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Large-time behavior of solutions to the equations of a viscous heat-conducting flow with shear viscosity in unbounded domains

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ABSTRACT

We consider the initial and initial–boundary value problems for a viscous heat-conducting flow with shear viscosity in unbounded domains with general large initial data. We prove that the temperature is bounded from below and above uniformly in time and space and that the global solution is asymptotically stable as the time tends to infinity. Our approach relies upon the energy-estimate method.

1. Introduction

1.1. The Eulerian description

We study the initial and initial–boundary value problems for the equations describing a viscous compressible heat-conducting flow between two horizontal plates:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 + P\right)_x = (\mu u_x)_x, \\ (\rho \boldsymbol{w})_t + (\rho u \boldsymbol{w})_x = (\lambda \boldsymbol{w}_x)_x, \\ (\rho E)_t + (\rho u E + P u)_x = (\kappa \theta_x + \mu u u_x + \lambda \boldsymbol{w} \cdot \boldsymbol{w}_x)_x, \end{cases} \tag{1.1}$$

where $t > 0$ and $x \in \mathbb{R}$ are the time variable and the spatial variable, respectively, and the primary dependent variables are the fluid density ρ, the longitudinal velocity $u \in \mathbb{R}$, the transverse velocity $\boldsymbol{w} \in \mathbb{R}^2$, and the temperature θ. The specific total energy is $E = e + \frac{1}{2}u^2 + \frac{1}{2}|\boldsymbol{w}|^2$ with e being the specific internal energy.

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The pressure P and the specific initial energy e are related with the density and temperature by means of constitutive relations

$$(P, e) = (P(\rho, \theta), e(\rho, \theta)).$$

The viscosity coefficient μ, the shear viscosity coefficient λ and the heat conductivity κ may depend on both ρ and θ generally. One can easily derive (1.1) from the three-dimensional compressible Navier–Stokes equations for a Newtonian fluid (see Wang [15]).

In this article, we focus on the ideal polytropic gases, which are identified by the constitutive relations

$$P = R\rho\theta, \quad e = c_v\theta, \tag{1.2}$$

where R is the gas constant and c_v is the specific heat at constant volume. And we assume that μ, λ, κ and c_v are positive constants.

The system (1.1) is supplemented with the initial data

$$(\rho(0, x), u(0, x), \boldsymbol{w}(0, x), \theta(0, x)) = (\rho_0(x), u_0(x), \boldsymbol{w}_0(x), \theta_0(x)) \quad \text{for } x \in \mathbb{R}, \tag{1.3}$$

and one type of the following far-field and boundary conditions:

1) far-field conditions for $\Omega = \mathbb{R}$,

$$\lim_{x\to\pm\infty}(\rho_0(x), u_0(x), \boldsymbol{w}_0(x), \theta_0(x)) = (1, 0, 0, 1); \tag{1.4}$$

2) boundary and far-field conditions for $\Omega = (0, \infty)$,

$$(u(t, 0), \boldsymbol{w}(t, 0), \theta(t, 0)) = (0, 0, 1), \quad \lim_{x\to\infty}(\rho_0(x), u_0(x), \boldsymbol{w}_0(x), \theta_0(x)) = (1, 0, 0, 1); \tag{1.5}$$

3) boundary and far-field conditions for $\Omega = (0, \infty)$,

$$(u(t, 0), \boldsymbol{w}(t, 0), \theta_x(t, 0)) = (0, 0, 0), \quad \lim_{x\to\infty}(\rho_0(x), u_0(x), \boldsymbol{w}_0(x), \theta_0(x)) = (1, 0, 0, 1). \tag{1.6}$$

Let us mention some results on the system (1.1) without any restrictions on the smallness of the initial data. Wang [15] investigated the existence, uniqueness and regularity of global solutions to system (1.1) in bounded domains under certain assumptions on the constitutive relations. Qin and Hu [14] improved and extended the results in [15] for more general constitutive relations. We note here that the case of ideal polytropic gases (1.2) is not included in the class of fluids established in [15] and [14].

In this paper we shall show the global existence and uniqueness of solutions to the Cauchy problem (1.1)–(1.4) and the initial–boundary value problems (1.1)–(1.3), (1.5) and (1.1)–(1.3), (1.6) for general initial data. To investigate the global solvability of the system (1.1), some difficulties arise for the case of unbounded domains, where the imbedding $L^2 \hookrightarrow L^1$ is not valid any longer. We will introduce the Lagrangian variable and transform the problems (1.1)–(1.4), (1.1)–(1.3), (1.5), and (1.1)–(1.3), (1.6) in the Eulerian coordinates into corresponding problems in the Lagrangian coordinates. The local existence and uniqueness of solutions can be shown by using the Banach theorem and the contractivity of the operator defined by the linearization of the problems on a small time-interval. The global solvability is proved by using the continuation argument to extend the local solutions globally in time based on the global a priori estimates of solutions.

1.2. The main result

We transform the problems (1.1)–(1.4), (1.1)–(1.3), (1.5), and (1.1)–(1.3), (1.6) into Lagrangian variables. To this end, we introduce the Lagrangian variable

$$y = \int_{x(t)}^{x} \rho(t,z)dz,$$

where $x(t)$ is a well-defined particle path satisfying $x'(t) = u(t, x(t))$. The Lagrangian version of the system (1.1) can be written as

$$\begin{cases} v_t - u_y = 0, \\ u_t + P_y = \left(\mu \dfrac{u_y}{v}\right)_y, \\ \boldsymbol{w}_t = \left(\lambda \dfrac{\boldsymbol{w}_y}{v}\right)_y, \\ E_t + (Pu)_y = \left(\kappa \dfrac{\theta_y}{v} + \mu \dfrac{u u_y}{v} + \lambda \dfrac{\boldsymbol{w} \cdot \boldsymbol{w}_y}{v}\right)_y, \end{cases} \tag{1.7}$$

where $v = 1/\rho$ is the specific volume and

$$P = R\frac{\theta}{v}, \quad E = c_v \theta + \frac{1}{2}u^2 + \frac{1}{2}|\boldsymbol{w}|^2. \tag{1.8}$$

The initial and boundary conditions (1.3)–(1.6) can be translated into similar conditions:

$$(v(0,y), u(0,y), \boldsymbol{w}(0,y), \theta(0,y)) = (v_0(y), u_0(y), \boldsymbol{w}_0(y), \theta_0(y)) \quad \text{for } y \in \mathbb{R}, \tag{1.9}$$

and

$$\lim_{y\to\pm\infty} (v_0(y), u_0(y), \boldsymbol{w}_0(y), \theta_0(y)) = (1,0,0,1), \quad \text{if } \Omega = \mathbb{R}, \tag{1.10}$$

$$\begin{cases} (u(t,0), \boldsymbol{w}(t,0), \theta(t,0)) = (0,0,1), \\ \lim\limits_{y\to\infty} (v_0(y), u_0(y), \boldsymbol{w}_0(y), \theta_0(y)) = (1,0,0,1), \end{cases} \quad \text{if } \Omega = (0,\infty), \tag{1.11}$$

$$\begin{cases} (u(t,0), \boldsymbol{w}(t,0), \theta_y(t,0)) = (0,0,0), \\ \lim\limits_{y\to\infty} (v_0(y), u_0(y), \boldsymbol{w}_0(y), \theta_0(y)) = (1,0,0,1), \end{cases} \quad \text{if } \Omega = (0,\infty). \tag{1.12}$$

We now present our main result as follows.

Theorem 1. *Assume that the initial data $(v_0, u_0, \boldsymbol{w}_0, \theta_0)$ satisfy*

$$(v_0 - 1, u_0, \boldsymbol{w}_0, \theta_0 - 1) \in H^1(\Omega), \quad \inf_{y\in\Omega} v_0(y) > 0, \quad \inf_{y\in\Omega} \theta_0(y) > 0, \tag{1.13}$$

and are compatible with (1.11), (1.12) when $\Omega = (0,\infty)$. Then there exists a unique global solution $(v, u, \boldsymbol{w}, \theta)$ to (1.7)–(1.10), or (1.7)–(1.9), (1.11), or (1.7)–(1.9), (1.12) such that

$$C_0^{-1} \le v(t,y) \le C_0, \quad C_0^{-1} \le \theta(t,y) \le C_0 \quad \textit{for all } (t,y) \in [0,\infty) \times \overline{\Omega}, \tag{1.14}$$

$$\sup_{0\le t<\infty} \|(v-1,u,\boldsymbol{w},\theta-1)(t)\|_{H^1(\Omega)}^2 + \int_0^\infty \left[\|v_y(t)\|_{L^2(\Omega)}^2 + \|(u_y,\boldsymbol{w}_y,\theta_y)(t)\|_{H^1(\Omega)}^2\right] dt \le C_0, \tag{1.15}$$

where the positive constant C_0 depends solely on μ, λ, κ, R, c_v, $\|(v_0-1,u_0,\boldsymbol{w}_0,\theta_0-1)\|_{H^1(\Omega)}$, $\inf_{x\in\Omega} v_0(x)$ and $\inf_{x\in\Omega}\theta_0(x)$. We have in addition the large time behavior

$$\lim_{t\to\infty} \left(\|(v-1,u,\boldsymbol{w},\theta-1)(t)\|_{L^p(\Omega)} + \|(v_y,u_y,\boldsymbol{w}_y,\theta_y)(t)\|_{L^2(\Omega)}\right) = 0, \tag{1.16}$$

for any $p \in (2,\infty]$.

Our main result shows that both the specific volume and the temperature are bounded from below and above uniformly in t and y. Then the results for the problems (1.7)–(1.10), (1.7)–(1.9), (1.11) and (1.7)–(1.9), (1.12) in Theorem 1 in Lagrangian coordinates can easily be converted to equivalent statements for the corresponding results for the problems (1.1)–(1.4), (1.1)–(1.3), (1.5) and (1.1)–(1.3), (1.6) (see Chen [1]).

The main interest of Theorem 1 is to show that the global solution converges to the constant steady state uniformly in the Lagrangian variable y as time goes to infinity for large initial data. We hope that in the near future, our approach will yield the time-asymptotic behavior towards steady state or rarefaction waves of solutions to the system (1.7) in unbounded domains with more general constitutive relations for large initial data. As far as we know, no result concerned with the stability of the constant state or rarefaction waves has been proved yet for (1.7) with general pressure and internal energy depending nonlinearly on the temperature.

For 1D compressible Navier–Stokes equations (i.e., the system (1.1) with $\boldsymbol{w} \equiv 0$), Kazhikhov and Shelukhin [9] first obtained the global existence and uniqueness of solutions in bounded domains without any smallness conditions of the initial data. Since then, significant progress has been made on the mathematical aspect of the initial and initial–boundary value problems. Especially, Kazhikhov [8] established global solvability for the 1D Navier–Stokes equations in unbounded domains. The time-asymptotic behavior of the global solution has been studied under some smallness conditions on the initial data, see [3,7,12,13] and the references therein. In the case of large initial data, Jiang [4,5] first showed the large-time behavior of the solutions in any fixed bounded domain by proving that the specific volume is pointwise bounded from below and above uniformly in time and y, and that for all t the temperature is bounded from below and above locally in y. Li and Liang [10] recently succeeded in proving that the temperature is pointwise bounded from below and above independently of both the time and the Lagrangian variable and further showing that the global solution is asymptotically stable as time tends to infinity for general large initial data.

As pointed out in [7,8], the key point to the global solvability of the problems (1.7)–(1.10), (1.7)–(1.9), (1.11), and (1.7)–(1.9), (1.12) with large initial data is to obtain the positive lower and upper bounds for the specific volume v and the temperature θ. We will first obtain an entropy-type energy estimate involving the dissipative effects of viscosity and heat diffusion, which yields the uniform lower and upper bounds of the specific volume v. Then one can deduce a positive lower bound for θ by employing the standard maximum principle. We deduce the uniform upper bound of the temperature and all required a priori estimates by using the entropy-type energy estimate and the nonlinear structure of the system itself in Section 2.

The existence of local solution is known from the standard argument based on the operator defined by the linearization of the problem on a small time-interval (see Nash [11] and Kawashima [6]). The global existence of solutions will be proved by the method of extending the local solution with respect to time based on the a priori global estimates in Section 3.

2. A priori estimates

We turn to deduce certain a priori estimates on the solutions $(v, u, \boldsymbol{w}, \theta)(t, x)$ in terms of the initial data. In this section, we will assume $(v, u, \boldsymbol{w}, \theta) \in X(0, T)$ with $T > 0$, where $X(0, T)$ is the function space

$$X(0,T) := \{(v,u,\boldsymbol{w},\theta) | (v-1,u,\boldsymbol{w},\theta-1) \in C([0,T];H^1(\Omega)),$$
$$v_x \in L^2(0,T;L^2\Omega), (u_x,\boldsymbol{w}_x,\theta_x) \in L^2(0,T;H^1(\Omega))\}.$$

We will use C to denote some generic positive constant which depends only on μ, λ, κ, R, c_v, $\|(v_0 - 1, u_0, \boldsymbol{w}_0, \theta_0 - 1)\|_{H^1(\Omega)}$, $\inf_{x\in\Omega} v_0(x)$ and $\inf_{x\in\Omega} \theta_0(x)$.

First, we have the following entropy-type energy estimate, which is essential to deduce the positive lower and upper bounds of the specific volume.

Lemma 2.1. *The following estimate holds:*

$$\sup_{0\le t\le T} \int_\Omega \eta(v,u,\boldsymbol{w},\theta)dy + \int_0^t \int_\Omega \left[\frac{\mu u_y^2}{v\theta} + \frac{\lambda|\boldsymbol{w}_y|^2}{v\theta} + \frac{\kappa\theta_y^2}{v\theta^2}\right] dyd\tau \le C, \tag{2.1}$$

where

$$\eta(v,u,\boldsymbol{w},\theta) = R\phi(v) + \frac{1}{2}u^2 + \frac{1}{2}|\boldsymbol{w}|^2 + c_v\phi(\theta) \tag{2.2}$$

with

$$\phi(z) = z - \ln z - 1 \quad \text{for } z > 0. \tag{2.3}$$

Proof. We first obtain from (1.7) and (1.8) that the temperature satisfies

$$c_v\theta_t + R\frac{\theta}{v}u_y = \left(\kappa\frac{\theta_y}{v}\right)_y + \mu\frac{u_y^2}{v} + \lambda\frac{|\boldsymbol{w}_y|^2}{v}. \tag{2.4}$$

Multiply $(1.7)_1$ by $R(1 - v^{-1})$, $(1.7)_2$ by u, (2.4) by $(1 - \theta^{-1})$, take the scalar product of $(1.7)_3$ with $\boldsymbol{w}$, and add the resulting identities altogether to find

$$\begin{aligned} &\eta(v,u,\boldsymbol{w},\theta)_t + \frac{\mu u_y^2}{v\theta} + \frac{\lambda|\boldsymbol{w}_y|^2}{v\theta} + \frac{\kappa\theta_y^2}{v\theta^2} \\ &= \left[\mu\frac{uu_y}{v} + \lambda\frac{\boldsymbol{w}\cdot\boldsymbol{w}_y}{v} + (1-\theta^{-1})\kappa\frac{\theta_y}{v} + R\left(1-\frac{\theta}{v}\right)u\right]_y. \end{aligned}$$

Integrating this last identity over $[0, t] \times \Omega$, we can deduce (2.1) from (1.10), (1.11) or (1.12). □

With the entropy-type energy estimate (2.1) in hand, we can obtain the uniform bounds on the specific volume v by using the similar approach in [4,5]. We omit the proof here for simplicity.

Lemma 2.2. *There is a positive constant C_1, independent of t, such that*

$$C_1^{-1} \le v(t,y) \le C_1 \quad \text{for all } y \in \overline{\Omega} \text{ and } t \ge 0. \tag{2.5}$$

We are now in position to get the uniform upper bound of the temperature, which will be achieved by the following two lemmas. The first one is on the L^2_y-norm estimate of $(\theta-1)(t,y)$ uniformly in the time t.

Lemma 2.3. *There exists some positive constant C, independent of T, such that*

$$\sup_{0\le t\le T}\int_\Omega \left[(\theta-1)^2+u^4+|\boldsymbol{w}|^4\right]+\int_0^T\int_\Omega \left[\theta_y^2+\left(1+\theta+u^2+|\boldsymbol{w}|^2\right)\left(u_y^2+|\boldsymbol{w}_y|^2\right)\right]\le C. \tag{2.6}$$

Proof. We divide the proof of Lemma 2.3 into five steps.

Step 1. For $t\ge 0$ and $a>1$, we denote

$$\Omega_a(t):=\{y\in\Omega:\ \theta(t,y)>a\}.$$

Then it follows from (2.1) and (2.5) that

$$\begin{aligned}&\int_\Omega (v-1)^2dy+\int_{\Omega\backslash\Omega_a(t)}(\theta-1)^2dy+\int_0^T\int_{\Omega\backslash\Omega_a(t)}\theta_y^2dydt\\ &\le C\int_\Omega \phi(v)dy+C(a)\int_\Omega\phi(\theta)dy+C(a)\int_0^T\int_\Omega\frac{\kappa\theta_y^2}{v\theta^2}dydt\le C(a).\end{aligned} \tag{2.7}$$

Step 2. We now estimate the integral $\int_0^T\int_{\Omega_2(t)}\theta_y^2dydt$. For this purpose, we multiply (2.4) by $(\theta-2)_+:=\max\{\theta-2,0\}$ and integrate it over $(0,t)\times\Omega$ to find

$$\begin{aligned}\frac{c_v}{2}\int_\Omega(\theta-2)_+^2+\kappa\int_0^t\int_{\Omega_2(\tau)}\frac{\theta_y^2}{v}&=\frac{c_v}{2}\int_\Omega(\theta_0-2)_+^2-R\int_0^t\int_\Omega\frac{\theta}{v}u_y(\theta-2)_+\\ &\quad+\mu\int_0^t\int_\Omega\frac{u_y^2}{v}(\theta-2)_++\lambda\int_0^t\int_\Omega\frac{|\boldsymbol{w}_y|^2}{v}(\theta-2)_+.\end{aligned} \tag{2.8}$$

To estimate the last two terms on the right-hand side of (2.8), we multiply $(1.7)_2$ by $2u(\theta-2)_+$, take the scalar product of $(1.7)_3$ with $2\boldsymbol{w}(\theta-2)_+$, and integrate the resulting identities over $(0,t)\times\Omega$ to get

$$\begin{aligned}&\int_\Omega\left(u^2+|\boldsymbol{w}|^2\right)(\theta-2)_++\int_0^t\int_\Omega\frac{2}{v}\left(\mu u_y^2+\lambda|\boldsymbol{w}_y|^2\right)(\theta-2)_+\\ &=\int_\Omega\left(u_0^2+|\boldsymbol{w}_0|^2\right)(\theta-2)_++2R\int_0^t\int_\Omega\frac{\theta}{v}u_y(\theta-2)_++2R\int_0^t\int_{\Omega_2(\tau)}\frac{\theta}{v}u\theta_y\\ &\quad-\int_0^t\int_{\Omega_2(\tau)}2\left(\mu uu_y+\lambda\boldsymbol{w}\cdot\boldsymbol{w}_y\right)\frac{\theta_y}{v}+\int_0^t\int_{\Omega_2(\tau)}\left(u^2+|\boldsymbol{w}|^2\right)\theta_t.\end{aligned} \tag{2.9}$$

Plugging (2.9) into (2.8), we obtain from (2.4) that

$$
\begin{aligned}
&\int_\Omega \left[\frac{c_v}{2}(\theta-2)_+^2 + \left(u^2+|\boldsymbol{w}|^2\right)(\theta-2)_+\right] \\
&\quad + \kappa\int_0^t\int_{\Omega_2(\tau)} \frac{\theta_y^2}{v} + \int_0^t\int_\Omega \frac{1}{v}\left(\mu u_y^2+\lambda|\boldsymbol{w}_y|^2\right)(\theta-2)_+ \\
&= \int_\Omega \left[\frac{c_v}{2}(\theta_0-2)_+^2 + \left(u_0^2+|\boldsymbol{w}_0|^2\right)(\theta_0-2)_+\right] + \underbrace{R\int_0^t\int_\Omega \frac{\theta}{v}u_y(\theta-2)_+}_{I_1} \\
&\quad + \underbrace{2R\int_0^t\int_{\Omega_2(\tau)} \frac{\theta}{v}u\theta_y}_{I_2} \underbrace{-2\int_0^t\int_{\Omega_2(\tau)} \left(\mu u u_y + \lambda \boldsymbol{w}\cdot\boldsymbol{w}_y\right)\frac{\theta_y}{v}}_{I_3} \\
&\quad + \underbrace{\frac{1}{c_v}\int_0^t\int_{\Omega_2(\tau)} \left(u^2+|\boldsymbol{w}|^2\right)\left(\mu\frac{u_y^2}{v} - R\frac{\theta}{v}u_y + \lambda\frac{|\boldsymbol{w}_y|^2}{v}\right)}_{I_4} \\
&\quad + \underbrace{\frac{\kappa}{c_v}\int_0^t\int_{\Omega_2(\tau)} \left(u^2+|\boldsymbol{w}|^2\right)\left(\frac{\theta_y}{v}\right)_y}_{I_5}.
\end{aligned}
\tag{2.10}
$$

We next estimate I_i $(i=1,\cdots,5)$ term by term. First, we have from Cauchy's inequality and (2.5) that

$$
\begin{aligned}
|I_1| &\le \epsilon\int_0^t\int_\Omega \frac{u_y^2}{v}(\theta-2)_+ + C(\epsilon)\int_0^t\int_\Omega \theta^2(\theta-2)_+ \\
&\le \epsilon\int_0^t\int_\Omega \frac{u_y^2}{v}(\theta-2)_+ + C(\epsilon)\int_0^t\int_\Omega \theta(\theta-3/2)_+^2 \\
&\le \epsilon\int_0^t\int_\Omega \frac{u_y^2}{v}(\theta-2)_+ + C(\epsilon)\int_0^t \sup_{y\in\Omega}(\theta-3/2)_+^2,
\end{aligned}
\tag{2.11}
$$

where in the last inequality we have used

$$
\int_{\Omega_{3/2}(\tau)} \theta \le C\int_\Omega \phi(\theta) \le C. \tag{2.12}
$$

Next, it follows from Cauchy's inequality and (2.5) that

$$
\begin{aligned}
|I_2|+|I_3| &\le \epsilon\int_0^t\int_\Omega \theta_y^2 + C(\epsilon)\int_0^t\int_{\Omega_2(\tau)} \left(u^2\theta^2+u^2u_y^2+|\boldsymbol{w}|^2|\boldsymbol{w}_y|^2\right) \\
&\le \epsilon\int_0^t\int_\Omega \theta_y^2 + C(\epsilon)\int_0^t \sup_{y\in\Omega}(\theta-3/2)_+^2 + C(\epsilon)\int_0^t\int_{\Omega_2(\tau)} \left(u^2u_y^2+|\boldsymbol{w}|^2|\boldsymbol{w}_y|^2\right),
\end{aligned}
\tag{2.13}
$$

where we have used (2.1). Then, we deduce from Cauchy's inequality and (2.1) that

$$
\begin{aligned}
|I_4| &\leq C\int_0^t\int_\Omega (u^2+|\boldsymbol{w}|^2)(u_y^2+|\boldsymbol{w}_y|^2) + C\int_0^t\int_{\Omega_2(\tau)} (u^2+|\boldsymbol{w}|^2)\theta^2 \\
&\leq C\int_0^t\int_\Omega (u^2+|\boldsymbol{w}|^2)(u_y^2+|\boldsymbol{w}_y|^2) + C\int_0^t \sup_{y\in\Omega}(\theta-3/2)_+^2. \qquad (2.14)
\end{aligned}
$$

Finally, Lebesgue's dominated convergence theorem yields that

$$
\begin{aligned}
I_5 &= \frac{\kappa}{c_v}\int_0^t\int_\Omega \chi_{\Omega_2(\tau)}\left(u^2+|\boldsymbol{w}|^2\right)\left(\frac{\theta_y}{v}\right)_y \\
&= \frac{\kappa}{c_v}\lim_{\nu\to 0^+}\int_0^t\int_\Omega \varphi_\nu(\theta)\left(u^2+|\boldsymbol{w}|^2\right)\left(\frac{\theta_y}{v}\right)_y \\
&= \frac{\kappa}{c_v}\lim_{\nu\to 0^+}\int_0^t\int_\Omega \left[-2\varphi_\nu(\theta)\left(uu_y+\boldsymbol{w}\cdot\boldsymbol{w}_y\right)\frac{\theta_y}{v} - \varphi_\nu'(\theta)\left(u^2+|\boldsymbol{w}|^2\right)\frac{\theta_y^2}{v}\right] \\
&\leq -\frac{2\kappa}{c_v}\lim_{\nu\to 0^+}\int_0^t\int_\Omega \varphi_\nu(\theta)\left(uu_y+\boldsymbol{w}\cdot\boldsymbol{w}_y\right)\frac{\theta_y}{v} \\
&\leq \epsilon\int_0^t\int_\Omega \theta_y^2 + C(\epsilon)\int_0^t\int_\Omega \left(u^2u_y^2+|\boldsymbol{w}|^2|\boldsymbol{w}_y|^2\right), \qquad (2.15)
\end{aligned}
$$

where the approximate scheme $\varphi_\nu(\theta)$ is defined by

$$
\varphi_\nu(\theta) := \begin{cases} 1, & \theta-2\geq\nu, \\ (\theta-2)/\nu, & 0\leq\theta-2<\nu, \\ 0, & \theta-2<0. \end{cases}
$$

Plugging (2.11)–(2.15) into (2.10), we get from (2.5) that

$$
\begin{aligned}
&\int_\Omega(\theta-2)_+^2 + \int_0^t\int_{\Omega_2(\tau)}\left[\theta_y^2+u_y^2(\theta-2)_+ + |\boldsymbol{w}_y|^2(\theta-2)_+\right] \\
&\leq C + C\epsilon\int_0^t\int_\Omega\theta_y^2 + C(\epsilon)\int_0^t\sup_{y\in\Omega}(\theta-3/2)_+^2 + C(\epsilon)\int_0^t\int_\Omega\left(u^2+|\boldsymbol{w}|^2\right)\left(u_y^2+|\boldsymbol{w}_y|^2\right). \qquad (2.16)
\end{aligned}
$$

Step 3. We estimate the terms on the right-hand side of (2.16). We note from (2.7) and (2.1) that

$$
\int_0^t\int_\Omega\left(\theta_y^2+u_y^2\theta+|\boldsymbol{w}_y|^2\theta\right) = \int_0^t\left[\int_{\Omega_3(\tau)} + \int_{\Omega\backslash\Omega_3(\tau)}\right]\left(\theta_y^2+u_y^2\theta+|\boldsymbol{w}_y|^2\theta\right)
$$

$$\le C + C\int_0^t \int_{\Omega_2(\tau)} \left[\theta_y^2 + (u_y^2 + |\boldsymbol{w}_y|^2)(\theta-2)_+\right] + C\int_{\Omega\backslash\Omega_3(\tau)} \left(\frac{u_y^2}{v\theta} + \frac{|\boldsymbol{w}_y|^2}{v\theta}\right)$$

$$\le C + C\int_0^t \int_{\Omega_2(\tau)} \left[\theta_y^2 + (u_y^2 + |\boldsymbol{w}_y|^2)(\theta-2)_+\right]. \tag{2.17}$$

Plugging (2.17) into (2.16) and choosing ϵ sufficiently small, we obtain

$$\int_\Omega (\theta-2)_+^2 + \int_0^T\int_\Omega \left(\theta_y^2 + u_y^2\theta + |\boldsymbol{w}_y|^2\theta\right)$$

$$\le C + C\int_0^t \sup_{y\in\Omega}(\theta-3/2)_+^2 + C\int_0^t\int_\Omega \left(u^2+|\boldsymbol{w}|^2\right)\left(u_y^2+|\boldsymbol{w}_y|^2\right). \tag{2.18}$$

Step 4. To estimate the last term on the right-hand side of (2.18), we multiply $(1.7)_2$ by u^3, take the scalar product of $(1.7)_3$ with $|\boldsymbol{w}|^2\boldsymbol{w}$, and then integrate the resulting identities over $(0,t)\times\Omega$ to find

$$\frac{1}{4}\int_\Omega \left(u^4+|\boldsymbol{w}|^4\right) + \int_0^t\int_\Omega \frac{1}{v}\left(3\mu u^2u_y^2 + \lambda|\boldsymbol{w}|^2|\boldsymbol{w}_y|^2 + 2\lambda|\boldsymbol{w}\cdot\boldsymbol{w}_y|^2\right)$$

$$= \frac{1}{4}\int_\Omega \left(u_0^4+|\boldsymbol{w}_0|^4\right) + 3R\int_0^t\int_\Omega u^2u_y\left(\frac{\theta}{v}-1\right). \tag{2.19}$$

Multiplying $(1.7)_2$ by $u|\boldsymbol{w}|^2$, taking the scalar product of $(1.7)_3$ with $u^2\boldsymbol{w}$, and then integrating the resulting identities over $(0,t)\times\Omega$, we get

$$\frac{1}{2}\int_\Omega \left(u^4+|\boldsymbol{w}|^4\right) + \int_0^t\int_\Omega \frac{1}{v}\left(\mu|\boldsymbol{w}|^2u_y^2 + \lambda u^2|\boldsymbol{w}_y|^2\right)$$

$$= \frac{1}{2}\int_\Omega u_0^2|\boldsymbol{w}_0|^2 + R\int_0^t\int_\Omega (u_y|\boldsymbol{w}|^2 + u\boldsymbol{w}\cdot\boldsymbol{w}_y)\left(\frac{\theta}{v}-1\right) - (\mu+\lambda)\int_0^t\int_\Omega \frac{1}{v}uu_y\boldsymbol{w}\cdot\boldsymbol{w}_y. \tag{2.20}$$

We use (2.5) and then take the proper linear combination of (2.19) and (2.20) to obtain

$$\int_\Omega \left(u^4+|\boldsymbol{w}|^4\right) + \int_0^t\int_\Omega \left(u^2+|\boldsymbol{w}|^2\right)\left(u_y^2+|\boldsymbol{w}_y|^2\right)$$

$$\le C + C\int_0^t\int_\Omega \left(|u_y|\,u^2 + |u_y|\,|\boldsymbol{w}|^2 + |u||\boldsymbol{w}||\boldsymbol{w}_y|\right)\left|\frac{\theta}{v}-1\right|. \tag{2.21}$$

We denote the last term on the right-hand side of (2.21) by I_6 and have from Hölder's and Cauchy's inequalities and (2.7) that

$$
\begin{aligned}
I_6 \leq & \int_0^t \int_\Omega \left(|u_y|\, u^2 + |u_y|\, |\boldsymbol{w}|^2 + |u||\boldsymbol{w}||\boldsymbol{w}_y|\right) \frac{|1-v|}{v} \\
& + \int_0^t \left[\int_{\Omega\setminus\Omega_2(\tau)} + \int_{\Omega_2(\tau)} \right] \left(|u_y|\, u^2 + |u_y|\, |\boldsymbol{w}|^2 + |u||\boldsymbol{w}||\boldsymbol{w}_y|\right) \frac{|\theta-1|}{v} \\
\leq & \, C \int_0^t \|(u,\boldsymbol{w})\|^2_{L^\infty(\Omega)} \|(u_y,\boldsymbol{w}_y)\|_{L^2(\Omega)} \left[\|v-1\|_{L^2(\Omega)} + \|\theta-1\|_{L^2(\Omega\setminus\Omega_2(\tau))}\right] \\
& + \epsilon \int_0^t \int_\Omega \left(u^2 + |\boldsymbol{w}|^2\right)\left(u_y^2 + |\boldsymbol{w}_y|^2\right) + C(\epsilon) \int_0^t \int_{\Omega_2(\tau)} \left(u^2 + |\boldsymbol{w}|^2\right)(\theta-1)^2 \\
\leq & \, C \int_0^t \|(u,\boldsymbol{w})\|_{L^2(\Omega)} \|\left(u_y,\boldsymbol{w}_y\right)\|^2_{L^2(\Omega)} + \epsilon \int_0^t \int_\Omega \left(u^2 + |\boldsymbol{w}|^2\right)\left(u_y^2 + |\boldsymbol{w}_y|^2\right) \\
& + C(\epsilon) \int_0^t \sup_{y\in\Omega} (\theta - 3/2)_+^2 \|(u,\boldsymbol{w})\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{2.22}
$$

We note from (2.1) and (2.5) that

$$
\begin{aligned}
\int_0^t \|(u_y,\boldsymbol{w}_y)\|^2_{L^2(\Omega)} & \leq \nu \int_0^t \int_\Omega \theta\left(u_y^2 + |\boldsymbol{w}_y|^2\right) + C(\nu) \int_0^t \int_\Omega \theta^{-1} u_y^2 \\
& \leq \nu \int_0^t \int_\Omega \theta\left(u_y^2 + |\boldsymbol{w}_y|^2\right) + C(\nu).
\end{aligned}
\tag{2.23}
$$

Choosing positive constant ϵ suitable small, we have from (2.21)–(2.23) that

$$
\begin{aligned}
& \int_\Omega (u^4 + |\boldsymbol{w}|^4) + \int_0^t \int_\Omega \left(1 + u^2 + |\boldsymbol{w}|^2\right)\left(u_y^2 + |\boldsymbol{w}_y|^2\right) \\
& \leq C(\nu) + C\nu \int_0^T \int_\Omega \theta\left(u_y^2 + |\boldsymbol{w}_y|^2\right) + C \int_0^T \sup_{y\in\Omega} (\theta - 3/2)_+^2.
\end{aligned}
\tag{2.24}
$$

We plug (2.24) into (2.18) and choose ν suitable small to find

$$
\begin{aligned}
& \int_\Omega \left[(\theta-2)_+^2 + u^4 + |\boldsymbol{w}|^4\right] + \int_0^t \int_\Omega \left[\theta_y^2 + (1+\theta+u^2+|\boldsymbol{w}|^2)(u_y^2 + |\boldsymbol{w}_y|^2)\right] \\
& \qquad \leq C + C \int_0^t \sup_{y\in\Omega} (\theta - 3/2)_+^2.
\end{aligned}
\tag{2.25}
$$

Step 5. It remains to control the last term on the right hand side of (2.25). In light of (2.12), we apply standard calculations to yield for any $\epsilon > 0$ that

$$\int_0^T \sup_{y\in\Omega}(\theta-3/2)_+^2 = \int_0^T \sup_{y\in\Omega}\left(\int_y^\infty \partial_y(\theta-3/2)_+\right)^2$$
$$\le \int_0^T \left(\int_{\Omega_{3/2}(t)} |\theta_y|\right)^2 \le \int_0^T \int_{\Omega_{3/2}(t)} \frac{\theta_y^2}{\theta} \int_{\Omega_{3/2}(t)} \theta$$
$$\le C\int_0^T \int_{\Omega_{3/2}(t)} \frac{\theta_y^2}{\theta} \le \epsilon \int_0^T\int_\Omega \theta_y^2 + C(\epsilon)\int_0^T\int_\Omega \frac{u\theta_y^2}{v\theta^2}$$
$$\le \epsilon \int_0^T\int_\Omega \theta_y^2 + C(\epsilon). \tag{2.26}$$

We obtain (2.6) by plugging (2.26) into (2.25), choosing $\epsilon > 0$ suitable small and using (2.7) and the fact that $\theta - 1 \le 2(\theta - 2)$ in $\Omega_3(t)$. □

The next lemma is concerned with the L_y^2-norm estimate of $\theta_y(t,y)$ for any $t \in (0,T)$.

Lemma 2.4. *The following estimates hold:*

$$\theta(t,y) \le C \quad \textit{for all } (t,y) \in [0,T]\times\overline{\Omega}, \tag{2.27}$$

$$\sup_{0\le t\le T}\int_\Omega \left[v_y^2 + u_y^2 + |\boldsymbol{w}_y|^2 + \theta_y^2\right] + \int_0^T\int_\Omega \left[\theta v_y^2 + u_{yy}^2 + |\boldsymbol{w}_{yy}|^2 + \theta_{yy}^2\right] \le C. \tag{2.28}$$

Proof.

Step 1. Noticing in our case that

$$\left(\mu\frac{u_y}{v}\right)_y = \left(\mu\frac{v_t}{v}\right)_y = \left(\mu\frac{v_y}{v}\right)_t,$$

we multiply $(1.7)_2$ by $\frac{v_y}{v}$ and integrate the resulting identity over $(0,t)\times\Omega$ to find

$$\frac{\mu}{2}\int_\Omega \left(\frac{v_y}{v}\right)^2 + R\int_0^t\int_\Omega \frac{\theta v_y^2}{v^3}$$
$$= \frac{\mu}{2}\int_\Omega\left(\frac{v_{0y}}{v_0}\right)^2 + \int_\Omega u\frac{v_y}{v} - \int_\Omega u_0\frac{v_{0y}}{v_0} + R\int_0^t\int_\Omega \frac{\theta_y v_y}{v} + \int_0^t\int_\Omega \frac{u_y^2}{v}$$
$$\le C + \frac{\mu}{4}\int_\Omega\left(\frac{v_y}{v}\right)^2 + C\int_\Omega u^2 + \frac{R}{2}\int_0^t\int_\Omega \frac{\theta v_y^2}{v^3} + C\int_0^t\int_\Omega\left[\frac{\theta_y^2}{v\theta} + \frac{u_y^2}{v}\right]$$
$$\le C + \frac{\mu}{4}\int_\Omega\left(\frac{v_y}{v}\right)^2 + \frac{R}{2}\int_0^t\int_\Omega \frac{\theta v_y^2}{v^3} + C\int_0^t\int_\Omega\left[\frac{\theta_y^2}{v\theta^2} + \theta_y^2 + u_y^2\right],$$

which implies from (2.1), (2.5) and (2.6) that

$$\sup_{0\le t\le T}\int_\Omega v_y^2+\int_0^T\int_\Omega \theta v_y^2\le C. \tag{2.29}$$

Step 2. Next, we integrate $(1.7)_2$ multiplied by u_{yy} over $(0,t)\times\Omega$ to obtain

$$\begin{aligned}
\frac12\int_\Omega u_y^2+\mu\int_0^t\int_\Omega \frac{u_{yy}^2}{v} &= \frac12\int_\Omega u_{0y}^2+R\int_0^t\int_\Omega\left(\frac{\theta}{v}\right)_y u_{yy}+\mu\int_0^t\int_\Omega \frac{u_y}{v^2}v_yu_{yy}\\
&\le C+\frac{\mu}{4}\int_0^t\int_\Omega\frac{u_{yy}^2}{v}+C\int_0^t\int_\Omega\left(\theta^2v_y^2+\theta_y^2+u_y^2v_y^2\right)\\
&\le C+\frac{\mu}{4}\int_0^t\int_\Omega\frac{u_{yy}^2}{v}+\sup_{[0,T]\times\Omega}\theta\int_0^t\int_\Omega\theta v_y^2+C\int_0^t\|u_y(\tau)\|_{L^\infty(\Omega)}^2\\
&\le C+\frac{\mu}{2}\int_0^t\int_\Omega\frac{u_{yy}^2}{v}+C\sup_{[0,T]\times\Omega}\theta+C\int_0^t\int_\Omega u_y^2\\
&\le C+\frac{\mu}{2}\int_0^t\int_\Omega\frac{u_{yy}^2}{v}+C\sup_{[0,T]\times\Omega}\theta,
\end{aligned}$$

where we have used Cauchy's inequality, (2.6) and (2.29). Thus, we have

$$\sup_{0\le t\le T}\int_\Omega u_y^2+\int_0^T\int_\Omega u_{yy}^2\le C+C\sup_{[0,T]\times\Omega}\theta. \tag{2.30}$$

Step 3. Integrating $(1.7)_3$ multiplied by $\boldsymbol{w}_{yy}$ over $(0,t)\times\Omega$ yields

$$\begin{aligned}
\frac12\int_\Omega |\boldsymbol{w}_y|^2+\lambda\int_0^t\int_\Omega\frac{|\boldsymbol{w}_{yy}|^2}{v} &= \frac12\int_\Omega|\boldsymbol{w}_{0y}|^2+\lambda\int_0^t\int_\Omega\frac{v_y}{v^2}\boldsymbol{w}_y\cdot\boldsymbol{w}_{yy}\\
&\le C+\frac{\lambda}{4}\int_0^t\int_\Omega\frac{|\boldsymbol{w}_{yy}|^2}{v}+C\int_0^t\int_\Omega v_y^2|\boldsymbol{w}_y|^2\\
&\le C+\frac{\lambda}{4}\int_0^t\int_\Omega\frac{|\boldsymbol{w}_{yy}|^2}{v}+C\int_0^t\|\boldsymbol{w}_y(\tau)\|_{L^\infty(\Omega)}^2\\
&\le C+\frac{\lambda}{2}\int_0^t\int_\Omega\frac{|\boldsymbol{w}_{yy}|^2}{v}+C\int_0^t\int_\Omega|\boldsymbol{w}_y|^2\\
&\le C+\frac{\lambda}{2}\int_0^t\int_\Omega\frac{|\boldsymbol{w}_{yy}|^2}{v},
\end{aligned}$$

which implies

$$\sup_{0\le t\le T}\int_\Omega |\boldsymbol{w}_y|^2 + \int_0^T\int_\Omega |\boldsymbol{w}_{yy}|^2 \le C. \tag{2.31}$$

Step 4. We multiply (2.4) by θ_{yy} and integrate the resulting identity over $(0,t)\times\Omega$ to deduce

$$\frac{c_v}{2}\int_\Omega \theta_y^2 + \int_0^t\int_\Omega \kappa\frac{\theta_{yy}^2}{v} = \frac{c_v}{2}\int_\Omega \theta_{0y}^2 + \int_0^t\int_\Omega \left[R\frac{\theta}{v}u_y + \kappa\frac{\theta_y}{v^2}v_y - \mu\frac{u_y^2}{v} - \lambda\frac{|\boldsymbol{w}_y|^2}{v}\right]\theta_{yy}. \tag{2.32}$$

We denote the last term on the right-hand side of (2.32) by I_7 and then we have from Hölder's and Young's inequalities, (2.5), (2.6), (2.29), (2.30) and (2.31) that

$$\begin{aligned} I_7 &\le \epsilon\int_0^t\int_\Omega \frac{\theta_{yy}^2}{v} + C(\epsilon)\int_0^t \left[\|\theta\|_{L^\infty}^2\|u_y\|_{L^2}^2 + \|\theta_y\|_{L^2}^2\|v_y\|_{L^2}^4 + \|(u_y,\boldsymbol{w}_y)\|_{H^1}^2\|(u_y,\boldsymbol{w}_y)\|_{L^2}^2\right] \\ &\le C + \epsilon\int_0^t\int_\Omega \frac{\theta_{yy}^2}{v} + C(\epsilon)\int_0^t \|\theta_y\|^2 + C\sup_{[0,T]\times\Omega}\theta^2. \end{aligned}$$

Here we have used

$$\|(u_y,\boldsymbol{w}_y)\|_{L^4}^4 \le \|(u_y,\boldsymbol{w}_y)\|_{L^\infty}^2\|(u_y,\boldsymbol{w}_y)\|_{L^2}^2 \le \|(u_y,\boldsymbol{w}_y)\|_{H^1}^2\|(u_y,\boldsymbol{w}_y)\|_{L^2}^2$$

and

$$\begin{aligned} \int_\Omega \kappa\frac{\theta_y}{v^2}v_y\theta_{yy} &\le C\|v_y\|_{L^2}\|\theta_{yy}\|_{L^2}\|\theta_y\|_{L^\infty} \\ &\le C\|v_y\|_{L^2}\|\theta_{yy}\|_{L^2}^{\frac{3}{2}}\|\theta_y\|_{L^2}^{\frac{1}{2}} \\ &\le \int_\Omega \frac{\theta_{yy}^2}{v} + C(\epsilon)\|\theta_y\|_{L^2}^2\|v_y\|_{L^2}^4. \end{aligned}$$

Choosing $\epsilon>0$ suitable small, we have

$$\sup_{0\le t\le T}\int_\Omega \theta_y^2 + \int_0^T\int_\Omega \theta_{yy}^2 \le C + C\sup_{[0,T]\times\Omega}\theta^2. \tag{2.33}$$

Step 5. Now we obtain from (2.6) that

$$\|(\theta-1)(t)\|_{L^\infty(\overline{\Omega})}^2 \le C\|(\theta-1)(t)\|_{L^2(\Omega)}\|\theta_y(t)\|_{L^2(\Omega)} \le C\|\theta_y(t)\|_{L^2(\Omega)}. \tag{2.34}$$

Plugging (2.33) into (2.34), we deduce

$$\|(\theta-1)(t)\|_{L^\infty(\overline{\Omega})}^2 \le C + C\sup_{[0,T]\times\overline{\Omega}}\theta,$$

which yields (2.27) from Cauchy's inequality. Then we can get (2.28) from (2.29)–(2.33). □

The next estimate is concerned with a lower bound estimate on the temperature θ.

Lemma 2.5. *For each $t \geq 0$ and $y \in \overline{\Omega}$, the following estimate holds:*

$$\theta(t,y) \geq (C_1 + C_2 t)^{-1}. \tag{2.35}$$

Proof. Multiply the equation (2.4) by $1/\theta^2$ to have

$$\begin{aligned} c_v\left(\frac{1}{\theta}\right)_t &= \left[\frac{\kappa}{v}\left(\frac{1}{\theta}\right)_y\right]_y - \frac{2\kappa\theta_y^2}{v\theta^3} \\ &\quad - \frac{\mu}{v\theta^2}\left[u_y - \frac{R\theta}{2\mu}\right]^2 + \frac{R^2}{4v\mu} - \frac{\lambda|\boldsymbol{w_y}|^2}{v\theta^2} \\ &\leq \left[\frac{\kappa}{v}\left(\frac{1}{\theta}\right)_y\right]_y + \frac{R^2}{4\mu}\left\|\frac{1}{v}\right\|_{L^\infty([0,T]\times\Omega)}. \end{aligned}$$

Take

$$h(t,y) := \frac{1}{\theta(t,y)} - \frac{R^2 t}{4\mu}\left\|\frac{1}{v}\right\|_{L^\infty([0,T]\times\Omega)}.$$

Then we have

$$\begin{cases} c_v h_t \leq \left(\frac{\kappa}{v} h_y\right)_y & \text{for } (t,y) \in [0,T]\times\Omega, \\ h(0,y) = \theta_0^{-1}(y) & \text{for } y \in \Omega. \end{cases} \tag{2.36}$$

In view of Lemma 2.2, we have the uniform lower and upper bounds of v. Hence the standard maximum principle for parabolic equations (see [2, Section 7.1]) can be applied to the problem (2.36) to yield $h(t,y) \leq (\inf\theta_0)^{-1}$ for each $(t,y) \in [0,T]\times\overline{\Omega}$. Applying the Lemma 2.2 again yields (2.35). This finishes the proof of the lemma. □

3. Proof of Theorem 1

With the a priori estimates (2.1), (2.5), (2.6) and (2.28) in hand, we can extend the local solution to the global one for the problems (1.7)–(1.10), (1.7)–(1.9), (1.11), or (1.7)–(1.9), (1.12) by using the standard continuation argument. For the global solution $(v, u, \boldsymbol{w}, \theta)$, we can easily deduce from (2.1), (2.5), (2.6) and (2.28) that

$$\sup_{0\leq t<\infty}\|(v-1, u, \boldsymbol{w}, \theta-1)(t)\|_{L^2(\Omega)}^2 + \int_0^\infty\int_\Omega \theta v_y^2 + \int_0^\infty \|(u_y, \boldsymbol{w}_y, \theta_y)(t)\|_{H^1(\Omega)}^2 \leq C. \tag{3.1}$$

To complete the proof of Theorem 1, it suffices to show the large-time behavior (1.16) and the uniform lower bound of the temperature θ. We define

$$E(t) := \|(u_y, \boldsymbol{w}_y, \theta_y)(t)\|_{L^2(\Omega)}^2.$$

Then we find from (3.1) that

$$E(t) \in L^1(0,\infty). \tag{3.2}$$

In light of (1.7) and (2.4), we can get from (3.1) that

$$E'(t) \in L^1(0,\infty). \tag{3.3}$$

According to (3.2)–(3.3), we have

$$\lim_{t\to\infty} \|(u_y, \boldsymbol{w}_y, \theta_y)(t)\|_{L^2} = \lim_{t\to\infty} E(t)^{\frac{1}{2}} = 0. \tag{3.4}$$

Hence, we have

$$\lim_{t\to\infty} \|(u, \boldsymbol{w}, \theta - 1)(t)\|_{L^\infty(\overline{\Omega})} \le \lim_{t\to\infty} \|(u, \boldsymbol{w}, \theta - 1)(t)\|_{L^2}^{\frac{1}{2}} \|(u_y, \boldsymbol{w}_y, \theta_y)(t)\|_{L^2}^{\frac{1}{2}} = 0. \tag{3.5}$$

Then we can find $T_0 > 0$ such that

$$\frac{1}{2} \le \theta(t,y) \le \frac{3}{2} \quad \text{for all } (t,y) \in [T_0,\infty) \times \overline{\Omega},$$

which combined with (2.35) gives a uniform lower bound of θ. Therefore, we get (1.14) and (1.15) from (3.1). Then we find from (1.15) that

$$E_1(t) := \|v_y(t)\|_{L^2(\Omega)}^2 \in L^1(0,\infty)$$

and

$$E_1'(t) = 2\int_\Omega v_y u_{yy} \le \int_\Omega u_{yy}^2 + \int_\Omega v_y^2 \in L^1(0,\infty).$$

Hence we have

$$\lim_{t\to\infty} \|v_y(t)\|_{L^2(\Omega)} = 0, \tag{3.6}$$

$$\lim_{t\to\infty} \|(v-1)(t)\|_{L^\infty(\Omega)} = 0. \tag{3.7}$$

The combination of (3.1)–(3.7) directly yields (1.16). This completes the proof of Theorem 1.

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