

ONE-DIMENSIONAL VISCOUS RADIATIVE GAS  
WITH TEMPERATURE DEPENDENT VISCOSITY\*

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## In Memory of Professor Xiaqi DING

**Abstract** This paper is concerned with the construction of global, large amplitude solutions to the Cauchy problem of the one-dimensional compressible Navier–Stokes system for a viscous radiative gas when the viscosity and heat conductivity coefficients depend on both specific volume and absolute temperature. The data are assumed to be without vacuum, mass concentrations, or vanishing temperatures, and the same is shown to hold for the global solution constructed. The proof is based on some detailed analysis on uniform positive lower and upper bounds of the specific volume and absolute temperature.

**Key words** compressible Navier–Stokes system; temperature-dependent viscosity; viscous radiative gas; global solution; asymptotic behavior

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## 1 Introduction

The dynamics of one-dimensional compressible viscous and heat-conducting flow can be described in the Lagrangian coordinates by the compressible Navier–Stokes system:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v, \theta)_x = \left( \frac{\mu(v, \theta)u_x}{v} \right)_x, \\ \left( e(v, \theta) + \frac{u^2}{2} \right)_t + (up(v, \theta))_x = \left( \frac{\mu(v, \theta)uu_x}{v} \right)_x + \left( \frac{\kappa(v, \theta)\theta_x}{v} \right)_x. \end{cases} \quad (1.1)$$

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Here  $t > 0$  and  $x \in \mathbb{R}$  are the time and Lagrangian spatial variables, respectively. The specific volume  $v$ , velocity  $u$ , and absolute temperature  $\theta$  are unknown functions of  $t$  and  $x$ . The pressure  $p$ , internal energy  $e$ , viscosity coefficient  $\mu > 0$ , and heat conductivity coefficient  $\kappa > 0$  are prescribed through constitutive relations as functions of  $v$  and  $\theta$ . The thermodynamic variables are related through Gibbs' equation  $de = \theta ds - pdv$  with  $s$  being the specific entropy.

This paper concerns system (1.1) with prescribed initial data

$$(v(t, x), u(t, x), \theta(t, x))|_{t=0} = (v_0(x), u_0(x), \theta_0(x)), \quad x \in \mathbb{R}, \quad (1.2)$$

which are assumed to satisfy the far-field condition:

$$\lim_{x \rightarrow \pm\infty} (v_0(x), u_0(x), \theta_0(x)) = (1, 0, 1). \quad (1.3)$$

Our main purpose is devoted to the construction of global, smooth, large amplitude, non-vacuum solution  $(v(t, x), u(t, x), \theta(t, x))$  to Cauchy problem (1.1)–(1.3) with temperature-dependent transport coefficients  $\mu$  and  $\kappa$  when the thermodynamic variables  $v$ ,  $p$ ,  $e$ ,  $\theta$ , and  $s$  do not satisfy the equations of state for ideal polytropic gases. In fact, our choice of constitutive relations are motivated by the following system describing the motion of compressible radiative and reactive gas:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v, \theta)_x = \left(\frac{\mu u_x}{v}\right)_x, \\ \left(e + \frac{u^2}{2}\right)_t + (up(v, \theta))_x = \left(\frac{\mu u u_x}{v}\right)_x + \left(\frac{\kappa(v, \theta) \theta_x}{v}\right)_x + \lambda \phi z, \\ z_t = \left(\frac{dz_x}{v^2}\right)_x - \phi z, \end{cases} \quad (1.4)$$

where  $z = z(t, x)$  represents the reactant mass fraction. The positive constants  $d$  and  $\lambda$  are the species diffusion coefficient and the difference in the heat between the reactant and the product, respectively. The reaction rate function  $\phi = \phi(\theta)$  is defined by the first-order Arrhenius law (cf. [11]):

$$\phi(\theta) = K\theta^\beta \exp\left(-\frac{A}{\theta}\right), \quad (1.5)$$

where positive constants  $K$  and  $A$  are the coefficients of the rate of the reactant and the activation energy, respectively, and  $\beta$  is a non-negative number.

When the radiation is treated as a continuous field and both the wave and photonic effect are considered, the high-temperature radiation is at thermal equilibrium with the fluid. Then pressure  $p$  consists of a linear term in  $\theta$  corresponding to the perfect polytropic contribution and a fourth-order radiative part due to the Stefan–Boltzmann radiative law (see [36] for instance) so that

$$p = p(v, \theta) = \frac{R\theta}{v} + \frac{a\theta^4}{3}, \quad e = e(v, \theta) = c_v\theta + av\theta^4, \quad (1.6)$$

where positive constants  $R$ ,  $c_v$ , and  $a$  are the perfect gas constant, the specific heat, and the Stefan–Boltzmann constant, respectively.

Since the energy producing process inside the medium is taken into account in system (1.4), where the gas consists of a reacting mixture and the combustion process is current at the high temperature stage, the experimental results for gases at high temperatures in [54] show that

both  $\mu$  and  $\kappa$  may depend on the specific volume  $v$  and/or the absolute temperature  $\theta$ . In this paper, we focus on the case when the heat conductivity  $\kappa$  takes the following form (cf. [7])

$$\kappa = \kappa_1 + \kappa_2 v \theta^b \tag{1.7}$$

for some positive constants  $\kappa_1, \kappa_2$ , and  $b$ . As for the viscosity coefficient  $\mu$ , motivated by the work [52] for the one-dimensional viscous, heat-conducting ideal polytropic gas, we assume that

$$\mu = \mu(v, \theta) = h(v) \theta^\alpha, \quad h(v) \sim \begin{cases} v^{-\ell_1}, & v \rightarrow 0^+, \\ v^{\ell_2}, & v \rightarrow \infty, \end{cases} \quad v |h'(v)|^2 \leq Ch^3(v), \tag{1.8}$$

where  $h(v)$  is a smooth function of  $v$  for  $v > 0$ , and  $\alpha, \ell_1, \ell_2$ , and  $C$  are positive constants. Here and in the rest of this paper,  $f(x) \sim g(x)$  ( $x \rightarrow x_0$ ) means that there exists a constant  $C \geq 1$  such that  $C^{-1}g(x) \leq f(x) \leq Cg(x)$  holds in a neighborhood of  $x_0$ .

Before stating our main result, it is worth to pointing out that the study on the global solvability and large time behaviors of the global solutions to the initial value problem and/or the initial-boundary value problems of the one-dimensional compressible Navier–Stokes equations (1.1) is one of the hottest topics in the field of nonlinear partial differential equations and many results have been obtained up to now. To go to the theme of this paper, we will only review some results on the compressible Navier–Stokes type equations (1.4)–(1.7) describing the motion of compressible radiative and reactive gas as follows:

When the viscosity coefficient  $\mu$  is a positive constant, the results obtained can be summarized as in the following.

For the initial-boundary value problem of (1.4)–(1.7) in the bounded interval  $(0, 1)$  with the free boundary conditions

$$\sigma(t, 0) = \sigma(t, 1) = -p_e < 0, \quad \sigma = -p(v, \theta) + \frac{\mu u_x}{v}$$

on the stress  $\sigma(t, x)$  and homogeneous Neumann condition

$$(\theta_x(t, 0), z_x(t, 0)) = (\theta_x(t, 1), z_x(t, 1)) = (0, 0)$$

on both  $\theta(t, x)$  and  $z(t, x)$ , Umehara–Tani [46] proved the global existence, uniqueness of a classical solutions under the assumptions  $4 \leq b \leq 16$  and  $0 \leq \beta \leq 13/2$ . Later on, they improved the results in [47] to the case of  $b \geq 3$  and  $0 \leq \beta < b + 9$ . Qin–Hu–Wang [42] strengthened the results to the case  $(b, \beta) \in E_1 \cup E_2$  with

$$E_1 = \{(b, \beta) \in \mathbb{R}^2 : 2 \leq b < 3, 0 \leq \beta < 2b + 6\},$$

$$E_2 = \{(b, \beta) \in \mathbb{R}^2 : 3 \leq b, 0 \leq \beta < b + 9\}.$$

See also the result by Jiang–Zheng [17] for more general assumptions on  $\kappa$ ;

For the initial-boundary value problem of (1.4)–(1.7) in  $(0, 1)$  with homogeneous Dirichlet boundary condition  $u(t, 0) = u(t, 1) = 0$  and homogeneous Neumann condition

$$(\theta_x(t, 0), z_x(t, 0)) = (\theta_x(t, 1), z_x(t, 1)) = (0, 0)$$

on  $\theta(t, x)$  and  $z(t, x)$ , Documet [7] established the global existence and exponential decay in  $H^1(0, 1)$  of solutions for  $b \geq 4$  and  $\beta > 0$ . Recently, Jiang–Zheng [18] improved this result to the case of  $b \geq 2$  and  $0 \leq \beta < b + 9$ . We also refer the readers to [5, 8–10] and the references therein for related studies;

For the Cauchy problem of (1.4)–(1.7), results on the global solvability and the precise description of large time behavior of the global solution constructed were established very recently by Liao–Zhao in [33] for  $b > 11/3$  and  $0 \leq \beta < b + 9$ ;

For spherically symmetric motions of compressible radiative and reactive gases, the global existence, uniqueness and exponential stability of spherically symmetric solutions in the bounded annular domain  $\Omega = \{x \in \mathbb{R}^n : 0 < a < |x| < b\}$  were obtained in [43], while the corresponding result concerning the global solvability and large time behavior in an exterior domain  $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$  was showed by Liao–Wang–Zhao in [29].

For the case when the viscosity coefficient  $\mu$  is a smooth, possible degenerate function of the specific volume  $v$  for positive  $v$ , the two types of initial-boundary value problems of (1.4)–(1.7) in the bounded interval  $(0, 1)$  mentioned above were studied in [31, 32], while the Cauchy problem was treated in Liao–Xu–Zhao [30]. It is worth to pointing out that all the estimates obtained in [30–32] depend on the time variable  $t$ , and thus the problem on the large time behavior of global solutions constructed in [30–32] remains unsolved.

Even so, to the best of our knowledge, no result is available for the case when the viscosity coefficient  $\mu$  depends on the absolute temperature up to now. As pointed out before, since the physical phenomena described by system (1.4) involve high temperature process and the experimental results for gases at high temperatures in [54] show that the viscosity coefficient  $\mu$  may also depend on the specific volume  $v$  and/or the absolute temperature  $\theta$ , a natural and interesting question is: Whether can we obtain a global solvability result for the Cauchy problem (1.4)–(1.7) with large initial data or not for a class of density and temperature dependent viscosity coefficient  $\mu$  satisfying (1.8)? The main goal of this paper is devoted to such a problem. Since the appearance of the reaction equation, i.e., the fourth one in (1.4), does not cause any essential difficulty in our analysis, we will focus on the Cauchy problem of (1.1), (1.6)–(1.8) with prescribed large initial data (1.2) satisfying the far field condition (1.3) in the rest of this paper.

Now we are in a position to state our main result. To do so, for each given positive constant  $0 < w \leq 1$ , we first introduce the notation

$$H(\omega) := \sup_{\omega \leq \sigma \leq \omega^{-1}} |(h(\sigma), h'(\sigma), h''(\sigma), h'''(\sigma))| \tag{1.9}$$

and then our result can be stated as follows.

**Theorem 1.1** Suppose that

- (i) The viscosity coefficient  $\mu$  satisfies (1.8);
- (ii) The parameters  $b, \ell_1$ , and  $\ell_2$  are assumed to satisfy:

$$b \geq 7, \quad \ell_1 > 1, \quad \ell_2 > 1, \tag{1.10}$$

$$\left(6 - \frac{3}{\ell_1} - \frac{8}{2\ell_2 + 1}\right)b + 18 - \frac{21}{2\ell_1} + \frac{3}{4\ell_1^2} + \frac{12}{(2\ell_2 + 1)^2} - \frac{36}{2\ell_2 + 1} + \frac{8}{\ell_1(2\ell_2 + 1)} > 0; \tag{1.11}$$

- (iii) The initial data  $(v_0(x), u_0(x), \theta_0(x))$  satisfy

$$(v_0(x) - 1, u_0(x), \theta_0(x) - 1) \in H^3(\mathbb{R}), \quad \|(v_0 - 1, u_0, \theta_0 - 1)\|_{H^3(\mathbb{R})} \leq \Pi_0, \tag{1.12}$$

$$V_0 \leq v_0(x) \leq V_0^{-1}, \quad \theta_0(x) \geq V_0, \quad \forall x \in \mathbb{R}, \tag{1.13}$$

where  $\Pi_0$  and  $V_0 \leq 1$  are given positive constants.

Then there exists  $\epsilon_0 > 0$ , which depends only on  $\Pi_0, V_0$  and  $H(C_0)$ , such that if

$$|\alpha| \leq \epsilon_0,$$

the Cauchy problem (1.1)–(1.3), (1.6)–(1.8) admits a unique solution  $(v(t, x), u(t, x), \theta(t, x))$  satisfying

$$\begin{aligned} &(v(t, x) - 1, u(t, x), \theta(t, x) - 1) \in C([0, \infty), H^3(\mathbb{R})), \\ &v_x(t, x) \in L^2([0, \infty); H^2(\mathbb{R})), \quad (u_x(t, x), \theta_x(t, x)) \in L^2([0, \infty); H^3(\mathbb{R})), \end{aligned}$$

and

$$\inf_{(t,x) \in [0, \infty) \times \mathbb{R}} \{v(t, x), \theta(t, x)\} > 0, \quad \sup_{(t,x) \in [0, \infty) \times \mathbb{R}} \{v(t, x), \theta(t, x)\} < +\infty.$$

In addition, we have the following large time behavior

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |(v(t, x) - 1, u(t, x), \theta(t, x) - 1)| = 0.$$

**Remark 1.2** Several remarks concerning our main result are listed below:

- (i) It is easy to see that for each  $b \geq 7$ , one can easily find  $\ell_1 > 1$  and  $\ell_2 > 1$  sufficiently large such that the assumption (1.11) holds. Since our main purpose is to show that we can indeed obtain a global solvability result for the Cauchy problem (1.1)–(1.3), (1.6)–(1.7) for a class of density and temperature dependent viscosity  $\mu$  which satisfies (1.8), assumptions (1.10) and (1.11) that we imposed on the parameters  $b, \ell_1$ , and  $\ell_2$  are far from being optimal;
- (ii) If we take  $\alpha = 0$ , then our main result Theorem 1.1 tells us that one can obtain a result on the global solvability together with the precise description of the large time behaviors of solutions to the Cauchy problem (1.1)–(1.3), (1.6)–(1.7) for a class of density dependent viscosity  $\mu$  which satisfies

$$\mu = \mu(v) \sim \begin{cases} v^{-\ell_1}, & v \rightarrow 0^+, \\ v^{\ell_2}, & v \rightarrow \infty, \end{cases} \quad v |\mu'(v)|^2 \leq C\mu^3(v).$$

Here  $C > 0, \ell_1$  and  $\ell_2$  are some positive constants satisfying (1.10) and (1.11). Recall that for the case when the viscosity coefficient  $\mu$  is a degenerate function of  $v$  (cf.  $\mu = v^{-\alpha}, \alpha \in [1/3, 1/2)$ ), although a global solvability result is obtained in [30], we do not know how to deduce the desired large time behaviors of the global solutions constructed there due to the lack of uniform-in-time estimates. We note, however, that our main result, i.e., Theorem 1.1, yields the large time behaviors of the global solutions for a class of nondegenerate density dependent viscosity coefficient  $\mu$ .

Now we outline the main difficulties encountered in the proof of Theorem 1.1. As is well known, the key point to deduce the global solvability result of the Cauchy problem (1.1)–(1.3), (1.6)–(1.8) is to derive the desired positive lower and upper bounds on the specific volume  $v(t, x)$  and the absolute temperature  $\theta(t, x)$ .

For the case when the viscosity coefficient  $\mu$  is a positive constant, motivated by the work of Jiang [19–21] for the viscous, heat-conducting ideal polytropic gas, Liao–Zhao [33] have used

the following cut-off function

$$\varphi(x) = \begin{cases} 1, & x \leq k + 1, \\ k + 2 - x, & k + 1 \leq x \leq k + 2, \\ 0, & x \geq k + 2 \end{cases}$$

to derive the following explicit expression of the specific volume  $v(t, x)$

$$v(t, x) = B(t, x)Q(t) + \frac{1}{\mu} \int_0^t \frac{B(t, x)Q(t)v(\tau, x)p(\tau, x)}{B(\tau, x)Q(\tau)} d\tau, \quad x \in \bar{\Omega}_k, \quad t \geq 0, \quad (1.14)$$

where

$$\begin{aligned} B(t, x) &:= v_0(x) \exp \left\{ \frac{1}{\mu} \int_x^\infty (u_0(y) - u(t, y)) \varphi(y) dy \right\}, \\ Q(t) &:= \exp \left\{ \frac{1}{\mu} \int_0^t \int_{k+1}^{k+2} \sigma(\tau, y) dy d\tau \right\}, \\ \sigma &:= -p + \frac{\mu u_x}{v}, \quad \Omega_k := (-k - 1, k + 1), \quad k \in \mathbf{Z}. \end{aligned}$$

By virtue of (1.14), one can deduce the uniform-in-time positive lower and upper bound of  $v(t, x)$ .

As for the uniform positive lower and upper estimate on  $\theta(t, x)$ , motivated by the work of [25] for the initial-boundary value problem of the one-dimensional compressible Navier–Stokes equation in bounded interval for general gas, the following auxiliary functions

$$\begin{aligned} \tilde{X}(t) &:= \int_0^t \int_{\mathbb{R}} (1 + \theta^{b+3}(\tau, x)) \theta_t^2(\tau, x) dx d\tau, \\ \tilde{Y}(t) &:= \max_{0 \leq \tau \leq t} \left\{ \int_{\mathbb{R}} (1 + \theta^{2b}(\tau, x)) \theta_x^2(\tau, x) dx \right\}, \\ \tilde{Z}(t) &:= \max_{\tau \in [0, t]} \left\{ \int_{\mathbb{R}} u_{xx}^2(\tau, x) dx \right\}, \\ \tilde{W}(t) &:= \int_0^t \int_{\mathbb{R}} u_{xt}^2(\tau, x) dx d\tau \end{aligned}$$

are introduced in [33] to derive the uniform upper bound of the absolute temperature  $\theta(t, x)$ . It is worth to emphasizing that the argument used in [33] to deduce the desired upper bound estimate on  $\theta(t, x)$  relies highly on the uniform bounds on  $v(t, x)$  obtained before.

When the viscosity coefficient  $\mu$  is not a positive constant but depends only on the specific volume  $v$ , the above argument can not be used any longer. In fact, for such case we can not deduce a similar explicit repression for  $v(t, x)$  and consequently we can not deduce the desired positive lower and upper bounds on  $v(t, x)$  first. The main ideas used in [30] are the following:

- (i) Based on the following identity

$$\left( \frac{\mu(v)v_x}{v} \right)_t = u_t + p(v, \theta)_x, \quad (1.15)$$

which is observed first by Kanel' in [24] for isentropic viscous flow, one can deduce an estimate on the lower and upper bounds of the specific volume  $v$  in terms of  $\|\theta\|_{L^\infty([0, T] \times \mathbb{R})}$  by employing Kanel's argument provided that  $\mu$  satisfies suitable growth conditions as  $v \rightarrow 0^+$  and  $v \rightarrow +\infty$ ;

(ii) Noticing that  $h(t, x) = 1/\theta(t, x)$  satisfies

$$\begin{aligned}
 & e_\theta(v, \theta)h_t - \left( \frac{\kappa(v, \theta) h_x}{v} \right)_x \\
 &= \frac{vp_\theta^2(v, \theta)}{4\mu} - \left[ \frac{2\kappa(v, \theta) h_x^2}{vh} + \frac{\mu h^2}{v} \left( u_x - \frac{vp_\theta(v, \theta)}{2\mu h} \right)^2 + \lambda h^2 \phi z \right]
 \end{aligned} \tag{1.16}$$

with

$$e_\theta(v, \theta) = c_v + 4av\theta^3, \quad p_\theta(v, \theta) = \frac{R}{v} + \frac{4}{3}a\theta^3, \tag{1.17}$$

one can employ the standard maximum principle to yield an estimate on the lower bound of the absolute temperature in terms of  $\|1/v\|_{L^\infty([0, T] \times \mathbb{R})}$  and  $\|v\|_{L^\infty([0, T] \times \mathbb{R})}$ ;

(iii) By utilizing the argument used in [44] and [4, 25, 33], one can then deduce an estimate on  $\|\theta\|_{L^\infty([0, T] \times \mathbb{R})}$ . From which and the estimates on the lower and upper bounds on  $v(t, x)$ , the lower estimate on  $\theta(t, x)$ , one can then deduce the desired positive lower and upper estimates on both  $v(t, x)$  and  $\theta(t, x)$  provided that the parameter  $b$  appearing in (1.7) and growth rates of the viscosity coefficient  $\mu$  as  $v \rightarrow 0^+$  and  $v \rightarrow +\infty$  satisfy certain conditions and then the desired global solvability result follows immediately.

For the case considered in this paper, the viscosity coefficient  $\mu$  depends on both the specific volume  $v$  and the absolute temperature  $\theta$ . For such a case, the identity corresponding to (1.15) becomes

$$\left( \frac{\mu(v, \theta) v_x}{v} \right)_t = u_t + p_x + \frac{\mu_\theta(v, \theta)}{v} (v_x \theta_t - \theta_x u_x). \tag{1.18}$$

Since the last term in (1.18) is a highly nonlinear term, the temperature dependence of the viscosity  $\mu$  has a strong influence on the solution and leads to difficulties in mathematical analysis for global solvability with large data, and as pointed out in [16], such a dependence has turned out to be especially problematic and challenging.

A natural way to go on is to use certain smallness mechanism induced by the structure of the system to control the last term in (1.18) suitably. It was to do so that we need to ask the viscosity coefficient  $\mu$  to take the form (1.8) and our main idea is to the smallness of  $|\alpha|$  to control the last highly nonlinear term in (1.18). We note, however, that to close the analysis, or in other words to determine the upper bound of  $|\alpha|$  in terms of the initial data  $(v_0(x), u_0(x), \theta_0(x))$ , one had to deduce the uniform positive lower and upper bounds on the absolute temperature  $\theta$  which are independent of the time variable  $t$ . It is worth to pointing out that such a problem is considered by Wang–Zhao in [52] for the one-dimensional, compressible Navier–Stokes system for a viscous and heat conducting ideal polytropic gas for a class of density and temperature dependent viscosity coefficient  $\mu$  satisfying (1.8). We recall, however, that the argument of Wang–Zhao in [52] is first to use Kanel’s method [24] to obtain the lower and the upper bound of  $v(t, x)$  in terms of  $\|\theta\|_{L^\infty([0, T] \times \mathbb{R})}$  (see Lemma 2.3 in [52]), then to employ the technique used in Li–Liang in [27] to derive the uniform upper bound of  $\theta(t, x)$ . Note that the method used by Wang–Zhao in [52] to deduce the uniform upper bound of  $\theta(t, x)$  relies on the following Sobolev inequality (see also (2.72) in Wang–Zhao [52])

$$\|\theta(t) - 1\|_{L^\infty(\mathbb{R})}^2 \leq C \|\theta(t) - 1\|_{L^2(\mathbb{R})} \|\theta_x(t)\|_{L^2(\mathbb{R})} \leq C \left( 1 + \|\theta\|_{L^\infty([0, T] \times \mathbb{R})} \right).$$

For our problem  $\|\theta(t) - 1\|_{L^2(\mathbb{R})}$  is bounded due to Lemma 2.1, but the method employed in [52] to deduce the estimate on  $\|\theta_x(t)\|_{L^2(\mathbb{R})}$  loses its power in our case which is caused by the fourth-order radiative part in both  $p(v, \theta)$  and  $e(v, \theta)$ , cf. (1.6); Besides, one can see the assumption  $\mu = \kappa$  also plays an important role in the process of deriving the upper bound of  $\theta(t, x)$  in their discussion. Thus the story is different since  $\mu \neq \kappa$  in our case.

To overcome the above difficulties, we introduce some new auxiliary functions  $X(t)$ ,  $W(t)$ ,  $Z(t)$  and  $W(t)$  (see (3.1) in section 3) to deduce the uniform-in-time upper bound of  $\theta(t, x)$ . More precisely, our strategy to prove Theorem 1.1 can be stated as follows:

- (i) We first apply Kanel's method [24] to deduce the lower and the upper bounds of  $v(t, x)$  in terms of  $\|\theta\|_{L^\infty([0, T] \times \mathbb{R})}$  simultaneously in Lemma 2.3. Notice that the assumption (1.10) plays an important role in our discussion. To control the last term in (1.18), we will use the smallness of  $|\alpha|$ ;
- (ii) Due to (3.12) and (3.13), we introduce the auxiliary functions  $X(t)$ ,  $Y(t)$ ,  $Z(t)$  and  $W(t)$  (see (3.1)) to deduce the upper bound of  $\theta(t, x)$  in Lemmas 3.1–3.4. Thus the lower and the upper bound of  $v(t, x)$  follows from Lemma 2.3. We should emphasize that all the bounds obtained above are independent of the time variable  $t$ ; We then adopt the method in Liao–Zhao [33] to deduce the positive local-in-time lower bound of  $\theta(t, x)$  and notice that such a bound depends on the time variable  $t$ ;
- (iii) By using the dedicated energy method, we can derive energy type estimates of higher-order derivatives in Section 4 and Section 5; Then by using the continuation argument designed in Wang–Zhao [52], we can then prove Theorem 1.1.

Before concluding this section, we recall that there are also many results on the construction of global, smooth, large amplitude, non-vacuum solutions and on the precise description of the large time behaviors of the global solutions constructed to compressible Navier–Stokes system for a viscous and heat conducting ideal polytropic gas, cf. [1–3, 16, 26, 40] for the one-dimensional initial-boundary value problem in bounded interval, [1, 6, 12–15, 20–22, 27, 34, 38, 39, 41, 44, 45, 50–53] for the corresponding one-dimensional problem in unbounded domain and [19, 28, 37, 48, 49] for global symmetric flows of multi-dimensional compressible Navier–Stokes equations. For compressible Navier–Stokes equations with general constitutive relations and other related compressible Navier–Stokes type equations, see [4, 25, 35, 55] and the references therein.

The rest of the paper is organized as follows. We derive pointwise bounds on the specific volume in Section 2. Then pointwise bounds on the absolute temperature will be derived in Section 3. Some second-order and third-order energy type estimates and the proof of our main result will be given in Section 4 and Section 5, respectively.

**Notations** Throughout this paper,  $C \geq 1$  or  $C_i \geq 1$  ( $i = 1, 2, \dots$ ) is used to denote a generic positive constant which may depend only on  $\Pi_0$ ,  $V_0$  and  $H(V_0)$ , where  $\Pi_0$ ,  $V_0$  and  $H$  are given by (1.12), (1.13) and (1.9), respectively. Note that these constants may vary from line to line.  $C(\cdot, \cdot)$  stands for some generic constant depending only on the quantities listed in the parenthesis.  $\epsilon < 1$  represents some small positive constant.

For function spaces,  $L^q(\mathbb{R})$  ( $1 \leq q \leq \infty$ ) denotes the usual Lebesgue space on  $\mathbb{R}$  with norm  $\|\cdot\|_{L^q(\mathbb{R})}$ , while  $H^q(\mathbb{R})$  denotes the usual Sobolev space in the  $L^2$  sense with norm  $\|\cdot\|_{H^q(\mathbb{R})}$ . We denote by  $C(I; H^q(\mathbb{R}))$  the space of continuous functions on the interval  $I$  with values in  $H^q(\mathbb{R})$  and  $L^2(I; H^q(\mathbb{R}))$  stands for the space of  $L^2$ -functions on  $I$  with values in  $H^q(\mathbb{R})$ . For simplicity, we use  $\|\cdot\|_\infty$  to denote the norm in  $L^\infty([0, T] \times \mathbb{R})$  with  $T > 0$  being some given positive constant,  $\|\cdot\|$  and  $\|\cdot\|_q$  are used to denote the norm  $\|\cdot\|_{L^2(\mathbb{R})}$  and the norm  $\|\cdot\|_{H^q(\mathbb{R})}$ , respectively.

Finally,  $A \lesssim B$  (or  $B \gtrsim A$ ) means that  $A \leq CB$  holds uniformly for some generic positive constant  $C$ .

## 2 Pointwise Bounds for the Specific Volume

We define the set

$$X(s, t; m_1, m_2, N) := \left\{ (v(\tau, x), u(\tau, x), \theta(\tau, x)) : \begin{cases} (v(\tau, x) - 1, u(\tau, x), \theta(\tau, x) - 1) \in C([s, t]; H^3(\mathbb{R})), \\ v_x(\tau, x) \in L^2(s, t; H^2(\mathbb{R})), \\ (u_x(\tau, x), \theta_x(\tau, x)) \in L^2(s, t; H^3(\mathbb{R})), \\ \mathcal{E}(s, t) \leq N^2, \\ v(\tau, x) \geq m_1, \theta(\tau, x) \geq m_2 \quad \forall (\tau, x) \in [s, t] \times \mathbb{R} \end{cases} \right\}$$

for some positive constants  $N, m_i, s$ , and  $t$  ( $i = 1, 2, t \geq s$ ), where

$$\mathcal{E}(s, t) := \sup_{s \leq \tau \leq t} \{ \|(v - 1, u, \theta - 1)(\tau)\|_3^2 \} + \int_s^t [\|v_x(\tau)\|_2^2 + \|(u_x, \theta_x)(\tau)\|_3^2] d\tau.$$

Since the existence and uniqueness of solution  $(v(t, x), u(t, x), \theta(t, x))$  to the Cauchy problem (1.1)–(1.3) with constitutive relations (1.6), (1.7) and (1.8) in the set of functions  $X(0, t_1; m_1, m_2, N)$  for some sufficiently small  $t_1 > 0$  and certain positive constants  $m_1, m_2$  and  $N$  is guaranteed by the well-established local existence result for hyperbolic-parabolic system, cf. [23], suppose that the local solution  $(v(t, x), u(t, x), \theta(t, x))$  to the Cauchy problem (1.1)–(1.3) with constitutive relations (1.6)–(1.8) has been extended to the time step  $t = T$  for some positive constant  $T > 0$  and  $(v(t, x), u(t, x), \theta(t, x)) \in X(0, T; m_1, m_2, N)$  for some positive constants  $T, m_i \leq 1$  ( $i = 1, 2$ ) and  $N \geq 1$ , then in order to prove Theorem 1.1, we only need to derive certain a priori estimates on the solution  $(v(t, x), u(t, x), \theta(t, x))$  in terms of the initial data  $(v_0(x), u_0(x), \theta_0(x))$  but independent of the constants  $m_i \leq 1$  ( $i = 1, 2$ ) and  $N \geq 1$ .

Applying Sobolev’s inequality yields

$$m_1 \leq v(t, x) \leq 4N, \quad m_2 \leq \theta(t, x) \leq 4N, \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \tag{2.1}$$

This section is devoted to deducing lower and upper bounds on the specific volume  $v(t, x)$  in terms of  $\|\theta\|_\infty$ . In the next lemma, we present the basic energy estimate.

**Lemma 2.1** Under the assumptions listed in Theorem 1.1, for any  $0 \leq t \leq T$ , we have

$$\int_{\mathbb{R}} \eta(v, u, \theta)(t, x) dx + \int_0^t \int_{\mathbb{R}} \left[ \frac{\mu u_x^2}{v\theta} + \frac{\kappa \theta_x^2}{v\theta^2} \right] dx d\tau \lesssim 1, \tag{2.2}$$

where

$$\eta(v, u, \theta) := R\tilde{\phi}(v) + \frac{u^2}{2} + c_v\tilde{\phi}(\theta) + \frac{a}{3}v(\theta - 1)^2(3\theta^2 + 2\theta + 1) \tag{2.3}$$

with  $\tilde{\phi}(z) := z - \ln z - 1$ .

**Proof** According to [38], the function  $\eta$  is nothing but the normalized entropy around  $(v, u, \theta) = (1, 0, 1)$  for system (1.1), (1.6), (1.7) and (1.8) (see [33] for the derivation).

Multiply (1.1)<sub>2</sub> and (1.1)<sub>3</sub> with  $u$  and  $(1 - \theta^{-1})$ , respectively, add the resulting identities, and use (1.1)<sub>1</sub> to discover

$$\eta_t + \frac{\mu u_x^2}{v\theta} + \frac{\kappa\theta_x^2}{v\theta^2} = \left[ \left( R + \frac{a}{3} - p \right) u + \frac{\mu u u_x}{v} + \left( 1 - \frac{1}{\theta} \right) \frac{\kappa\theta_x}{v} \right]_x.$$

The lemma follows by integrating the above identity over  $[0, t] \times \mathbb{R}$ . □

The derivation of pointwise bounds for  $v(t, x)$  relies on the following lemma.

**Lemma 2.2** Suppose that the conditions listed in Theorem 1.1 hold. Then there is a constant  $0 < \epsilon_1 \leq 1$ , depending only on  $\Pi_0$  and  $V_0$ , such that if

$$m_2^{-|\alpha|} \leq 2, \quad N^{|\alpha|} \leq 2, \quad \Xi(m_1, m_2, N)|\alpha| \leq \epsilon_1, \tag{2.4}$$

where

$$\Xi(m_1, m_2, N) := \left[ m_1^{-1} + m_2^{-1} + N + \sup_{m_1 \leq \sigma \leq 4N} h(\sigma) + 1 \right]^{800},$$

then we can get for  $0 \leq t \leq T$  that

$$\left\| \frac{\mu v_x}{v}(t) \right\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\mu \theta v_x^2}{v^3} dx d\tau \lesssim 1 + \|\theta\|_{\infty} + \|v\|_{\infty}^{\ell_2+2} + \left\| \frac{1}{v} \right\|_{\infty}^{\ell_1+1}. \tag{2.5}$$

**Proof** We use the chain rule to get

$$\left( \frac{\mu v_x}{v} \right)_t = \left( \frac{\mu v_t}{v} \right)_x + \frac{\mu \theta}{v} (v_x \theta_t - \theta_x v_t), \tag{2.6}$$

which combined with (1.1)<sub>1</sub> and (1.1)<sub>2</sub> imply

$$\left( \frac{\mu v_x}{v} \right)_t = u_t + p_x + \frac{\mu \theta}{v} (v_x \theta_t - \theta_x u_x). \tag{2.7}$$

Multiply this last equation with  $\mu v_x/v$  and use (2.6) to find

$$\begin{aligned} & \left[ \frac{1}{2} \left( \frac{\mu v_x}{v} \right)_t^2 \right] + \frac{R\mu\theta v_x^2}{v^3} + \left( \frac{\mu u u_x}{v} \right)_x - \left( \frac{\mu v_x u}{v} \right)_t \\ &= \frac{\mu u_x^2}{v} + \frac{R\mu v_x \theta_x}{v^2} + \frac{4a\mu\theta^3 v_x \theta_x}{3v} + \frac{\mu\theta}{v^2} (\mu v_x - uv)(v_x \theta_t - \theta_x u_x). \end{aligned}$$

Integrating the above identity over  $[0, t] \times \mathbb{R}$ , we obtain from Cauchy's inequality and (2.2) the following estimate:

$$\left\| \frac{\mu v_x}{v}(t) \right\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\mu \theta v_x^2}{v^3} dx d\tau \lesssim 1 + \sum_{j=1}^3 \mathcal{H}_j, \tag{2.8}$$

where

$$\begin{aligned} \mathcal{H}_1 &:= \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v} dx d\tau, \\ \mathcal{H}_2 &:= \int_0^t \int_{\mathbb{R}} \left( \frac{\mu}{v\theta} + \mu v \theta^5 \right) \theta_x^2 dx d\tau, \end{aligned}$$

$$\mathcal{H}_3 := \left| \int_0^t \int_{\mathbb{R}} \frac{\mu\theta}{v^2} (\mu v_x - uv)(v_x \theta_t - \theta_x u_x) dx d\tau \right|.$$

It follows from (2.2) that

$$\mathcal{H}_1 \leq \|\theta\|_{\infty} \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v\theta} dx d\tau \lesssim \|\theta\|_{\infty}, \tag{2.9}$$

and

$$\mathcal{H}_2 \leq \left\| \frac{\mu\theta}{\kappa} + \frac{\mu v^2 \theta^7}{\kappa} \right\|_{\infty} \int_0^t \int_{\mathbb{R}} \frac{\kappa \theta_x^2}{v\theta^2} dx d\tau \lesssim \left\| \frac{\mu\theta}{\kappa} \right\|_{\infty} + \left\| \frac{\mu v^2 \theta^7}{\kappa} \right\|_{\infty}. \tag{2.10}$$

The a priori assumptions (2.1) and (2.4) yield

$$\|\theta^{\alpha} + \theta^{-\alpha}\|_{\infty} \lesssim \max \{m_2^{-|\alpha|}, N^{|\alpha|}\} \lesssim 1. \tag{2.11}$$

Since  $b \geq 7$ , we have

$$\frac{\mu\theta}{\kappa} \lesssim \frac{h(v)\theta}{1+v\theta^b} \lesssim \begin{cases} h(v) & \text{if } \theta \leq 1, \\ h(v)v^{-1} & \text{if } \theta > 1, \end{cases}$$

which combined with (1.8) implies

$$\left\| \frac{\mu\theta}{\kappa} \right\|_{\infty} \lesssim 1 + \|v\|_{\infty}^{\ell_2} + \left\| \frac{1}{v} \right\|_{\infty}^{\ell_1+1}. \tag{2.12}$$

Similarly, we infer that

$$\frac{\mu v^2 \theta^7}{\kappa} \lesssim \begin{cases} h(v)v^2 & \text{if } \theta \leq 1, \\ h(v)v & \text{if } \theta > 1, \end{cases}$$

and consequently,

$$\begin{aligned} \left\| \frac{\mu v^2 \theta^7}{\kappa} \right\|_{\infty} &\lesssim 1 + \|v\|_{\infty}^{\ell_2+2} + \left\| \frac{1}{v} \right\|_{\infty}^{(\ell_1-2)+} + \|v\|_{\infty}^{\ell_2+1} + \left\| \frac{1}{v} \right\|_{\infty}^{\ell_1-1} \\ &\lesssim 1 + \|v\|_{\infty}^{\ell_2+2} + \left\| \frac{1}{v} \right\|_{\infty}^{\ell_1-1}. \end{aligned} \tag{2.13}$$

From (1.1)<sub>3</sub>, we get

$$\theta_t = \frac{1}{e_{\theta}} \left[ \left( \frac{\kappa_v \theta_x}{v} - \frac{\kappa \theta_x}{v^2} \right) v_x + \frac{\kappa \theta_x^2}{v} + \frac{\kappa \theta_{xx}}{v} + \frac{\mu u_x^2}{v} - \theta p_{\theta} u_x \right], \tag{2.14}$$

where

$$\begin{aligned} \kappa_v &= \kappa_2 \theta^b, \quad \kappa_{\theta} = \kappa_2 b v \theta^{b-1} \\ e_{\theta}(v, \theta) &= c_v + 4av\theta^3, \quad p_{\theta}(v, \theta) = \frac{R}{v} + \frac{4}{3}a\theta^3. \end{aligned} \tag{2.15}$$

Hence we have

$$(\mu v_x - uv)(v_x \theta_t - \theta_x u_x) = uv \theta_x u_x + \mathcal{R}_1 v_x + \mathcal{R}_2 v_x^2,$$

where

$$\begin{aligned} \mathcal{R}_1 &:= -\mu \theta_x u_x - \frac{u}{e_{\theta}} \left[ \kappa_{\theta} \theta_x^2 + \kappa \theta_{xx} + \mu u_x^2 - (p + a\theta^4) v u_x \right], \\ \mathcal{R}_2 &:= \mu \theta_t - \frac{u}{e_{\theta}} \left( \kappa_v \theta_x - \frac{\kappa \theta_x}{v} \right). \end{aligned}$$

Similar to the proof of Lemma 2.2 in [52], we can obtain

$$\mathcal{H}_3 \leq \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\theta v_x^2}{v^3} dx d\tau + C(\epsilon) \left[ 1 + |\alpha| \int_0^t \int_{\mathbb{R}} \frac{\kappa\theta_{xx}^2}{v} dx d\tau \right], \tag{2.16}$$

under the a priori assumption (2.4). Multiplying (2.14) by  $\theta_{xx}$  yields

$$\left( \frac{\theta_x^2}{2} \right)_t - (\theta_x \theta_t)_x + \frac{\kappa\theta_{xx}^2}{ve_\theta} = \frac{\theta_{xx}}{e_\theta} \left[ \theta p_\theta u_x - \frac{\kappa_v v_x \theta_x}{v} + \frac{\kappa\theta_x v_x}{v^2} - \frac{\kappa\theta\theta_x^2}{v} - \frac{\mu u_x^2}{v} \right].$$

We integrate this last identity over  $[0, t] \times \mathbb{R}$  and employ Cauchy's inequality to infer

$$\|\theta_x(t)\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\kappa\theta_{xx}^2}{v} dx d\tau \lesssim 1 + \Xi(m_1, m_2, N)^{\frac{1}{8}}. \tag{2.17}$$

Plugging (2.17) into (2.16) and using (2.4), we obtain

$$\mathcal{H}_3 \leq \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu\theta v_x^2}{v^3} dx d\tau + C(\epsilon). \tag{2.18}$$

Then (2.5) follows by inserting (2.9)–(2.13) and (2.18) into (2.8).  $\square$

By applying the Kanel' technique (cf. [24]), we obtain pointwise bounds for the specific volume  $v(t, x)$  in the following lemma.

**Lemma 2.3** Assume that the conditions listed in Lemma 2.2 hold. Then

$$\left\| \frac{1}{v} \right\|_\infty \lesssim 1 + \|\theta\|_\infty^{\varsigma_1}, \quad \|v\|_\infty \lesssim 1 + \|\theta\|_\infty^{\varsigma_2}, \tag{2.19}$$

where

$$\varsigma_1 := \frac{1}{2\ell_1}, \quad \varsigma_2 := \frac{1}{2\ell_2 + 1}. \tag{2.20}$$

**Proof** Set

$$\Phi(v) := \int_1^v \frac{\sqrt{\tilde{\phi}(z)}}{z} h(z) dz.$$

In light of (1.8), we have

$$\|v\|_\infty^{\frac{1}{2}+\ell_2} + \left\| \frac{1}{v} \right\|_\infty^{\ell_1} \lesssim 1 + \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\tilde{\phi}(v(t, x))|. \tag{2.21}$$

Estimates (2.11) and (2.2) imply

$$|\Phi(v)(t, x)| = \left| \int_x^\infty \frac{\partial}{\partial y} \tilde{\phi}(v(t, y)) dy \right| \lesssim \left\| \sqrt{\tilde{\phi}(v(t))} \right\| \left\| \frac{h(v)v_x(t)}{v} \right\| \lesssim \left\| \frac{\mu v_x(t)}{v} \right\|.$$

As a consequence, we have

$$\|v\|_\infty^{\frac{1}{2}+\ell_2} + \left\| \frac{1}{v} \right\|_\infty^{\ell_1} \lesssim 1 + \sup_{t \in [0,T]} \left\| \frac{\mu v_x(t)}{v} \right\|. \tag{2.22}$$

In view of  $\ell_1 > 1$  and  $\ell_2 > 1$ , we plug (2.5) into (2.22) and utilize Young's inequality to conclude the estimates (2.19).  $\square$

A direct corollary follows from Lemmas 2.2 and 2.3.

**Corollary 2.4** Assume that the conditions listed in Lemma 2.2 hold. Then for any  $0 \leq t \leq T$ , we have

$$\left\| \frac{\mu v_x(t)}{v} \right\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\mu\theta v_x^2}{v^3} dx d\tau \lesssim 1 + \|\theta\|_\infty. \tag{2.23}$$

### 3 Pointwise Bounds for the Absolute Temperature

In this section, we will obtain a uniform-in-time upper bound and a local-in-time lower bound for the absolute temperature  $\theta$ . For this purpose, we set

$$\begin{aligned}
 X(t) &:= \int_0^t \int_{\mathbb{R}} \left( \frac{1}{1 + \|\theta\|_{\infty}^{2\varsigma_2}} + \frac{\theta^{b+3}(\tau, x)}{1 + \|\theta\|_{\infty}^{\varsigma_1}} \right) \theta_t^2(\tau, x) dx d\tau, \\
 \bar{Y}(t) &:= \int_{\mathbb{R}} \left( \frac{1}{1 + \|\theta\|_{\infty}^{2\varsigma_2}} + \theta^{2b}(t, x) \right) \theta_x^2(t, x) dx, \quad Y(t) := \sup_{0 \leq \tau \leq t} \{\bar{Y}(\tau)\}, \\
 Z(t) &:= \sup_{0 \leq \tau \leq t} \int_{\mathbb{R}} u_{xx}^2(\tau, x) dx, \\
 W(t) &:= \int_0^t \int_{\mathbb{R}} \frac{\mu(v(\tau, x), \theta(\tau, x)) u_{xt}^2(\tau, x)}{v(\tau, x)} dx d\tau.
 \end{aligned} \tag{3.1}$$

We first employ the basic energy estimate (2.2) to derive the following lemma.

**Lemma 3.1** Assume that the conditions listed in Theorem 1.1 hold, then we can get that

$$\|\theta\|_{\infty} \lesssim 1 + Y(T)^{\frac{1}{2b+6-\varsigma_1-2\varsigma_2}}. \tag{3.2}$$

**Proof** Noticing from (2.2) that  $\tilde{\phi}(\theta)(t, \cdot) \in L^1(\mathbb{R})$ , one can employ Jensen’s inequality to find  $\xi(t) \in [0, 1]$  such that  $\theta(t, \xi(t)) \sim 1$ . Then

$$\begin{aligned}
 (\theta - 1)^{2b+6}(t, x) &= (\theta(t, \xi(t)) - 1)^{2b+6} + \int_{\xi(t)}^x (2b + 6)(\theta - 1)^{2b+5} \theta_x(t, x) dx \\
 &\lesssim 1 + \int_{\mathbb{R}} |\theta - 1|^{2b+5} |\theta_x| dx \\
 &\lesssim 1 + \|\theta - 1\|_{\infty}^{b+3} \|v^{-1}\|_{\infty}^{\frac{1}{2}} \left[ \int_{\mathbb{R}} v(\theta - 1)^4 dx \right]^{\frac{1}{2}} \left[ \int_{\mathbb{R}} (\theta - 1)^{2b} \theta_x^2 dx \right]^{\frac{1}{2}}.
 \end{aligned}$$

Combining this last estimate with (2.2) and (2.19) yields

$$\|\theta - 1\|_{\infty}^{2b+6} \lesssim 1 + \|\theta - 1\|_{\infty}^{b+3} (1 + \|\theta - 1\|_{\infty}^{2\varsigma_2+\varsigma_1})^{\frac{1}{2}} Y(T)^{\frac{1}{2}}.$$

Since  $2b + 6 > \varsigma_1 + 2\varsigma_2$ , we deduce (3.2) from Young’s inequality. This completes the proof of Lemma 3.1.  $\square$

The next lemma follows directly from Gagliardo–Nirenberg and Sobolev’s inequalities.

**Lemma 3.2** Assume that the conditions listed in Theorem 1.1 hold. Then one can get for each  $0 \leq t \leq T$  that

$$\sup_{0 \leq \tau \leq t} \{\|u_x(\tau)\|\} \lesssim 1 + Z(t)^{\frac{1}{4}}, \quad \|u_x\|_{\infty} \lesssim 1 + Z(T)^{\frac{3}{8}}. \tag{3.3}$$

With the above preparations in hand, our next result is to show that  $X(T)$  and  $Y(T)$  can be controlled by  $Z(T)$  and  $W(T)$ .

**Lemma 3.3** Under the assumptions listed in Theorem 1.1, we have

$$X(T) + Y(T) \lesssim 1 + \frac{\epsilon}{1 + \|\theta\|_{\infty}^{\varsigma_1} + \|\theta\|_{\infty}^{2\varsigma_2}} W(T) + Z(T)^{\lambda}, \tag{3.4}$$

where

$$\lambda = \max \left\{ \frac{2b + 6 - \varsigma_1 - 2\varsigma_2}{2[b + 5 - 2\varsigma_2 - \varsigma_1(\ell_1 + 2)]}, \frac{2b + 6 - \varsigma_1 - 2\varsigma_2}{2[b + 5 - \varsigma_1 - \varsigma_2(\ell_2 + 1)]} \right\} \quad (0 < \lambda < 1).$$

**Proof** In the same manner as in [25] an [46], if we set

$$K(v, \theta) = \int_0^\theta \frac{\kappa(v, \xi)}{v} d\xi, \tag{3.5}$$

then it is easy to verify that

$$K_t = K_v u_x + \frac{\kappa(v, \theta) \theta_t}{v}, \tag{3.6}$$

$$K_{xt} = \left[ \frac{\kappa \theta_x}{v} \right]_t + K_v u_{xx} + K_{vv} v_x u_x + \left( \frac{\kappa}{v} \right)_v v_x \theta_t, \tag{3.7}$$

$$|K(v, \theta)| \lesssim \frac{\theta}{v} + \theta^{b+1}, \quad |K_v(v, \theta)| \lesssim \frac{\theta}{v^2}, \quad \left| \left( \frac{\kappa}{v} \right)_v \right| \lesssim \frac{1}{v^2}. \tag{3.8}$$

We first rewrite (1.1)<sub>3</sub> in the following form

$$e_\theta \theta_t + \theta p_\theta u_x = \frac{\mu u_x^2}{v} + \left( \frac{\kappa \theta_x}{v} \right)_x. \tag{3.9}$$

Then multiplying (3.9) by  $K_t$  and integrating the resulting identity over  $(0, t) \times \mathbb{R}$ , we arrive at

$$\int_0^t \int_{\mathbb{R}} \left( e_\theta \theta_t + \theta p_\theta u_x - \frac{\mu u_x^2}{v} \right) K_t dx d\tau + \int_0^t \int_{\mathbb{R}} \frac{\kappa}{v} \theta_x K_{tx} dx d\tau = 0. \tag{3.10}$$

Combining (3.6)–(3.10), we have

$$\int_0^t \int_{\mathbb{R}} \frac{e_\theta \kappa \theta_t^2}{v} dx d\tau + \int_0^t \int_{\mathbb{R}} \frac{\kappa \theta_x}{v} \left( \frac{\kappa \theta_x}{v} \right)_t dx d\tau \lesssim 1 + \sum_{k=4}^9 \mathcal{H}_k, \tag{3.11}$$

where the definition of  $\mathcal{H}_k$  ( $4 \leq k \leq 9$ ) will be given below.

We now turn to control  $\mathcal{H}_k$  ( $k = 4, 5, \dots, 9$ ) term by term. To do so, we can infer from (2.19) that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \frac{e_\theta \kappa \theta_t^2}{v} dx d\tau &\gtrsim \int_0^t \int_{\mathbb{R}} \frac{(1 + v\theta^3)(1 + v\theta^b) \theta_t^2}{v} dx d\tau \\ &\gtrsim \int_0^t \int_{\mathbb{R}} \left( \frac{1}{v} + \theta^b + \theta^3 + v\theta^{b+3} \right) \theta_t^2 dx d\tau \\ &\gtrsim \int_0^t \int_{\mathbb{R}} \left( \frac{1}{1 + \|\theta\|_\infty^2} + \frac{\theta^{b+3}}{1 + \|\theta\|_\infty^1} \right) \theta_t^2 dx d\tau \\ &:= X(t) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \frac{\kappa \theta_x}{v} \left( \frac{\kappa \theta_x}{v} \right)_t dx d\tau &= \frac{1}{2} \int_{\mathbb{R}} \left( \frac{\kappa \theta_x}{v} \right)^2 dx - \frac{1}{2} \int_{\mathbb{R}} \left( \frac{\kappa \theta_x}{v} \right)^2 (0, x) dx \\ &\gtrsim \int_{\mathbb{R}} \left( \frac{1}{v^2} + \theta^{2b} \right) \theta_x^2 dx - 1 \\ &\gtrsim \int_{\mathbb{R}} \left( \frac{1}{1 + \|\theta\|_\infty^{2s_2}} + \theta^{2b} \right) \theta_x^2 dx - 1 \\ &= \bar{Y}(t) - 1. \end{aligned} \tag{3.13}$$

For the term  $\mathcal{H}_4$ ,

$$|\mathcal{H}_4| = \left| \int_0^t \int_{\mathbb{R}} e_\theta \theta_t K_v u_x dx d\tau \right|$$

$$\begin{aligned}
 &\lesssim \int_0^t \int_{\mathbb{R}} \left( \frac{|\theta|u_x\theta_t|}{v^2} + \frac{\theta^4|u_x\theta_t|}{v} \right) dx d\tau \\
 &\lesssim \epsilon X(T) + C(\epsilon) (1 + \|\theta\|_{\infty}^{\varsigma_1} + \|\theta\|_{\infty}^{\varsigma_2}) \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v\theta} \left( \frac{1}{\mu v^3} + \frac{1}{\mu v} \right) dx d\tau \\
 &\leq \epsilon X(T) + C(\epsilon) \left( 1 + Y(T)^{\frac{3\varsigma_1}{2b+6-\varsigma_1-2\varsigma_2}} + Y(T)^{\frac{2\varsigma_1+\varsigma_2}{2b+6-\varsigma_1-2\varsigma_2}} \right) \\
 &\leq \epsilon (X(T) + Y(T)) + C(\epsilon),
 \end{aligned} \tag{3.14}$$

where we have used (1.8), (1.10), (3.8) and Young’s inequality.

Next, by virtue of (1.8), (1.10), (2.2) and (3.8), we can conclude

$$\begin{aligned}
 |\mathcal{H}_5| &= \left| \int_0^t \int_{\mathbb{R}} \theta p_{\theta} u_x K_v u_x dx d\tau \right| \\
 &\lesssim \int_0^t \int_{\mathbb{R}} \left( \frac{\mu u_x^2}{v\theta} \cdot \frac{\theta^3}{\mu v^2} + \frac{\mu u_x^2}{v\theta} \cdot \frac{\theta^6}{\mu v} \right) dx d\tau \\
 &\lesssim 1 + Y(T)^{\frac{6}{2b+6-\varsigma_1-2\varsigma_2}} \\
 &\leq \epsilon Y(T) + C(\epsilon).
 \end{aligned} \tag{3.15}$$

As for the term  $\mathcal{H}_6$ , it follows from (1.8), (1.10), (2.2), (2.15) and (3.14) that

$$\begin{aligned}
 |\mathcal{H}_6| &= \left| \int_0^t \int_{\mathbb{R}} \frac{\theta p_{\theta} \kappa(v, \theta) u_x \theta_t}{v} dx d\tau \right| \\
 &\lesssim \int_0^t \int_{\mathbb{R}} \left( \frac{|\theta|u_x\theta_t|}{v^2} + \frac{\theta^4|u_x\theta_t|}{v} + \frac{\theta^{b+1}|u_x\theta_t|}{v} + \theta^{b+4}|u_x\theta_t| \right) dx d\tau \\
 &\lesssim \epsilon (X(T) + Y(T)) + C(\epsilon) (1 + \|\theta\|_{\infty}^{\varsigma_1} + \|\theta\|_{\infty}^{\varsigma_2}) \\
 &\quad \times \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v\theta} \left[ \frac{\theta(1 + \theta^{b-1})}{\mu v} + \frac{v(1 + \theta^{b+6})}{\mu} \right] dx d\tau \\
 &\leq \epsilon (X(T) + Y(T)) + C(\epsilon) (1 + \|\theta\|_{\infty}^{\varsigma_1} + \|\theta\|_{\infty}^{\varsigma_2}) (1 + \|\theta\|_{\infty}^{b+6}) \\
 &\leq \epsilon (X(T) + Y(T)) + C(\epsilon) \left( 1 + Y(T)^{\frac{b+6+\varsigma_1}{2b+6-\varsigma_1-2\varsigma_2}} + Y(T)^{\frac{b+6+\varsigma_2}{2b+6-\varsigma_1-2\varsigma_2}} \right) \\
 &\leq \epsilon (X(T) + Y(T)) + C(\epsilon).
 \end{aligned} \tag{3.16}$$

Now we deal with the term  $\mathcal{H}_7$ . For this purpose, we have by integration by parts that

$$\begin{aligned}
 |\mathcal{H}_7| &= \left| \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2 K_t}{v} dx d\tau \right| \\
 &= \left| \int_{\mathbb{R}} \frac{\mu u_x^2 K}{v} dx \Big|_{s=0}^{s=t} - \int_0^t \int_{\mathbb{R}} \left( \frac{\mu u_x^2}{v} \right)_t K dx d\tau \right| \\
 &\leq \left| \int_{\mathbb{R}} \frac{\mu u_x^2 K}{v} dx \right| + \left| \int_{\mathbb{R}} \left( \frac{\mu u_x^2 K}{v} \right) (0, x) dx \right| \\
 &\quad + \left| \int_0^t \int_{\mathbb{R}} \left( \frac{2\mu u_x u_{xt}}{v} - \frac{\mu u_x^3}{v^2} + \frac{\mu_t u_x^2}{v} \right) K dx d\tau \right|.
 \end{aligned} \tag{3.17}$$

For the first term on the right-hand side of (3.17), it follows from (1.8), (1.10) and (2.19) that

$$\left| \int_{\mathbb{R}} \frac{\mu u_x^2 K}{v} dx \right| \lesssim \int_{\mathbb{R}} \left( \frac{h(v)\theta u_x^2}{v^2} + \frac{h(v)\theta^{b+1}u_x^2}{v} \right) dx$$

$$\begin{aligned}
 &\lesssim \|\theta\|_\infty \left( 1 + \left\| \frac{1}{v} \right\|_\infty^{\ell_1+2} + \|v\|_\infty^{(\ell_2-2)+} \right) \|u_x\|^2 + \|\theta\|_\infty^{b+1} \left( 1 + \left\| \frac{1}{v} \right\|_\infty^{\ell_1+1} + \|v\|_\infty^{\ell_2-1} \right) \|u_x\|^2 \\
 &\lesssim \left( 1 + Z(T)^{\frac{1}{2}} \right) \left( 1 + \|\theta\|_\infty^{(\ell_1+1)\varsigma_1+b+1} + \|\theta\|_\infty^{(\ell_2-1)\varsigma_2+b+1} \right) \\
 &\lesssim \left( 1 + Z(T)^{\frac{1}{2}} \right) \left( 1 + Y(T)^{\frac{(\ell_1+1)\varsigma_1+b+1}{2b+6-\varsigma_1-2\varsigma_2}} + Y(T)^{\frac{(\ell_2-1)\varsigma_2+b+1}{2b+6-\varsigma_1-2\varsigma_2}} \right) \\
 &\leq \epsilon Y(T) + C(\epsilon) \left( 1 + Z(T)^{\frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-2\varsigma_2-\varsigma_1(\ell_1+2)]}} + Z(T)^{\frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-\varsigma_1-\varsigma_2(\ell_2+1)]}} \right). \tag{3.18}
 \end{aligned}$$

It is worth to pointing out that we can deduce from (1.10) and (2.20) that

$$0 < \frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-2\varsigma_2-\varsigma_1(\ell_1+2)]} < 1, \quad 0 < \frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-\varsigma_1-\varsigma_2(\ell_2+1)]} < 1.$$

Moreover, the combination of (1.10), (2.2), (3.8) and Hölder’s inequality gives

$$\begin{aligned}
 &\left| \int_0^t \int_{\mathbb{R}} \frac{2\mu u_x u_{xt} K}{v} dx d\tau \right| \\
 &\lesssim \int_0^t \int_{\mathbb{R}} \frac{\mu |u_x u_{xt}|}{v} \left( \frac{\theta}{v} + \theta^{b+1} \right) dx d\tau \\
 &\lesssim \left( \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xt}^2}{v} dx d\tau \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v\theta} \cdot \theta \left( \frac{\theta}{v} + \theta^{b+1} \right)^2 dx d\tau \right)^{\frac{1}{2}} \\
 &\leq \frac{\epsilon}{1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}} W(T) + C(\epsilon) (1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}) (1 + \|\theta\|_\infty^{2\varsigma_1+3} + \|\theta\|_\infty^{2b+3}) \\
 &\leq \frac{\epsilon}{1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}} W(T) + C(\epsilon) \left( 1 + Y(T)^{\frac{2b+3+\varsigma_1}{2b+6-\varsigma_1-2\varsigma_2}} + Y(T)^{\frac{2b+3+2\varsigma_2}{2b+6-\varsigma_1-2\varsigma_2}} \right) \\
 &\leq \frac{\epsilon}{1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}} W(T) + \epsilon Y(T) + C(\epsilon). \tag{3.19}
 \end{aligned}$$

Furthermore, (1.10), (3.3) and (3.8) tell us that

$$\begin{aligned}
 \left| \int_0^t \int_{\mathbb{R}} \frac{\mu |u_x^3| K}{v^2} dx d\tau \right| &\lesssim \|u_x\|_\infty \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v\theta} \cdot \frac{\theta}{v} \left( \frac{\theta}{v} + \theta^{b+1} \right) dx d\tau \\
 &\lesssim \left( 1 + Z(T)^{\frac{3}{8}} \right) \left( 1 + Y(T)^{\frac{b+2+\varsigma_1}{2b+6-\varsigma_1-2\varsigma_2}} \right) \\
 &\leq \epsilon Y(T) + C(\epsilon) \left( 1 + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{8(b+4-2\varsigma_1-2\varsigma_2)}} \right). \tag{3.20}
 \end{aligned}$$

For the last term on the right-hand side of (3.17), one can get that

$$\begin{aligned}
 &\int_0^t \int_{\mathbb{R}} \frac{|\mu_t| u_x^2 K}{v} dx d\tau \\
 &\lesssim \int_0^t \int_{\mathbb{R}} \left| \frac{\alpha \mu \theta_t}{\theta} + h'(v) \theta^\alpha u_x \right| \frac{u_x^2}{v} \left( \frac{\theta}{v} + \theta^{b+1} \right) dx d\tau \\
 &\lesssim 1 + \|u_x\|_\infty \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v\theta} \cdot \frac{|h'(v)|}{h(v)} \left( \frac{\theta}{v} + \theta^{b+1} \right) \theta dx d\tau \\
 &\lesssim 1 + \left( 1 + Z(T)^{\frac{3}{8}} \right) \left\| \frac{h'(v)}{h(v)} \right\|_\infty \left\| \frac{\theta^2}{v} + \theta^{b+2} \right\|_\infty \\
 &\lesssim \left( 1 + Z(T)^{\frac{3}{8}} \right) \left( 1 + Y(T)^{\frac{(\ell_1+1)\varsigma_1}{2(2b+6-\varsigma_1-2\varsigma_2)}} + Y(T)^{\frac{(\ell_2-1)\varsigma_2}{2(2b+6-\varsigma_1-2\varsigma_2)}} \right) \left( 1 + Y(T)^{\frac{b+2}{2b+6-\varsigma_1-2\varsigma_2}} \right) \\
 &\lesssim \left( 1 + Z(T)^{\frac{3}{8}} \right) \left( 1 + Y(T)^{\frac{2(b+2)+(\ell_1+1)\varsigma_1}{2(2b+6-\varsigma_1-2\varsigma_2)}} + Y(T)^{\frac{2(b+2)+(\ell_2-1)\varsigma_2}{2(2b+6-\varsigma_1-2\varsigma_2)}} \right)
 \end{aligned}$$

$$\leq \epsilon Y(T) + C(\epsilon) \left( 1 + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[2b+8-(\ell_1+3)\varsigma_1-4\varsigma_2]}} + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[2b+8-2\varsigma_1-(\ell_2+3)\varsigma_2]}} \right), \tag{3.21}$$

where we have used (1.8), (2.4), (2.19), (3.2), (3.3) and the fact that

$$\begin{aligned} \left| \frac{h'(v)}{h(v)} \right| &\lesssim \sqrt{\frac{h(v)}{v}} \\ &\lesssim 1 + \left\| \frac{1}{v} \right\|_{\infty}^{\frac{\ell_1+1}{2}} + \|v\|_{\infty}^{\frac{\ell_2-1}{2}} \\ &\lesssim 1 + \|\theta\|_{\infty}^{\frac{(\ell_1+1)\varsigma_1}{2}} + \|\theta\|_{\infty}^{\frac{(\ell_2-1)\varsigma_2}{2}} \\ &\lesssim 1 + Y(T)^{\frac{(\ell_1+1)\varsigma_1}{2(2b+6-\varsigma_1-2\varsigma_2)}} + Y(T)^{\frac{(\ell_2-1)\varsigma_2}{2(2b+6-\varsigma_1-2\varsigma_2)}}. \end{aligned} \tag{3.22}$$

Then it follows from (3.17)–(3.21) that

$$\begin{aligned} |\mathcal{H}_7| &\leq \epsilon Y(T) + \frac{\epsilon}{1 + \|\theta\|_{\infty}^{\varsigma_1} + \|\theta\|_{\infty}^{2\varsigma_2}} W(T) + C(\epsilon) \\ &\times \left( 1 + Z(T)^{\frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-2\varsigma_2-\varsigma_1(\ell_1+2)]}} + Z(T)^{\frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-\varsigma_1-\varsigma_2(\ell_2+1)]}} + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{8(b+4-2\varsigma_1-2\varsigma_2)}} \right. \\ &\left. + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[2b+8-(\ell_1+3)\varsigma_1-4\varsigma_2]}} + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[2b+8-2\varsigma_1-(\ell_2+3)\varsigma_2]}} \right). \end{aligned} \tag{3.23}$$

Now we are in a position to control the term  $\mathcal{H}_8$ . Noticing that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \frac{\kappa}{v} \theta_x K_v u_{xx} dx d\tau &= - \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa \theta_x}{v} \right)_x K_v u_x dx d\tau \\ &\quad - \int_0^t \int_{\mathbb{R}} \frac{\kappa}{v} \theta_x u_x \left( K_{vv} v_x + \left( \frac{\kappa}{v} \right)_v \theta_x \right) dx d\tau, \end{aligned} \tag{3.24}$$

thus

$$\begin{aligned} \mathcal{H}_8 &= \left| \int_0^t \int_{\mathbb{R}} \frac{\kappa}{v} \theta_x (K_{vv} v_x u_x + K_v u_{xx}) dx d\tau \right| \\ &\leq \left| \int_0^t \int_{\mathbb{R}} \frac{\kappa}{v} \theta_x^2 \left( \frac{\kappa}{v} \right)_v u_x dx d\tau \right| + \left| \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa \theta_x}{v} \right)_x K_v u_x dx d\tau \right| \\ &\lesssim \|u_x\|_{\infty} \int_0^t \int_{\mathbb{R}} \frac{\kappa \theta_x^2}{v \theta^2} \cdot \frac{\theta^2}{v^2} dx d\tau + \int_0^t \int_{\mathbb{R}} \left| \left( \frac{\kappa \theta_x}{v} \right)_x \frac{\theta}{v^2} u_x \right| dx d\tau \\ &\lesssim \left( 1 + Z(T)^{\frac{3}{8}} \right) \left( 1 + Y(T)^{\frac{2\varsigma_1+2}{2b+6-\varsigma_1-2\varsigma_2}} \right) \\ &\quad + \left( \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v \theta} dx d\tau \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}} \frac{\theta^3}{\mu v^3} \left| \left( \frac{\kappa \theta_x}{v} \right)_x \right|^2 dx d\tau \right)^{\frac{1}{2}} \\ &\leq \epsilon Y(T) + C(\epsilon) \left( 1 + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{8(2b+4-3\varsigma_1-2\varsigma_2)}} \right) \\ &\quad + \underbrace{\left[ \int_0^t \int_{\mathbb{R}} \frac{\theta^3}{\mu v^3} \left( e_{\theta}^2 \theta_t^2 + \theta^2 p_{\theta}^2 u_x^2 + \frac{\mu^2 u_x^4}{v^2} \right) dx d\tau \right]^{\frac{1}{2}}}_{\mathcal{H}_8^1}, \end{aligned} \tag{3.25}$$

where we have used (1.1)<sub>3</sub>, (2.2), (3.2), (3.3), (3.8) and Hölder’s inequality. Consequently, to yield an estimate on  $\mathcal{H}_8$ , it suffices to bound the term  $\mathcal{H}_8^1$ . To this end, we can get that

$$\mathcal{H}_8^1 \lesssim \left\{ \int_0^t \int_{\mathbb{R}} \left[ \left( \frac{\theta^3}{\mu v^3} + \frac{\theta^9}{\mu v} \right) \theta_t^2 + \frac{\mu u_x^2}{v \theta} \left( \frac{\theta^6}{\mu^2 v^4} + \frac{\theta^{12}}{\mu^2 v^2} + \frac{u_x^2 \theta^4}{v^4} \right) \right] dx d\tau \right\}^{\frac{1}{2}}$$

$$\begin{aligned}
 &\lesssim \left[ (1 + \|\theta\|_\infty^{3\varsigma_1} + \|\theta\|_\infty^{2\varsigma_1+\varsigma_2}) X(T) + \left\| \frac{\theta^6}{v^2} \right\|_\infty + \|\theta\|_\infty^{12} + \left( 1 + Z(T)^{\frac{3}{4}} \right) \left( 1 + \|\theta\|_\infty^{4\varsigma_1+4} \right) \right]^{\frac{1}{2}} \\
 &\lesssim \left[ 1 + X(T) + X(T)Y(T)^{\frac{3\varsigma_1}{2b+6-\varsigma_1-2\varsigma_2}} + X(T)Y(T)^{\frac{2\varsigma_1+\varsigma_2}{2b+6-\varsigma_1-2\varsigma_2}} \right. \\
 &\quad \left. + Y(T)^{\frac{12}{2b+6-\varsigma_1-2\varsigma_2}} + Z(T)^{\frac{3}{4}} + Y(T)^{\frac{4\varsigma_1+4}{2b+6-\varsigma_1-2\varsigma_2}} + Z(T)^{\frac{3}{4}} Y(T)^{\frac{4\varsigma_1+4}{2b+6-\varsigma_1-2\varsigma_2}} \right]^{\frac{1}{2}} \\
 &\lesssim 1 + X(T)^{\frac{1}{2}} + X(T)^{\frac{1}{2}} Y(T)^{\frac{3\varsigma_1}{2(2b+6-\varsigma_1-2\varsigma_2)}} + X(T)^{\frac{1}{2}} Y(T)^{\frac{2\varsigma_1+\varsigma_2}{2(2b+6-\varsigma_1-2\varsigma_2)}} \\
 &\quad + Y(T)^{\frac{6}{2b+6-\varsigma_1-2\varsigma_2}} + Z(T)^{\frac{3}{8}} + Y(T)^{\frac{2\varsigma_1+2}{2b+6-\varsigma_1-2\varsigma_2}} + Z(T)^{\frac{3}{8}} Y(T)^{\frac{2\varsigma_1+2}{2b+6-\varsigma_1-2\varsigma_2}} \\
 &\leq \epsilon (X(T) + Y(T)) + C(\epsilon) \left( 1 + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{8(2b+4-3\varsigma_1-2\varsigma_2)}} \right), \tag{3.26}
 \end{aligned}$$

where we have used (1.8), (1.10), (2.2), (2.19), (2.20), (3.2), (3.3) and Young’s inequality. Thus we can deduce from (3.25) and (3.26) that

$$\mathcal{H}_8 \leq \epsilon (X(T) + Y(T)) + C(\epsilon) \left( 1 + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{8(2b+4-3\varsigma_1-2\varsigma_2)}} \right). \tag{3.27}$$

As for the term  $\mathcal{H}_9$ , we can deduce from (1.1)<sub>3</sub>, (2.2), (2.23), (3.8), Hölder’s inequality and the following fact

$$\sup_{x \in \mathbb{R}} \left( \frac{\kappa \theta_x}{v} \right)^2 (t) \lesssim \int_{\mathbb{R}} \left| \frac{\kappa \theta_x}{v} \right| \left| \left( \frac{\kappa \theta_x}{v} \right)_x \right| dx$$

that

$$\begin{aligned}
 |\mathcal{H}_9| &= \left| \int_0^t \int_{\mathbb{R}} \frac{\kappa \theta_x}{v} \left( \frac{\kappa}{v} \right)_v v_x \theta_t dx d\tau \right| \\
 &\lesssim \int_0^t \int_{\mathbb{R}} \left( \frac{|v_x \theta_x \theta_t|}{v^3} + \frac{\theta^b |v_x \theta_x \theta_t|}{v^2} \right) dx d\tau \\
 &\leq \epsilon X(T) + C(\epsilon) (1 + \|\theta\|_\infty^{\varsigma_2}) \int_0^t \left\| \frac{\kappa \theta_x}{v} \right\|_{L^\infty}^2 \int_{\mathbb{R}} \frac{\mu^2 v_x^2}{v^2} \cdot \frac{1}{\mu^2 v^2} dx d\tau \\
 &\leq \epsilon X(T) + C(\epsilon) (1 + \|\theta\|_\infty^{\varsigma_2+1}) \int_0^t \int_{\mathbb{R}} \left| \frac{\kappa \theta_x}{v} \right| \left| \left( \frac{\kappa \theta_x}{v} \right)_x \right| dx d\tau \\
 &\leq \epsilon X(T) + C(\epsilon) \left( 1 + Y(T)^{\frac{\varsigma_2+1}{2b+6-\varsigma_1-2\varsigma_2}} \right) \\
 &\quad \times \left( \int_0^t \int_{\mathbb{R}} \frac{\kappa \theta_x^2}{v \theta^2} dx d\tau \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}} \frac{\kappa \theta^2}{v} \left| \left( \frac{\kappa \theta_x}{v} \right)_x \right|^2 dx d\tau \right)^{\frac{1}{2}} \\
 &\leq \epsilon X(T) + C(\epsilon) \left( 1 + Y(T)^{\frac{\varsigma_2+1}{2b+6-\varsigma_1-2\varsigma_2}} \right) \\
 &\quad \times \underbrace{\left[ \int_0^t \int_{\mathbb{R}} \left( \frac{\theta^2}{v} + \theta^{b+2} \right) \left( e_\theta^2 \theta_t^2 + \theta^2 p_\theta^2 u_x^2 + \frac{\mu^2 u_x^4}{v^2} \right) dx d\tau \right]^{\frac{1}{2}}}_{\mathcal{H}_9^1}. \tag{3.28}
 \end{aligned}$$

Similar to the term  $\mathcal{H}_8^1$ , the term  $\mathcal{H}_9^1$  can be bounded as follows

$$\begin{aligned}
 \mathcal{H}_9^1 &\lesssim \left\{ \int_0^t \int_{\mathbb{R}} \left( \frac{\theta^2}{v} + v \theta^8 + \theta^{b+2} + v^2 \theta^{b+8} \right) \theta_t^2 + \frac{\mu u_x^2}{v \theta} \left( \frac{\theta^5}{\mu v^2} + \frac{\theta^{11}}{\mu} + \frac{\theta^{b+5}}{\mu v} + \frac{v \theta^{b+11}}{\mu} \right) \right. \\
 &\quad \left. + \frac{\mu u_x^2}{v \theta} \left( \frac{h(v) \theta^3}{v^2} + \frac{h(v) \theta^{b+3}}{v} \right) |u_x|^2 dx d\tau \right\}^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 &\lesssim \left\{ (1 + \|\theta\|_\infty^{\varsigma_1+2\varsigma_2+5} + \|\theta\|_\infty^{3\varsigma_2+5}) X(T) + \|\theta\|_\infty^{b+11} \right. \\
 &\quad \left. + \left(1 + Z(T)^{\frac{3}{4}}\right) \left(1 + \|\theta\|_\infty^{\varsigma_1(\ell_1+1)+b+3} + \|\theta\|_\infty^{\varsigma_2(\ell_2-1)+b+3}\right) \right\}^{\frac{1}{2}} \\
 &\lesssim \left[ 1 + \left(1 + Y(T)^{\frac{\varsigma_1+2\varsigma_2+5}{2b+6-\varsigma_1-2\varsigma_2}} + Y(T)^{\frac{3\varsigma_2+5}{2b+6-\varsigma_1-2\varsigma_2}}\right) X(T) + Y(T)^{\frac{b+11}{2b+6-\varsigma_1-2\varsigma_2}} \right. \\
 &\quad \left. + Z(T)^{\frac{3}{4}} Y(T)^{\frac{\varsigma_1(\ell_1+1)+b+3}{2b+6-\varsigma_1-2\varsigma_2}} + Z(T)^{\frac{3}{4}} Y(T)^{\frac{\varsigma_2(\ell_2-1)+b+3}{2b+6-\varsigma_1-2\varsigma_2}} \right]^{\frac{1}{2}} \\
 &\lesssim 1 + X(T)^{\frac{1}{2}} + X(T)^{\frac{1}{2}} Y(T)^{\frac{\varsigma_1+2\varsigma_2+5}{2(2b+6-\varsigma_1-2\varsigma_2)}} + X(T)^{\frac{1}{2}} Y(T)^{\frac{3\varsigma_2+5}{2(2b+6-\varsigma_1-2\varsigma_2)}} \\
 &\quad + Y(T)^{\frac{b+11}{2(2b+6-\varsigma_1-2\varsigma_2)}} + Z(T)^{\frac{3}{8}} Y(T)^{\frac{\varsigma_1(\ell_1+1)+b+3}{2(2b+6-\varsigma_1-2\varsigma_2)}} + Z(T)^{\frac{3}{8}} Y(T)^{\frac{\varsigma_2(\ell_2-1)+b+3}{2(2b+6-\varsigma_1-2\varsigma_2)}}. \tag{3.29}
 \end{aligned}$$

Plugging (3.29) into (3.28), we find that

$$\begin{aligned}
 |\mathcal{H}_9| &\leq \epsilon X(T) + C(\epsilon) \left(1 + Y(T)^{\frac{\varsigma_2+1}{2b+6-\varsigma_1-2\varsigma_2}}\right) \left(1 + X(T)^{\frac{1}{2}} + X(T)^{\frac{1}{2}} Y(T)^{\frac{\varsigma_1+2\varsigma_2+5}{2(2b+6-\varsigma_1-2\varsigma_2)}} \right. \\
 &\quad \left. + X(T)^{\frac{1}{2}} Y(T)^{\frac{3\varsigma_2+5}{2(2b+6-\varsigma_1-2\varsigma_2)}} + Y(T)^{\frac{b+11}{2(2b+6-\varsigma_1-2\varsigma_2)}} + Z(T)^{\frac{3}{8}} Y(T)^{\frac{\varsigma_1(\ell_1+1)+b+3}{2(2b+6-\varsigma_1-2\varsigma_2)}} \right. \\
 &\quad \left. + Z(T)^{\frac{3}{8}} Y(T)^{\frac{\varsigma_2(\ell_2-1)+b+3}{2(2b+6-\varsigma_1-2\varsigma_2)}}\right) \\
 &\leq \epsilon (X(T) + Y(T)) \\
 &\quad + C(\epsilon) \left(1 + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[3b-(\ell_1+3)\varsigma_1-6\varsigma_2+7]}} + Z(T)^{\frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[3b-(\ell_2+5)\varsigma_2-2\varsigma_1+7]}}\right). \tag{3.30}
 \end{aligned}$$

Finally, if we set

$$\begin{aligned}
 \lambda &= \max \left\{ \frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-2\varsigma_2-\varsigma_1(\ell_1+2)]}, \frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-\varsigma_1-\varsigma_2(\ell_2+1)]}, \frac{3(2b+6-\varsigma_1-2\varsigma_2)}{8(b+4-2\varsigma_1-2\varsigma_2)}, \right. \\
 &\quad \frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[2b+8-(\ell_1+3)\varsigma_1-4\varsigma_2]}, \frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[2b+8-2\varsigma_1-(\ell_2+3)\varsigma_2]}, \frac{3(2b+6-\varsigma_1-2\varsigma_2)}{8(2b+4-3\varsigma_1-2\varsigma_2)}, \\
 &\quad \left. \frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[3b-(\ell_1+3)\varsigma_1-6\varsigma_2+7]}, \frac{3(2b+6-\varsigma_1-2\varsigma_2)}{4[3b-(\ell_2+5)\varsigma_2-2\varsigma_1+7]} \right\} \\
 &= \max \left\{ \frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-2\varsigma_2-\varsigma_1(\ell_1+2)]}, \frac{2b+6-\varsigma_1-2\varsigma_2}{2[b+5-\varsigma_1-\varsigma_2(\ell_2+1)]} \right\}, \tag{3.31}
 \end{aligned}$$

then combining all the above estimates and choosing  $\epsilon > 0$  small enough, we can complete the proof of our lemma.  $\square$

Our next result in this section is to show that  $Z(T)$  can be bounded by  $X(T)$  and  $Y(T)$ .

**Lemma 3.4** Under the assumptions listed in Theorem 1.1, we have

$$Z(T) \lesssim 1 + (1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}) (X(T) + Y(T)). \tag{3.32}$$

**Proof** Differentiating (1.1)<sub>2</sub> with respect to  $t$  and multiplying the resulting identity with  $u_t$ , we have

$$\left(\frac{1}{2}u_t^2\right)_t + \frac{\mu u_{xt}^2}{v} = \left[u_t \left(\frac{\mu u_x}{v}\right)_t - u_t p_t\right]_x + u_{xt} \left[p_t - \left(\frac{\mu}{v}\right)_t u_x\right].$$

Integrating this last identity over  $(0, t) \times \mathbb{R}$  and by using Cauchy's inequality yield

$$\|u_t(t)\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xt}^2}{v} dx d\tau \lesssim 1 + \int_0^t \int_{\mathbb{R}} \frac{v}{\mu} \left[p_t^2 + \left|\left(\frac{\mu}{v}\right)_t\right|^2 u_x^2\right] dx d\tau. \tag{3.33}$$

Since

$$p_t = \left( \frac{R}{v} + \frac{4}{3}a\theta^3 \right) \theta_t - \frac{R\theta u_x}{v^2}, \quad (3.34)$$

thus it follows from (1.8) and (2.2) that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \frac{vp_t^2}{\mu} dx d\tau &\lesssim \int_0^t \int_{\mathbb{R}} \left( \frac{\theta_t^2}{\mu v} + \frac{v\theta^6\theta_t^2}{\mu} + \frac{\theta^2 u_x^2}{\mu v^3} \right) dx d\tau \\ &\lesssim (1 + \|\theta\|_{\infty}^{s_1} + \|\theta\|_{\infty}^{s_2}) X(T) + \|\theta\|_{\infty}^3 \\ &\lesssim 1 + Y(T) + (1 + \|\theta\|_{\infty}^{s_1} + \|\theta\|_{\infty}^{s_2}) X(T). \end{aligned} \quad (3.35)$$

On the other hand, according to (1.8), we have

$$\begin{aligned} \left| \left( \frac{\mu}{v} \right)_t \right|^2 &= \left[ \frac{\theta^\alpha h'(v) u_x}{v} + \frac{\alpha \theta^{\alpha-1} h(v) \theta_t}{v} - \frac{h(v) \theta^\alpha u_x}{v^2} \right]^2 \\ &\lesssim \frac{\mu^2 u_x^2}{v^4} + \frac{\theta^{2\alpha} |h'(v)|^2 u_x^2}{v^2} + \frac{\alpha^2 \mu^2 \theta_t^2}{\theta^2 v^2}. \end{aligned} \quad (3.36)$$

Then it follows from (2.2), (2.4), (3.2), (3.3) and (3.22) that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \frac{v u_x^2}{\mu} \left| \left( \frac{\mu}{v} \right)_t \right|^2 dx d\tau &\lesssim \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v\theta} \left( \frac{\theta u_x^2}{v^2} + \frac{\theta u_x^2 |h'(v)|^2}{h^2(v)} + \frac{\alpha^2 \theta_t^2}{\theta} \right) dx d\tau \\ &\lesssim 1 + \left\| \frac{\theta u_x^2}{v^2} \right\|_{\infty} + \left\| \frac{\theta u_x^2 |h'(v)|^2}{h^2(v)} \right\|_{\infty} \\ &\lesssim \left( 1 + Z(T)^{\frac{3}{4}} \right) \left( 1 + \|\theta\|_{\infty}^{1+2s_1} + \|\theta\|_{\infty}^{(\ell_1+1)s_1+1} \right) \\ &\lesssim \left( 1 + Z(T)^{\frac{3}{4}} \right) \left( 1 + Y(T)^{\frac{(\ell_1+1)s_1+1}{2b+6-s_1-2s_2}} \right) \\ &\lesssim 1 + Y(T) + Z(T)^{\frac{3(2b+6-s_1-2s_2)}{4[2b+5-(\ell_1+2)s_1-2s_2]}}. \end{aligned} \quad (3.37)$$

Thus combining (3.33)–(3.37), we obtain

$$\begin{aligned} \|u_t(t)\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\mu u_{tx}^2}{v} dx d\tau \\ \lesssim 1 + Y(T) + (1 + \|\theta\|_{\infty}^{s_1} + \|\theta\|_{\infty}^{s_2}) X(T) + Z(T)^{\frac{3(2b+6-s_1-2s_2)}{4[2b+5-(\ell_1+2)s_1-2s_2]}}. \end{aligned} \quad (3.38)$$

Moreover, we can conclude from (1.1)<sub>2</sub> that

$$u_{xx} = \frac{v}{\mu} \left[ u_t + p_x + \frac{\mu u_x v_x}{v^2} \right], \quad (3.39)$$

hence

$$\begin{aligned} \|u_{xx}\|^2 &\lesssim \int_{\mathbb{R}} \left( \frac{v^2 u_t^2}{\mu^2} + \frac{v^2 p_x^2}{\mu^2} + \frac{\mu^2 v_x^2}{v^2} \cdot \frac{u_x^2}{\mu^2} \right) dx \\ &\lesssim \|u_t(t)\|^2 + \int_{\mathbb{R}} \left[ \left( \frac{1}{\mu^2} + \frac{v^2 \theta^6}{\mu^2} \right) \theta_x^2 + \frac{\mu^2 v_x^2}{v^2} \cdot \frac{\theta^2}{\mu^4} \right] dx + \|u_x\|_{\infty}^2 \left\| \frac{\mu v_x}{v} \right\|^2 \\ &\lesssim \|u_t(t)\|^2 + (1 + \|\theta\|_{\infty}^{2s_2}) Y(T) + \|\theta\|_{\infty}^3 + \left( 1 + Z(T)^{\frac{3}{4}} \right) \|\theta\|_{\infty} \\ &\lesssim 1 + (1 + \|\theta\|_{\infty}^{s_1} + \|\theta\|_{\infty}^{2s_2}) (X(T) + Y(T)) + Z(T)^{\frac{3(2b+6-s_1-2s_2)}{4[2b+5-(\ell_1+2)s_1-2s_2]}}. \end{aligned} \quad (3.40)$$

Here we have used (2.23), (3.38) and the fact that

$$p_x = \left( \frac{R\theta}{v} + \frac{a\theta^4}{3} \right)_x = \frac{R\theta_x}{v} - \frac{R\theta v_x}{v^2} + \frac{4}{3}a\theta^3 \theta_x. \quad (3.41)$$

In view of (1.10), one can deduce that

$$0 < \frac{3(2b + 6 - \varsigma_1 - 2\varsigma_2)}{4[2b + 5 - (\ell_1 + 2)\varsigma_1 - 2\varsigma_2]} < 1.$$

Then by virtue of (3.1) and Young’s inequality, we can complete the proof of our lemma.  $\square$

We are in a position to deduce the upper bound of  $\theta(t, x)$  now. In fact, (3.4), (3.32) and (3.38) tell us that

$$\begin{aligned} Z(T) + W(T) &\lesssim 1 + (1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}) (X(T) + Y(T)) \\ &\lesssim 1 + (1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}) \left( 1 + \frac{\epsilon}{1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}} W(T) + Z(T)^\lambda \right) \\ &\lesssim 1 + \epsilon W(T) + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2} + (1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}) Z(T)^\lambda \\ &\lesssim 1 + \epsilon W(T) + Y(T)^{\frac{\varsigma_1}{2b+6-\varsigma_1-2\varsigma_2}} + Y(T)^{\frac{2\varsigma_2}{2b+6-\varsigma_1-2\varsigma_2}} \\ &\quad + Y(T)^{\frac{\varsigma_1}{2b+6-\varsigma_1-2\varsigma_2}} Z(T)^\lambda + Y(T)^{\frac{2\varsigma_2}{2b+6-\varsigma_1-2\varsigma_2}} Z(T)^\lambda \\ &\lesssim 1 + \epsilon W(T) + Y(T) + Z(T)^{\lambda^*}, \end{aligned} \tag{3.42}$$

where

$$\lambda^* = \max \left\{ \frac{2b + 6 - \varsigma_1 - 2\varsigma_2}{2b + 6 - 2\varsigma_1 - 2\varsigma_2} \lambda, \frac{2b + 6 - \varsigma_1 - 2\varsigma_2}{2b + 6 - \varsigma_1 - 4\varsigma_2} \lambda \right\} \tag{3.43}$$

and we have used the fact that

$$\begin{aligned} Y(T)^{\frac{\varsigma_1}{2b+6-\varsigma_1-2\varsigma_2}} Z(T)^\lambda &\lesssim Y(T) + Z(T)^{\frac{2b+6-\varsigma_1-2\varsigma_2}{2b+6-2\varsigma_1-2\varsigma_2} \lambda}, \\ Y(T)^{\frac{2\varsigma_2}{2b+6-\varsigma_1-2\varsigma_2}} Z(T)^\lambda &\lesssim Y(T) + Z(T)^{\frac{2b+6-\varsigma_1-2\varsigma_2}{2b+6-\varsigma_1-4\varsigma_2} \lambda}. \end{aligned}$$

Noticing that under the condition (1.10) and (1.11), we have

$$0 < \lambda^* < 1. \tag{3.44}$$

Thus with the hand of Young’s inequality and Lemma 3.3, we can deduce from (3.42) that

$$\begin{aligned} Z(T) + W(T) &\lesssim 1 + \epsilon W(T) + Y(T) \\ &\lesssim 1 + \epsilon W(T) + \frac{\epsilon}{1 + \|\theta\|_\infty^{\varsigma_1} + \|\theta\|_\infty^{2\varsigma_2}} W(T) + Z(T)^\lambda \\ &\lesssim 1 + \epsilon W(T) + Z(T)^\lambda. \end{aligned} \tag{3.45}$$

Finally, choosing  $\epsilon > 0$  small enough then using (3.4) and Young’s inequality again, we immediately obtain

$$X(T) + Y(T) + Z(T) + W(T) \lesssim 1. \tag{3.46}$$

Recalling the definition of  $X(T)$ ,  $Y(T)$ ,  $Z(T)$  and  $W(T)$ , then combining Lemmas 2.1–3.4, we have the following lemma.

**Lemma 3.5** Under the assumptions listed in Theorem 1.1, there exist positive constants  $C_1$  and  $C_2$ , which depend only on  $\Pi_0$  and  $V_0$ , such that

$$\theta(t, x) \leq C_1, \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \tag{3.47}$$

$$C_2^{-1} \leq v(t, x) \leq C_2, \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \tag{3.48}$$

Moreover, we have

$$\|(v - 1, u, \theta - 1, v_x, u_t, u_x, \theta_x, u_{xx})(t)\|^2 + \int_0^t \left\| \left( \sqrt{\theta} v_x, u_x, \theta_x, \theta_t, u_{xt} \right) (\tau) \right\|^2 d\tau \lesssim 1,$$

$$\int_0^t \int_{\mathbb{R}} u_x^4(\tau, x) dx d\tau \lesssim 1, \quad \|u_x\|_{\infty} \lesssim 1. \tag{3.49}$$

Before concluding this section, let us deduce uniform bounds on  $\int_0^t \int_{\mathbb{R}} u_{xx}^2 dx d\tau$  and  $\int_0^t \int_{\mathbb{R}} \theta_{xx}^2 dx d\tau$ , which will be used later on. In fact, we have the following lemma.

**Lemma 3.6** Under the assumptions listed in Theorem 1.1, for any  $0 \leq t \leq T$ , we have

$$\|u_x(t)\|^2 + \int_0^t \int_{\mathbb{R}} u_{xx}^2(\tau, x) dx d\tau \lesssim 1 \tag{3.50}$$

and

$$\|\theta_x(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta_{xx}^2(\tau, x) dx d\tau \lesssim 1. \tag{3.51}$$

**Proof** First, multiplying (1.1)<sub>2</sub> by  $u_{xx}$ , we get

$$\partial_t (u_x^2) + \frac{\mu u_{xx}^2}{v} = (u_t u_x)_x + p_x u_{xx} - \frac{\mu_x u_x u_{xx}}{v} + \frac{\mu u_x v_x u_{xx}}{v^2}.$$

Integrating the above identity with respect to  $t$  and  $x$  over  $(0, t) \times \mathbb{R}$  and taking advantage of (3.41), we arrive at

$$\begin{aligned} & \|u_x(t)\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xx}^2}{v} dx d\tau \\ & \lesssim 1 + \int_0^t \int_{\mathbb{R}} \left( \frac{|\mu_x u_x u_{xx}|}{v} + \left| \left( \frac{\theta}{v} \right)_x u_{xx} \right| + \theta^3 |\theta_x u_{xx}| + \frac{|\mu u_x u_{xx} v_x|}{v^2} \right) dx d\tau. \end{aligned} \tag{3.52}$$

The first term on the right-hand side of (3.52) can be estimated as follows

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \frac{|\mu_x u_x u_{xx}|}{v} dx d\tau &= \int_0^t \int_{\mathbb{R}} \left( \frac{\alpha \mu |\theta_x u_x u_{xx}|}{v \theta} + \frac{\theta^\alpha |h'(v) v_x u_x u_{xx}|}{v} \right) dx d\tau \\ &\leq \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xx}^2}{v} dx d\tau + C(\epsilon) \int_0^t \int_{\mathbb{R}} \left( \frac{\alpha^2 \mu u_x^2 \theta^2}{v \theta^2} + \frac{u_x^2 v_x^2 |h'(v)|^2}{v h(v)} \right) dx d\tau \\ &\leq \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xx}^2}{v} dx d\tau + C(\epsilon) \left( 1 + \int_0^t \|u_x(s)\|_{L^\infty}^2 ds \right) \\ &\leq \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xx}^2}{v} dx d\tau + C(\epsilon) \left( 1 + \int_0^t \|u_x(s)\| \|u_{xx}(s)\| ds \right) \\ &\leq 2\epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xx}^2}{v} dx d\tau + C(\epsilon) \left( 1 + \int_0^t \int_{\mathbb{R}} \frac{\mu u_x^2}{v \theta} \cdot \theta dx d\tau \right) \\ &\leq 2\epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xx}^2}{v} dx d\tau + C(\epsilon). \end{aligned} \tag{3.53}$$

Here we have used (2.2), Lemma 3.5 and Sobolev’s inequality.

Moreover, it follows from (2.2), Lemma 3.5 and Cauchy’s inequality that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} \left( \left| \left( \frac{\theta}{v} \right)_x u_{xx} \right| + \theta^3 |\theta_x u_{xx}| + \frac{|\mu u_x u_{xx} v_x|}{v^2} \right) dx d\tau \\ & \lesssim \int_0^t \int_{\mathbb{R}} \left( \frac{|\theta_x u_{xx}|}{v} + \frac{|\theta v_x u_{xx}|}{v^2} + \theta^3 |\theta_x u_{xx}| + \frac{|\mu u_x u_{xx} v_x|}{v^2} \right) dx d\tau \\ & \leq \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xx}^2}{v} dx d\tau + C(\epsilon) \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa \theta_x^2}{v \theta^2} \cdot \frac{\theta^2 + \theta^8}{1 + v \theta^b} + \theta^2 v_x^2 + u_x^2 v_x^2 \right) dx d\tau \\ & \leq \epsilon \int_0^t \int_{\mathbb{R}} \frac{\mu u_{xx}^2}{v} dx d\tau + C(\epsilon). \end{aligned} \tag{3.54}$$

Then combining (3.52)–(3.54) and choosing  $\epsilon > 0$  small enough, we can get (3.50).

Secondly, multiplying (3.9) by  $-\frac{\theta_{xx}}{e_\theta}$  and integrating the result identity with respect to  $x$  over  $\mathbb{R}$ , one has

$$\frac{1}{2} \frac{d}{dt} \|\theta_x(t)\|^2 + \int_{\mathbb{R}} \frac{\kappa \theta_{xx}^2}{ve_\theta} dx = \int_{\mathbb{R}} \left\{ \theta p_\theta u_x - \frac{\mu u_x^2}{v} - \frac{\kappa_v v_x + \kappa_\theta \theta_x}{v} \theta_x + \frac{\kappa \theta_x v_x}{v^2} \right\} \frac{\theta_{xx}}{e_\theta} dx.$$

Then

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\theta_x(t)\|^2 + \int_{\mathbb{R}} \frac{\kappa \theta_{xx}^2}{ve_\theta} dx \\ & \leq \epsilon \int_{\mathbb{R}} \frac{\kappa \theta_{xx}^2}{ve_\theta} dx + C(\epsilon) \int_{\mathbb{R}} \left( \frac{\mu u_x^2}{v\theta} \cdot \frac{v^2 \theta^3 p_\theta^2}{\mu \kappa e_\theta} + \frac{\mu^2 u_x^4}{\kappa v e_\theta} + \frac{\kappa_v^2 v_x^2 \theta_x^2}{ve_\theta \kappa} + \frac{\kappa_\theta^2 \theta_x^4}{v \kappa e_\theta} + \frac{\kappa \theta_x^2 v_x^2}{v^3 e_\theta} \right) dx \\ & \leq \epsilon \int_{\mathbb{R}} \frac{\kappa \theta_{xx}^2}{ve_\theta} dx + C(\epsilon) \left( V(t) + \int_{\mathbb{R}} u_x^4(t, x) dx + \|\theta_x(t)\| \|\theta_{xx}(t)\| \right) \\ & \leq 2\epsilon \int_{\mathbb{R}} \frac{\kappa \theta_{xx}^2}{ve_\theta} dx + C(\epsilon) \left( V(t) + \int_{\mathbb{R}} u_x^4(t, x) dx \right), \end{aligned} \tag{3.55}$$

where

$$V(t) = \int_{\mathbb{R}} \left( \frac{\mu u_x^2}{v\theta} + \frac{\kappa \theta_x^2}{v\theta^2} \right) (t, x) dx,$$

and we have used (2.2), (2.15), Lemma 3.5, Sobolev’s inequality and the following fact

$$\begin{aligned} \|\theta_x(t)\| \|\theta_{xx}(t)\| &= \left( \int_{\mathbb{R}} \frac{\kappa \theta_x^2}{v\theta^2} \cdot \frac{v\theta^2}{\kappa} dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \frac{\kappa \theta_{xx}^2}{ve_\theta} \cdot \frac{ve_\theta}{\kappa} dx \right)^{\frac{1}{2}} \\ &\lesssim V^{\frac{1}{2}}(t) \left( \int_{\mathbb{R}} \frac{\kappa \theta_{xx}^2}{ve_\theta} dx \right)^{\frac{1}{2}} \\ &\leq \epsilon \int_{\mathbb{R}} \frac{\kappa \theta_{xx}^2}{ve_\theta} dx + C(\epsilon) V(t). \end{aligned}$$

Finally, choosing  $\epsilon > 0$  small enough, integrating (3.55) with respect to  $t$  over  $(0, t)$  and using (2.2) as well as (3.49), we can obtain (3.51).  $\square$

As a result of Lemmas 2.1–3.6, we can obtain the following corollary immediately.

**Corollary 3.7** Under the assumptions listed in Theorem 1.1, there exists a positive constants  $C_3$ , such that

$$\|(v - 1, u, \theta - 1)(t)\|_1^2 + \int_0^t \left[ \|\sqrt{\theta} v_x(\tau)\|^2 + \|(\theta_x, u_x)(\tau)\|_1^2 \right] d\tau \leq C_3^2. \tag{3.56}$$

The next estimate is concerned with the local-in-time estimate on the lower bound on the absolute temperature  $\theta(t, x)$ . To this end, we can deduce by repeating the method used in [33] that

**Lemma 3.8** Under the assumptions stated in Theorem 1.1, for each  $0 \leq s \leq t \leq T$  and  $x \in \mathbb{R}$ , there exist a positive constant  $C_4$ , such that

$$\theta(t, x) \geq \frac{C_4 \inf_{x \in \mathbb{R}} \{\theta(s, x)\}}{1 + (t - s) \inf_{x \in \mathbb{R}} \{\theta(s, x)\}}. \tag{3.57}$$

### 4 Estimates of Second-order Derivatives

In the following sections, to simplify the presentation, we introduce  $A \lesssim_h B$  if  $A \leq C_h B$  holds uniformly for some constant  $C_h$ , depending only on  $\Pi_0, V_0$  and  $H(C_2)$  with  $C_2$  given in Lemma 3.5. The letter  $C(m_2)$  will be employed to denote some positive constant which depends only on  $m_2, \Pi_0, V_0$  and  $H(C_2)$ . We note from (1.9) and (3.48) that

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |(h(v(t,x)), h'(v(t,x)), h''(v(t,x)), h'''(v(t,x)))| \leq H(C_2). \tag{4.1}$$

We estimate the second-order derivatives of  $(u(t,x), \theta(t,x))$  with respect to the space variable  $x$  in the next lemma.

**Lemma 4.1** Under the assumptions listed in Theorem 1.1, for any  $0 \leq t \leq T$ , we have

$$\begin{aligned} & \| (u_{xx}, \theta_{xx}) (t) \|^2 + \int_0^t \| (u_{xxx}, \theta_{xxx}) (\tau) \|^2 d\tau \\ & \lesssim_h C(m_2) + \int_0^t \| v_{xx} (\tau) \|^2 d\tau + \sup_{0 \leq t \leq T} \{ \| v_{xx} (t) \|^2 \}. \end{aligned} \tag{4.2}$$

**Proof** First, differentiating (1.1)<sub>2</sub> with respect to  $x$ , and multiplying the resulting identity by  $u_{xxx}$ , we have

$$\left[ \frac{1}{2} u_{xx}^2 \right]_t - [u_{xt} u_{xx}]_x + \frac{\mu u_{xxx}^2}{v} = p_{xx} u_{xxx} + \left[ \frac{\mu u_{xxx}}{v} - \left( \frac{\mu u_x}{v} \right)_{xx} \right] u_{xxx}.$$

Integrating the above identity over  $[0, t] \times \mathbb{R}$  and by using (2.4), (3.48) and Cauchy's inequality, we can deduce that

$$\| u_{xx} (t) \|^2 + \int_0^t \int_{\mathbb{R}} u_{xxx}^2 (\tau, x) dx d\tau \lesssim 1 + \int_0^t \int_{\mathbb{R}} \left( |p_{xx}|^2 + \left| \frac{\mu u_{xxx}}{v} - \left( \frac{\mu u_x}{v} \right)_{xx} \right|^2 \right) dx d\tau. \tag{4.3}$$

Since

$$p_{xx} = \frac{R\theta_{xx}}{v} - \frac{2R\theta_x v_x}{v^2} - \frac{R\theta v_{xx}}{v^2} + \frac{2R\theta v_x^2}{v^3} + 4a\theta^2 \theta_x^2 + \frac{4}{3} a\theta^3 \theta_{xx}, \tag{4.4}$$

we have

$$p_{xx}^2 \lesssim |(v_{xx}, \theta_{xx})|^2 + |(v_x, \theta_x)|^4.$$

From (2.1) and (3.56), we have

$$\int_0^t \| v_x (\tau) \|^2 d\tau \leq C(m_2),$$

which combined with (3.56) implies

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} |(v_x, u_x, \theta_x) (\tau, x)|^4 dx d\tau \lesssim \int_0^t \| (v_{xx}, u_{xx}, \theta_{xx}) (\tau) \| \| (v_x, u_x, \theta_x) (\tau) \|^3 d\tau \\ & \lesssim \sup_{0 \leq t \leq T} \{ \| (v_x, u_x, \theta_x) (t) \|^2 \} \int_0^t \| (v_x, u_x, \theta_x) (\tau) \|_1^2 d\tau \\ & \lesssim C(m_2) + \int_0^t \| v_{xx} (\tau) \|^2 d\tau. \end{aligned} \tag{4.5}$$

Consequently, we have

$$\int_0^t \int_{\mathbb{R}} p_{xx}^2 (\tau, x) dx d\tau \lesssim C(m_2) + \int_0^t \| v_{xx} (\tau) \|^2 d\tau. \tag{4.6}$$

To estimate the last term in (4.3), we first make some estimate of  $\theta^\alpha$ . It follows from (2.4) that

$$\begin{aligned} |(\theta^\alpha)_x| &\lesssim |\theta_x|, \\ |(\theta^\alpha)_{xx}| &\lesssim |\theta_{xx}| + \theta_x^2, \\ |(\theta^\alpha)_{xxx}| &\lesssim |\theta_{xxx}| + |\theta_{xx}|\theta_x^2 + |\theta_x|^3. \end{aligned} \tag{4.7}$$

Since  $v(t, x)$  is bounded, for general smooth function  $f(v)$ , we have

$$\begin{aligned} |f(v)_x| &\lesssim |f'(v)||v_x|, \\ |(f(v)_{xx})| &\lesssim |(f', f'')(v)| (|v_{xx}| + v_x^2), \\ |(f(v)_{xxx})| &\lesssim |(f', f'', f''')(v)| (|v_{xxx}| + |v_{xx}v_x^2 + |v_x^3|). \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we have

$$\begin{aligned} |(f(v)\theta^\alpha)_x| &\lesssim |(f, f')(v)||v_x, \theta_x|, \\ |(f(v)\theta^\alpha)_{xx}| &\lesssim |(f, f', f'')(v)| [|(v_x, \theta_{xx})| + |(v_x, \theta_x)|^2], \\ |(f(v)\theta^\alpha)_{xxx}| &\lesssim |(f, f', f'', f''')(v)| [|(v_{xxx}, \theta_{xxx})| \\ &\quad + |(v_{xx}, \theta_{xx})||v_x, \theta_x|^2 + |(v_x, \theta_x)|^3]. \end{aligned} \tag{4.9}$$

Taking  $f(v) = \frac{h(v)}{v}$ , we can combine the identity

$$\left(\frac{\mu u_x}{v}\right)_{xx} = \left(\frac{\mu}{v}\right)_{xx} u_x + 2\left(\frac{\mu}{v}\right)_x u_{xx} + \frac{\mu}{v} u_{xxx}$$

and the assumption (1.8) to conclude

$$\left|\left(\frac{\mu u_x}{v}\right)_{xx} - \frac{\mu}{v} u_{xxx}\right| \lesssim_h |(v_x, u_x, \theta_x)|^3 + |v_{xx}||u_x| + |(u_{xx}, \theta_{xx})||v_x, u_x, \theta_x|.$$

From this estimate, we derive

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \left|\left(\frac{\mu u_x}{v}\right)_{xx} - \frac{\mu}{v} u_{xxx}\right|^2 dx d\tau &\lesssim_h \int_0^t \int_{\mathbb{R}} |(v_x, u_x, \theta_x)|^6 dx d\tau + \int_0^t \int_{\mathbb{R}} u_x^2 v_{xx}^2 dx d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}} |(u_{xx}, \theta_{xx})|^2 |v_x, u_x, \theta_x|^2 dx d\tau. \end{aligned} \tag{4.10}$$

Employ Sobolev’s inequality and (3.56) to get

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} |(v_x, u_x, \theta_x)(\tau, x)|^6 dx d\tau &\lesssim \int_0^t \|(v_x, u_x, \theta_x)(\tau)\|_{L^\infty(\mathbb{R})}^4 \|(v_x, u_x, \theta_x)(\tau)\|^2 d\tau \\ &\lesssim \int_0^t \|(v_{xx}, u_{xx}, \theta_{xx})(\tau)\|^2 \|(v_x, u_x, \theta_x)(\tau)\|^4 d\tau \\ &\lesssim 1 + \int_0^t \|v_{xx}(\tau)\|^2 d\tau, \end{aligned} \tag{4.11}$$

$$\int_0^t \int_{\mathbb{R}} u_x^2 v_{xx}^2 dx d\tau \leq \sup_{0 \leq t \leq T} \{\|(v_{xx}(t)\|^2\} \int_0^t \|u_x(\tau)\|_1^2 d\tau \leq \sup_{0 \leq t \leq T} \{\|v_{xx}(t)\|^2\}, \tag{4.12}$$

and

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} |(u_{xx}, \theta_{xx})|^2 |v_x, u_x, \theta_x|^2 dx d\tau &\lesssim \int_0^t \|(u_{xx}, \theta_{xx})(\tau)\|_{L^\infty(\mathbb{R})}^2 \|(v_x, u_x, \theta_x)(\tau)\|^2 d\tau \\ &\lesssim \int_0^t \|(u_{xx}, \theta_{xx})(\tau)\| \|(u_{xxx}, \theta_{xxx})(\tau)\| d\tau \end{aligned}$$

$$\lesssim C(\delta) + \delta \int_0^t \|(u_{xxx}, \theta_{xxx})(\tau)\|^2 d\tau. \tag{4.13}$$

We plug (4.11), (4.12) and (4.13) into (4.10), and use (4.3)–(4.10) to deduce that

$$\begin{aligned} & \|u_{xx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} u_{xxx}^2(\tau, x) dx d\tau \\ & \lesssim hC(m_2) + \int_0^t \|v_{xx}(s)\|^2 ds + \sup_{0 \leq t \leq T} \{ \|v_{xx}(t)\|^2 \} + \delta \int_0^t \|(u_{xxx}, \theta_{xxx})(\tau)\|^2 d\tau. \end{aligned} \tag{4.14}$$

Next, We differentiate (3.9) with respect to  $x$  and multiply the result by  $\theta_{xxx}$  to find

$$\begin{aligned} & \frac{1}{2}(\theta_{xx}^2)_t - [\theta_{tx}\theta_{xxx}]_x + \frac{\kappa\theta_{xxx}^2}{ve_\theta} \\ & = \left(\frac{\theta p_\theta u_x}{e_\theta}\right)_x \theta_{xxx} - \left(\frac{\mu u_x^2}{ve_\theta}\right)_x \theta_{xxx} + \left[\frac{\kappa\theta_{xxx}}{ve_\theta} - \left(\frac{1}{e_\theta} \left(\frac{\kappa\theta_x}{v}\right)_x\right)_x\right] \theta_{xxx}. \end{aligned}$$

Integrating the above identity over  $[0, t] \times \mathbb{R}$ , we obtain from Cauchy’s inequality, (2.4) and (3.48) that

$$\begin{aligned} & \|\theta_{xx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta_{xxx}^2(\tau, x) dx d\tau \\ & \lesssim 1 + \int_0^t \int_{\mathbb{R}} \left| \left(\frac{\theta p_\theta u_x}{e_\theta}\right)_x \right|^2 dx d\tau + \int_0^t \int_{\mathbb{R}} \left| \left(\frac{\mu u_x^2}{ve_\theta}\right)_x \right|^2 dx d\tau \\ & \quad + \int_0^t \int_{\mathbb{R}} \left| \frac{\kappa\theta_{xxx}}{ve_\theta} - \left(\frac{1}{e_\theta} \left(\frac{\kappa\theta_x}{v}\right)_x\right)_x \right|^2 dx d\tau. \end{aligned} \tag{4.15}$$

(2.15) together with Lemma 3.5 yield

$$\begin{aligned} & \left| \left(\frac{\theta p_\theta u_x}{e_\theta}\right)_x \right| \lesssim |(v_x, \theta_x)| |u_x| + |u_{xx}|, \\ & \left| \left(\frac{\mu u_x^2}{ve_\theta}\right)_x \right| \lesssim h|(v_x, \theta_x)| u_x^2 + |u_x u_{xx}|, \\ & \left| \frac{\kappa\theta_{xxx}}{ve_\theta} - \left(\frac{1}{e_\theta} \left(\frac{\kappa\theta_x}{v}\right)_x\right)_x \right| \lesssim |\theta_x| |(v_x^2, v_x \theta_x, \theta_x^2, v_{xx})| + |\theta_{xx}| |(v_x, \theta_x)|, \end{aligned}$$

which implies

$$\begin{aligned} & \left| \left(\frac{\theta p_\theta u_x}{e_\theta}\right)_x \right|^2 \lesssim |(v_x, u_x, \theta_x)|^4 + u_{xx}^2, \\ & \left| \left(\frac{\mu u_x^2}{ve_\theta}\right)_x \right|^2 \lesssim h|(v_x^2, \theta_x^2)| u_x^4 + |u_x^2 u_{xx}^2|, \\ & \left| \frac{\kappa\theta_{xxx}}{ve_\theta} - \left(\frac{1}{e_\theta} \left(\frac{\kappa\theta_x}{v}\right)_x\right)_x \right|^2 \lesssim |\theta_x^2| |(v_x^4, v_x^2 \theta_x^2, \theta_x^4, v_{xx}^2)| + |\theta_{xx}^2| |(v_x^2, \theta_x^2)|. \end{aligned}$$

Then by employing (3.56), we obtain

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} \left| \left(\frac{\theta p_\theta u_x}{e_\theta}\right)_x \right|^2 dx d\tau \lesssim \int_0^t \int_{\mathbb{R}} (|(v_x, u_x, \theta_x)|^4 + u_{xx}^2) dx d\tau \\ & \quad \lesssim C(m_2) + \int_0^t \|v_{xx}(\tau)\|^2 d\tau, \\ & \int_0^t \int_{\mathbb{R}} \left| \left(\frac{\mu u_x^2}{ve_\theta}\right)_x \right|^2 dx d\tau \lesssim h \int_0^t \int_{\mathbb{R}} (|(v_x^2, \theta_x^2)| u_x^4 + |u_x^2 u_{xx}^2|) dx d\tau \end{aligned} \tag{4.16}$$

$$\lesssim hC(\delta) + \int_0^t \|v_{xx}(\tau)\|^2 d\tau + \delta \int_0^t \|u_{xxx}(\tau)\|^2 d\tau, \tag{4.17}$$

and

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} \left| \frac{\kappa \theta_{xxx}}{ve_\theta} - \left( \frac{1}{e_\theta} \left( \frac{\kappa \theta_x}{v} \right)_x \right)_x \right|^2 dx d\tau \\ & \lesssim \int_0^t \int_{\mathbb{R}} (|\theta_x^2| (|v_x^4|, |v_x^2 \theta_x^2|, |\theta_x^4|, |v_{xx}^2|) + |\theta_{xx}^2| (|v_x^2|, |\theta_x^2|)) dx d\tau \\ & \lesssim C(\delta) + \sup_{0 \leq t \leq T} \{ \|v_{xx}(t)\|^2 \} + \delta \int_0^t \|\theta_{xxx}(\tau)\|^2 d\tau. \end{aligned} \tag{4.18}$$

Putting (4.15)–(4.18) together, we arrive at

$$\begin{aligned} & \|\theta_{xx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta_{xxx}^2(\tau, x) dx d\tau \\ & \lesssim hC(m_2) + \int_0^t \|v_{xx}(\tau)\|^2 d\tau + \delta \int_0^t \|(u_{xxx}, \theta_{xxx})(\tau)\|^2 d\tau. \end{aligned} \tag{4.19}$$

Combining (4.14) and (4.19) and taking  $\delta > 0$  small enough, we can obtain (4.2).  $\square$

We next obtain a  $m_2$ -dependent bound for the second-order derivatives with respect to  $x$  of the solution  $(v(t, x), u(t, x), \theta(t, x))$ .

**Lemma 4.2** Under the assumptions listed in Theorem 1.1, for any  $0 \leq t \leq T$ , we have

$$\|(v_{xx}, u_{xx}, \theta_{xx})(t)\|^2 + \int_0^t \|(v_{xx}, u_{xxx}, \theta_{xxx})(\tau)\|^2 d\tau \leq C(m_2). \tag{4.20}$$

**Proof** Differentiate (2.7) with respect to  $x$  and multiply the result by  $(\frac{\mu v_x}{v})_x$  to find

$$\begin{aligned} & \left[ \frac{1}{2} \left( \frac{\mu v_x}{v} \right)_x^2 \right]_t - \left[ u_x \left( \frac{\mu v_x}{v} \right)_x \right]_t + \left[ u_x \left( \frac{\mu v_x}{v} \right)_t \right]_x \\ & = u_{xx} \left( \frac{\mu v_x}{v} \right)_t + p_{xx} \left( \frac{\mu v_x}{v} \right)_x + \left( \frac{\mu v_x}{v} \right)_x \left[ \frac{\mu \theta}{v} (v_x \theta_t - \theta_x u_x) \right]_x. \end{aligned}$$

We integrate the above identity over  $[0, t] \times \mathbb{R}$ , and Cauchy’s inequality to obtain

$$\begin{aligned} \left\| \left( \frac{\mu v_x}{v} \right)_x(t) \right\|^2 & \lesssim h1 + \int_0^t \int_{\mathbb{R}} \left\{ u_{xx} \left( \frac{\mu v_x}{v} \right)_t + p_{xx} \left( \frac{\mu v_x}{v} \right)_x \right. \\ & \quad \left. + \left( \frac{\mu v_x}{v} \right)_x \left[ \frac{\mu \theta}{v} (v_x \theta_t - \theta_x u_x) \right]_x \right\} dx d\tau. \end{aligned} \tag{4.21}$$

On the other hand, one can deduce from (2.14) that

$$|\theta_t| \lesssim u_x^2 + |u_x| + |v_x \theta_x| + |\theta_x^2| + |\theta_{xx}|. \tag{4.22}$$

Thus combining (2.4), (4.22) and the fact

$$\left( \frac{\mu v_x}{v} \right)_t = \frac{v \mu_v - \mu}{v^2} v_x u_x + \frac{\mu \theta \theta_t v_x}{v} + \frac{\mu u_{xx}}{v},$$

we have

$$\left| \left( \frac{\mu v_x}{v} \right)_t \right| \lesssim_h |v_x u_x| + Nm_2^{-1} |\alpha| |v_x| + |u_{xx}|.$$

We then deduce from Cauchy’s inequality and (3.56) that

$$\int_0^t \int_{\mathbb{R}} u_{xx} \left( \frac{\mu v_x}{v} \right)_t dx d\tau \leq C(m_2). \tag{4.23}$$

In view of (4.4) and

$$\left(\frac{\mu v_x}{v}\right)_x = \frac{\mu}{v} v_{xx} + \left(\frac{\mu}{v}\right)_x v_x, \tag{4.24}$$

we have

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} p_{xx} \left(\frac{\mu v_x}{v}\right)_x dx d\tau \\ & \leq - \int_0^t \int_{\mathbb{R}} \frac{\mu \theta v_{xx}^2}{2v^3} dx d\tau + C(m_2) \int_0^t \int_{\mathbb{R}} [\theta_{xx}^2 + |(\theta_x, v_x)|^4] dx d\tau. \end{aligned} \tag{4.25}$$

With the aid of Sobolev’s inequality, we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} |(\theta_x, v_x)|^4 dx d\tau & \lesssim \int_0^t \|(\theta_x, v_x)(\tau)\|^2 \|(\theta_x, v_x)(\tau)\|_1^2 d\tau \\ & \lesssim C(m_2) + \int_0^t \|(\theta_x, v_x)(\tau)\|^2 \|v_{xx}(\tau)\|^2 d\tau. \end{aligned} \tag{4.26}$$

Combining (4.25)–(4.26), we obtain

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} p_{xx} \left(\frac{\mu v_x}{v}\right)_x dx d\tau \\ & \leq - \int_0^t \int_{\mathbb{R}} \frac{\mu \theta v_{xx}^2}{2v^3} dx d\tau + C(m_2) + C(m_2) \int_0^t \|(\theta_x, v_x)(\tau)\|^2 \|v_{xx}(\tau)\|^2 d\tau. \end{aligned} \tag{4.27}$$

For the last term in (4.21), we use (4.24),

$$\begin{aligned} \left[\frac{\mu \theta}{v} (v_x \theta_t - \theta_x u_x)\right]_x & = \frac{\mu \theta}{v} (v_{xx} \theta_t + v_x \theta_{xt} - \theta_{xx} u_x - \theta_x u_{xx}) \\ & \quad + \left[\frac{\mu \theta \theta_x + \mu \theta_v v_x}{v} - \frac{\mu \theta v_x}{v^2} (v_x \theta_t - \theta_x u_x)\right], \end{aligned}$$

and (2.4) to derive

$$\begin{aligned} \left|\left(\frac{\mu v_x}{v}\right)_x\right| & \lesssim h |v_{xx}| + |(v_x, \theta_x)|^2, \\ \left[\frac{\mu \theta}{v} (v_x \theta_t - \theta_x u_x)\right]_x & \lesssim h |\alpha| (|\theta_t v_{xx}| + |\theta_{xt} v_x| + |\theta_{xx} u_x| + |\theta_x u_{xx}|) \\ & \quad + |\alpha| |(v_x, \theta_x)| (|v_x \theta_t| + |\theta_x u_x|). \end{aligned}$$

It follows from the identity

$$\theta_{tx} = \left(\frac{\mu u_x^2}{v e_\theta}\right)_x - \left(\frac{\theta p_\theta u_x}{e_\theta}\right)_x + \left(\frac{\kappa_v v_x \theta_x + \kappa_\theta \theta_x^2}{v e_\theta}\right)_x - \left(\frac{\kappa \theta_{xx}}{v e_\theta}\right)_x - \left(\frac{\kappa \theta_x v_x}{v^2 e_\theta}\right)_x$$

and (3.47)–(3.48) that

$$|\theta_{tx}| \lesssim |\theta_{xx}| + |(v_x, \theta_x)| |\theta_{xx}| + (1 + |u_x|) |u_{xx}| + |\theta_x v_{xx}| + (1 + |(u_x, v_x, \theta_x)|) |(u_x, v_x, \theta_x)|^2.$$

Hence applying Cauchy’s inequality yields

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} \left|\left(\frac{\mu v_x}{v}\right)_x \left[\frac{\mu \theta}{v} (v_x \theta_t - \theta_x u_x)\right]_x\right| dx d\tau \\ & \lesssim h \left(1 + C(\epsilon) \alpha^2 \|(\theta_t, \theta_x v_x)\|_\infty^2\right) \int_0^t \int_{\mathbb{R}} v_{xx}^2 dx d\tau + C(\epsilon) \alpha^2 \|v_x\|_\infty^2 \int_0^t \int_{\mathbb{R}} \theta_{xxx}^2 dx d\tau \\ & \quad + C(\epsilon) \alpha^2 \int_0^t \int_{\mathbb{R}} |(v_x, \theta_x)|^2 |(v_x \theta_t, \theta_x u_x)|^2 dx d\tau + C(\epsilon) \int_0^t \int_{\mathbb{R}} |(v_x, \theta_x)|^4 dx d\tau \end{aligned}$$

$$\begin{aligned}
 &+C(\epsilon)\alpha^2 \int_0^t \int_{\mathbb{R}} [|(v_x, u_x, \theta_x)|^4 \theta_{xx}^2 + (1 + |(u_x, v_x, \theta_x)|^2) |(u_x, v_x, \theta_x)|^6] dx d\tau \\
 &+C(\epsilon)\alpha^2 \int_0^t \int_{\mathbb{R}} [(1 + |u_x|^2) u_{xx}^2 v_x^2 + \theta_x^2 u_{xx}^2] dx d\tau \\
 &\lesssim h\epsilon \int_0^t \int_{\mathbb{R}} v_{xx}^2 dx d\tau + C(\epsilon, m_2).
 \end{aligned} \tag{4.28}$$

Here we have used (2.4) and

$$\int_0^t \int_{\mathbb{R}} \theta_{xxx}^2 dx d\tau \lesssim N^2.$$

Combining (4.21), (4.23), (4.27) and (4.28), we obtain

$$\|v_{xx}(t)\|^2 + \int_0^t \|v_{xx}(\tau)\|^2 d\tau \lesssim hC(m_2) + \int_0^t \|(v_x, \theta_x)(\tau)\|^2 \|v_{xx}(\tau)\|^2 d\tau.$$

We apply Gronwall’s inequality to the above estimate to obtain

$$\|v_{xx}(t)\|^2 + \int_0^t \|v_{xx}(\tau)\|^2 d\tau \lesssim C(m_2),$$

which combined with (4.2) implies (4.20). □

### 5 Estimates of Third-order Derivatives

Estimates on the third-order derivatives of  $(v(t, x), u(t, x), \theta(t, x))$  with respect to  $x$  will be proved in this subsection. We first give an estimate on the third-order derivatives of  $u(t, x)$  and  $\theta(t, x)$  with respect to  $x$  in the following lemma.

**Lemma 5.1** Under the assumptions listed in Theorem 1.1, for any  $0 \leq t \leq T$ , we have

$$\begin{aligned}
 &\|(u_{xxx}, \theta_{xxx})(t)\|^2 + \int_0^t \|(u_{xxx}, \theta_{xxx})(\tau)\|^2 d\tau \\
 &\lesssim hC(m_2) + \int_0^t \|v_{xxx}(\tau)\|^2 d\tau + \sup_{0 \leq t \leq T} \left\{ \|v_{xxx}(t)\|^2 \right\}.
 \end{aligned} \tag{5.1}$$

**Proof** Firstly, differentiating (1.1)<sub>2</sub> with respect to  $x$  twice and multiplying the resulting identity by  $u_{xxxx}$ , we have

$$\left[ \frac{1}{2} u_{xxx}^2 \right]_t - [u_{xxt} u_{xxx}]_x + \frac{\mu u_{xxxx}^2}{v} = P_{xxx} u_{xxxx} + \left[ \frac{\mu u_{xxxx}}{v} - \left( \frac{\mu u_x}{v} \right)_{xxx} \right] u_{xxxx}.$$

Integrating the above identity over  $[0, t] \times \mathbb{R}$  and using Cauchy’s inequality, we obtain

$$\|u_{xxx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} u_{xxxx}^2(\tau, x) dx d\tau \lesssim 1 + \int_0^t \int_{\mathbb{R}} \left( |p_{xxx}|^2 + \left| \frac{\mu u_{xxxx}}{v} - \left( \frac{\mu u_x}{v} \right)_{xxx} \right|^2 \right) dx d\tau. \tag{5.2}$$

We compute from (4.4) that

$$\begin{aligned}
 p_{xxx} = &\frac{R\theta_{xxx}}{v} - \frac{3R\theta_{xx}v_x}{v^2} - \frac{3R\theta_x v_{xx}}{v^2} + \frac{6R\theta_x v_x^2}{v^3} - \frac{R\theta v_{xxx}}{v^2} + \frac{6R\theta v_x v_{xx}}{v^3} \\
 &- \frac{6R\theta v_x^3}{v^4} + 8a\theta\theta_x^3 + 12a\theta^2\theta_x\theta_{xx} + \frac{4}{3}a\theta^3\theta_{xxx},
 \end{aligned}$$

hence we have

$$p_{xxx}^2 \lesssim |(v_{xxx}, \theta_{xxx})|^2 + |(v_x, \theta_x)|^6 + |(v_x, \theta_x)|^2 |(v_{xx}, \theta_{xx})|^2 + \theta_x^2 v_x^4,$$

and it follows from (4.11)–(4.13), (4.20) that

$$\int_0^t \int_{\mathbb{R}} p_{xxx}^2 dx d\tau \lesssim C(m_2) + \int_0^t \|v_{xxx}(\tau)\|^2 d\tau. \tag{5.3}$$

From (4.9) with  $f(v) = \frac{h(v)}{v}$  and

$$\left(\frac{\mu u_x}{v}\right)_{xxx} = \left(\frac{\mu}{v}\right)_{xxx} u_x + 3\left(\frac{\mu}{v}\right)_{xx} u_{xx} + 3\left(\frac{\mu}{v}\right)_x u_{xxx} + \frac{\mu}{v} u_{xxxx},$$

we have

$$\begin{aligned} \left| \left(\frac{\mu u_x}{v}\right)_{xxx} - \frac{\mu}{v} u_{xxxx} \right| &\lesssim h|(v_x, u_x, \theta_x)|^4 + |(v_x, u_x, \theta_x)|^2 |(v_{xx}, u_{xx}, \theta_{xx})| \\ &\quad + |v_{xxx}| |u_x| + |(u_{xxx}, \theta_{xxx})| |(v_x, u_x, \theta_x)| + |u_{xx}| |(v_{xx}, \theta_{xx})|. \end{aligned}$$

In view of (3.56) and (4.20), we deduce that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \left| \left(\frac{\mu u_x}{v}\right)_{xxx} - \frac{\mu}{v} u_{xxxx} \right|^2 dx d\tau \\ &\lesssim hC(\delta, m_2) + \sup_{0 \leq t \leq T} \{ \|v_{xxx}\|^2 \} + \delta \int_0^t \|(u_{xxxx}, \theta_{xxxx})(\tau)\|^2 d\tau. \end{aligned} \tag{5.4}$$

Combining (5.2)–(5.4), we get

$$\begin{aligned} &\|u_{xxx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} u_{xxxx}^2(\tau, x) dx d\tau \\ &\lesssim hC(\delta, m_2) + \int_0^t \|v_{xxx}(\tau)\|^2 d\tau + \sup_{0 \leq t \leq T} \{ \|v_{xxx}(t)\|^2 \} + \delta \int_0^t \|(u_{xxxx}, \theta_{xxxx})(\tau)\|^2 d\tau. \end{aligned} \tag{5.5}$$

Next, we differentiate (3.9) with respect to  $x$  twice and multiply the result by  $\theta_{xxxx}$  to obtain

$$\begin{aligned} &\frac{1}{2} (\theta_{xxx}^2)_t - [\theta_{txx} \theta_{xxxx}]_x + \frac{\kappa \theta_{xxxx}^2}{v e_\theta} \\ &= \left(\frac{\theta p_\theta u_x}{e_\theta}\right)_{xx} \theta_{xxxx} - \left(\frac{\mu u_x^2}{v e_\theta}\right)_{xx} \theta_{xxxx} + \left[ \frac{\kappa \theta_{xxxx}}{v e_\theta} - \left(\frac{1}{e_\theta} \left(\frac{\kappa \theta_x}{v}\right)_x\right)_{xx} \right] \theta_{xxxx}. \end{aligned}$$

Integrating the above identity over  $[0, t] \times \mathbb{R}$ , we obtain from Cauchy’s inequality that

$$\begin{aligned} &\|\theta_{xxx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta_{xxxx}^2(\tau, x) dx d\tau \\ &\lesssim 1 + \int_0^t \int_{\mathbb{R}} \left| \left(\frac{\theta p_\theta u_x}{e_\theta}\right)_{xx} \right|^2 dx d\tau + \int_0^t \int_{\mathbb{R}} \left| \left(\frac{\mu u_x^2}{v e_\theta}\right)_{xx} \right|^2 dx d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}} \left| \frac{\kappa \theta_{xxxx}}{v e_\theta} - \left(\frac{1}{e_\theta} \left(\frac{\kappa \theta_x}{v}\right)_x\right)_{xx} \right|^2 dx d\tau. \end{aligned} \tag{5.6}$$

It follows from (2.15) and Lemma 3.5 that

$$\begin{aligned} \left| \left(\frac{\theta p_\theta u_x}{e_\theta}\right)_{xx} \right| &\lesssim |(\theta_{xx}, v_x \theta_x, \theta_x^2, v_x^2, v_{xx})| |u_x| + |u_{xx}| |(v_x, \theta_x)| + |u_{xxx}|, \\ \left| \left(\frac{\mu u_x^2}{v e_\theta}\right)_{xx} \right| &\lesssim h|(v_{xx}, \theta_{xx}, v_x^2, \theta_x^2, v_x \theta_x)| u_x^2 + |u_x u_{xx}| |(v_x, \theta_x)| + u_{xx}^2 + |u_x u_{xxx}|, \\ \left| \frac{\kappa \theta_{xxxx}}{v e_\theta} - \left(\frac{1}{e_\theta} \left(\frac{\kappa \theta_x}{v}\right)_x\right)_{xx} \right| &\lesssim |v_x| |(v_x^2 \theta_x, \theta_x^3)| + |v_x|^2 |(\theta_x^2, \theta_{xx})| + |v_{xx}| |(\theta_x^2, v_x \theta_x, \theta_{xx})| \\ &\quad + |v_{xxx} \theta_x| + |\theta_{xx}| |(v_x \theta_x, \theta_x^2)| + |\theta_{xxx}| |(v_x, \theta_x)| + \theta_x^4 + \theta_{xx}^2, \end{aligned}$$

which implies

$$\begin{aligned} \left| \left( \frac{\theta p_\theta u_x}{e_\theta} \right)_{xx} \right|^2 &\lesssim |(\theta_{xx}^2, v_x^2 \theta_x^2, \theta_x^4, v_x^4, v_{xx}^2)| |u_x^2| + |u_{xx}^2| |(v_x^2, \theta_x^2)| + |u_{xxx}^2|, \\ \left| \left( \frac{\mu u_x^2}{v e_\theta} \right)_{xx} \right|^2 &\lesssim h |(v_{xx}^2, \theta_{xx}^2, v_x^4, \theta_x^4, v_x^2 \theta_x^2)| u_x^4 + |u_x^2 u_{xx}^2| |(v_x^2, \theta_x^2)| + u_{xx}^4 + |u_x^2 u_{xxx}^2|, \\ \left| \frac{\kappa \theta_{xxx}}{v e_\theta} - \left( \frac{1}{e_\theta} \left( \frac{\kappa \theta_x}{v} \right)_x \right)_{xx} \right|^2 &\lesssim v_x^2 |(v_x^4 \theta_x^2, \theta_x^6)| + |v_x|^4 |(\theta_x^4, \theta_{xx}^2)| + v_{xx}^2 |(\theta_x^4, v_x^2 \theta_x^2, \theta_{xx}^2)| \\ &\quad + v_{xxx}^2 \theta_x^2 + \theta_{xx}^2 |(v_x^2 \theta_x^2, \theta_x^4)| + \theta_{xxx}^2 |(v_x^2, \theta_x^2)| + \theta_x^8 + \theta_{xx}^4. \end{aligned}$$

Then one can infer from (3.56) and (4.20) that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \left| \left( \frac{\theta p_\theta u_{xx}}{e_\theta} \right)_x \right|^2 dx d\tau \\ &\lesssim \int_0^t \int_{\mathbb{R}} (|(\theta_{xx}^2, v_x^2 \theta_x^2, \theta_x^4, v_x^4, v_{xx}^2)| |u_x^2| + |u_{xx}^2| |(v_x^2, \theta_x^2)| + |u_{xxx}^2|) dx d\tau \\ &\lesssim C(m_2), \end{aligned} \tag{5.7}$$

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \left| \left( \frac{\mu u_x^2}{v e_\theta} \right)_{xx} \right|^2 dx d\tau \\ &\lesssim h \int_0^t \int_{\mathbb{R}} (|(v_{xx}^2, \theta_{xx}^2, v_x^4, \theta_x^4, v_x^2 \theta_x^2)| u_x^4 + |u_x^2 u_{xx}^2| |(v_x^2, \theta_x^2)| + u_{xx}^4 + |u_x^2 u_{xxx}^2|) dx d\tau \\ &\lesssim C(m_2), \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \left| \frac{\kappa \theta_{xxx}}{v e_\theta} - \left( \frac{1}{e_\theta} \left( \frac{\kappa \theta_x}{v} \right)_x \right)_{xx} \right|^2 dx d\tau \\ &\lesssim \int_0^t \int_{\mathbb{R}} \left( v_x^2 |(v_x^4 \theta_x^2, \theta_x^6)| + |v_x|^4 |(\theta_x^4, \theta_{xx}^2)| + v_{xx}^2 |(\theta_x^4, v_x^2 \theta_x^2, \theta_{xx}^2)| \right. \\ &\quad \left. + v_{xxx}^2 \theta_x^2 + \theta_{xx}^2 |(v_x^2 \theta_x^2, \theta_x^4)| + \theta_{xxx}^2 |(v_x^2, \theta_x^2)| + \theta_x^8 + \theta_{xx}^4 \right) dx d\tau \\ &\lesssim C(m_2, \delta) + \int_0^t \|v_{xxx}(\tau)\|^2 d\tau + \delta \int_0^t \|\theta_{xxx}(\tau)\|^2 d\tau. \end{aligned} \tag{5.9}$$

Putting (5.6)–(5.9) together, we obtain

$$\|\theta_{xxx}(t)\|^2 + \int_0^t \int_{\mathbb{R}} \theta_{xxx}^2(\tau, x) dx d\tau \lesssim_h C(m_2, \delta) + \int_0^t \|v_{xxx}(\tau)\|^2 d\tau + \delta \int_0^t \|\theta_{xxx}(\tau)\|^2 d\tau. \tag{5.10}$$

Combining (5.5) and (5.10) and taking  $\delta > 0$  small enough, we can obtain (5.1).  $\square$

By using (2.4) and Gronwall’s inequality, we can deduce the  $m_2$ -dependent bound for the third-order derivatives of  $(v(t, x), u(t, x), \theta(t, x))$  with respect to  $x$ . The proof is similar to that of Lemma 4.2 and hence we omit the details for brevity.

**Lemma 5.2** Under the assumptions listed in Theorem 1.1, for any  $0 \leq t \leq T$ , we have

$$\|(v_{xxx}, u_{xxx}, \theta_{xxx})(t)\|^2 + \int_0^t \|(v_{xxx}, u_{xxx}, \theta_{xxx})(\tau)\|^2 d\tau \leq C(m_2). \tag{5.11}$$

By virtue of Lemma 2.1–Lemma 5.2, we can get the following corollary.

**Corollary 5.3** Under the assumptions listed in Theorem 1.1, there exists a positive constants  $C(m_2) > 0$  which depends only on  $m_2, \Pi_0, V_0$  and  $H(C_2)$  with  $C_2$  being given in Lemma 3.5, such that for all  $t \in [0, T]$ ,

$$\|(v - 1, u, \theta - 1)(t)\|_3^2 + \int_0^t \left[ \|v_x(\tau)\|_2^2 + \|(u_x, \theta_x)(\tau)\|_3^2 \right] d\tau \leq C(m_2). \quad (5.12)$$

With Corollary 5.3 in hand, Theorem 1.1 follows by the combing the well-established local existence of solution  $(v(t, x), u(t, x), \theta(t, x))$  of the Cauchy problem (1.1)–(1.3) with constitutive relations (1.6), (1.7) and (1.8) which is without vacuum, mass concentrations, or vanishing temperatures, cf. [23], and the continuation argument designed in [52] and we omit the details for brevity.

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