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ONE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH LARGE DENSITY OSCILLATION

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Abstract. This paper is concerned with nonlinear stability of viscous shock profiles for the one-dimensional isentropic compressible Navier-Stokes equations. For the case when the diffusion wave introduced in [6, 7] is excluded, such a problem has been studied in [5, 11] and local stability of weak viscous shock profiles is well-established, but for the corresponding result with large initial perturbation, fewer results have been obtained. Our main purpose is to deduce the corresponding nonlinear stability result with large initial perturbation by exploiting the elementary energy method. As a first step toward this goal, we show in this paper that for certain class of "large" initial perturbation which can allow the initial density to have large oscillation, similar stability result still holds. Our analysis is based on the continuation argument and the technique developed by Kanel' in [4].

1. **Introduction.**

This paper is concerned with the precise description of the large time behaviors of global solutions of the Cauchy problem of the one-dimensional isentropic compressible Navier-Stokes equations in Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, & t>0,\ x\in\mathbb{R}, \\ u_t + p(v)_x = \left(\mu \frac{u_x}{v}\right)_x, & t>0,\ x\in\mathbb{R}, \end{cases} \tag{1}$$

with prescribed initial data

$$(v(0,x),u(0,x)) = (v_0(x),u_0(x)), \quad \lim_{x\to\pm\infty}(v_0(x),u_0(x)) = (v_\pm,u_\pm). \tag{2}$$

Here $v>0$, u, and $p(v)$ denote, respectively, the specific volume, the velocity, and the pressure of the gas, while $v_\pm>0,\ u_\pm$, and the viscosity coefficient $\mu>0$ are given constants. To simplify the presentation, we assume that the gas is polytropic and in such a case

$$p(v) = av^{-\gamma}, \tag{3}$$

where $\gamma>1$ is the adiabatic exponent and a is a positive constant.

It is well-known that the large time behavior of the global solutions $(v(t,x)$, $u(t,x))$ of the Cauchy problem (1)-(2) is determined by the structure of the unique

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global entropy solution $(v^r(x/t), u^r(x/t))$ of the resulting Riemann problem

$$\begin{cases} v_t - u_x = 0, & t>0,\ x\in\mathbb{R},\\ u_t + p(v)_x = 0, & t>0,\ x\in\mathbb{R},\\ (v(0,x),u(0,x)) = \begin{cases} (v_-,u_-), & x<0,\\ (v_+,u_+), & x>0. \end{cases} \end{cases} \tag{4}$$

Throughout this paper, we assume that the unique global entropy solution $(v^r(x/t), u^r(x/t))$ to the Riemann problem (4) is a superposition of a shock wave $(v^{S_1}(x/t), u^{S_1}(x/t))$ of the first family and a shock wave $(v^{S_2}(x/t), u^{S_2}(x/t))$ of the second family. That is, $(v_+,u_+)\in S_1S_2(v_-,u_-)$, where $S_1S_2(v_-,u_-) := \{(v,u);\ u<u_- - (v-v_-)s_i(v,v_-),\ i=1,2\}$ with the speed $s_i(v,v_-) = (-1)^i\sqrt{(p(v)-p(v_-))/(v_- - v)}$. By the standard arguments (cf. [17]), we see that there is a unique $(\bar v,\bar u)\in S_1(v_-,u_-)$ so that $(v_+,u_+)\in S_2(\bar v,\bar u)$, where $S_i(v_-,u_-) := \{(v,u);\ u = u_- - (v-v_-)s_i(v,v_-),\ u<u_-\}$ is the i-shock curve passing through (v_-,u_-).

For such a case, the large time behavior of solutions of the Cauchy problem (1)-(2) is described by the superposition of the suitably shifted viscous shock waves of different family of the one-dimensional compressible Navier-Stokes equations (1). Recall that the viscous shock wave, which connects the states (v_l,u_l) and (v_r,u_r) smoothly with the speed s, is a special solution of (1) which has the form

$$(v(t,x),u(t,x)) := (V(\xi),U(\xi)),\quad \xi = x-st, \tag{5}$$

and satisfies

$$(V(-\infty),U(-\infty)) = (v_l,u_l),\quad (V(+\infty),U(+\infty)) = (v_r,u_r). \tag{6}$$

Plugging (5) into (1) yields

$$\begin{cases} -sV_\xi - U_\xi = 0, & \xi\in\mathbb{R},\\ -sU_\xi + p(V)_\xi = \left(\mu\frac{U_\xi}{V}\right)_\xi, & \xi\in\mathbb{R}. \end{cases} \tag{7}$$

Integrating (7) over $(\pm\infty,\xi)$ under the Rankine-Hugoniot condition

$$\begin{cases} -s(v_r-v_l) - (u_r-u_l) = 0,\\ -s(u_r-u_l) + p(v_r) - p(v_l) = 0, \end{cases} \tag{8}$$

we can then reduce the problem (7) to

$$\begin{cases} U = u_l - s(V-v_l) \equiv u_r - s(V-v_r), & \xi\in\mathbb{R},\\ \mu s V_\xi/V = p(v_l) - p(V) + s^2(v_l - V)\\ \qquad\qquad \equiv p(v_r) - p(V) + s^2(v_r-V), & \xi\in\mathbb{R}. \end{cases} \tag{9}$$

Since $(v_+,u_+)\in S_2(\bar v,\bar u)$ for some $(\bar v,\bar u)\in S_1(v_-,u_-)$ and $p(v)$ is a convex function of v, it is easy to see that (cf. [5]) the problem (1) admits a 1-viscous shock wave $(V_1(x-s_1t), U_1(x-s_1t))$ connecting (v_-,u_-) with $(\bar v,\bar u)$ and a 2-viscous shock wave $(V_2(x-s_2t),U_2(x-s_2t))$ connecting $(\bar v,\bar u)$ with (v_+,u_+), and both of them are unique up to a shift. Here $s_1 = s_1(v_-,\bar v)<0$, $s_2 = s_2(v_+,\bar v)>0$ and the strengths of the 1-viscous shock wave $(V_1(x-s_1t),U_1(x-s_1t))$ and the 2-viscous shock wave $(V_2(x-s_2t),U_2(x-s_2t))$ are denoted by $\delta_1 := |v_- - \bar v|$ and $\delta_2 := |\bar v - v_+|$, respectively. Moreover, if we set $\delta := |u_+ - u_-|$, we can easily deduce that

$$\delta_1+\delta_2 \le C\delta, \tag{10}$$

where C is some positive constant depending only on v_- and v_+.

In this manuscript, we are concerned with the case when δ_i $(i = 1, 2)$, the strengths of the $i-$viscous shock waves $(V_i(x - s_it), U_i(x - s_it))$ $(i = 1, 2)$, are of the same order and in such a case, if we assume that

$$A := \int_{\mathbb{R}} (v_0(x) - V_1(x) - V_2(x) + \bar{v})\mathrm{d}x < +\infty \tag{11}$$

and

$$B := \int_{\mathbb{R}} (u_0(x) - U_1(x) - U_2(x) + \bar{u})\mathrm{d}x < +\infty, \tag{12}$$

then we can uniquely determine two constants α_1 and α_2

$$\alpha_1 = \frac{s_2A + B}{\delta_1(s_1 - s_2)}, \quad \alpha_2 = \frac{s_1A + B}{\delta_2(s_1 - s_2)} \tag{13}$$

such that

$$\begin{cases} \int_{\mathbb{R}} (v_0(x) - V_1(x + \alpha_1) - V_2(x + \alpha_2) + \bar{v})\,\mathrm{d}x = 0, \\ \int_{\mathbb{R}} (u_0(x) - U_1(x + \alpha_1) - U_2(x + \alpha_2) + \bar{u})\,\mathrm{d}x = 0. \end{cases} \tag{14}$$

From (14), it is expected that the large time behavior of global solutions of the Cauchy problem (1)-(2) is described by the superposition of the 1-viscous shock wave $(V_1, U_1)(x - s_1t + \alpha_1)$ and the 2-viscous shock wave $(V_2, U_2)(x - s_2t + \alpha_2)$:

$$(V, U)(t, x; \alpha_1, \alpha_2) := (V_1, U_1)(x - s_1t + \alpha_1) + (V_2, U_2)(x - s_2t + \alpha_2) - (\bar{v}, \bar{u}). \tag{15}$$

Note that $(V_i, U_i)(x - s_it + \alpha_i)$ $(i = 1, 2)$ are exact solutions of the compressible Navier-Stokes equation (1), while $(V, U)(t, x; \alpha_1, \alpha_2)$ just satisfies (1) approximately as in the following

$$\begin{cases} V_t - U_x = 0, & t > 0,\ x \in \mathbb{R}, \\ U_t + p(V)_x = \left(\mu \frac{U_x}{V}\right)_x - g_x, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{16}$$

where

$$g = \mu\frac{U_x}{V} - \mu\frac{U_{1x}}{V_1} - \mu\frac{U_{2x}}{V_2} - p(V) + p(V_1) + p(V_2) - p(\bar{v}). \tag{17}$$

To use the compressibility of the viscous shock profiles, we need to use the anti-derivative technique as in [2, 3, 5, 7, 12]. In fact, the conservative form of (1) and (16) together with (14) tell us that we can define $(\phi(t, x), \psi(t, x))$ by

$$(\phi(t, x), \psi(t, x)) := \int_{-\infty}^{x} (v(t, y) - V(t, y; \alpha_1, \alpha_2), u(t, y) - U(t, y; \alpha_1, \alpha_2))\, dy, \tag{18}$$

and from (1) and (16), the original problem can be reformulated as

$$\begin{cases} \phi_t - \psi_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \psi_t + p(V + \phi_x) - p(V) = \mu\left(\frac{\psi_{xx} + U_x}{\phi_x + V} - \frac{U_x}{V}\right) + g, & t > 0,\ x \in \mathbb{R}, \\ (\phi(0, x), \psi(0, x)) = (\phi_0(x), \psi_0(x)), & x \in \mathbb{R}, \end{cases} \tag{19}$$

Here

$$(\phi_0(x), \psi_0(x)) = \int_{-\infty}^{x} (v_0(y) - V(0, y; \alpha_1, \alpha_2), u_0(y) - U(0, y; \alpha_1, \alpha_2))\,\mathrm{d}x. \tag{20}$$

Under the above preparation in hand, our original problem can be transferred into a stability problem: If the initial data $(v_0(x), u_0(x))$ of the Cauchy problem (1)-(2) is a suitable perturbation of $(V(0,x;\alpha_1,\alpha_2), U(0,x;\alpha_1,\alpha_2))$, does the Cauchy problem (1)-(2) admit a unique global solution $(v(t,x), u(t,x))$ which tends to $(V(t,x;\alpha_1,\alpha_2), U(t,x;\alpha_1,\alpha_2))$ as $t\to\infty$? Or equivalently, if $(\phi_0(x), \psi_0(x))$ belongs to the Sobolev space $H^2(\mathbb{R})$, does the Cauchy problem (19) admit a unique global solution $(\phi(t,x), \psi(t,x))$ whose $L^\infty_x(\mathbb{R})$-norm tends to zero as $t\to\infty$? Recall that according to whether $H^2(\mathbb{R})-$norm of the initial perturbation $(\phi_0(x), \psi_0(x))$ and/or δ_i $(i=1,2)$, the strengths of the viscous shock waves, are assumed to be small or not, the stability results are classified into global (or local) stability of strong (or weak) viscous shock waves.

To deduce the desired nonlinear stability result by the elementary energy method as in [5, 10, 12], it is sufficient to deduce certain uniform (with respect to the time variable t) energy type estimates on the solutions $(\phi(t,x), \psi(t,x))$ and the main difficulty to do so lies in how to control the possible growth of $(\phi(t,x), \psi(t,x))$ caused by the nonlinearity of the equation $(19)_2$. For general $\gamma>1$, the arguments employed in [5, 10, 12] is to use the smallness of both $N(T) := \sup_{0\le t\le T}\|(\phi(t),\psi(t))\|_{H^2(\mathbb{R})}$ and δ_i $(i=1,2)$ to overcome such a difficulty. One of the key points in such an argument is that, based on the *a priori* assumption that $N(T)$ is sufficiently small, one can deduce a uniform lower and upper positive bounds on the specific volume $v(t,x)$. With such a bound on $v(t,x)$ in hand, one can thus deduce certain *a priori* $H^2(\mathbb{R})$ energy type estimates on $(\phi(t,x), \psi(t,x))$ in terms of the initial perturbation $(\phi_0(x), \psi_0(x))$ provided that the strengths of the viscous shock waves are suitably small. The combination of the above analysis with the standard continuation argument yields the local stability of weak viscous shock waves for the one-dimensional compressible Navier-Stokes equations. It is easy to see that in such a result, for all $t\in\mathbb{R}$, Osc $v(t) := \sup_{x\in\mathbb{R}} v(t,x) - \inf_{x\in\mathbb{R}} v(t,x)$, the oscillation of the specific volume $v(t,x)$, should be sufficiently small.

For the global stability of viscous shock waves, the story is quite different and as pointed out in [1, 13, 14, 16] where the global stability of rarefaction waves for the one-dimensional compressible Navier-Stokes equations is investigated, the main difficulty lies in how to deduce the uniform lower and upper bounds on the specific volume v under large initial perturbation.

As a first step to achieve this goal, we will show that the weak viscous shock waves are nonlinear stable for a class of large initial perturbation which can allow the specific volume $v(t,x)$ to have large oscillation. This type of result is motivated by the special structure of the system (1), which suggests that the nonlinearities involved are mainly caused by the specific volume v. Hence, when we deal with such a problem by exploiting the energy method, the key step lies in how to deduce a uniform positive lower bound and a uniform upper bound for the specific volume v. It is worth pointing out that the argument developed by Kanel' in [4] plays an essential role in our analysis.

Now we turn to state our main result. First we list some assumptions on the initial data $(v_0(x), u_0(x))$, the strengths of the viscous shock waves δ_1, δ_2, and the shifts α_1, α_2 as in the folowing:

(H_0) there exist $\delta-$independent constants $\ell\ge 0$ and $C_1>0$ such that

$$C_1^{-1}\delta^\ell \le v_0(x) \le C_1(1+\delta^{-\ell}); \tag{21}$$

(H_1) $(v_+, u_+)\in S_1S_2(v_-, u_-)$ and $(\bar v, \bar u)\in S_1(v_-, u_-)$ such that $(v_+, u_+)\in S_2(\bar v, \bar u)$;

(H$_2$) v_- and v_+ are positive constants independent of δ;

(H$_3$) the strengths of the viscous shock waves $\delta_1,\ \delta_2$, the shifts $\alpha_1,\ \alpha_2$ defined by (13) and the initial data $(v_0(x), u_0(x))$ are assumed to satisfy

$$(v_0(x) - V(0, x; \alpha_1, \alpha_2), u_0(x) - U(0, x; \alpha_1, \alpha_2)) \in H^1(\mathbb{R}) \cap L^1(\mathbb{R}), \tag{22}$$

$$(\phi_0(x), \psi_0(x)) \in L^2(\mathbb{R}), \tag{23}$$

and for some positive constant C_0 independent of δ,

$$C_0^{-1}\delta_2 \le \delta_1 \le C_0\delta_2, \quad \alpha_2 - \alpha_1 \le C_0\delta^{-1}, \quad \text{as } \delta \to 0_+. \tag{24}$$

Under the above assumptions, we have

Theorem 1.1. *In addition to the assumptions (H$_0$)-(H$_3$), we assume further that*

$$\|(\phi_0, \psi_0)\|_{H^1(\mathbb{R})} \le C_2\delta^\alpha, \quad \|\phi_{0xx}\|_{L^2(\mathbb{R})} \le C_2(1 + \delta^{-\beta}) \tag{25}$$

hold for some $\delta-$independent positive constants C_2, α and β. If the parameters ℓ, α and β are assumed to satisfy

$$\begin{cases} (3\gamma + 5)\ell < \min\{2, \alpha\}, \\ \min\left\{\alpha - \frac{\gamma+1}{2}\ell, \frac{1}{4}\right\} \le \ell + \beta, \\ \beta + \ell < \frac{4\gamma^2+3\gamma+1}{4\gamma^2+2\gamma+2}\min\left\{\alpha - \frac{\gamma+1}{2}\ell, \frac{1}{4}\right\}, \end{cases} \tag{26}$$

then there exists a suitably small $\delta_0 > 0$ such that if $0 < \delta \le \delta_0$, the Cauchy problem (1)-(2) has a unique solution $(v(t, x), u(t, x))$ satisfying

$$(v(t, x) - V(t, x; \alpha_1, \alpha_2), u(t, x) - U(t, x; \alpha_1, \alpha_2)) \in C\left([0, \infty); H^1(\mathbb{R})\right),$$
$$v(t, x) - V(t, x; \alpha_1, \alpha_2) \in L^2\left(0, \infty; H^1(\mathbb{R})\right),$$
$$u(t, x) - U(t, x; \alpha_1, \alpha_2) \in L^2\left(0, \infty; H^2(\mathbb{R})\right),$$

and

$$C_3^{-1}\delta^{\frac{2}{1-\gamma}\left(\min\left\{\alpha - \frac{\gamma+1}{2}\ell, \frac{1}{4}\right\} - (\beta+\ell)\right)} \le v(t, x) \le C_3(1 + \delta^{\min\left\{2\alpha - (\gamma+1)\ell, \frac{1}{2}\right\} - 2(\beta+\ell)}) \tag{27}$$

for some positive constant C_3 independent of δ. Furthermore, it holds that

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|(v(t, x), u(t, x)) - (V(t, x; \alpha_1, \alpha_2, U(t, x; \alpha_1, \alpha_2))| = 0. \tag{28}$$

Remark 1. Several remarks concerning Theorem 1.1 are listed below:

- It is easy to see that the set of the parameters $\alpha > 0,\ \beta > 0,\ \ell \ge 0$ which satisfy the assumption (26) is not empty. In fact, let $\ell = 0,\ \alpha \le \frac{1}{4}$, one can deduce that (26) is equivalent to $\alpha \le \beta < \frac{4\gamma^2+3\gamma+1}{4\gamma^2+2\gamma+2}\alpha$, and the existence of such α and β is easy to verify.
- If the parameter $\alpha,\ \beta,\ \ell$ satisfy $\min\{2\alpha - (\gamma + 1)\ell, 1/2\} < 2(\beta + \ell)$, then for $\delta > 0$ sufficiently small, we can deduce from (27) that for each fixed $t \ge 0$, Osc $v(t) = \sup_{x\in\mathbb{R}} v(t, x) - \inf_{x\in\mathbb{R}} v(t, x)$, the oscillation of $v(t, x)$, can be large.
- For the nonlinear stability of the superposition of viscous shock profiles of different families, Theorem 1.1 asks that δ_i $(i = 1, 2)$, the strengths of the two viscous shock profiles, are of the same order and $\alpha_2 - \alpha_1$, the difference of the shifts of the two viscous shock profiles, is bounded from above by δ^{-1}, *i.e.*, the assumption (24) holds. Such a condition is motivated by our study on the

nonlinear stability of viscous shock profile to the impermeable wall problem for the system (1) on the half line

$$\begin{cases} v_t - u_x = 0, & x \in \mathbb{R}_+,\ t \in \mathbb{R}_+, \\ u_t + p(v)_x = \left(\mu \frac{u_x}{v}\right)_x, & x \in \mathbb{R}_+,\ t \in \mathbb{R}_+, \\ (v(0,x), u(0,x)) = (v_0(x), u_0(x)), & x \geq 0, \\ \lim_{x\to\infty} (v_0(x), u_0(x)) = (v_+, u_+), & v_+ > 0,\ u_+ < 0, \\ u(t,0) = 0, & t \geq 0. \end{cases} \tag{29}$$

For details, see Section 4.

- For the simplicity of presentation, we assume that v_- and v_+, the far fields of $v_0(x)$, are independent of δ. For the case when the large time behavior of the global solution of (1) is described by a single viscous shock profile, similar result can also be obtained even when the far fields of $v_0(x)$ depend on δ. An outline of the result will be given in Section 4.
- Similar result for the non-isentropic but ideal polytropic one-dimensional compressible Navier-Stokes equation holds provided that γ, the adiabatic exponent, is sufficiently close to 1.
- In our results, we only consider the cases when either the initial perturbation is general but the asymptotics is described by the superposition of viscous shock profiles of different families whose strengths are of the same order, or the asymptotics is determined by a single viscous shock profile but with zero mass condition. In both cases, the diffusion wave introduced by T.-P. Liu in [6, 7] is excluded. For the corresponding nonlinear stability results with general but small perturbation for the system (1), the interested readers are referred to [3, 6, 7, 8, 18] and the references cited therein.

This paper is arranged as follows. After listing some notations in the rest of this section, the properties of the viscous shock wave will be stated in Section 2, the proof of Theorem 1.1 is given in Section 3 and some remarks concerning the third and forth points listed in Remark 1 are given in Section 4.

Notations. Throughout this paper, c and C are used to denote some generic positive constants which are independent of δ, the strength of the viscous shock wave. Note that these constants may vary from line to line. $C(\cdot,\cdot)$ stand for some generic constants depending only on the quantities listed in the parenthesis.

For function spaces, $L^q(\Omega)(1 \leq q \leq \infty)$ denotes the usual Lebesgue space on $\Omega \subset \mathbb{R}$ with norm $\|\cdot\|_{L^q(\Omega)}$, while $H^q(\Omega)$ denotes the usual Sobolev space in the L^2 sense with norm $\|\cdot\|_{H^q(\Omega)}$. To simplify the presentation, we use $\|\cdot\|$ and $\|\cdot\|_q$ to denote $\|\cdot\|_{L^2(\mathbb{R})}$ and $\|\cdot\|_{H^q(\mathbb{R})}$, respectively.

Finally, $(V(t,x), U(t,x))$ will be used to denote $(V(t,x;\alpha_1,\alpha_2), U(t,x;\alpha_1,\alpha_2))$ in the rest of this manuscript for notational simplicity.

2. **Viscous shock waves.**

This section is devoted to collecting some basic properties of the viscous shock waves $(V_i(t,x),\ U_i(t,x))$ $(i = 1, 2)$ and their superposition $(V(t,x),\ U(t,x))$.

The first result is concerned with the existence of the viscous shock profiles $(V_i(x - s_it), U_i(x - s_it))$ $(i = 1, 2)$ together with their decay estimates as $x - s_it \to \pm\infty$.

Proposition 1. *Assume that the assumptions (H_0)-(H_3) hold, then (1) admits a viscous shock wave $(V_1(x-s_1t), U_1(x-s_2t))$ of the first family connecting (v_-, u_-) with $(\bar v, \bar u)$ with speed s_1 and a viscous shock wave $(V_2(x-s_2t), U_2(x-s_2t))$ of the second family connecting $(\bar v, \bar u)$ with (v_+, u_+) with speed s_2, and both of them are unique up to a shift. Moreover, there exist positive constants c and C which depend only on v_- and v_+, such that, for $i = 1, 2$,*

$$\begin{aligned}
|(V_1(\xi)-\bar v, U_1(\xi)-\bar u)| &\le C\delta_1 e^{-c\delta_1|\xi|}, && \forall\ \xi > 0,\\
|(V_2(\xi)-\bar v, U_2(\xi)-\bar u)| &\le C\delta_2 e^{-c\delta_2|\xi|}, && \forall\ \xi < 0,\\
|V_1'(\xi)| &\le C\,|V_1(\xi)-v_-|\,|V_1(\xi)-\bar v|\,, && \forall\ \xi \in \mathbb{R},\\
|V_2'(\xi)| &\le C\,|V_2(\xi)-\bar v|\,|V_2(\xi)-v_+|\,, && \forall\ \xi \in \mathbb{R},\\
U_i'(\xi) &< 0, && \forall\ \xi \in \mathbb{R},\\
|(U_i'(\xi), V_i''(\xi), U_i''(\xi))| &\le C\,|V_i'(\xi)| \le C\delta_i^2 e^{-c\delta_i|\xi|}, && \forall\ \xi \in \mathbb{R}.
\end{aligned} \tag{30}$$

Although in our case, $v_0(x)$ may depend on δ, the assumption (H_2) implies that v_- and v_+ are independent of δ and hence the proof of Proposition 1 follows essentially the argument used in [3]. We thus omit the details for brevity.

Our next lemma is concerned with an estimate on $g(t,x)$ which will play an important role in performing the energy estimates.

Lemma 2.1. *Under the assumption (24), we have that there exists a δ-independent constant $C > 0$ such that*

$$\int_0^\infty \|g(t)\|\mathrm{d}t \le C\delta^{\frac12}, \qquad \int_0^\infty \left(\|g_x(t)\| + \|g_{xx}(t)\|\right)\mathrm{d}t \le C\delta^{\frac32}. \tag{31}$$

Proof. We only consider the case $\alpha_1 < \alpha_2$ in the following since the other case can be treated similarly. We note that

$$\begin{aligned}
|g(t,x)| &\le C\,|V_1(x-s_1t+\alpha_1)-\bar v|\,|V_2(x-s_2t+\alpha_2)-\bar v|\,,\\
|(g_x, g_{xx})(t,x)| &\le C\,(\delta_1+\delta_2)\,|V_1(x-s_1t+\alpha_1)-\bar v|\,|V_2(x-s_2t+\alpha_2)-\bar v|\,.
\end{aligned} \tag{32}$$

Then we divide the upper plane $\mathbb{R}_+ \times \mathbb{R}$ into the following six regions

$$\begin{aligned}
\Omega_1 &= \{(t,x):\ 0 \le t \le t_0,\ x \le s_2t-\alpha_2\},\\
\Omega_2 &= \{(t,x):\ 0 \le t \le t_0,\ s_2t-\alpha_2 < x \le s_1t-\alpha_1\},\\
\Omega_3 &= \{(t,x):\ 0 \le t \le t_0,\ x \ge s_1t-\alpha_1\},\\
\Omega_4 &= \{(t,x):\ t > t_0,\ x \le s_1t-\alpha_1\},\\
\Omega_5 &= \{(t,x):\ t > t_0,\ s_1t-\alpha_1 < x \le s_2t-\alpha_2\},\\
\Omega_6 &= \{(t,x):\ t > t_0,\ x \ge s_2t-\alpha_2\},
\end{aligned}$$

with $t_0 = (\alpha_2-\alpha_1)/(s_2-s_1)$. With such a partition of the upper plane $\mathbb{R}_+ \times \mathbb{R}$ in hand, (31) can be verified easily by exploiting Proposition 1 and (32). This completes the proof of Lemma 2.1. □

3. **The proof of Theorem 1.1.** This section is devoted to proving Theorem 1.1. To this end, we first define the function space in which we find the solutions

$$X_{m,M}(0,T)=\left\{(\phi(t,x),\psi(t,x))\;\middle|\;\begin{array}{c}(\phi(t,x),\psi(t,x))\in C\left([0,T];H^2(\mathbb{R})\right),\\ \phi_x(t,x)\in L^2\left(0,T;H^1(\mathbb{R})\right),\\ \psi_x(t,x)\in L^2\left(0,T;H^2(\mathbb{R})\right),\\ m\leq V(t,x)+\phi_x(t,x)\leq M\end{array}\right\},$$

and the local solvability of the Cauchy problem (19) in such a function space can be stated as in the following

Proposition 2. *Let $(\phi_0(x),\psi_0(x))$ be in $H^2(\mathbb{R})$ and assume that $m\leq V(0,x)+\phi_{0x}(x)\leq M$ holds for each $x\in\mathbb{R}$, then there exists $t_0>0$ depending only on m, M and $\|(\phi_0,\psi_0)\|_2$ such that (19) has a unique solution $(\phi(t,x),\psi(t,x))\in X_{m/2,2M}(0,t_0)$ which satisfies for each $0\leq t\leq t_0$ that*

$$\|\psi(t)\|\leq 2\|\psi_0\|,\quad \|\psi_x(t)\|\leq 2\|\psi_{0x}\|,\quad \|(\phi,\psi)(t)\|_2\leq 2\|(\phi_0,\psi_0)\|_2. \tag{33}$$

Assume that the local solution $(\phi(t,x),\psi(t,x))$ constructed in Proposition 2 has been extended to the time step $t=T\geq t_0$. In order to show that $T=\infty$, we now turn to deduce certain energy type *a priori* estimates on $(\phi(t,x),\psi(t,x))$ based on the *a priori* assumption that $(\phi(t,x),\psi(t,x))\in X_{1/m,M}(0,T)$ for some positive constants m and M and consequently

$$\frac{1}{m}\leq V(t,x)+\phi_x(t,x)=v(t,x)\leq M,\quad \forall\ (t,x)\in[0,T]\times\mathbb{R}. \tag{34}$$

Before doing so, we first recall that throughout this manuscript c and C are used to denote some generic positive constants independent of T, m, M and δ. Besides, we will often use the notation $(v,u)=(V+\phi_x,U+\psi_x)$, though the unknown functions are ϕ and ψ. Moreover, we denote here $N_\psi(T):=\sup_{[0,T]}\|\psi(t)\|_{L^\infty}$, or by N_ψ for simplicity. Without loss of generality, we can assume that $m\geq 1$ and $M\geq 1$.

Our first result is concerned with the basic energy estimate, which is stated in the following lemma.

Lemma 3.1. *There exists a sufficiently small positive constant ϵ_1 independent of δ such that if $0<\delta\leq\epsilon_1$, then it holds for each $0\leq t\leq T$,*

$$\begin{aligned}\|(\phi,\psi)(t)\|^2+\int_0^t\int_{\mathbb{R}}\left(|V_t|\psi^2+\psi_x^2\right)\mathrm{d}x\mathrm{d}\tau\leq\ & C\Big\{\|(\phi_0,\psi_0)\|^2+\delta^{\frac{1}{2}}\\ &+\int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+\left(m^{\gamma+2}N_\psi+m^2\delta^2\right)\int_0^t\|\phi_x(\tau)\|^2d\tau\Big\}.\end{aligned} \tag{35}$$

Proof. Firstly, rewrite $(19)_2$ (second equation of (19)) as

$$\begin{aligned}&\psi_t+p'(V)\phi_x-\mu\frac{\psi_{xx}}{V}\\ =&-\left(p(v)-p(V)-p'(V)\phi_x\right)-\mu\frac{\phi_x\psi_{xx}}{Vv}-\mu\frac{\phi_xU_x}{Vv}+g.\end{aligned} \tag{36}$$

Multiplying $(19)_1$ and (36) by ϕ and $-p'(V)^{-1}\psi$, respectively and adding these two equations, we discover

$$\left\{\frac{1}{2}\phi^2-\frac{1}{2p'(V)}\psi^2\right\}_t-\frac{p''(V)}{2p'(V)^2}V_t\psi^2-\frac{\mu}{p'(V)V}\psi_x^2-\left\{\phi\psi-\frac{\mu\psi\psi_x}{p'(V)V}\right\}_x$$
$$=\left(\frac{\mu}{p'(V)V}\right)'V_x\psi\psi_x+\left(p(v)-p(V)-p'(V)\phi_x\right)\frac{\psi}{p'(V)}$$
$$+\mu\frac{\psi\phi_x\psi_{xx}}{p'(V)Vv}+\mu\frac{\psi\phi_xU_x}{p'(V)Vv}-\frac{\psi g}{p'(V)}.$$

Integrating the above identity with respect to t and x over $[0,t]\times\mathbb{R}$ yields

$$\begin{aligned}&\left\|\left(\phi,V^{\frac{\gamma+1}{2}}\psi\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}V^\gamma\left(|V_t|\psi^2+\psi_x^2\right)dxd\tau\\ \le&C\left\|\left(\phi_0,V^{\frac{\gamma+1}{2}}(0,\cdot)\psi_0\right)\right\|^2+C\underbrace{\int_0^t\int_{\mathbb{R}}\left|V^{\gamma-1}V_x\psi\psi_x\right|dxd\tau}_{I_1}\\ &+C\underbrace{\int_0^t\int_{\mathbb{R}}\left|V^{\gamma+1}\left(p(v)-p(V)-p'(V)\phi_x\right)\psi\right|dxd\tau}_{I_2}\\ &+C\underbrace{\int_0^t\int_{\mathbb{R}}\frac{V^\gamma\left(|\psi\phi_x\psi_{xx}|+|\psi\phi_xU_x|\right)}{v}dxd\tau}_{I_3}+C\underbrace{\int_0^t\int_{\mathbb{R}}V^{\gamma+1}|\psi g|dxd\tau}_{I_4}.\end{aligned}\tag{37}$$

Noticing that $C\delta^2\ge -V_t=-U_x>0$ and

$$\begin{aligned}|p(v)-p(V)-p'(V)\phi_x|&=\phi_x^2\int_0^1\int_0^1p''(\theta_1\theta_2v+(1-\theta_1\theta_2)V)\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &\le Cm^{\gamma+2}\phi_x^2,\end{aligned}\tag{38}$$

we have from Cauchy's and Hölder's inequalities and (31) that

$$\begin{aligned}I_1\le&\epsilon\int_0^t\int_{\mathbb{R}}V^\gamma\psi_x^2dxd\tau+C(\epsilon)\int_0^t\int_{\mathbb{R}}V^{\gamma-2}V_x^2\psi^2dxd\tau,\\ I_2\le&CN_\psi m^{\gamma+2}\int_0^t\int_{\mathbb{R}}V^{\gamma+1}\phi_x^2dxd\tau,\\ I_3\le&\int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v}dxd\tau+\epsilon\int_0^t\int_{\mathbb{R}}V^\gamma|U_x|\psi^2dxd\tau\\ &+CN_\psi m\int_0^t\int_{\mathbb{R}}V^{2\gamma}\phi_x^2dxd\tau+Cm^2\int_0^t\int_{\mathbb{R}}V^\gamma|U_x|\phi_x^2dxd\tau\\ I_4\le&\int_0^t\|V\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}^{\frac{\gamma+1}{2}}\left\|V^{\frac{\gamma+1}{2}}\psi(\tau)\right\|\|g(\tau)\|\mathrm{d}\tau\\ \le&\delta^{1/2}\|V\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}^{\gamma+1}+C\int_0^t\left\|V^{\frac{\gamma+1}{2}}\psi(\tau)\right\|^2\|g(\tau)\|\mathrm{d}\tau.\end{aligned}\tag{39}$$

Here $\epsilon>0$ is a sufficiently small positive constant.

Since v_- and v_+ are independent of δ and δ is assumed to be sufficiently small, we can deduce that $V(t,x)$ can be bounded from both below and above by some positive constants independent of δ. With such an observation in hand, we can deduce that, for some sufficiently small positive constant $\epsilon_1 > 0$, if δ is suitably chosen such that the condition listed in Lemma 3.1 is satisfied, (35) can be proved by substituting the above estimates on I_j $(j = 1, 2, 3, 4)$ into (37) and by employing the Gronwall inequality. This completes the proof of Lemma 3.1. □

Next, differentiating (19) with respect to x once and multiplying the two resulting identities by $(p(V) - p(v))$ and ψ_x, respectively, we can get by adding the final results that

$$\begin{aligned}&\left\{\Phi + \frac{1}{2}\psi_x^2\right\}_t + \mu\frac{\psi_{xx}^2}{v} + \left\{\psi_x\left(p(v) - p(V) - \mu\frac{\psi_{xx}}{v} + \mu\frac{U_x\phi_x}{vV}\right)\right\}_x\\ =&\mu\frac{U_x\phi_x\psi_{xx}}{vV} + \psi_x g_x - V_t\left(p(v) - p(V) - p'(V)\phi_x\right),\end{aligned} \tag{40}$$

where

$$\Phi = \Phi(v, V) = p(V)(v - V) - \int_V^v p(\eta)\mathrm{d}\eta. \tag{41}$$

Integrating (40) over $[0, t] \times \mathbb{R}$, we have from (38) and Cauchy's inequality that

$$\begin{aligned}&\left\|\left(\sqrt{\Phi}, \psi_x\right)(t)\right\|^2 + \int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau\\ \leq&C\left\{\left\|\left(\sqrt{\Phi_0}, \psi_{0x}\right)\right\|^2 + \int_0^t\int_{\mathbb{R}}|\psi_x||g_x|\mathrm{d}x\mathrm{d}\tau\right.\\ &\left.+\int_0^t\int_{\mathbb{R}}\left(\frac{U_x^2}{vV^2} + |V_t|m^{\gamma+2}\right)\phi_x^2\mathrm{d}x\mathrm{d}\tau\right\}\end{aligned} \tag{42}$$

with $\Phi_0 = \Phi|_{t=0}$. Noting that

$$\Phi(v, V) = -\phi_x^2\int_0^1\int_0^1\theta_1 p'((1 - \theta_2\theta_1)V + \theta_2\theta_1 v)\mathrm{d}\theta_1\mathrm{d}\theta_2 \geq CM^{-\gamma-1}\phi_x^2 \tag{43}$$

and

$$\int_0^t\int_{\mathbb{R}}|\psi_x||g_x|\mathrm{d}x\mathrm{d}\tau \leq \int_0^t\|g_x(\tau)\|\|\psi_x(\tau)\|^2 d\tau + \int_0^t\|g_x(\tau)\|d\tau,$$

we can deduce from (42), (31) and Gronwall's inequality that

$$\begin{aligned}&\left\|\left(\sqrt{\Phi}, \psi_x, M^{-\frac{\gamma+1}{2}}\phi_x\right)(t)\right\|^2 + \int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau\\ \leq&C\left\{\left\|\left(\sqrt{\Phi_0}, \psi_{0x}\right)\right\|^2 + \left(m^{\gamma+2}\delta^2 + m\delta^4\right)\int_0^t\|\phi_x(\tau)\|^2 d\tau + \delta^{\frac{3}{2}}\right\}.\end{aligned}$$

Since $m \geq 1$, a suitably linear combination of the above inequality with (35) yields

Lemma 3.2. *If δ is suitably small, then it holds for each $0 \leq t \leq T$ that*

$$\begin{aligned}&\left\|\left(\phi, \psi, \psi_x, \sqrt{\Phi}, M^{-\frac{\gamma+1}{2}}\phi_x\right)(t)\right\|^2 + \int_0^t\int_{\mathbb{R}}\left(\psi_x^2 + \frac{\psi_{xx}^2}{v}\right)\mathrm{d}x\mathrm{d}\tau\\ \leq&C\left\{\left\|\left(\phi_0, \psi_0, \sqrt{\Phi_0}, \psi_{0x}\right)\right\|^2 + \delta^{\frac{1}{2}} + m^{\gamma+2}\left(N_\psi + \delta^2\right)\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau\right\}.\end{aligned} \tag{44}$$

We now turn to deal with the term $\int_0^t \|\phi_x(\tau)\|^2 \mathrm{d}\tau$. For this purpose, we have from $(19)_2$ that

$$\begin{aligned}&(\phi_x\psi)_t + \psi_x^2 - \frac{\mu\phi_x\psi_{xx}}{v} - \phi_x g - (\psi\phi_t)_x\\ =& -\int_0^1 p'(V+\theta\phi_x)d\theta\phi_x^2 - \frac{\mu U_x\phi_x^2}{Vv}\\ \geq& -\int_0^1 p'(V+\theta\phi_x)d\theta\phi_x^2 \geq CM^{-\gamma-1}\phi_x^2.\end{aligned}$$

Here we have used the *a priori* assumption (34) and the fact that $U_x(t,x)<0$.

Integrating the above inequality with respect to t and x over $[0,t]\times\mathbb{R}$, we have from Cauchy's and Hölder's inequalities that

$$\begin{aligned}M^{-\gamma-1}\int_0^t \|\phi_x(\tau)\|^2 d\tau \leq C\Big\{&\|\phi_x(t)\|\|\psi(t)\| + \|\phi_{0x}\|\|\psi_0\| + \int_0^t \|\psi_x(\tau)\|^2 d\tau\\ &+mM^{\gamma+1}\int_0^t\int_{\mathbb{R}} \frac{\psi_{xx}^2}{v} dxd\tau + \sup_{0\leq\tau\leq t}\|\phi_x(\tau)\|\int_0^t \|g(\tau)\| d\tau\Big\},\end{aligned} \tag{45}$$

which implies from (31) and (44) that

$$\begin{aligned}\int_0^t \|\phi_x(\tau)\|^2 \mathrm{d}\tau \leq& CmM^{2\gamma+2}\left\{\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2 + \delta^{\frac12}\right\}\\ &+ Cm^{\gamma+3}M^{2\gamma+2}\left(N_\psi+\delta^2\right)\int_0^t \|\phi_x(\tau)\|^2 \mathrm{d}\tau.\end{aligned} \tag{46}$$

And we have the following result

Lemma 3.3. *There exists a $\delta-$independent positive constant ϵ_2 such that if*

$$m^{\gamma+3}M^{2\gamma+2}\left(N_\psi+\delta^2\right) \leq \epsilon_2 \tag{47}$$

holds true, then we have for each $0\leq t\leq T$ that

$$\int_0^t \|\phi_x(\tau)\|^2 \mathrm{d}\tau \leq CmM^{2\gamma+2}\left\{\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2 + \delta^{\frac12}\right\} \tag{48}$$

and

$$\begin{aligned}&\left\|\left(\phi,\psi,\psi_x,\sqrt{\Phi},M^{-\frac{\gamma+1}{2}}\phi_x\right)(t)\right\|^2 + \int_0^t\int_{\mathbb{R}}\left(\psi_x^2+\frac{\psi_{xx}^2}{v}\right)\mathrm{d}x\mathrm{d}\tau\\ \leq& C\left\{\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2 + \delta^{\frac12}\right\}.\end{aligned} \tag{49}$$

To deduce a lower bound and an upper bound on $v(t,x)$, as in [13], we set $\tilde{v} := v/V$. Then $\Phi(v,V)$ can be reformulated as

$$\Phi(v,V) = aV^{-\gamma+1}\tilde{\Phi}(\tilde{v}), \quad \tilde{\Phi}(\tilde{v}) = \tilde{v} - 1 + \frac{1}{\gamma-1}\left(\tilde{v}^{-\gamma+1}-1\right). \tag{50}$$

Moreover we can rewrite $(19)_2$ as

$$\left(\mu\frac{\tilde{v}_x}{\tilde{v}} - \psi_x\right)_t - (p(v)-p(V))_x = -g_x. \tag{51}$$

Multiplying (51) by $\tilde{v}_x/\tilde{v}$, we have

$$\begin{aligned}&\left\{\frac{\mu}{2}\left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2-\psi_x\frac{\tilde{v}_x}{\tilde{v}}\right\}_t-V^2\frac{p'(v)}{v}\tilde{v}_x^2+\left(\left(\frac{u_x}{v}-\frac{U_x}{V}\right)\psi_x\right)_x\\=&\frac{V_x\tilde{v}_x}{v}\left(p'(v)v-p'(V)V\right)-\frac{\tilde{v}_x}{\tilde{v}}g_x+\frac{\psi_{xx}^2}{v}-\frac{U_x\phi_x\psi_{xx}}{vV}.\end{aligned}\tag{52}$$

Noting that for any $\epsilon>0$,

$$\begin{aligned}&\frac{V_x\tilde{v}_x}{v}\left(p'(v)v-p'(V)V\right)\\\le&\epsilon\frac{|p'(v)|}{v}\tilde{v}_x^2+C(\epsilon)V_x^2v^\gamma\left|p'(v)v-p'(V)V\right|^2\\\le&\epsilon\frac{|p'(v)|}{v}\tilde{v}_x^2+C(\epsilon)V_x^2M^\gamma m^{2\gamma+2}\phi_x^2,\end{aligned}\tag{53}$$

we can get by integrating (52) with respect to t and x over $[0,t]\times\mathbb{R}$ and by employing Lemma 3.3, (53) and Gronwall's inequality that

$$\begin{aligned}&\left\|\frac{\tilde{v}_x}{\tilde{v}}(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\tilde{v}_x^2}{v^{\gamma+2}}\mathrm{d}x\mathrm{d}\tau\\\le&C\left\|\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right\|^2+C\left(1+M^{3\gamma+2}m^{2\gamma+3}\delta^4\right)\left\{\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2+\delta^{\frac12}\right\}.\end{aligned}\tag{54}$$

Since the assumption (47) implies that $M^{3\gamma+2}m^{2\gamma+3}\delta^4$ is bounded by some generic positive constant independent of m, M, T, x and δ, we have from (54) that

Lemma 3.4. *If δ is suitably small such that (47) holds, it follows that*

$$\left\|\frac{\tilde{v}_x}{\tilde{v}}(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\tilde{v}_x^2}{v^{\gamma+2}}\mathrm{d}x\mathrm{d}\tau\le C\left\{\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x},\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right)\right\|^2+\delta^{\frac12}\right\}.\tag{55}$$

Now we turn to deduce the desired lower and upper bounds on $v(t,x)$ in terms of the initial perturbation. Following Kanel' [4], we now have the key lemma.

Lemma 3.5. *Under the assumption that (47) holds, we have for each $(t,x)\in[0,T]\times\mathbb{R}$ that*

$$C^{-1}B_0^{\frac{2}{1-\gamma}}\le v(t,x)\le CB_0^2\tag{56}$$

with

$$B_0=\left(\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|+\delta^{\frac14}\right)\left(\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x},\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right)\right\|+\delta^{\frac14}\right).\tag{57}$$

Proof. Note that

$$\tilde{\Phi}(z)\sim\begin{cases}z, & z\to\infty,\\ z^{-\gamma+1}, & z\to 0.\end{cases}\tag{58}$$

We set

$$\Psi(\tilde{v}):=\int_1^{\tilde{v}}\sqrt{\tilde{\Phi}(z)}\frac{dz}{z}\tag{59}$$

and it is easy to see that

$$|\Psi(\tilde{v})|\ge C\left(\tilde{v}^{\frac12}+\tilde{v}^{\frac{1-\gamma}{2}}-1\right)\tag{60}$$

holds for all $0 \leq t \leq T$ and thus we have from (49) and (55) that

$$
\begin{aligned}
|\Psi(\tilde{v}(t,x))| &= \left| \int_{-\infty}^{x} \frac{\partial \Psi(\tilde{v})}{\partial y}(t,y)dy \right| \\
&\leq \int_{\mathbb{R}} \sqrt{\tilde{\Phi}(\tilde{v})} \left| \frac{\tilde{v}_x}{\tilde{v}} \right| (t,x)dx \\
&\leq \left\| \sqrt{\tilde{\Phi}(\tilde{v})}(t) \right\| \left\| \frac{\tilde{v}_x}{\tilde{v}}(t) \right\|.
\end{aligned} \tag{61}
$$

Having obtained (60), (61) and noticing that $v_\pm$ and $u_\pm$ are assumed to be independent of δ and $\min\{v_-, \bar{v}, v_+\} \leq V(t,x) \leq \max\{v_-, \bar{v}, v_+\}$, (56) follows easily from (49) and (54). □

For the second order energy type estimates, since

$$
\frac{\tilde{v}_x}{\tilde{v}} = \frac{\phi_{xx}}{v} - \frac{V_x \phi_x}{Vv}, \tag{62}
$$

we have from Lemmas 3.3-3.5 that

$$
\|\phi_{xx}(t)\|^2 + \int_0^t \|\phi_{xx}(\tau)\|^2 \mathrm{d}\tau \leq C\left(\delta, \|(\phi_0, \psi_0)\|_2\right). \tag{63}
$$

It is worth to pointing out that, unlike the positive constants in Lemmas 3.3-3.5, the constant $C\left(\delta, \|(\phi_0, \psi_0)\|_2\right)$ here in the right hand side of (63) depends on δ and $\|(\phi_0, \psi_0)\|_2$.

As to the estimate on $\|\psi_{xx}(t)\|$, we have by multiplying $\partial_x(19)_2$ by $-\psi_{xxx}$ that

$$
\begin{aligned}
&\left\{ \frac{1}{2}\psi_{xx}^2 \right\}_t + \left\{ \psi_{xx} g_x - \psi_{xx}\psi_{xt} \right\}_x + \mu \frac{\psi_{xxx}^2}{v} \\
=& \psi_{xxx} \left(p(v) - p(V)\right)_x + \mu \psi_{xxx}\psi_{xx} \frac{V_x}{v^2} + \mu \frac{\psi_{xxx} U_{xx} \phi_x}{Vv} + \mu \frac{\psi_{xxx} U_x \phi_{xx}}{Vv} \\
&- \mu \psi_{xxx} U_x \phi_x \left(\frac{V_x + \phi_{xx}}{Vv^2} + \frac{V_x}{V^2 v} \right) + \psi_{xx} g_{xx} + \mu \psi_{xxx}\psi_{xx} \frac{\phi_{xx}}{v^2}.
\end{aligned} \tag{64}
$$

We only show how to estimate the term corresponding to the last one in (64) in the following, since the rest is easier. To do so, by Sobolev's inequality and Young's inequality with ϵ, it holds that

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \left| \psi_{xx}\psi_{xxx} \frac{\phi_{xx}}{v^2} \right| \mathrm{d}x \mathrm{d}\tau \\
\leq& m^2 \int_0^t \|\psi_{xxx}(\tau)\|^{\frac{3}{2}} \|\psi_{xx}(\tau)\|^{\frac{1}{2}} \|\phi_{xx}(\tau)\| \mathrm{d}\tau \\
\leq& \frac{1}{2M} \int_0^t \|\psi_{xxx}(\tau)\|^2 \mathrm{d}\tau + C(m, M, \delta) \sup_{0 \leq \tau \leq t} \|\phi_{xx}(\tau)\|^4 \int_0^t \|\psi_{xx}(\tau)\|^2 \mathrm{d}\tau,
\end{aligned} \tag{65}
$$

where the last term on the right-hand side of (65) can be controlled by (44) and (63).

Thus, integrating (64) with respect to t and x over $[0,t] \times \mathbb{R}$, we have from Lemmas 3.3-3.5 that

Lemma 3.6. *If δ is suitably small such that (47) holds, it follows that*

$$
\|\psi_{xx}(t)\|^2 + \int_0^t \|\psi_{xxx}(\tau)\|^2 \mathrm{d}\tau \leq C\left(\delta, \|(\phi_0, \psi_0)\|_2\right). \tag{66}
$$

With the above preparations in hand, we now turn to prove Theorem 1.1. Noticing (62) and the fact $\Phi_0(x) \le C(|V(0,x)|^{-\gamma-1} + |v_0(x)|^{-\gamma-1})\phi_{0x}^2$, we first deduce from (30) and the assumptions (H_0) and (H_2) that

$$\left\|\sqrt{\Phi_0}\right\| \le C(1+\delta^{-(\gamma+1)\ell/2})\|\phi_{0x}\|, \quad \left\|\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right\| \le C\delta^{-\ell}(\|\phi_{0xx}\| + \delta^2\|\phi_{0x}\|).$$

Hence, if (25) and $(26)_2$ hold, then for $0<\delta<1$,

$$\left\|\left(\phi_0, \psi_0, \sqrt{\Phi_0}, \psi_{0x}\right)\right\| + \delta^{\frac14} \le C\delta^{\min\left\{\alpha - \frac{\gamma+1}{2}\ell, \frac14\right\}}$$

and

$$\left\|\left(\phi_0, \psi_0, \sqrt{\Phi_0}, \psi_{0x}, \frac{\tilde{v}_{0x}}{\tilde{v}_0}\right)\right\| + \delta^{\frac14} \le C\delta^{-(\beta+\ell)}.$$

Now we prove Theorem 1.1 by exploiting the continuation argument. Applying Proposition 2, we find a positive constant t_1, which depends only on δ and $\|(\phi_0,\psi_0)\|_2$, such that the Cauchy problem (19) admits a unique solution $(\phi,\psi) \in X_{m_0,M_0}(0,t_1)$ with $m_0 = 2^{-1}C_1^{-1}\delta^{\ell}$ and $M_0 = 2C_1(1+\delta^{-\ell})$, which satisfies (33) for each $0 \le t \le t_1$. Hence we have from (25) and Sobolev's inequality that

$$N_\psi(t_1) = \sup_{[0,t_1]} \|\psi(t)\|_{L^\infty(\mathbb{R})} \le \sup_{[0,t_1]} \|\psi(t)\|^{\frac12}\|\psi_x(t)\|^{\frac12} \le 2C_2\delta^{\alpha}.$$

Consequently,

$$m_0^{-\gamma-3}M_0^{2\gamma+2}\left(N_\psi(t_1)+\delta^2\right) \le C\delta^{\min\{\alpha,2\}-(3\gamma+5)\ell}.$$

Thus if $(26)_1$ holds, we can choose a sufficiently small constant $\delta_1<1$ such that if $0<\delta\le\delta_1$, the assumptions imposed in Lemmas 3.1-3.6 hold with $T=t_1$, $m=m_0^{-1}$ and $M=M_0$. Thus we have from (56) that

$$C_4^{-1}\delta^{\frac{2}{1-\gamma}\left(\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell)\right)} \le v(t,x) \le C_4\delta^{2\left(\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell)\right)} \tag{67}$$

holds for each $0\le t\le t_1$, and from (48), (49), (63) and (66) that

$$\|\psi(t)\|_1 \le C_5\delta^{\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}} \tag{68}$$

and

$$\|(\phi,\psi)(t)\|_2^2 + \int_0^t \left(\|\phi_x(\tau)\|_1^2 + \|\psi_x(\tau)\|_2^2\right)d\tau \le C\left(\delta, \|(\phi_0,\psi_0)\|_2\right) \tag{69}$$

hold for each $0\le t\le t_1$. Next if we take $(\phi(t_1,x),\psi(t_1,x))$ as the initial data, we can deduce by employing Proposition 2 again that the unique local solution $(\phi(t,x),\psi(t,x))$ constructed above can be extended to the time internal $[0,t_1+t_2]$ and satisfies

$$\|\psi(t)\|_{L^\infty(\mathbb{R})} \le \|\psi(t)\|_1 \le 2\|\psi(t_1)\|_1 \le 2C_5\delta^{\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}}$$

and

$$2^{-1}C_4^{-1}\delta^{\frac{2}{1-\gamma}\left(\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell)\right)} \le v(t,x) \le 2C_4\delta^{2\left(\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell)\right)}$$

for each $t_1\le t\le t_1+t_2$. Thus,

$$N_\psi(t_1+t_2) \le \max\left\{N_\psi(t_1), 2C_5\delta^{\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}}\right\} \le C_6\delta^{\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}}.$$

Set

$$m_1 = 2^{-1}C_4^{-1}\delta^{\frac{2}{1-\gamma}\left(\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell)\right)}$$

and

$$M_1 = 2C_4\delta^{2\left(\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell)\right)}.$$

Then one can easily deduce that if the parameters $\alpha > 0$, β and ℓ satisfy (26)$_3$, then there exists a sufficiently small $\delta_2 > 0$ such that if $0 < \delta \leq \delta_2$, the assumptions listed in Lemmas 3.1-3.6 are satisfied with $T = t_1 + t_2$, $m = m_1^{-1}$ and $M = M_1$. Consequently, (67), (68) and (69) hold for each $0 \leq t \leq t_1 + t_2$. If we take $(\phi(t_1+t_2,x), \psi(t_1+t_2,x))$ as the initial data and employ Proposition 2 again, we can then extend the above solution $(\phi(t,x), \psi(t,x))$ to the time step $t = t_1 + 2t_2$. Repeating the above procedure, we thus extend $(\phi(t,x), \psi(t,x))$ step by step to the unique global solution and (67), (68) and (69) hold for all $t \geq 0$. This completes the proof of Theorem 1.1.

4. **Some remarks.** This section is concerned with two remarks on Theorem 1.1. The first is an application of Theorem 1.1 to the initial-boundary value problem of the one-dimensional compressible Navier-Stokes equations with impermeable boundary condition and the second is concerned with the nonlinear stability of a single viscous shock profile for the case when the far fields of the initial data may depend on the strength of the viscous shock wave.

To make the presentation easy to read, we divide this section into two subsections and the first one is devoted to the initial-boundary value problem (29).

4.1. **An application to the impermeable wall problem.** For the impermeable wall problem (29), since $u_+ < 0$, there exists a unique $v_m > 0$ such that $(v_+, u_+) \in S_2(v_m, 0)$ and the large time behavior of its global solutions $(v(t,x), u(t,x))$ can be described by the viscous shock wave of the second family. Such an expectation is justified rigorously by Matsumura and Mei in [10] and it is shown in [10] that the solution tends to the 2-viscous shock wave connecting $(v_m, 0)$ with (v_+, u_+) as t goes to infinity, where the constant v_m is uniquely determined by the Rankine-Hugoniot condition (8) so that $(v_m, 0)$ is located on the 2-shock curve passing through (v_+, u_+), provided that the viscous shock wave is initially far away from the boundary and both the $H^2(\mathbb{R}_+)-$norm of the initial perturbation and the strength of the viscous shock wave are sufficiently small. The main purpose of this subsection is to show that our main result Theorem 1.1 can be applied to this problem and we can deduce a similar nonlinear stability result without assuming that the viscous shock wave is initially far away from the boundary and for a class of large initial perturbation which allow that specific volume $v(t,x)$ to have large oscillation.

To this end, as in [15], we reformulate the half space problem (29) into a special case of the initial value problem (1)-(2) with $(v(0,x), u(0,x)) = (\bar{v}_0(x), \bar{u}_0(x))$, $(v_-, u_-) = (v_+, -u_+)$ and $(\bar{v}, \bar{u}) = (v_m, 0)$. Here

$$\bar{v}_0(x) = \begin{cases} v_0(x), & x \geq 0, \\ v_0(-x), & x \leq 0, \end{cases} \qquad \bar{u}_0(x) = \begin{cases} u_0(x), & x \geq 0, \\ -u_0(-x), & x \leq 0. \end{cases} \tag{70}$$

Since $(v_+, u_+) \in S_2(v_m, 0)$ implies $(v_m, 0) \in S_1(v_+, -u_+)$, the system (1) admits a viscous shock wave $(V_i(x - s_i t), U_i(x - s_i t))$ of the $i-$family, which connects $(v_-, -u_+)$ with $(v_m, 0)$ for $i = 1$ and $(v_m, 0)$ with (v_+, u_+) for $i = 2$, respectively. The strengths of these two viscous shock waves satisfy $\delta_1 = \delta_2 = |v_m - v_+|$ with speeds

$$s_i = (-1)^i \frac{u_+}{v_m - v_+} = (-1)^i \sqrt{\frac{v_+ - v_m}{p(v_m) - p(v_+)}}, \quad i = 1, 2.$$

It is easy to see that if v_+ is independent of $\delta = |u_+|$, one can easily deduce that

$$C^{-1}\delta \leq \delta_1 = \delta_2 \leq C\delta \tag{71}$$

holds for some $\delta-$independent positive constant C.

Noticing that

$$V_1(\xi)=V_2(-\xi),\quad U_1(\xi)=-U_2(-\xi),\quad \forall\ \xi\in\mathbb{R}, \tag{72}$$

we have

$$\begin{aligned}A:&=\int_{\mathbb{R}}(\bar v_0(x)-V_1(x)-V_2(x)+v_m)\,dx\\&=\int_{\mathbb{R}}(\bar v_0(x)-V_2(-x)-V_2(x)+v_m)\,dx\\&=2\int_{\mathbb{R}_+}(\bar v_0(x)-V_2(-x)-V_2(x)+v_m)\,dx,\end{aligned} \tag{73}$$

$$\begin{aligned}B:&=\int_{\mathbb{R}}(\bar u_0(x)-U_1(x)-U_2(x))\,dx\\&=\int_{\mathbb{R}}(\bar u_0(x)+U_2(-x)-U_2(x))\,dx\\&=0.\end{aligned} \tag{74}$$

If we choose the shifts α_i $(i=1,2)$ as

$$\alpha_1=-\alpha_2=-\frac{A}{2\delta_1} \tag{75}$$

and define

$$\begin{cases}V(t,x;\alpha_1,\alpha_2)=V_1(x-s_1t+\alpha_1)+V_2(x-s_2t+\alpha_2)-v_m,\\ U(t,x;\alpha_1,\alpha_2)=U_1(x-s_1t+\alpha_1)+U_2(x-s_2t+\alpha_2),\end{cases} \tag{76}$$

we can deduce that

$$\int_{\mathbb{R}}((\bar v_0,\bar u_0)(x)-(V,U)(0,x;\alpha_1,\alpha_2))\,dx=0. \tag{77}$$

(77) together with the conservative form of the equations for both $(v(t,x),U(t,x))$ and $(V(t,x;\alpha_1,\alpha_2),U(t,x;\alpha_1,\alpha_2))$ imply that we can define the anti-derivative of the perturbation

$$(\phi(t,x),\psi(t,x))=-\int_x^\infty((v,u)(t,x)-(V,U)(t,x;\alpha_1,\alpha_2))\,dx, \tag{78}$$

and set

$$(\phi_0(x),\psi_0(x))=(\phi(0,x),\psi(0,x)),\quad \forall\ x\in\mathbb{R}. \tag{79}$$

Then if we assume that

(H_1') There exists a $\delta-$independent constant $C>0$ such that $|A|\le C$ and consequently the shifts α_i $(i=1,2)$ and δ_i, the strengths of the $i-$viscous shock profiles satisfy

$$C^{-1}\le\delta_1=\delta_2\le C\delta,\quad \alpha_2-\alpha_1=\frac{A}{\delta_1}\le C\delta^{-1} \tag{80}$$

for some $\delta-$independent constant C,

$$(\bar v_0(x)-V(0,x;\alpha_1,\alpha_2),\bar u_0(x)-U(0,x;\alpha_1,\alpha_2))\in H^1(\mathbb{R})\cap L^1(\mathbb{R}), \tag{81}$$

$$(\phi_0(x),\psi_0(x))\in L^2(\mathbb{R}); \tag{82}$$

(H_2') There exist $\delta-$independent constants $\ell\ge0$ and $C>0$ such that

$$C^{-1}\delta^\ell\le\bar v_0(x)\le C(1+\delta^{-\ell}); \tag{83}$$

(H_3') v_+ is a positive constant independent of δ;

(H_4') There exists $\delta-$independent positive constants C, α, and β such that

$$\|(\phi_0,\psi_0)\|_{H^1(\mathbb{R})}\leq C_2\delta^{\alpha},\quad \|\phi_{0xx}\|_{L^2(\mathbb{R})}\leq C_2\left(1+\delta^{-\beta}\right); \tag{84}$$

(H_5') The parameters $\alpha,\ \beta$, and ℓ satisfies (26).

Then from Theorem 1.1, we can deduce that there exists a sufficiently small positive constant $\delta_0>0$ such that if $0<\delta\leq\delta_0$, the system (1) with initial data $(v(0,x),u(0,x))=(\bar{v}_0(x),\bar{u}_0(x))$ admits a unique global solution $(\bar{v}(t,x),\bar{u}(t,x))$ which satisfies

$$\bar{v}(t,x)=\bar{v}(t,-x),\quad \bar{u}(t,x)=-\bar{u}(t,-x),\quad \forall x\in\mathbb{R} \tag{85}$$

and

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|(\bar{v}(t,x)-V(t,x;\alpha_1,\alpha_2),\bar{u}(t,x)-U(t,x;\alpha_1,\alpha_2))|=0. \tag{86}$$

Setting

$$v(t,x)=\bar{v}(t,x)|_{(t,x)\in\mathbb{R}^2_+},\quad u(t,x)=\bar{u}(t,x)|_{(t,x)\in\mathbb{R}^2_+}, \tag{87}$$

it is easy to see that $(v(t,x),u(t,x))$ is a solution of the initial-boundary value problem of (29) and the only thing left is to show that such a solution tends to $(V_2(x-s_2t+\alpha_2),U_2(x-s_2t+\alpha_2))$ time asymptotically.

In fact, we have from (86) and the fact

$$|(V_1-v_m,U_1)(x-s_1t+\alpha_1)|\leq|(V_1-v_m,U_1)(-s_1t+\alpha_1)| \tag{88}$$

that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}_+}|(v,u)(t,x)-(V_2,U_2)(x-s_2t+\alpha_2)|\\
\leq&\sup_{x\in\mathbb{R}_+}|(v,u)(t,x)-(V,U)(t,x;\alpha_1,\alpha_2)|\\
&+\sup_{x\in\mathbb{R}_+}|(V_1-v_m,U_1)(x-s_1t+\alpha_1)|\\
\leq&\sup_{x\in\mathbb{R}_+}|(v,u)(t,x)-(V,U)(t,x;\alpha_1,\alpha_2)|\\
&+|(V_1-v_m,U_1)(-s_1t+\alpha_1)|\to 0,\quad \text{as } t\to\infty,
\end{aligned} \tag{89}$$

which gives the desired stability result.

Remark 2. Although the assumptions (H_1')-(H_4') are imposed on the initial data $(\bar{v}_0(x),\bar{u}_0(x))$, it can be translated into similar conditions on $(v_0(x),u_0(x))$ which are only defined on $x\in\mathbb{R}_+$.

4.2. **Nonlinear stability of single viscous shock wave.** This subsection is devoted to the nonlinear stability of a single viscous shock wave whose far fields may depend on its strength. To deal with this case, we need a careful analysis of the viscous shock waves. In fact, by mimicking the argument used in [11], we have

Lemma 4.1. *Suppose that $v_-\sim v_+\sim\delta^{\ell_1}$ ($\ell_1\leq 1$) ($\delta:=|v_+-v_-|$) and $(v_+,u_+)\in S_i(v_-,u_-)$, then there exists a viscous shock wave $(V(x-s_it),U(x-s_it))$ of (7) which satisfies $(V(\pm\infty),U(\pm\infty))=(v_\pm,u_\pm)$ and is unique up to a shift. Moreover, it holds that*

$$|s_i|\leq C\delta^{-\frac{\gamma+1}{2}\ell_1},\quad |V_\xi(\xi)|\leq C\delta^{2-\frac{\gamma+1}{2}\ell_1},\quad |U_\xi(\xi)|\leq C\delta^{2-(\gamma+1)\ell_1}. \tag{90}$$

Here $C>0$ is some positive constant independent of δ and throughout this paper, $a\sim b$ means that there exists some $\delta-$independent positive constant $C>0$ such that $C^{-1}b\leq a\leq Cb$ as $\delta\to 0_+$.

We assume without loss of generality that

$$(H_4)\qquad \begin{cases} (v_+,u_+)\in S_2(v_-,u_-),\\ (v_0(x)-V(x-s_2t),u_0(x)-U(x-s_2t))\in H^1(\mathbb{R})\cap L^1(\mathbb{R}),\\ \displaystyle\int_{\mathbb{R}}(v_0(x)-V(x),u_0(x)-U(x))dx=0,\\ m_0^{-1}\le v_0(x)\le m_0,\quad m_0\sim\delta^{-\ell_2},\quad |\ell_1|\le\ell_2. \end{cases}$$

Remark 3. The assumption $(H_4)_3$ is nothing but the zero mass assumption.

Setting

$$\begin{cases} (\phi(t,x),\psi(t,x))=\displaystyle\int_{-\infty}^{x}(v(t,y)-V(y-s_2t),u(t,y)-U(y-s_2t))\,dy,\\ (\phi_0(x),\psi_0(x))=\displaystyle\int_{-\infty}^{x}(v_0(y)-V(y),u_0(y)-U(y))\,dy, \end{cases}\tag{91}$$

it is easy to see that $(\phi(t,x),\psi(t,x))$ solves

$$\begin{cases} \phi_t-\psi_x= & 0, & t>0,\ x\in\mathbb{R},\\ \psi_t+p(V+\phi_x)-p(V)= & \mu\left(\frac{\psi_{xx}+U_x}{\phi_x+V}-\frac{U_x}{V}\right), & t>0,\ x\in\mathbb{R},\\ (\phi(0,x),\psi(0,x))= & (\phi_0(x),\psi_0(x)), & x\in\mathbb{R}. \end{cases}\tag{92}$$

Before giving our main result, we first list some assumptions in the following:

(H_5) In addition to the above assumptions, we assume further that the initial perturbation $(\phi_0,\psi_0)\in H^2(\mathbb{R})$ satisfy

$$\|(\phi_0,\psi_0)\|_{H^1(\mathbb{R})}\le C_{11}\delta^{\alpha},\quad \|\phi_{0xx}\|_{L^2(\mathbb{R})}\le C_{11}(1+\delta^{-\beta})\tag{93}$$

for some δ-independent positive constants C_{11}, α and β;

(H_6) ℓ_1, ℓ_2, α and β satisfy

$$\begin{cases} |\ell_1|\le\ell_2,\quad \ell_1<1,\\ -(3\gamma+5)\ell_2+\min\{\alpha+(\gamma+1)\ell_1,2-(\gamma+1)\ell_1\}>0,\\ \alpha+\frac{\gamma-1}{2}\ell_1-\frac{\gamma+3}{2}\ell_2-\beta\le\min\{(\gamma-1)\ell_1,-\ell_1\},\\ \alpha+\frac{\gamma-1}{2}\ell_1-\frac{\gamma+3}{2}\ell_2-\beta>\frac{1-\gamma}{4\gamma^2+2\gamma+2}\min\left\{\alpha-\frac{\gamma+1}{2}\ell_2+\frac{7\gamma-1}{4}\ell_1,2-2\ell_1\right\}. \end{cases}\tag{94}$$

Our main result can be stated as in the following

Theorem 4.2. *Assume that the conditions (H_4)-(H_6) hold, then there exists a sufficiently small $\delta_0>0$ such that if $0<\delta\le\delta_0$, the Cauchy problem (1), (2) has a unique global solution $(v(t,x),u(t,x))$ which satisfies*

$$\begin{aligned} (v(t,x)-V(x-s_2t),u(t,x)-U(x-s_2t))&\in C\left([0,\infty);H^1(\mathbb{R})\right),\\ v(t,x)-V(x-s_2t)&\in L^2(0,\infty;H^1(\mathbb{R})),\\ u(t,x)-U(x-s_2t)&\in L^2\left(0,\infty;H^2(\mathbb{R})\right), \end{aligned}$$

and there exists a positive constant C_{12}, which is independent of δ, such that for each $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$C_{12}^{-1}\delta^{\frac{2\alpha-(\gamma+3)\ell_2-2\beta}{1-\gamma}}\le v(t,x)\le C_{12}\left(1+\delta^{2\alpha-(\gamma+3)\ell_2+\gamma\ell_1-2\beta}\right).\tag{95}$$

Furthermore it holds that

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|(v(t,x),u(t,x))-(V(x-s_2t),U(x-s_2t))|=0. \tag{96}$$

Now we outline the main steps to prove Theorem 4.2. Since in this case $(V(x-s_2t),U(x-s_2t))$ is an exact solution of (16), $(\phi(t,x),\psi(t,x))$ solves (19) with $g\equiv 0$. Assume that such a Cauchy problem has a solution $(\phi(t,x),\psi(t,x))\in X_{1/m,M}(0,T)$ for some $T>0$. Without loss of generality, we assume that $m,\ M\geq C\delta^{-|\ell_1|}$. To use the continuation argument to extend it to a global one, we now turn to deduce certain energy type *a priori* estimates on $(\phi(t,x),\psi(t,x))$.

The key steps to prove Theorem 4.2 is to deduce certain estimates similar to those obtained in Lemmas 3.1-3.6. Since the arguments to deduce these estimates are completely similar to those used to derive Lemmas 3.1-3.6, we just outline the main differences. Firstly, we have from $|V_x|=s_2^{-1}|V_t|$ and (90) that $V^{\gamma-2}V_x^2\psi^2\leq C\delta^{2-2\ell_1}V^\gamma|V_t|\psi^2$. Hence if $\ell_1<1$ and if $\delta>0$ is sufficiently small, the second term on the right-hand side of $(39)_1$ can be controlled by the terms on the left-hand side of (37). Then noting here $g\equiv 0$ and the fact that $m,M\geq C\delta^{-|\ell_1|}$ implies (38) and (43), similar to the proof of Lemma 3.1, we have the following lemma.

Lemma 4.3. *If $l_1<1$and if $0<\delta<\epsilon_3$ for some $\delta-$independent positive constant ϵ_3, it holds that for each $0\leq t\leq T$,*

$$\begin{aligned}&\left\|\left(\phi,V^{\frac{\gamma+1}{2}}\psi\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}V^\gamma\left(|V_t|\psi^2+\psi_x^2\right)\mathrm{d}x\mathrm{d}\tau\\ \leq{}& C\left\{\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0\right)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau\right.\\ &\left.+\left(m^{\gamma+2}N_\psi\delta^{(\gamma+1)\ell_1}+m^2\delta^{2-\ell_1}\right)\int_0^t\|\phi_x(\tau)\|^2d\tau\right\}.\end{aligned} \tag{97}$$

Next, we have from (42), (90) and (43) that

$$\begin{aligned}&\left\|\left(\sqrt{\Phi},\psi_x,M^{-\frac{\gamma+1}{2}}\phi_x\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau\\ \leq{}& C\left\{\left\|\left(\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2+m^{\gamma+2}\delta^{2-(\gamma+1)\ell_1}\int_0^t\|\phi_x(\tau)\|^2d\tau\right\},\end{aligned}$$

which together with (97) and $m\geq\delta^{-|\ell_1|}$ yields

$$\begin{aligned}&\left\|\left(\phi,V^{\frac{\gamma+1}{2}}\psi,\psi_x,\sqrt{\Phi},M^{-\frac{\gamma+1}{2}}\phi_x\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\left(V^\gamma\psi_x^2+\frac{\psi_{xx}^2}{v}\right)\mathrm{d}x\mathrm{d}\tau\\ \leq{}& C\left\{\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2\right.\\ &\left.+m^{\gamma+2}\left(N_\psi\delta^{(\gamma+1)\ell_1}+\delta^{2-(\gamma+1)\ell_1}\right)\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau\right\}.\end{aligned}$$

Secondly, similar to the proof of (46), we can get that

$$\begin{aligned}&\int_0^t \|\phi_x(\tau)\|^2 \mathrm{d}\tau\\ \leq& CmM^{2\gamma+2}\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2\\ &+Cm^{\gamma+3}M^{2\gamma+2}\left(N_\psi\delta^{(\gamma+1)\ell_1}+\delta^{2-(\gamma+1)\ell_1}\right)\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau.\end{aligned}$$

Consequently, we have the following lemma.

Lemma 4.4. *There is a positive constant ϵ_4 independent of δ such that if*

$$m^{\gamma+3}M^{2\gamma+2}\left(N_\psi\delta^{(\gamma+1)\ell_1}+\delta^{2-(\gamma+1)\ell_1}\right)\leq\epsilon_4, \tag{98}$$

then it follows that

$$\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau\leq CmM^{2\gamma+2}\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2, \tag{99}$$

$$\begin{aligned}&\left\|\left(\phi,V^{\frac{\gamma+1}{2}}\psi,\psi_x,\sqrt{\Phi},M^{-\frac{\gamma+1}{2}}\phi_x\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\left(V^\gamma\psi_x^2+\frac{\psi_{xx}^2}{v}\right)\mathrm{d}x\mathrm{d}\tau\\ \leq& C\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2,\end{aligned} \tag{100}$$

and

$$\left\|\frac{\tilde{v}_x}{\tilde{v}}(t)\right\|^2+\int_0^t\int_{\mathbb{R}}V^2\frac{\tilde{v}_x^2}{v^{\gamma+2}}\mathrm{d}x\mathrm{d}\tau\leq C\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0,\sqrt{\Phi_0},\psi_{0x},\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right)\right\|^2. \tag{101}$$

Noticing that (50) implies that

$$\left\|\sqrt{\tilde{\Phi}(\tilde{v})}\right\|\leq\|V\|^{\frac{\gamma-1}{2}}_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\left\|\sqrt{\Phi(v)}\right\|\leq C\delta^{\frac{(\gamma-1)\ell_1}{2}}\left\|\sqrt{\Phi(v)}\right\|, \tag{102}$$

the above three estimates together with the argument of Kanel' [4], the proof of Lemma 3.5, yield the following desired lower and upper bounds on v in terms of the initial perturbation.

Lemma 4.5. *If (98) holds, we have*

$$\begin{aligned}&\left(\frac{v(t,x)}{V(x-s_2t)}\right)^{\frac{1-\gamma}{2}}+\left(\frac{v(t,x)}{V(x-s_2t)}\right)^{\frac{1}{2}}\\ \leq& C\delta^{\frac{\gamma-1}{2}\ell_1}\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0,\sqrt{\Phi_0},\psi_{0x},\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right)\right\|.\end{aligned} \tag{103}$$

With the above results in hand, we now prove Theorem 4.2. Firstly note that if

$$\alpha+2-\frac{\gamma+3}{2}\ell_1-\ell_2\geq-\ell_2-\beta,\quad \alpha-\frac{\gamma+1}{2}\ell_2\geq-\ell_2-\beta, \tag{104}$$

and $|\ell_1|\leq\ell_2$, then we have from (62) and (90) that

$$\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|\leq C\delta^{\alpha-\frac{\gamma+1}{2}\ell_2},$$

$$\left\|\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right\|\leq C\delta^{-\ell_2}\|\phi_{0xx}\|+\delta^{2-\frac{(\gamma+3)}{2}\ell_1-\ell_2}\|\phi_{0x}\|\leq C\delta^{-\ell_2-\beta},$$

and

$$\left\|\left(\phi_0,\delta^{\frac{\gamma+1}{2}\ell_1}\psi_0,\sqrt{\Phi_0},\psi_{0x},\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right)\right\|\leq C\delta^{-\ell_2-\beta}.$$

Applying Proposition 2, we can find $t_0 > 0$ such that (92) has a unique solution $(\phi, \psi) \in X_{1/m_0, M_0}(0, t_0)$, which satisfies that $m_0, M_0 \leq C\delta^{-\ell_2}$, and

$$N_\psi(t_0) \leq \sup_{[0,t_0]} \|\psi(t)\|^{\frac{1}{2}} \|\psi_x(t)\|^{\frac{1}{2}} \leq 2\|\psi_0\|^{\frac{1}{2}} \|\psi_{0x}\|^{\frac{1}{2}} \leq C\delta^{\alpha}.$$

So we have

$$m_0^{\gamma+3} M_0^{2\gamma+2} \left(N_\psi(t_0)\delta^{(\gamma+1)\ell_1} + \delta^{2-(\gamma+1)\ell_1} \right) \leq C\delta^{-(3\gamma+5)\ell_2 + \min\{\alpha+(\gamma+1)\ell_1, 2-(\gamma+1)\ell_1\}}.$$

Thus, if δ is sufficiently small, $(94)_2$ implies (98) with $T = t_0$, $m = m_0$ and $M = M_0$. We have from (100) and (103) that

$$\|\psi(t_0)\|_{L^\infty} \leq \|\psi(t_0)\|^{\frac{1}{2}} \|\psi_x(t_0)\|^{\frac{1}{2}} \leq C\delta^{\alpha - \frac{\gamma+1}{2}\ell_2 - \frac{\gamma+1}{4}\ell_1}$$

and

$$C^{-1}\delta^{\frac{2}{1-\gamma}\left(\alpha + \frac{\gamma-1}{2}\ell_1 - \frac{\gamma+3}{2}\ell_2 - \beta\right) + \ell_1} \leq v(t_0, x) \leq C\delta^{2\left(\alpha + \frac{\gamma-1}{2}\ell_1 - \frac{\gamma+3}{2}\ell_2 - \beta\right) + \ell_1}.$$

By applying Proposition 2 again, there exists $t_1 > 0$ such that (19) has a unique solution $(\phi, \psi) \in X_{1/m, M}(0, t_0 + t_1)$, which satisfies

$$m \leq C\delta^{\frac{2}{\gamma-1}\left(\alpha + \frac{\gamma-1}{2}\ell_1 - \frac{\gamma+3}{2}\ell_2 - \beta\right) - \ell_1}, \quad M \leq C\delta^{2\left(\alpha + \frac{\gamma-1}{2}\ell_1 - \frac{\gamma+3}{2}\ell_2 - \beta\right) + \ell_1},$$

and

$$N_\psi(t_0 + t_1) \leq C\delta^{\alpha - \frac{\gamma+1}{2}\ell_2 - \frac{\gamma+1}{4}\ell_1}.$$

Thus $(94)_3$ implies $m, M \geq \delta^{-|\ell_1|}$. Here, we deduce from $(94)_2$ and $(94)_3$ that (104) holds. Hence, it is easy to check that if (94) holds, we can choose $\delta > 0$ so small that (98) holds with $T = t_0 + t_1$. Therefore, the standard continuation argument can be applied and our theorem will be proved.

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