\epsilon_1\int_{\Omega}r^4\left(\frac{v}{r}\right)_x^2+\epsilon_1^{-1}\sup_{\Omega}\left[v^4+\frac{|v|^3}{r}\right].
\end{aligned} \tag{2.67}$$

According to (2.11), we have

$$\begin{aligned}\frac{|v|^3}{r}(t,x) &= \left|\frac{3}{r(t,x)}\int_0^x v^2 v_x\right| \\ &\le \sup_\Omega v^4(t,\cdot)\frac{C(\epsilon_1)}{r(t,x)^2}\int_0^x \frac{1}{r^2} + \epsilon_1^2\int_0^x r^2 v_x^2 \\ &\le C(\epsilon_1)\sup_\Omega v^4(t,\cdot) + \epsilon_1^2\int_\Omega r^2 v_x^2 \\ &\le C(\epsilon_1)\sup_\Omega v^4(t,\cdot) + \epsilon_1^2\int_\Omega \left[(rv)_x^2 + r^4\left(\frac{v}{r}\right)_x^2\right]. \end{aligned} \tag{2.68}$$

Plugging (2.61) and (2.68) into (2.67) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^4 + c\int_\Omega r^4 v^2\left(\frac{v}{r}\right)_x^2 \lesssim \epsilon_1\int_\Omega r^4\left(\frac{v}{r}\right)_x^2 + C(\epsilon_1)\int_\Omega r^2(rv)_x^2. \tag{2.69}$$

Multiplying (1.10b) by u^3 and integrating the resulting identity over Ω gives

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^4 + \int_\Omega \frac{3\nu}{\tau}u^2(ru)_x^2 &= \int_\Omega \frac{u^3v^2}{r} + \int_\Omega \frac{2\nu}{r}u^3(ru)_x - \int_\Omega ru^3\left(\frac{\theta}{\tau}\right)_x \\ &\le C\sup_\Omega\frac{|u|^3}{r} + \epsilon_1\int_\Omega (ru)_x^2 + \frac{C}{\epsilon_1}\sup_\Omega u^4 + \int_\Omega (ru^3)_x\left[\frac{\theta-1}{\tau} + \frac{1-\tau}{\tau}\right]. \end{aligned} \tag{2.70}$$

Similar to the derivation of (2.68), we deduce from (2.9) that

$$\frac{|u|^3}{r}(t,x) \lesssim \sup_\Omega u^4(t,\cdot) + \int_\Omega \frac{r^2u_x^2}{\theta}. \tag{2.71}$$

Using (2.1), (2.59) and

$$\int_{\Omega\setminus\Omega_2(t)}(\theta-1)^2 \lesssim \int_{\Omega\setminus\Omega_2(t)}\phi(\theta) \lesssim 1, \tag{2.72}$$

we obtain, for $0<\epsilon_1\le 1$, that

$$\begin{aligned}\int_\Omega (ru^3)_x\frac{\theta-1}{\tau} &= \left[\int_{\Omega_2(t)} + \int_{\Omega\setminus\Omega_2(t)}\right]\left(3u^2(ru)_x - 2r^{-1}u^3\tau\right)\frac{\theta-1}{\tau} \\ &\le \int_{\Omega_2(t)}\left[\epsilon_1u^2(ru)_x^2 + C(\epsilon_1)(\theta-1)^2u^2 + (\theta-1)^2 + u^6\right] \\ &\quad + \int_{\Omega\setminus\Omega_2(t)}\left[\epsilon_1(ru)_x^2 + \frac{C}{\epsilon_1}(\theta-1)^2u^4 + \frac{u^2}{r^2\theta} + (\theta-1)^2u^4\right] \\ &\le \epsilon_1\int_\Omega (1+u^2)(ru)_x^2 + C(\epsilon_1)\sup_{\Omega_2(t)}(\theta-1)^2 + \frac{C}{\epsilon_1}\sup_\Omega u^4 + \int_\Omega\frac{u^2}{r^2\theta}. \end{aligned} \tag{2.73}$$

Noting that (2.1) and (2.14) imply

$$\int_\Omega (1-\tau)^2 \lesssim \int_\Omega \phi(\tau) \lesssim 1, \tag{2.74}$$

we have from (2.14) and (2.59) that

$$\int_\Omega \theta(1-\tau)^2 \lesssim \sup_\Omega(1-\tau)^2\int_{\Omega_2(t)}\theta + \int_{\Omega\setminus\Omega_2(t)}(1-\tau)^2 \lesssim 1.$$

Hence we deduce

$$\begin{aligned}\int_\Omega (ru^3)_x \frac{1-\tau}{\tau} &= \int_\Omega \left(3u^2(ru)_x - 2r^{-1}u^3\tau\right)\frac{1-\tau}{\tau}\\ &\lesssim \epsilon_1 \int_\Omega (ru)_x^2 + \frac{C}{\epsilon_1}\sup_\Omega u^4 + \int_\Omega \theta(1-\tau)^2 + \int_\Omega \frac{u^2}{r^2\theta}\\ &\lesssim \epsilon_1 \int_\Omega (ru)_x^2 + \frac{C}{\epsilon_1}\sup_\Omega u^4 + \int_\Omega \frac{u^2}{r^2\theta}. \end{aligned} \tag{2.75}$$

Plugging (2.71), (2.73) and (2.75) into (2.70), we obtain, for $\epsilon_1 > 0$ sufficiently small, that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^4 + c\int_\Omega u^2(ru)_x^2\\ &\lesssim \epsilon_1^{-1}\sup_\Omega u^4 + C(\epsilon_1)\sup_{\Omega_2(t)}(\theta-1)^2 + \epsilon_1\int_\Omega (ru)_x^2 + \int_\Omega \left[\frac{r^2u_x^2}{\theta} + \frac{u^2}{r^2\theta}\right]. \end{aligned} \tag{2.76}$$

Combining (2.66), (2.69) and (2.76), we use (2.1), (2.12) and (2.64) with $\epsilon = \epsilon_1^4$ to derive

$$\begin{aligned}&\int_\Omega \left[u^4+v^4+w^4\right] + \int_0^t\int_\Omega \left[u^2(ru)_x^2 + r^4v^2\left(\frac{v}{r}\right)_x^2 + r^2w^2w_x^2\right]\\ &\lesssim C(\epsilon_1) + \epsilon_1^{-1}\int_0^t \sup_\Omega u^4 + C(\epsilon_1)\int_0^t \sup_{\Omega_2(s)}(\theta-1)^2 + \epsilon_1\int_0^t\int_\Omega\left[r^4\left(\frac{v}{r}\right)_x^2 + (ru)_x^2\right]\\ &\lesssim C(\epsilon_1) + \epsilon_1\int_0^t\int_\Omega\left[u^2(ru)_x^2 + w^2w_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + (ru)_x^2\right] + C(\epsilon_1)\int_0^t \sup_{\Omega_2(s)}(\theta-1)^2. \end{aligned}$$

Noting that

$$\sup_{\Omega_2(t)}(\theta-1)^2 \lesssim \sup_\Omega(\theta-3/2)_+^2(t,\cdot), \tag{2.77}$$

we find that for $\epsilon_1 > 0$ suitably small,

$$\begin{aligned}&\int_\Omega \left[u^4+v^4+w^4\right] + \int_0^t\int_\Omega\left[u^2(ru)_x^2 + r^4v^2\left(\frac{v}{r}\right)_x^2 + r^2w^2w_x^2\right]\\ &\lesssim C(\epsilon_1) + C(\epsilon_1)\int_0^t \sup_\Omega(\theta-3/2)_+^2 + \epsilon_1\int_0^t\int_\Omega\left[r^4\left(\frac{v}{r}\right)_x^2 + (ru)_x^2\right]. \end{aligned} \tag{2.78}$$

Step 5. Multiply (1.10b), (2.46), and (1.10d) with $(u^2+v^2+w^2)u$, $(u^2+v^2+w^2)v$, and $(u^2+v^2+w^2)w$, respectively, and integrate the resulting identities to have

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(u^2+v^2+w^2\right)^2 = \sum_{q=1}^5 \mathcal{K}_q, \tag{2.79}$$

where the term $\mathcal{K}_q$ in the decomposition will be defined below. We define the first term $\mathcal{K}_1$ as

$$\mathcal{K}_1 := \int_\Omega (u^2+v^2+w^2)\left[2\mu rv\left(\frac{v}{r}\right)_x + \mu ww_x\right],$$

which can be estimated by

$$\begin{aligned}\mathcal{K}_1 &\lesssim \epsilon_1\int_\Omega\left[r^2w_x^2 + r^4\left(\frac{v}{r}\right)_x^2\right] + \epsilon_1^{-1}\int_\Omega r^{-2}\left(u^6+v^6+w^6\right)\\ &\lesssim \epsilon_1\int_\Omega\left[r^2w_x^2 + r^4\left(\frac{v}{r}\right)_x^2\right] + \epsilon_1^{-1}\sup_\Omega\left(u^4+v^4+w^4\right). \end{aligned} \tag{2.80}$$

For the term

$$\mathcal{K}_2 := -\int_\Omega \left[\left(u^2+v^2+w^2\right) rv\right]_x \frac{\mu r^2}{\tau}\left(\frac{v}{r}\right)_x,$$

we have from (1.9) and (2.1) that

$$\begin{aligned}
&\mathcal{K}_2 + \int_\Omega \left(u^2+v^2+w^2\right)\frac{\mu r^4}{\tau}\left(\frac{v}{r}\right)_x^2 = -\int_\Omega \left[\left(u^2+v^2+w^2\right) r^2\right]_x \frac{v}{r}\frac{\mu r^2}{\tau}\left(\frac{v}{r}\right)_x \\
&= -2\int_\Omega \left[ru(ru)_x + r^4\frac{v}{r}\left(\frac{v}{r}\right)_x + 2r^3 r_x\frac{v^2}{r^2} + r^2 w w_x + r r_x w^2\right]\frac{v}{r}\frac{\mu r^2}{\tau}\left(\frac{v}{r}\right)_x \\
&\lesssim \int_\Omega \left[u^2(ru)_x^2 + r^4v^2\left(\frac{v}{r}\right)_x^2 + r^2w^2w_x^2\right] + \epsilon_1\int_\Omega r^4\left(\frac{v}{r}\right)_x^2 + \epsilon_1^{-1}\int_\Omega \frac{v^6+v^2w^4}{r^2} \\
&\lesssim \int_\Omega \left[u^2(ru)_x^2 + r^4v^2\left(\frac{v}{r}\right)_x^2 + r^2w^2w_x^2\right] + \epsilon_1\int_\Omega r^4\left(\frac{v}{r}\right)_x^2 + \epsilon_1^{-1}\sup_\Omega\left(v^4+w^4\right).
\end{aligned} \tag{2.81}$$

The terms

$$\begin{aligned}
\mathcal{K}_3 &:= -\int_\Omega \left[\left(u^2+v^2+w^2\right) ru\right]_x \frac{\nu(ru)_x}{\tau}, \\
\mathcal{K}_4 &:= -\int_\Omega \left[\left(u^2+v^2+w^2\right) rw\right]_x \frac{\mu r w_x}{\tau}
\end{aligned}$$

can be estimated in a similar way, and we can get

$$\begin{aligned}
&\mathcal{K}_3 + \int_\Omega \left(u^2+v^2+w^2\right)\frac{\nu}{\tau}(ru)_x^2 \\
&\quad\lesssim \int_\Omega \left[u^2(ru)_x^2 + r^4v^2\left(\frac{v}{r}\right)_x^2 + r^2w^2w_x^2\right] + \epsilon_1\int_\Omega (ru)_x^2 + \epsilon_1^{-1}\sup_\Omega\left(v^4+u^4+w^4\right),
\end{aligned} \tag{2.82}$$

$$\begin{aligned}
&\mathcal{K}_4 + \int_\Omega \left(u^2+v^2+w^2\right)\frac{\nu}{\tau}r^2w_x^2 \\
&\quad\lesssim \int_\Omega \left[u^2(ru)_x^2 + r^4v^2\left(\frac{v}{r}\right)_x^2 + r^2w^2w_x^2\right] + \epsilon_1\int_\Omega r^2w_x^2 + \epsilon_1^{-1}\sup_\Omega\left(v^4+u^4+w^4\right).
\end{aligned} \tag{2.83}$$

Similar to the derivation of (2.73) and (2.75), we estimate the last term $\mathcal{K}_5$ as

$$\begin{aligned}
\mathcal{K}_5 &:= \int_\Omega \left[\left(u^2+v^2+w^2\right) ru\right]_x \left[\frac{\theta-1}{\tau} + \frac{1-\tau}{\tau}\right] \\
&\lesssim \epsilon_1\int_\Omega \left(1+u^2+v^2+w^2\right)\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] \\
&\quad + C(\epsilon_1)\sup_{\Omega_2(t)}(\theta-1)^2 + \frac{C}{\epsilon_1}\sup_\Omega\left(u^4+v^4+w^4\right) + \int_\Omega \frac{u^2}{r^2\theta}.
\end{aligned} \tag{2.84}$$

Integrating (2.79) over $[0,t]$, we use (2.80)–(2.84) to get, for $\epsilon_1 > 0$ sufficiently small, that

$$\begin{aligned}
&\int_\Omega \left(u^2+v^2+w^2\right)^2 + \int_0^t\int_\Omega \left(u^2+v^2+w^2\right)\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] \\
&\lesssim C(\epsilon_1) + \epsilon_1^{-1}\int_0^t \sup_\Omega\left(u^4+v^4+w^4\right) + C(\epsilon_1)\int_0^t \sup_{\Omega_2(t)}(\theta-1)^2 \\
&\quad + \epsilon_1\int_0^t\int_\Omega \left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] + \int_0^t\int_\Omega \left[u^2(ru)_x^2 + r^4v^2\left(\frac{v}{r}\right)_x^2 + r^2w^2w_x^2\right].
\end{aligned}$$

By virtue of (2.61), (2.64) with $\epsilon = \epsilon_1^4$, (2.1), and (2.77), we have, for $\epsilon_1 > 0$ suitably small, that

$$\begin{aligned} &\int_\Omega \left(u^2+v^2+w^2\right)^2 + \int_0^t\int_\Omega (u^2+v^2+w^2)\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] \\ &\lesssim C(\epsilon_1) + \epsilon_1\int_0^t\int_\Omega \left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] \\ &\quad + C(\epsilon_1)\int_0^t \sup_\Omega(\theta-3/2)_+^2 + \int_\Omega\left[u^2(ru)_x^2 + r^4v^2\left(\frac{v}{r}\right)_x^2 + r^2w^2w_x^2\right]. \end{aligned} \tag{2.85}$$

Step 6. A suitable linear combination of (2.65), (2.78) and (2.85) yields, for $\epsilon_1 > 0$ sufficiently small, that

$$\begin{aligned} &\int_\Omega \left[(\theta-2)_+^2 + \left(u^2+v^2+w^2\right)^2\right] + \int_0^t\int_\Omega r^2\theta_x^2 \\ &\quad + \int_0^t\int_\Omega (1+\theta+u^2+v^2+w^2)\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] \\ &\lesssim C(\epsilon_1) + \epsilon_1\int_0^t\int_\Omega\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] + C(\epsilon_1)\int_0^t\sup_\Omega(\theta-3/2)_+^2. \end{aligned} \tag{2.86}$$

Noting that

$$r^2v_x^2 + r^{-2}v^2 \lesssim (rv)_x^2 + r^4\left(\frac{v}{r}\right)_x^2,$$

and choosing ϵ_1 to be sufficiently small, we can get

$$\begin{aligned} &\int_\Omega \left[(\theta-2)_+^2 + \left(u^2+v^2+w^2\right)^2\right] + \int_0^t\int_\Omega \left[r^2\theta_x^2 + r^2v_x^2 + r^{-2}v^2\right] \\ &+ \int_0^t\int_\Omega (1+\theta+u^2+v^2+w^2)\left[(ru)_x^2 + r^4\left(\frac{v}{r}\right)_x^2 + r^2w_x^2\right] \lesssim 1 + \int_0^t\sup_\Omega(\theta-3/2)_+^2. \end{aligned} \tag{2.87}$$

It remains to control the last term on the right hand side of (2.87). In light of (2.59), we apply standard calculations to obtain for any $\epsilon > 0$ that

$$\begin{aligned} \int_0^T \sup_\Omega(\theta-3/2)_+^2 &= \int_0^T \sup_{x\in\Omega}\left[\int_x^\infty \partial_x(\theta-3/2)_+\right]^2 \\ &\le \int_0^T\left[\int_{\Omega_{3/2}(t)}|\theta_x|\right]^2 \le \int_0^T\int_{\Omega_{3/2}(t)}\frac{\theta_x^2}{\theta}\int_{\Omega_{3/2}(t)}\theta \\ &\lesssim \int_0^T\int_{\Omega_{3/2}(t)}\frac{\theta_x^2}{\theta} \le \epsilon\int_0^T\int_\Omega\theta_x^2 + C(\epsilon)\int_0^T\int_\Omega\frac{r^2\theta_x^2}{\tau\theta^2} \\ &\lesssim \epsilon\int_0^T\int_\Omega\theta_x^2 + C(\epsilon). \end{aligned} \tag{2.88}$$

Noting

$$\int_\Omega(\theta-1)^2 = \left[\int_{\Omega_3(t)} + \int_{\Omega\setminus\Omega_3(t)}\right](\theta-1)^2 \lesssim \int_{\Omega_3(t)}(\theta-2)_+^2 + 1,$$

we plug (2.88) into (2.87) to conclude (2.44). □

In the following lemma we deduce the uniform-in-time estimate for L_x^2-norm of the first-order derivative τ_x.

Lemma 2.8. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$\int_\Omega \tau_x^2 \mathrm{d}x + \int_0^t \int_\Omega (1+\theta)\tau_x^2 \lesssim 1. \tag{2.89}$$

Proof. Multiplying (2.16) by τ_x/τ and integrating the resulting identity over Ω yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{\nu}{2}\left(\frac{\tau_x}{\tau}\right)^2 - \frac{\tau_x}{\tau}\frac{u}{r}\right] + \int_\Omega \frac{\theta\tau_x^2}{\tau^3} = -\int_\Omega \frac{u}{r}\left(\frac{\tau_x}{\tau}\right)_t + \int_\Omega \frac{u^2-v^2}{r^2}\frac{\tau_x}{\tau} + \int_\Omega \frac{\theta_x}{\tau}\frac{\tau_x}{\tau}. \tag{2.90}$$

We now estimate the terms on the right-hand side of (2.90). By virtue of (1.10a) and (1.13), we get

$$-\int_\Omega \frac{u}{r}\left(\frac{\tau_x}{\tau}\right)_t = \int_\Omega \frac{(ru)_x^2}{r^2\tau} - \int_\Omega \frac{2u(ru)_x}{r^3} \lesssim \int_\Omega \left[(1+\theta)(ru)_x^2 + \frac{\tau u^2}{r^2\theta}\right], \tag{2.91}$$

where in the last inequality we have used (2.11) and (2.14). It follows from (2.31) and (2.14) that

$$\theta(t,x) + \int_\Omega r^2\theta_x^2 \gtrsim 1 \quad \text{for all } t \geq 0,\ x \in \Omega, \tag{2.92}$$

which implies

$$\int_\Omega \tau_x^2 \lesssim \int_\Omega \theta\tau_x^2 + \int_\Omega r^2\theta_x^2 \int_\Omega \tau_x^2. \tag{2.93}$$

We utilize (2.93) and (2.11) to obtain

$$\begin{aligned}\int_\Omega \frac{u^2-v^2}{r^2}\frac{\tau_x}{\tau} &\leq C(\epsilon)\sup_\Omega\left(u^4+v^4\right)\int_\Omega r^{-4} + \epsilon\int_\Omega \tau_x^2 \\ &\lesssim C(\epsilon)\sup_\Omega\left(u^4+v^4\right) + \epsilon\int_\Omega \theta\tau_x^2 + \epsilon\int_\Omega r^2\theta_x^2\int_\Omega \tau_x^2.\end{aligned} \tag{2.94}$$

For the last term in (2.90), we get

$$\int_\Omega \frac{\theta_x}{\tau}\frac{\tau_x}{\tau} \lesssim \epsilon\int_\Omega \theta\tau_x^2 + C(\epsilon)\int_\Omega \left[\frac{\theta_x^2}{\theta^2} + \theta_x^2\right]. \tag{2.95}$$

Integrating (2.90) over $[0,t]$, we apply Cauchy's inequality and use (2.91)–(2.95), (2.1), (2.44), (2.61), (2.64) to have

$$\begin{aligned}\int_\Omega \tau_x^2 + \int_0^t\int_\Omega \theta\tau_x^2 &\lesssim 1 + \int_0^t \sup_\Omega\left(u^4+v^4\right) + \int_0^t\left[\int_\Omega r^2\theta_x^2\int_\Omega \tau_x^2\right] \\ &\lesssim 1 + \int_0^t\left[\int_\Omega r^2\theta_x^2\int_\Omega \tau_x^2\right].\end{aligned}$$

Applying Grönwall's inequality to this last estimate, we deduce from (2.44) that

$$\int_\Omega \tau_x^2 \mathrm{d}x + \int_0^t\int_\Omega \theta\tau_x^2 \lesssim 1. \tag{2.96}$$

By virtue of (2.93) and (2.96), we have

$$\int_0^t\int_\Omega \tau_x^2 \lesssim 1 + \int_0^t\left[\int_\Omega r^2\theta_x^2\int_\Omega \tau_x^2\right].$$

We employ Grönwall's inequality again to have from (2.44) that

$$\int_0^t\int_\Omega \tau_x^2 \lesssim 1. \tag{2.97}$$

The proof of this lemma is completed by combining (2.96) and (2.97). □

The next lemma concerns the estimate for the L^2_x-norm of (u_x, v_x, w_x).

Lemma 2.9. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$\int_\Omega w_x^2 \mathrm{d}x + \int_0^t \int_\Omega r^2 w_{xx}^2 \lesssim 1, \tag{2.98}$$

$$\int_\Omega \left(u_x^2 + v_x^2\right) \mathrm{d}x + \int_0^t \int_\Omega r^2 \left(u_{xx}^2 + v_{xx}^2\right) \lesssim 1 + \sup_{[0,t]\times\Omega} \theta. \tag{2.99}$$

Proof. Multiply (1.10d) by w_{xx} and integrate the resulting identity over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w_x^2 + \int_\Omega \frac{\mu r^2 w_{xx}^2}{\tau} &= -2\int_\Omega \mu w_x w_{xx} + \int_\Omega \frac{r^2 \tau_x w_x w_{xx}}{\tau^2}. \\ &\lesssim \epsilon \int_\Omega r^2 w_{xx}^2 + C(\epsilon)\int_\Omega \left(w_x^2 + r^2 w_x^2 \tau_x^2\right) \\ &\lesssim \epsilon \int_\Omega r^2 w_{xx}^2 + C(\epsilon)\int_\Omega w_x^2 + C(\epsilon)\sup_\Omega \left(r^2 w_x^2\right) \\ &\lesssim \epsilon \int_\Omega r^2 w_{xx}^2 + C(\epsilon)\int_\Omega r^2 w_x^2,\end{aligned} \tag{2.100}$$

where we have used (2.89) and

$$\begin{aligned}\sup_\Omega \left(r^2 w_x^2\right) &\lesssim \left[\int_\Omega r^2 w_x^2\right]^{\frac{1}{2}} \left[\int_\Omega \left(r_x^2 w_x^2 + r^2 w_{xx}^2\right)\right]^{\frac{1}{2}} \\ &\lesssim \epsilon \int_\Omega r^2 w_{xx}^2 + C(\epsilon)\int_\Omega r^2 w_x^2.\end{aligned}$$

Integrating (2.100) over $[0, t]$ and using (2.44) imply (2.98).

Multiply (1.10b) by u_{xx} and integrate the resulting identity over $[0, t] \times \Omega$ to get

$$\begin{aligned}&\int_\Omega u_x^2 + \nu \int_0^t \int_\Omega \frac{r^2 u_{xx}^2}{\tau} \\ &= -\int_0^t \int_\Omega \frac{v^2 u_{xx}}{r} + \int_0^t \int_\Omega r\left(\frac{\theta}{\tau}\right)_x u_{xx} - \int_0^t \int_\Omega \nu r u_{xx}\left\{\left[\frac{(ru)_x}{\tau}\right]_x - \frac{r u_{xx}}{\tau}\right\} \\ &\lesssim \epsilon \int_0^t \int_\Omega r^2 u_{xx}^2 + C(\epsilon)\int_0^t \int_\Omega \left[\frac{v^4}{r^4} + \theta_x^2 + \theta^2 \tau_x^2 + (ru)_x^2 \tau_x^2 + \frac{u_x^2}{r^2} + r_{xx}^2 u^2\right] \\ &\lesssim \epsilon \int_0^t \int_\Omega r^2 u_{xx}^2 + C(\epsilon)\left[1 + \sup_{[0,t]\times\Omega} \theta\right].\end{aligned} \tag{2.101}$$

Here we have used (2.1), (2.44), (2.89), and

$$\begin{aligned}&\int_0^t \int_\Omega \left[\frac{v^4}{r^4} + (ru)_x^2 \tau_x^2 + r_{xx}^2 u^2\right] \\ &\lesssim \int_0^t \sup_\Omega \left[\frac{v^2}{r^2} + (ru)_x^2 + \frac{u^2}{r^2}\right] \int_\Omega \left(v^2 + \tau_x^2\right) + \int_0^t \int_\Omega \frac{u^2}{r^6} \\ &\lesssim \int_0^t \int_\Omega \left[\frac{v^2}{r^2} + \left(\frac{v}{r}\right)_x^2 + \epsilon (ru)_{xx}^2 + C(\epsilon)(ru)_x^2 + \frac{u^2}{r^2} + u_x^2\right] \\ &\lesssim \epsilon \int_0^t \int_\Omega r^2 u_{xx}^2 + C(\epsilon)\left[1 + \sup_{[0,t]\times\Omega} \theta\right].\end{aligned}$$

Similarly we can have

$$\int_\Omega v_x^2 + \mu \int_0^t \int_\Omega \frac{r^2 v_{xx}^2}{\tau} \lesssim \epsilon \int_0^t \int_\Omega r^2 v_{xx}^2 + C(\epsilon) \left[1 + \sup_{[0,t]\times\Omega} \theta\right]. \tag{2.102}$$

The estimate (2.99) follows from (2.101) and (2.102) by taking ϵ suitably small. □

In order to obtain the uniform-in-time upper bound for the temperature θ, we establish the estimate for the L_x^2-norm of θ_x in the following lemma.

Lemma 2.10. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$\theta(t,x) \lesssim 1 \quad \text{for all } x \in \Omega, \tag{2.103}$$

$$\int_\Omega \left(u_x^2 + v_x^2 + \theta_x^2\right) \mathrm{d}x + \int_0^t \int_\Omega r^2 \left(u_{xx}^2 + v_{xx}^2 + \theta_{xx}^2\right) \lesssim 1. \tag{2.104}$$

Proof. We multiply (1.10e) by θ_{xx} and integrate the resulting identity over $[0,t]\times\Omega$ to infer

$$\begin{aligned}
\frac{c_v}{2}\int_\Omega \theta_x^2 + \int_0^t \int_\Omega \frac{\kappa r^2 \theta_{xx}^2}{\tau} &= \int_0^t \int_\Omega \theta_{xx}\left[P(ru)_x - \kappa\theta_x\left(\frac{r^2}{\tau}\right)_x - \mathcal{Q}\right] \\
&\lesssim \epsilon \int_0^t \int_\Omega r^2\theta_{xx}^2 + C(\epsilon)\int_0^t \int_\Omega r^{-2}\left[\theta^2 (ru)_x^2 + \theta_x^2\left(\frac{r^2}{\tau}\right)_x^2 + \mathcal{Q}^2\right].
\end{aligned} \tag{2.105}$$

In view of Sobolev's inequality, we have from (2.44) and (2.89) that

$$\begin{aligned}
&\int_0^t \int_\Omega r^{-2}\theta_x^2 \left(\frac{r^2}{\tau}\right)_x^2 \\
&\lesssim \int_0^t \int_\Omega \left[\frac{\theta_x^2}{r^2} + r^2\theta_x^2\tau_x^2\right] \lesssim 1 + \int_0^t \sup_\Omega (r\theta_x)^2 \int_\Omega \tau_x^2 \\
&\lesssim 1 + \epsilon \int_0^t \int_\Omega r^2\theta_{xx}^2 + C(\epsilon)\int_0^t \int_\Omega r^2\theta_x^2 \lesssim C(\epsilon) + \epsilon\int_0^t \int_\Omega r^2\theta_{xx}^2.
\end{aligned} \tag{2.106}$$

Similarly, we apply Sobolev's inequality, (2.1), (2.44), (2.98), (2.99) to have

$$\begin{aligned}
\int_0^t \int_\Omega r^{-2}\mathcal{Q}^2 &\lesssim \int_0^t \int_\Omega \left[\frac{u^4+v^4}{r^6} + r^2\left(u_x^4 + v_x^4 + w_x^4\right)\right] \\
&\lesssim \left[1 + \sup_{[0,t]\times\Omega}\theta\right]\int_0^t \sup_\Omega \left[\frac{u^2+v^2}{r^2} + r^2\left(u_x^2+v_x^2+w_x^2\right)\right] \lesssim \left[1 + \sup_{[0,t]\times\Omega}\theta^2\right].
\end{aligned} \tag{2.107}$$

Plugging (2.106)–(2.107) into (2.105) and using (2.1), we take ϵ sufficiently small to derive

$$\int_\Omega \theta_x^2(t,x)\mathrm{d}x + \int_0^t \int_\Omega r^2\theta_{xx}^2 \lesssim 1 + \sup_{[0,t]\times\Omega}\theta^2. \tag{2.108}$$

By virtue of (2.51) and (2.108), we employ Sobolev's inequality to have

$$\begin{aligned}
\sup_{[0,t]\times\Omega}\theta^2 &\lesssim \sup_{s\in[0,t]}\left[\int_\Omega (\theta-1)^2(s,x)\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_\Omega \theta_x^2(s,x)\mathrm{d}x\right]^{\frac{1}{2}} \\
&\lesssim 1 + \sup_{[0,t]\times\Omega}\theta \lesssim C(\epsilon) + \epsilon\sup_{[0,t]\times\Omega}\theta^2.
\end{aligned}$$

Taking ϵ sufficiently small yields (2.103). The estimate (2.104) follows from (2.99), (2.108) and (2.103). The proof of this lemma is completed. □

In the following lemma we establish the uniform estimate for L^1-norm of $r^2\tau_x^2$.

Lemma 2.11. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$\int_\Omega r^2\tau_x^2 \mathrm{d}x + \int_0^t \int_\Omega (1+\theta) r^2 \tau_x^2 \lesssim 1. \tag{2.109}$$

Proof. Utilizing (1.9) and (1.10a), we reformulate Eq. (1.10b) into

$$\nu\Big(\frac{r\tau_x}{\tau}\Big)_t + \frac{r\theta\tau_x}{\tau^2} = u_t - \frac{v^2}{r} + \nu\frac{u\tau_x}{\tau} + \frac{r\theta_x}{\tau}. \tag{2.110}$$

Multiplying (2.110) by $r\tau_x/\tau$ and integrating the resulting identity over $[0,t]\times\Omega$ yield

$$\begin{aligned}&\int_\Omega \left[\frac{\nu r^2\tau_x^2}{2\tau^2} - \frac{ur\tau_x}{\tau}\right] - \int_\Omega\left[\frac{\nu r_0^2\tau_{0x}^2}{2\tau_0^2} - \frac{u_0r_0\tau_{0x}}{\tau_0}\right] + \int_0^t\int_\Omega \frac{r^2\theta\tau_x^2}{\tau^3}\\ &= \int_0^t\int_\Omega \frac{(ru)_x^2}{\tau} - \int_0^t\int_\Omega (u^2+v^2)\frac{\tau_x}{\tau} + \nu\int_0^t\int_\Omega \frac{ru\tau_x^2}{\tau^2} + \int_0^t\int_\Omega \frac{r^2\tau_x\theta_x}{\tau^2}.\end{aligned} \tag{2.111}$$

In view of (2.14) and (2.103), we have from (2.1) and (2.44) that

$$\begin{aligned}&\int_0^t\int_\Omega \frac{(ru)_x^2}{\tau} + \int_0^t\int_\Omega \frac{r^2\tau_x\theta_x}{\tau^2}\\ &\lesssim \epsilon\int_0^t\int_\Omega r^2\theta\tau_x^2 + C(\epsilon)\int_0^t\int_\Omega\left[\frac{(ru)_x^2 + r^2\theta_x^2}{\theta}\right] \lesssim \epsilon\int_0^t\int_\Omega r^2\theta\tau_x^2 + C(\epsilon).\end{aligned} \tag{2.112}$$

Applying integration by parts, we use (2.14), (2.103), (2.1) and (2.44) to get

$$\begin{aligned}-\int_0^t\int_\Omega (u^2+v^2)\frac{\tau_x}{\tau} &= 2\int_0^t\int_\Omega (uu_x + vv_x)\ln\tau\\ &\lesssim \int_0^t\int_\Omega \left[\frac{u^2}{r^2\theta} + \frac{r^2u_x^2}{\theta} + \frac{v^2}{r^2} + r^2v_x^2\right] \lesssim 1.\end{aligned} \tag{2.113}$$

It follows from (2.92) and (2.103) that

$$\begin{aligned}\nu\int_0^t\int_\Omega \frac{ru\tau_x^2}{\tau^2} &\lesssim \epsilon\int_0^t\int_\Omega r^2\tau_x^2 + C(\epsilon)\int_0^t\left[\sup_\Omega u^2\int_\Omega r^2\tau_x^2\right]\\ &\lesssim \epsilon\int_0^t\int_\Omega r^2\theta\tau_x^2 + C(\epsilon)\int_0^t\left[\int_\Omega \left(r^2\theta_x^2 + |uu_x|\right)\int_\Omega r^2\tau_x^2\right]\\ &\lesssim \epsilon\int_0^t\int_\Omega r^2\theta\tau_x^2 + C(\epsilon)\int_0^t\left[\int_\Omega\left(r^2\theta_x^2 + \frac{u^2}{r^2\theta} + \frac{r^2u_x^2}{\theta}\right)\int_\Omega r^2\tau_x^2\right].\end{aligned} \tag{2.114}$$

Here we have used

$$\int_0^t\int_\Omega r^2\tau_x^2 \lesssim \int_0^t\int_\Omega r^2\theta\tau_x^2 + \int_0^t\left[\int_\Omega r^2\theta_x^2\int_\Omega r^2\tau_x^2\right]. \tag{2.115}$$

Plugging (2.112)–(2.114) into (2.111), we choose ϵ sufficiently small and use (2.1), (2.115) to obtain

$$\int_\Omega r^2\tau_x^2\mathrm{d}x + \int_0^t\int_\Omega (1+\theta)r^2\tau_x^2 \lesssim 1 + \int_0^t\left[\int_\Omega\left(r^2\theta_x^2 + \frac{u^2}{r^2\theta} + \frac{r^2u_x^2}{\theta}\right)\int_\Omega r^2\tau_x^2\right].$$

The estimate (2.109) then follows by utilizing Grönwall's inequality, (2.1), and (2.44). □

We deduce that $(u_t^2, v_t^2, w_t^2, \theta_t^2)$ belongs to $L^1((0,\infty)\times\Omega)$ in Lemmas 2.12 and 2.13 for the purpose of obtaining the large-time behavior of the global solution $(\tau, u, v, w, \theta)(t,x)$.

Lemma 2.12. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$\int_\Omega \left[(ru)_x^2 + (rv)_x^2 + r^2 w_x^2\right] \mathrm{d}x + \int_0^t \int_\Omega |(u_t, v_t, w_t)|^2 \lesssim 1. \tag{2.116}$$

Proof. Multiplying (1.10d) by w_t, using (1.9),

$$(u_t, v_t, w_t)(t,0) = 0, \quad \sup_\Omega |(u,v,w)(t,x)| \lesssim 1 \quad \text{for } t \geq 0, \tag{2.117}$$

and (2.44), we get

$$\begin{aligned}
&\int_\Omega \frac{\mu r^2 w_x^2}{2\tau} - \int_\Omega \frac{\mu r_0^2 w_{0x}^2}{2\tau_0} + \int_0^t \int_\Omega w_t^2 \\
&= \int_0^t \int_\Omega \left[\mu w_x^2 - \frac{\mu r^2 \tau_x w_x^2}{2\tau^2} + \mu w_x w_t \left(1 - \frac{ru}{\tau}\right)\right] \\
&\lesssim C(\epsilon) + \int_0^t \sup_\Omega w_x \int_\Omega |r\tau_x r w_x| + \epsilon \int_0^t \int_\Omega w_t^2 \\
&\lesssim C(\epsilon) + \int_0^t \left[\int_\Omega r^2 w_x^2\right]^{\frac{3}{2}} \left[\int_\Omega w_{xx}^2\right]^{\frac{1}{2}} \left[\int_\Omega r^2 \tau_x^2\right]^{\frac{1}{2}} + \epsilon \int_0^t \int_\Omega w_t^2.
\end{aligned} \tag{2.118}$$

Taking ϵ sufficiently small and utilizing (2.109), Young's inequality, (2.44), and (2.98) imply

$$\int_\Omega r^2 w_x^2 \mathrm{d}x + \int_0^t \int_\Omega w_t^2 \lesssim 1. \tag{2.119}$$

Multiplying (1.10b) by u_t, integrating the resulting identity over $[0,t] \times \Omega$, and using (2.117) yield

$$\begin{aligned}
&\int_\Omega \frac{\nu (ru)_x^2}{2\tau} - \int_\Omega \frac{\nu (r_0 u_0)_x^2}{2\tau_0} + \int_0^t \int_\Omega u_t^2 \\
&= \int_0^t \int_\Omega \left[\frac{v^2 u_t}{r} - r u_t \left(\frac{\theta}{\tau}\right)_x - \frac{\nu (ru)_x^3}{2\tau^2} + \frac{2\nu u u_x (ru)_x}{\tau}\right] \\
&\leq \frac{1}{2} \int_0^t \int_\Omega u_t^2 + C \int_0^t \int_\Omega \left[\frac{v^2}{r^2} + r^2 \theta_x^2 + r^2 \theta^2 \tau_x^2 + (ru)_x^4 + u_x^2 + (ru)_x^2\right] \\
&\leq \frac{1}{2} \int_0^t \int_\Omega u_t^2 + C + C \int_0^t \left[\sup_\Omega (ru)_x^2 \int_\Omega (ru)_x^2\right].
\end{aligned}$$

Employing Grönwall's inequality to this last estimate, we have

$$\begin{aligned}
\int_\Omega (ru)_x^2 \mathrm{d}x + \int_0^t \int_\Omega u_t^2 &\lesssim 1 + \int_0^t \sup_\Omega (ru)_x^2 \\
&\lesssim 1 + \int_0^t \int_\Omega \left[(ru)_x^2 + r^2 u_{xx}^2 + r^{-2} u^2 + \tau_x^2\right] \lesssim 1.
\end{aligned} \tag{2.120}$$

For the estimate about $L^1_{t,x}$-norm of v_t^2, we can similarly get

$$\int_\Omega (rv)_x^2 \mathrm{d}x + \int_0^t \int_\Omega v_t^2 \lesssim 1. \tag{2.121}$$

Combining (2.119), (2.120) and (2.121), we can conclude the proof of this lemma. □

Lemma 2.13. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$\int_\Omega r^2 \theta_x^2 \mathrm{d}x + \int_0^t \int_\Omega \theta_t^2 \lesssim 1. \tag{2.122}$$

Proof. We multiply (1.10e) by θ_t and employ integration by parts to derive

$$\begin{aligned}
&\int_\Omega \frac{\kappa r^2\theta_x^2}{2\tau} - \int_\Omega \frac{\kappa r_0^2\theta_{0x}^2}{2\tau_0} + c_v\int_0^t\int_\Omega \theta_t^2 \\
&= \int_0^t\int_\Omega \left[-\frac{\theta(ru)_x\theta_t}{\tau} + \kappa\theta_x^2 - \frac{\kappa r^2(ru)_x}{2\tau^2}\theta_x^2 + \mathcal{Q}\theta_t\right] \\
&\le \frac{c_v}{2}\int_0^t\int_\Omega \theta_t^2 + C\int_0^t\int_\Omega \left[\theta^2(ru)_x^2 + r^2\theta_x^2 + r^2(ru)_x^2\theta_x^2 + \mathcal{Q}^2\right].
\end{aligned} \tag{2.123}$$

By virtue of Grönwall's and Sobolev's inequalities, we deduce

$$\begin{aligned}
&\int_\Omega r^2\theta_x^2 \mathrm{d}x + \int_0^t\int_\Omega \theta_t^2 \lesssim 1 + \int_0^t \sup_\Omega (ru)_x^2 + \int_0^t\int_\Omega \mathcal{Q}^2 \\
&\lesssim 1 + \int_0^t\int_\Omega \left[\frac{u^4+v^4}{r^2} + |r(u_x,v_x,w_x)|^4\right] \\
&\lesssim 1 + \int_0^t \left[\sup_\Omega |r(u_x,v_x,w_x)|^2 \int_\Omega |r(u_x,v_x,w_x)|^2\right] \lesssim 1,
\end{aligned}$$

where in the last inequality we have used (2.44) and (2.116). The proof is complete. □

3. Proof of Theorem 1.1

In order to show the global existence of the solutions to the problem (1.10)–(1.13) (i.e. (i) of Theorem 1.1), one can use the standard argument in [19], that is, first to construct an approximate problem in the bounded interval $(0,k)$ and to prove the a priori estimates independent of k similar to those obtained in Section 2, then to let k tend to infinity for the purpose of getting a global cylindrically symmetric solution as the limit. We omit the details for proving (i), while we deduce the assertion (ii) of Theorem 1.1 as follows.

Proof of (1.20). In view of (2.103), (2.1) and (2.44), we have

$$\int_0^\infty \|(\tau_x,u_x,v_x,w_x,\theta_x)(t)\|_{L^2(\Omega)}^2 \mathrm{d}t \le C. \tag{3.1}$$

By virtue of integration by parts, it follows from (2.98), (2.104), (2.116) and (2.122) that

$$\begin{aligned}
&\int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\|(\tau_x,u_x,v_x,w_x,\theta_x)(t)\|_{L^2(\Omega)}^2\right| \mathrm{d}t \\
&\le C\int_0^\infty \|(\tau_x,\tau_{xt},u_{xx},u_t,v_{xx},v_t,w_{xx},w_t,\theta_{xx},\theta_t)(t)\|_{L^2(\Omega)}^2 \mathrm{d}t \le C.
\end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2) gives

$$\lim_{t\to\infty} \|(\tau_x,u_x,v_x,w_x,\theta_x)(t)\|_{L^2(\Omega)}^2 = 0. \tag{3.3}$$

In light of Sobolev's inequality, the large-time behavior (1.20) is a consequence of (2.1), (2.44), (2.14), and (3.3).

Proof of (1.17). According to (1.20), one can find a positive constant T_1 such that

$$\theta(t,x) \ge \frac{1}{2} \quad \text{for all } t \ge T_1,\ x \in \overline{\Omega}. \tag{3.4}$$

The uniform boundedness (1.17) for τ and θ follows from (2.14), (2.42), (2.103) and (3.4).

Proof of (1.18)–(1.19). The estimates (1.18) and (1.19) follow easily from (1.17), (2.1), (2.12), (2.44), (2.109), (2.116), and (2.122).

Acknowledgments

The research of Ling Wan was supported in part by the Fundamental Research Funds for the Central Universities and the grants from National Natural Science Foundation of China under contracts 11601398 and 11671309. The research of Tao Wang was supported in part by the Fundamental Research Funds for the Central Universities, the Project funded by China Postdoctoral Science Foundation under contract 2016M592370, and the grant from National Natural Science Foundation of China under contract 11601398. The authors express much gratitude to Professor Huijiang Zhao for his support and advice. Tao Wang would like to warmly thank Alessandro Morando, Professor Paolo Secchi, and Paola Trebeschi for support and hospitality during his postdoctoral stay at University of Brescia.

References

[1] L.D. Landau, E.M. Lifshitz, Course of Theoretical Physics. Vol. 6, second ed. Pergamon Press, Oxford, ISBN: 0-08-033933-6, 1987, p. xiv+539. ISBN: 0-08-033932-8. Fluid mechanics, Translated from the third Russian edition by J.B. Sykes and W.H. Reid.

[2] P. Secchi, Existence theorems for compressible viscous fluids having zero shear viscosity, Rend. Semin. Mat. Univ. Padova (ISSN: 0041-8994) 71 (1984) 73–102. http://www.numdam.org/item?id=RSMUP_1984__71__73_0.

[3] A.V. Kazhikhov, V.V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, J. Appl. Math. Mech. 41 (2) (1977) 273–282. http://dx.doi.org/10.1016/0021-8928(77)90011-9.

[4] V.B. Nikolaev, Global solvability of the equations of motion of a viscous gas with axial and spherical symmetry, Dinamika Sploshn. Sredy (ISSN: 0420-0497) (63) (1983) 136–141, 162.

[5] G.-Q. Chen, M. Kratka, Global solutions to the Navier–Stokes equations for compressible heat-conducting flow with symmetry and free boundary, Comm. Partial Differential Equations (ISSN: 0360-5302) 27 (5–6) (2002) 907–943. http://dx.doi.org/10.1081/PDE-120004889.

[6] H. Frid, V. Shelukhin, Vanishing shear viscosity in the equations of compressible fluids for the flows with the cylinder symmetry, SIAM J. Math. Anal. (ISSN: 0036-1410) 31 (5) (2000) 1144–1156. http://dx.doi.org/10.1137/S003614109834394X. (electronic).

[7] D. Hoff, H.K. Jenssen, Symmetric nonbarotropic flows with large data and forces, Arch. Ration. Mech. Anal. (ISSN: 0003-9527) 173 (3) (2004) 297–343. http://dx.doi.org/10.1007/s00205-004-0318-5.

[8] H.K. Jenssen, T.K. Karper, One-dimensional compressible flow with temperature dependent transport coefficients, SIAM J. Math. Anal. (ISSN: 0036-1410) 42 (2) (2010) 904–930. http://dx.doi.org/10.1137/090763135.

[9] T. Wang, One dimensional pth power Newtonian fluid with temperature-dependent thermal conductivity, Commun. Pure Appl. Anal. (ISSN: 1534-0392) 15 (2) (2016) 477–494. http://dx.doi.org/10.3934/cpaa.2016.15.477.

[10] S. Jiang, J. Zhang, Boundary layers for the Navier–Stokes equations of compressible heat-conducting flows with cylindrical symmetry, SIAM J. Math. Anal. (ISSN: 0036-1410) 41 (1) (2009) 237–268. http://dx.doi.org/10.1137/07070005X.

[11] X. Qin, T. Yang, Z.-a. Yao, W. Zhou, Vanishing shear viscosity and boundary layer for the Navier–Stokes equations with cylindrical symmetry, Arch. Ration. Mech. Anal. (ISSN: 0003-9527) 216 (3) (2015) 1049–1086. http://dx.doi.org/10.1007/s00205-014-0826-x.

[12] S.N. Antontsev, A.V. Kazhikhov, V.N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids, in: Studies in Mathematics and its Applications, vol. 22, North-Holland Publishing Co., Amsterdam, ISBN: 0-444-88382-7, 1990, p. xii+309. Translated from the Russian.

[13] S. Jiang, Large-time behavior of solutions to the equations of a viscous polytropic ideal gas, Ann. Mat. Pura Appl. (4) (ISSN: 0003-4622) 175 (1998) 253–275. http://dx.doi.org/10.1007/BF01783686.

[14] H. Cui, Z.-a. Yao, Asymptotic behavior of compressible pth power Newtonian fluid with large initial data, J. Differential Equations (ISSN: 0022-0396) 258 (3) (2015) 919–953. http://dx.doi.org/10.1016/j.jde.2014.10.011.

[15] L. Wan, T. Wang, Symmetric flows for compressible heat-conducting fluids with temperature dependent viscosity coefficients, J. Differential Equations (ISSN: 0022-0396) 262 (12) (2017) 5939–5977. http://dx.doi.org/10.1016/j.jde.2017.02.022.

[16] A.V. Kazhikhov, Cauchy problem for viscous gas equations, Sib. Math. J. 23 (1) (1982) 44–49. http://dx.doi.org/10.1007/BF00971419.

[17] S. Jiang, Remarks on the asymptotic behaviour of solutions to the compressible Navier–Stokes equations in the half-line, Proc. Roy. Soc. Edinburgh Sect. A (ISSN: 0308-2105) 132 (3) (2002) 627–638. http://dx.doi.org/10.1017/S0308210500001815.

[18] J. Li, Z. Liang, Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier–Stokes system in unbounded domains with large data, Arch. Ration. Mech. Anal. (ISSN: 0003-9527) 220 (3) (2016) 1195–1208. http://dx.doi.org/10.1007/s00205-015-0952-0.

[19] S. Jiang, Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain, Comm. Math. Phys. (ISSN: 0010-3616) 178 (2) (1996) 339–374. http://projecteuclid.org/euclid.cmp/1104286655.

[20] T. Nakamura, S. Nishibata, Large-time behavior of spherically symmetric flow of heat-conductive gas in a field of potential forces, Indiana Univ. Math. J. (ISSN: 0022-2518) 57 (2) (2008) 1019–1054. http://dx.doi.org/10.1512/iumj.2008.57.3165.

[21] Z. Liang, Large-time behavior for spherically symmetric flow of viscous polytropic gas in exterior unbounded domain with large initial data, 2014. Preprint at arXiv:1405.0569.

[22] S. Jiang, Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains, Comm. Math. Phys. (ISSN: 0010-3616) 200 (1) (1999) 181–193. http://dx.doi.org/10.1007/s002200050526.

[23] L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, second ed. American Mathematical Society, Providence, RI, ISBN: 978-0-8218-4974-3, 2010, p. xxii+749. http://dx.doi.org/10.1090/gsm/019.