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Global spherically symmetric flows for a viscous radiative and reactive gas in an exterior domain

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Abstract

We consider the equations for a viscous, compressible, radiative and reactive gas (pressure $P = R\rho\theta + a\theta^4/3$, internal energy $e = c_v\theta + a\theta^4/\rho$) over an unbounded exterior domain in $\mathbb{R}^n$, where $n \geq 2$ is the space dimension. The existence, uniqueness, and large-time behavior of global spherically symmetric solutions are established for large initial data. The key point in the analysis is to deduce certain uniform *a priori* estimates on the solutions, especially on lower and upper bounds of the specific volume and temperature.

Keywords: Viscous radiative and reactive gas; Spherically symmetric solutions; Large initial data; Global well-posedness; Large-time behavior

1. Introduction

1.1. The Eulerian description

We are concerned with the existence, uniqueness, and large-time behavior of global spherically symmetric solutions to a model for the dynamic combustion of a viscous, compressible, radiative and reactive gas in the unbounded exterior domain $\Omega := \{\xi \in \mathbb{R}^n : |\xi| > 1\}$, where ξ is

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the spatial variable with space dimension $n \geq 2$. The motion of the gas is described by the compressible Navier–Stokes system coupled to the reaction–diffusion equation, which can be written in the Eulerian coordinates as (*cf.* Williams [28, Chapter 1])

$$\begin{cases} \rho_t + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla P = \nabla \cdot \mathbb{S}, \\ (\rho e)_t + \nabla \cdot (\rho e \boldsymbol{u}) + P \nabla \cdot \boldsymbol{u} = \nabla \cdot (\kappa \nabla \theta) + \mathbb{S} : \nabla \boldsymbol{u} + \lambda \phi \rho z, \\ (\rho z)_t + \nabla \cdot (\rho \boldsymbol{u} z) = \nabla \cdot (\rho d \nabla z) - \phi \rho z. \end{cases} \tag{1.1}$$

Here $t > 0$ is the time variable, and the primary dependent variables are density ρ, velocity $\boldsymbol{u}$, absolute temperature θ, and reactant mass fraction z.

We treat the radiation as a continuous field and take both the wave and photonic effects into account. We also assume that the high-temperature radiation is at thermal equilibrium with the fluid. As a result, the pressure P and the internal energy e consist of a linear term in θ corresponding to the perfect polytropic contribution and a fourth-order radiative part due to the Stefan–Boltzmann law (see, e.g., Mihalas and Mihalas [20, p. 320]):

$$P = P(\rho, \theta) = R\rho\theta + \frac{a}{3}\theta^4, \quad e = e(\rho, \theta) = c_v\theta + a\frac{\theta^4}{\rho}, \tag{1.2}$$

where R, c_v, and a are positive constants. In this paper we concentrate on Newtonian fluids for which the viscous stress tensor $\mathbb{S}$ reads as

$$\mathbb{S} = \mu \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathsf{T}} \right) + \zeta \nabla \cdot \boldsymbol{u}\, \mathbb{I},$$

where $(\nabla \boldsymbol{u})^{\mathsf{T}}$ is the transpose matrix of $\nabla \boldsymbol{u}$ and $\mathbb{I}$ is the $n \times n$ identity matrix. The viscosity coefficients μ and ζ, the species diffusion $d > 0$, and the heat release $\lambda > 0$ are supposed to be constants and satisfy

$$\mu > 0, \quad n\zeta + 2\mu > 0. \tag{1.3}$$

The heat conductivity κ is assumed to have the following form (*cf.* Ducomet [5]):

$$\kappa(\rho, \theta) = \kappa_1 + \kappa_2 \frac{\theta^b}{\rho}, \tag{1.4}$$

where κ_1, κ_2, and b are positive constants. The reaction function $\phi = \phi(\theta)$ is defined by the first-order Arrhenius law:

$$\phi(\theta) = \theta^{\beta} \mathrm{e}^{-A/\theta} \qquad (\beta \geq 0), \tag{1.5}$$

where positive constant A stands for the activation energy.

We shall consider the initial boundary value problem of system (1.1) in the region $(0, \infty) \times \Omega$ with the following initial and boundary conditions:

$$\begin{cases} (\rho, \boldsymbol{u}, \theta, z)|_{t=0} = (\rho_0, \boldsymbol{u}_0, \theta_0, z_0)(\xi), & \xi \in \overline{\Omega}, \\ (\boldsymbol{u}, \partial_{\boldsymbol{n}}\theta, \partial_{\boldsymbol{n}} z)|_{\partial\Omega} = 0, & t \geq 0. \end{cases} \tag{1.6}$$

Here $\boldsymbol{n}$ denotes the unit outer normal to $\partial\Omega$. We suppose that the boundary conditions are compatible with the initial data.

The global well-posedness of large solutions to initial/initial–boundary value problems of system (1.1)–(1.2) has been studied by many authors. In the non-radiative ($a = 0$) case, the existence of global strong solutions was proved by Chen [1], Chen et al. [2,3], and Li [16], among others. For the radiative ($a > 0$) case, the well-posedness of global classical solutions in one-dimensional bounded domains was established by [5,12,18] for the Dirichlet–Neumann boundary conditions, and by [11,18,21,23,24] for the free and pure Neumann boundary conditions. Also see Ducomet and Zlotnik [6,7] for exponential stabilization of solutions by constructing new global Lyapunov functionals. The global existence and large-time behavior of solutions to the Cauchy problem have been treated very recently by Liao and Zhao [19] for the constant viscosity case and by He et al. [9] for the temperature-dependent viscosity case.

In more than one dimension, the global existence of *variational solutions* was shown by Donatelli and Trivisa [4], Feireisl and Novotný [8] for bounded domains. As for the *classical solutions* to (1.12), the local existence was given in Jenssen et al. [10, Appendix] for sufficiently smooth initial data. Zhang [29] showed that spherically and cylindrically symmetric solutions exist globally-in-time for initial–boundary value problem of (1.1) in a bounded annular domain, which extended the results of [22,25] to large initial data. It is worth noting that the boundedness of domains is essential in the analysis of [22,25,29].

We are interested in global well-posedness of spherically symmetric large solutions to (1.1)–(1.6) in the *unbounded* domain Ω. Supplemented with spherically symmetric initial data, the solution $(\rho, \boldsymbol{u}, \theta)$ to problem (1.1)–(1.6) is also spherically symmetric, *i.e.*

$$(\rho(t,\xi),\, \boldsymbol{u}(t,\xi),\, \theta(t,\xi), z(t,\xi)) = \left(\hat{\rho}(t,r),\, \frac{\xi}{r}\hat{u}(t,r),\, \hat{\theta}(t,r),\, \hat{z}(t,r)\right), \quad r := |\xi| \geq 1.$$

The system for $(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{z})$ then takes the form:

$$\begin{cases} \hat{\rho}_t + \dfrac{(r^{n-1}\hat{\rho}\hat{u})_r}{r^{n-1}} = 0, \\ \hat{\rho}(\hat{u}_t + \hat{u}\hat{u}_r) + \hat{P}_r = \left[\dfrac{\alpha(r^{n-1}\hat{u})_r}{r^{n-1}}\right]_r, \\ \hat{\rho}(\hat{e}_t + \hat{u}\hat{e}_r) + \dfrac{\hat{P}(r^{n-1}\hat{u})_r}{r^{n-1}} = \dfrac{(\hat{\kappa} r^{n-1}\hat{\theta}_r)_r}{r^{n-1}} + \hat{\mathcal{Q}} + \lambda\phi\rho z, \\ \hat{\rho}(\hat{z}_t + \hat{u}\hat{z}_r) = \dfrac{(\hat{\rho} d r^{n-1}\hat{z}_r)_r}{r^{n-1}} - \phi\rho z, \end{cases} \tag{1.7}$$

where $\alpha = 2\mu + \zeta$ and

$$\hat{\mathcal{Q}} = \frac{\alpha[(r^{n-1}\hat{u})_r]^2}{r^{2n-2}} - \frac{2(n-1)\mu(r^{n-2}\hat{u}^2)_r}{r^{n-1}}.$$

The initial and boundary conditions (1.6) are reduced to

$$\begin{cases} (\hat{\rho}, \hat{u}, \hat{\theta}, \hat{z})|_{t=0} = (\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0, \hat{z}_0)(r), & r \geq 1, \\ (\hat{u}, \partial_r \hat{\theta}, \partial_r \hat{z})|_{r=1} = 0, & t \geq 0. \end{cases} \tag{1.8}$$

These boundary conditions are supposed to be compatible with the initial data.

1.2. Lagrangian coordinates and the main result

In order to achieve the global existence, it is more convenient to transform initial boundary value problem (1.7)–(1.8) into that in the Lagrangian coordinates. We introduce the Lagrangian variables (t, x) and denote

$$(\rho, u, \theta, z)(t, x) := (\hat{\rho}, \hat{u}, \hat{\theta}, \hat{z})(t, r),$$

where

$$r = r(t, x) = r_0(x) + \int_0^t \hat{u}(s, r(s, x))\mathrm{d}s, \tag{1.9}$$

and

$$r_0(x) := h^{-1}(x), \quad h(r) := \int_1^r y^{n-1} \hat{\rho}_0(y)\mathrm{d}y. \tag{1.10}$$

Then one can obtain the following identities:

$$r_t(t, x) = u(t, x), \quad r_x(t, x) = r^{1-n} v(t, x), \tag{1.11}$$

where $v := 1/\rho$ is the specific volume. In view of (1.11), system (1.7) can be reformulated to that for $(v, u, \theta, z)(t, x)$ as

$$v_t = (r^{n-1} u)_x, \tag{1.12a}$$

$$u_t = r^{n-1} \left[\frac{\alpha (r^{n-1} u)_x}{v} - P \right]_x, \tag{1.12b}$$

$$\begin{aligned} e_t = {} & \left[\frac{\kappa r^{2n-2} \theta_x}{v} \right]_x + \left[\frac{\alpha (r^{n-1} u)_x}{v} - P \right] (r^{n-1} u)_x \\ & - (2n-2)\mu (r^{n-2} u^2)_x + \lambda \phi z, \end{aligned} \tag{1.12c}$$

$$z_t = \left[\frac{d r^{2n-2} z_x}{v^2} \right]_x - \phi z, \tag{1.12d}$$

where $t > 0$ and $x \in \mathbb{R}_+ := (0, \infty)$. The corresponding initial and boundary conditions are

$$(v, u, \theta, z)|_{t=0} = (v_0, u_0, \theta_0, z_0) \quad \text{for } x \geq 0, \tag{1.13a}$$

$$(u, \partial_x \theta, \partial_x z)|_{x=0} = 0 \quad \text{for } t \geq 0. \tag{1.13b}$$

The initial data are supposed to be compatible with boundary conditions (1.13b) and satisfy the far-field condition:

$$\lim_{x \to \infty} (v_0, u_0, \theta_0, z_0)(x) = (1, 0, 1, 0). \tag{1.14}$$

We are now in a position to state the main result of the present paper, that is, the global well-posedness theorem for problem (1.12)–(1.13). Clearly, this result implies an equivalent statement for the corresponding problem (1.7)–(1.8) in the Eulerian coordinates.

Theorem 1.1. *Suppose that the transport coefficients μ, ζ, and κ satisfy* (1.3)–(1.4)*, and*

$$b > \frac{19}{4}, \quad 0 \leq \beta < b + 9. \tag{1.15}$$

Let the initial data $(v_0, u_0, \theta_0, z_0)$ be compatible with boundary conditions (1.13b) *and satisfy*

$$\begin{aligned} &(v_0 - 1, u_0, \theta_0 - 1, z_0, \partial_{xx} u_0) \in L^2(\mathbb{R}_+), \quad \inf_{y \in \mathbb{R}_+} \{v_0(y), \theta_0(y)\} > 0, \\ &\quad (r^{n-1} \partial_x v_0, r^{n-1} \partial_x u_0, r^{n-1} \partial_x \theta_0, r^{n-1} \partial_x z_0, r^{n-1} \partial_{xx} v_0) \in L^2(\mathbb{R}_+), \\ &\quad z_0 \in L^1(\mathbb{R}_+), \quad 0 \leq z_0(x) \leq 1 \quad \text{for all } x \in \mathbb{R}_+. \end{aligned} \tag{1.16}$$

Then there exist positive constants $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$, and C such that problem (1.12)–(1.13) *admits a unique global solution (v, u, θ, z) satisfying*

$$\underline{V} \leq v(t, x) \leq \overline{V}, \quad \underline{\Theta} \leq \theta(t, x) \leq \overline{\Theta}, \quad 0 \leq z(t, x) \leq 1, \tag{1.17}$$

and

$$\begin{aligned} &\|(v - 1, u, \theta - 1, z, u_{xx})(t)\|_{L^2(\mathbb{R}_+)}^2 + \left\| r^{n-1} (v_x, u_x, \theta_x, z_x, v_{xx})(t) \right\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad + \|z(t)\|_{L^1(\mathbb{R}_+)} + \int_{\mathbb{R}_+} \left\| r^{n-1} \left(v_x, u_x, \theta_x, z_x, v_{xx}, r^{n-1} u_{xx}, \theta_{xx}, z_{xx} \right)(s) \right\|_{L^2(\mathbb{R}_+)}^2 \mathrm{d}s \leq C, \end{aligned} \tag{1.18}$$

for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$. Moreover, the global solution $(v, u, \theta, z)(t, x)$ converges to the non-vacuum equilibrium state $(1, 0, 1, 0)$ as time goes to infinity:

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}_+} |(v - 1, u, \theta - 1, z)(t, x)| = 0. \tag{1.19}$$

1.3. Outline of the paper and our methodology

As pointed out by Liao and Zhao [19], the crucial step to construct global solutions of initial–boundary value problem (1.12)–(1.13) with large initial data is to obtain the positive lower and upper bounds of the specific volume $v(t,x)$ and temperature $\theta(t,x)$. In particular, in order to deduce the upper bound of $\theta(t,x)$, one has to derive certain estiamtes on some nonlinear terms, among which the most difficulty one is the higher-order term

$$\int_0^t \int_I r^{4n-4} u_x^4$$

for $I := \mathbb{R}_+$. This term can be controlled in Zhang [29] where radius $r(t,x)$ is assumed to satisfy $r_1 \le r(t,x) \le r_2$ with r_1 and r_2 being two positive constants so that domain I is bounded.

However, the domain under our consideration is unbounded. To overcome such difficulty, we will deduce bounds on $\|v_x(t)\|_{L^2(\mathbb{R}_+)}$,

$$\int_0^t \|r^{n-1}u_{xx}(s)\|^2_{L^2(\mathbb{R}_+)}\mathrm{d}s, \quad \int_0^t \|r^{n-1}u_x(s)\|^2_{L^\infty(\mathbb{R}_+)}\mathrm{d}s, \quad \|r^{n-1}v_x(t)\|_{L^2(\mathbb{R}_+)},$$

and $\|r^{n-1}u_x(t)\|_{L^2(\mathbb{R}_+)}$ in terms of $\|\theta\|_{L^\infty([0,T]\times\mathbb{R}_+)}$. Then the nonlinear term $\int_0^t\int_{\mathbb{R}_+} r^{4n-4}u_x^4$ can be estimated as follows (*cf.* (4.59) along with the definitions of l_2 and l_3 in (3.12) and (3.16)):

$$\int_0^t \int_{\mathbb{R}_+} r^{4n-4}u_x^4 \lesssim \int_0^t \|r^{n-1}u_x(s)\|^2_{L^\infty(\mathbb{R}_+)}\|r^{n-1}u_x(s)\|^2\mathrm{d}s \lesssim \|\theta\|^{2l_2+l_3}_{L^\infty([0,T]\times\mathbb{R}_+)}.$$

The above estimate turns out to play a central role in our analysis.

The rest of this paper is organized as follows. In Section 2, we derive the uniform-in-time upper and lower bounds of the specific volume $v(t,x)$. For this purpose, we first construct a normalized entropy η (*cf.* (2.4)) to system (1.12) and derive basic energy estimates for our problem. We emphasize that a different method from Zhang [29] needs to be applied because of the unboundedness of the domain under our consideration. Then we derive a local representation of $v(t,x)$ (*cf.* (2.11)) and obtain the uniform pointwise bounds of $v(t,x)$. This is motivated by the works of Jiang [13,14] on the one-dimensional, compressible Navier–Stokes system for a viscous and heat-conducting ideal polytropic gas.

Section 3 is devoted to the derivation of estimates on $\|v_x(t)\|_{L^2(\mathbb{R}_+)}$, $\int_0^t \|r^{n-1}u_{xx}(s)\|^2_{L^2(\mathbb{R}_+)}\mathrm{d}s$, $\int_0^t \|r^{n-1}u_x(s)\|^2_{L^\infty(\mathbb{R}_+)}\mathrm{d}s$, and $\|r^{n-1}(v_x,u_x)(t)\|_{L^2(\mathbb{R}_+)}$ in terms of $\|\theta\|_{L^\infty([0,T]\times\mathbb{R}_+)}$, which will be useful in getting the upper bound of $\theta(t,x)$.

Motivated by Kawohl [15] and [19], we introduce in Section 4 the auxiliary functions $X(t)$, $Y(t)$, and $Z(t)$ in order to deduce the upper bound of $\theta(t,x)$. We notice that our definition of $Y(t)$ is different from that in Liao and Zhao [19, (2.51)] as a result of (4.16) and the fact that $r(t,x) \ge 1$. Moreover, it is worth noting that the method used in Liang [17] does not work for our case owing to the fourth-order radiative parts in $P(v,\theta)$ and $e(v,\theta)$ (*cf.* (1.2)).

In Section 5, we derive a local-in-time estimate on the lower bound of $\theta(t,x)$. Even though such a bound depends on time variable t, it is sufficient to extend the local solution to a global one by combining these *a priori* estimates with the continuation argument introduced in [27]. Then the proof of Theorem 1.1 is completed.

Notations. To simplify the presentation, we employ C, c, and C_i $(i\in\mathbb{N})$ to denote various positive constants, depending only on $\mu, \lambda_1, \lambda, A, d, R, c_v, a, \kappa_1, \kappa_2, n$ and initial data $(v_0, u_0, \theta_0, z_0)$. Hence C, c, and C_i are independent of t. Symbol $A \lesssim B$ (or $B \gtrsim A$) means that $A \le CB$ holds uniformly for some uniform-in-time constant C.

2. Uniform bounds for the specific volume

The main purpose of this section is to deduce the uniform-in-time positive bounds for the specific volume $v(t,x)$ to initial–boundary value problem (1.12)–(1.13) in terms of initial data $(v_0, u_0, \theta_0, z_0)$. To this end, we assume $(v,u,\theta,z) \in X(0,T; M_1, M_2; N_1, N_2)$, where the set $X(t_1,t_2; M_1, M_2; N_1, N_2)$ is defined as:

$$
\begin{aligned}
X(t_1,t_2;M_1,M_2;N_1,N_2) := \Big\{ & (v,u,\theta,z):\ z\in C(t_1,t_2;L^2(\mathbb{R}_+)\cap L^1(\mathbb{R}_+)),\\
& (v-1,u,\theta-1,r^{n-1}v_x,r^{n-1}u_x,r^{n-1}\theta_x,r^{n-1}z_x,r^{n-1}v_{xx},u_{xx})\in C(t_1,t_2;L^2(\mathbb{R}_+)),\\
& r^{n-1}(v_x,u_x,\theta_x,z_x,v_{xx},u_{xx},\theta_{xx},z_{xx})\in L^2(t_1,t_2;L^2(\mathbb{R}_+)),\\
& M_1\le v(t,x)\le M_2,\ N_1\le \theta(t,x)\le N_2 \quad \forall\,(t,x)\in[t_1,t_2]\times\mathbb{R}_+ \Big\},
\end{aligned}
$$

for constants M, N, t_1, and t_2 $(t_1 \le t_2)$. For the sake of simplicity, we will use the following abbreviations:

$$\|\cdot\|_\infty := \|\cdot\|_{L^\infty([0,T]\times\mathbb{R}_+)}, \qquad \|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}_+)}.$$

In the next lemma we have the basic energy estimates, which will play a fundamental role in deducing the desired positive lower and upper bounds of the specific volume $v(t,x)$.

Lemma 2.1. *Assume that the conditions listed in Theorem 1.1 hold. Then for all $t\in[0,T]$,*

$$\int_{\mathbb{R}_+} z(t,x)\mathrm{d}x + \int_0^t\int_{\mathbb{R}_+}\phi z = \int_{\mathbb{R}_+} z_0(x)\mathrm{d}x, \tag{2.1}$$

$$\int_{\mathbb{R}_+} z^2(t,x)\mathrm{d}x + \int_0^t\int_{\mathbb{R}_+}\left[\frac{dr^{2n-2}z_x^2}{v^2}+\phi z^2\right] = \int_{\mathbb{R}_+} z_0^2(x)\mathrm{d}x, \tag{2.2}$$

$$\sup_{t\in[0,T]}\int_{\mathbb{R}_+}\eta(t,x)\mathrm{d}x + \int_0^T V(t)\mathrm{d}t \lesssim 1, \tag{2.3}$$

where the entropy η and the entropy dissipation rate functional $V(t)$ are defined respectively by

$$\eta(v,\theta):=R\psi(v)+\frac{u^2}{2}+c_v\psi(\theta)+\frac{av}{3}(\theta-1)^2(3\theta^2+2\theta+1),\quad \psi(y):=y-\ln y-1, \tag{2.4}$$

$$V(t):=\int_{\mathbb{R}_+}\left[\frac{\kappa\left(r^{n-1}\theta_x\right)^2}{v\theta^2}+\frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}+\frac{vu^2}{r^2\theta}+\frac{r^{2(n-1)}u_x^2}{v\theta}+\frac{\phi z}{\theta}\right](t,x)\mathrm{d}x. \tag{2.5}$$

Proof. Identities (2.1) and (2.2) follow directly from (1.12d) and integration by parts. According to Liao and Zhao [19], function $\eta(v,\theta)$ is the normalized entropy around $(v,u,\theta)=(1,0,1)$ for system (1.12) with constitutive relations (1.2). A straightforward calculation yields

$$\eta_t+\frac{\alpha\left|(r^{n-1}u)_x\right|^2}{v\theta}+\frac{\kappa\left(r^{n-1}\theta_x\right)^2}{v\theta^2}+\frac{\lambda\phi z}{\theta}+2\mu(n-1)\left(1-\frac{1}{\theta}\right)\left(r^{n-2}u^2\right)_x$$
$$=\left[r^{n-1}u\left(\frac{\alpha(r^{n-1}u)_x}{v}-\frac{R\theta}{v}\right)+\left(R+\frac{a}{3}-\frac{a}{3}\theta^4\right)r^{n-1}u+\left(1-\frac{1}{\theta}\right)\frac{\kappa r^{2n-2}\theta_x}{v}\right]_x+\lambda\phi z.$$

We integrate the above identity over $(0,t)\times\mathbb{R}_+$ and use (2.1) to obtain

$$\int_{\mathbb{R}_+}\eta(t,x)\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}\left[\frac{\kappa\left(r^{n-1}\theta_x\right)^2}{v\theta^2}+\frac{\lambda\phi z}{\theta}\right]$$
$$+\int_0^t\int_{\mathbb{R}_+}\left[\frac{\alpha\left|(r^{n-1}u)_x\right|^2}{v\theta}-2\mu(n-1)\frac{\left(r^{n-2}u^2\right)_x}{\theta}\right]\lesssim 1. \tag{2.6}$$

By virtue of (1.11), we can get (*cf.* [26, (2.14)]) that

$$\frac{\alpha\left|(r^{n-1}u)_x\right|^2}{v\theta}-2\mu(n-1)\frac{\left(r^{n-2}u^2\right)_x}{\theta}\geq\frac{vu^2}{Cr^2\theta}+\frac{r^{2(n-1)}u_x^2}{Cv\theta}.$$

Then we plug the above estimate into (2.6) to derive (2.3). □

Since the pointwise bounds of $z(t,x)$ can be established by a standard maximum principle, we omit the proof for brevity (see, for instance, Chen [1]).

Lemma 2.2. *Assume that the conditions listed in Theorem 1.1 hold. Then*

$$0\leq z(t,x)\leq 1\quad \textit{for all } (t,x)\in[0,T]\times\mathbb{R}_+. \tag{2.7}$$

The next lemma follows from utilizing (2.3) and Jensen's inequality.

Lemma 2.3. *Assume that the conditions listed in Theorem 1.1 hold. Then for all $k\in\mathbb{N}$ and $t\in[0,T]$, there exist $a_k(t),b_k(t)\in\Omega_k:=[k,k+1]$ such that*

$$\int_{\Omega_k}v(t,x)\mathrm{d}x\sim 1,\quad \int_{\Omega_k}\theta(t,x)\mathrm{d}x\sim 1,\quad v(t,a_k(t))\sim 1,\quad \theta(t,b_k(t))\sim 1. \tag{2.8}$$

In the following lemma we deduce a rough estimate on $\theta(t,x)$ in terms of $V(t)$ (*cf.* (2.5)).

Lemma 2.4. *Let* $0 \le m \le b+1$ *and* $x \in \mathbb{R}_+$. *Then*

$$1 - CV(t) \lesssim \theta(t,x)^m \lesssim 1 + V(t), \quad \textit{for all } 0 \le t \le T. \tag{2.9}$$

Proof. We assume $x \in \Omega_k$ without loss of generality. In light of (1.4), we have

$$\begin{aligned}\left|\theta(t,x)^{\frac{m}{2}} - \theta(t,b_k(t))^{\frac{m}{2}}\right| &\lesssim \left[\int_{\Omega_k} \frac{v\theta^m}{\left(1+v\theta^b\right) r^{2(n-1)}} \mathrm{d}x\right]^{\frac{1}{2}} \left[\int_{\Omega_k} \frac{\kappa\left(r^{n-1}\theta_x\right)^2}{v\theta^2} \mathrm{d}x\right]^{\frac{1}{2}} \\ &\lesssim \left[\int_{\Omega_k} \frac{v\theta^m}{1+v\theta^b} \mathrm{d}x\right]^{\frac{1}{2}} V(t)^{\frac{1}{2}} \lesssim V(t)^{\frac{1}{2}}.\end{aligned}$$

To derive the last inequality, we have used assumption $m \le b+1$, boundedness of Ω_k, and estimate (2.8). □

By defining the cut-off function $\varphi \in W^{1,\infty}(\mathbb{R}_+)$ as follows:

$$\varphi(x) = \begin{cases} 1, & 0 \le x \le k+1, \\ k+2-x, & k+1 \le x \le k+2, \\ 0, & x \ge k+2, \end{cases} \tag{2.10}$$

we can deduce a local representation of $v(t,x)$ in the next lemma.

Lemma 2.5. *Assume that the conditions listed in Theorem 1.1 hold. Then*

$$v(t,x) = B(t,x)Q(t,x) + \frac{1}{\alpha}\int_0^t \frac{B(t,x)Q(t,x)v(s,x)P(s,x)}{B(s,x)Q(s,x)} \mathrm{d}s, \quad (t,x) \in [0,T] \times \Omega_k, \tag{2.11}$$

where

$$B(t,x) := v_0(x) \exp\left\{\frac{1}{\alpha}\int_x^\infty \left(u_0 r_0^{1-n} - u r^{1-n}\right)\varphi(y)\mathrm{d}y\right\},$$

$$Q(t,x) := \exp\left\{\frac{1}{\alpha}\int_0^t \int_{\Omega_{k+1}} \left[-P + \frac{\alpha(r^{n-1}u)_x}{v}\right] - \frac{n-1}{\alpha}\int_0^t \int_x^\infty \varphi u^2 r^{-n}\right\}.$$

With basic energy estimates (2.1)–(2.3) and local formulation (2.11) in hand, we are ready to derive uniform-in-time pointwise bounds of $v(t,x)$.

Lemma 2.6. *Assume that the conditions listed in Theorem 1.1 hold. Then*

$$\underline{V} \le v(t,x) \le \overline{V} \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}_+. \tag{2.12}$$

Proof. First we use the second identity of (1.11) to infer $r(t,x) \ge 1$. Then it follows from (2.3) that

$$C^{-1} \le B(t,x) \le C \quad \text{for all } x \in \Omega_k. \tag{2.13}$$

By modifying the argument in [13, p. 186], we can get

$$-\int_s^t \inf_{x\in\Omega_k} \theta(\tau,x)\mathrm{d}\tau \le \begin{cases} 0, & 0 \le t-s \le 1, \\ -C(t-s), & t-s \ge 1. \end{cases}$$

Consequently, we utilize Hölder's and Jensen's inequalities to obtain

$$\int_s^t \int_{\Omega_{k+1}} \left[\frac{\alpha(r^{n-1}u)_x}{v} - \frac{R\theta}{v}\right] \le C \int_s^t \int_{\Omega_{k+1}} \frac{|(r^{n-1}u)_x|^2}{v\theta} - \frac{R}{2}\int_s^t \int_{\Omega_{k+1}} \frac{\theta}{v}$$
$$\le C - \frac{R}{2}\int_s^t \inf_{x\in\Omega_{k+1}} \theta(\tau,\cdot) \int_{\Omega_{k+1}} \frac{1}{v(\tau,x)}\mathrm{d}x\mathrm{d}\tau$$
$$\le C - \frac{R}{2}\int_s^t \inf_{x\in\Omega_{k+1}} \theta(\tau,\cdot) \left[\int_{\Omega_{k+1}} v\mathrm{d}x\right]^{-1} \mathrm{d}\tau \le C - C(t-s), \quad 0 \le s \le t \le T. \tag{2.14}$$

From the definition of $Q(t,x)$ and (2.14), we have for $0 \le s \le t \le T$ that

$$\frac{Q(t,x)}{Q(s,x)} \le \exp\left\{\frac{1}{\alpha}\int_s^t \int_{\Omega_{k+1}} \left[\frac{\alpha(r^{n-1}u)_x}{v} - \frac{R\theta}{v}\right]\right\} \le C\exp\{-C(t-s)\}. \tag{2.15}$$

Using (2.13)–(2.15) and Lemma 2.4, we have

$$v(t,x) \lesssim Q(t,x) + \int_0^t \frac{Q(t,x)}{Q(s,x)} v(s,x)P(s,x)\mathrm{d}s$$
$$\lesssim 1 + \int_0^t \exp\{-C(t-s)\}(1+v(s,x))(1+\theta^4(s,x))\mathrm{d}s$$
$$\lesssim 1 + \int_0^t \exp\{-C(t-s)\}(1+v(s,x))(1+V(s))\mathrm{d}s.$$

Then applying Gronwall's inequality yields the uniform upper bound of v:

$$v(t,x) \le C. \tag{2.16}$$

Now we turn to deduce a positive lower bound of $v(t,x)$. As in [17, Lemma 3.1], we can derive

$$\left| \int_0^t \int_x^\infty \varphi r^{-n} u^2 \right| \lesssim \int_0^t \left\| \frac{u}{r^{n/2}} \right\|_{L^\infty(\mathbb{R}_+)}^2 \mathrm{d}s \lesssim 1, \quad \left| \int_0^t \int_{\Omega_{k+1}} \frac{(r^{n-1}u)_x}{v} \right| \lesssim 1.$$

As a consequence, one has

$$\frac{Q(t,x)}{Q(s,x)} \sim \exp\left\{ -\frac{1}{\alpha} \int_s^t \int_{\Omega_{k+1}} P(v,\theta) \right\} =: \widetilde{Q}(s,t). \tag{2.17}$$

In view of (2.3) and (2.8), we have

$$\int_{\Omega_k} vP\mathrm{d}x \lesssim \int_{\Omega_k} \left[\theta + v\left(1 + (\theta^2 - 1)^2\right) \right] \mathrm{d}x \lesssim \int_{\Omega_k} [\theta + v + \eta] \,\mathrm{d}x \lesssim 1. \tag{2.18}$$

From (2.13), (2.17) and (2.18), we integrate (2.11) over Ω_k to conclude

$$\begin{aligned} 1 &\lesssim \int_{\Omega_k} Q(t,x)\mathrm{d}x + C \int_0^t \int_{\Omega_k} \frac{Q(t,x)}{Q(s,x)} vP(s,x)\mathrm{d}x\mathrm{d}s \\ &\lesssim \exp\{-Ct\} + \int_0^t \widetilde{Q}(s,t) \int_{\Omega_k} vP(s,x)\mathrm{d}x\mathrm{d}s \lesssim \exp\{-Ct\} + \int_0^t \widetilde{Q}(s,t)\mathrm{d}s. \end{aligned} \tag{2.19}$$

Plugging (2.9), (2.13), (2.15), (2.17), and (2.19) into (2.11) yields

$$\begin{aligned} v(t,x) &\gtrsim \int_0^t \frac{Q(t,x)}{Q(s,x)} \theta(s,x)\mathrm{d}s \gtrsim \int_0^t \frac{Q(t,x)}{Q(s,x)} (1 - CV(s))\mathrm{d}s \\ &\gtrsim \int_0^t \widetilde{Q}(s,t)\mathrm{d}s - C\int_0^t \frac{Q(t,x)}{Q(s,x)} V(s)\mathrm{d}s \\ &\gtrsim 1 - C\exp\{-Ct\} - C\left\{ \int_0^{t/2} + \int_{t/2}^t \right\} \exp\{-C(t-s)\} V(s)\mathrm{d}s \end{aligned}$$

$$\gtrsim 1 - C\exp\{-Ct\} - C\exp\left\{-\frac{Ct}{2}\right\} - \int_{\frac{t}{2}}^{t} V(s)\mathrm{d}s \gtrsim 1, \tag{2.20}$$

for $t \geq T_0$, where T_0 is a positive constant independent of t.

On the other hand, we use (2.13), (2.17), and integrate (2.11) over Ω_k to find

$$\widetilde{Q}(0,t)^{-1} \lesssim 1 + \int_0^t \widetilde{Q}(0,s)^{-1} \int_{\Omega_k} vP\mathrm{d}x\mathrm{d}s,$$

which combined with (2.18) implies $\widetilde{Q}(0,t) \gtrsim \exp\{-Ct\}$. Use again (2.11) to deduce that $v(t,x) \gtrsim \exp\{-Ct\}$. Then we combine with (2.20) to obtain the uniform lower bound of v and hence finish the proof of this lemma. □

The next corollary follows directly from (1.9) and Lemma 2.6.

Corollary 2.7. *Assume that the conditions listed in Theorem 1.1 hold. Then*

$$1 + C^{-1}x \leq r^n(t,x) \leq 1 + Cx \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}_+. \tag{2.21}$$

3. More energy estimates

Lemma 3.1. *Under the assumptions listed in Theorem 1.1, we have*

$$\int_0^t \left\| \frac{u}{r}(s) \right\|_{L^\infty(\mathbb{R}_+)}^2 \mathrm{d}s \lesssim 1 \quad \textit{for all } t \in [0,T]. \tag{3.1}$$

Proof. Estimate (3.1) for $n = 2$ can be proved by a similar argument in [17, Lemma 3.1]. Now let $n \geq 3$. Since

$$u^2 = \left(\int_x^\infty u_x \mathrm{d}x \right)^2 \lesssim \int_{\mathbb{R}_+} \frac{r^{2n-2}u_x^2}{v\theta}\mathrm{d}x \int_{\mathbb{R}_+} \frac{v\theta}{r^{2n-2}}\mathrm{d}x,$$

and

$$\int_{\mathbb{R}_+} \frac{v\theta}{r^{2n-2}}\mathrm{d}x = \sum_{k=1}^\infty \int_k^{k+1} \frac{\theta}{r^{n-1}}\mathrm{d}r \lesssim \sum_{k=1}^\infty \frac{1}{k^{n-1}} \int_k^{k+1} \theta\mathrm{d}r \lesssim \sum_{k=1}^\infty \frac{1}{k^{n-1}} \lesssim 1,$$

we use the fact that $r \geq 1$ to complete the proof of our lemma. □

Lemma 3.2. *Under the assumptions listed in Theorem 1.1, we have for any* $0 \le t \le T$ *that*

$$\|v_x(t)\|^2 + \int_0^t \|\sqrt{\theta} v_x(s)\|^2 \mathrm{d}s \lesssim 1 + C\|\theta\|_\infty^{l_1}, \tag{3.2}$$

where

$$l_1 = \max\{1, (7-b)_+\}. \tag{3.3}$$

Proof. We first rewrite (1.12b) as

$$r^{1-n} u_t + P_x = \alpha \left(\frac{v_t}{v}\right)_x = \alpha \left(\frac{v_x}{v}\right)_t. \tag{3.4}$$

Multiply the above equation by v_x/v to get

$$\left[\frac{\alpha}{2}\left(\frac{v_x}{v}\right)^2 - r^{1-n} u \frac{v_x}{v}\right]_t + \frac{R\theta v_x^2}{v^3} + \left[r^{1-n} u \frac{v_t}{v}\right]_x$$
$$= \frac{R\theta_x v_x}{v^2} + \frac{4a}{3}\frac{\theta^3 v_x \theta_x}{v} + (n-1) r^{-n} \frac{u^2 v_x}{v} + \left(r^{1-n} u\right)_x \frac{v_t}{v}.$$

Integrating this last identity, we utilize (2.3), (2.12), and Cauchy's inequality to deduce

$$\|v_x(t)\|^2 + \int_0^t \|\sqrt{\theta} v_x(s)\|^2 \mathrm{d}s \lesssim 1 + \int_0^t \int_{\mathbb{R}_+} \left[(\theta^{-1} + \theta^5)\theta_x^2 + r^{-n} u^2 |v_x| + |(r^{n-1} u)_x v_t|\right]. \tag{3.5}$$

We now estimate the terms on the right-hand side of (3.5). First, it follows from (2.3) that

$$\int_0^t \int_{\mathbb{R}_+} (\theta^{-1} + \theta^5)\theta_x^2 \lesssim \int_0^t \int_{\mathbb{R}_+} \frac{\kappa \theta_x^2}{\theta^2} \frac{\theta + \theta^7}{1 + \theta^b} \lesssim 1 + \|\theta\|_\infty^{(7-b)_+}. \tag{3.6}$$

By virtue of (2.3), we get

$$\int_0^t \int_{\mathbb{R}_+} r^{-n} u^2 |v_x| \lesssim \|\theta\|_\infty \int_0^t \int_{\mathbb{R}_+} \frac{u^2}{r^2 \theta} + \int_0^t \left\|\frac{u}{r}\right\|_{L^\infty(\mathbb{R}_+)}^2 \|v_x\|^2$$
$$\lesssim \|\theta\|_\infty + \int_0^t \left\|\frac{u}{r}\right\|_{L^\infty(\mathbb{R}_+)}^2 \|v_x\|^2. \tag{3.7}$$

Using (1.12a) and

$$\left(r^{1-n} u\right)_x = r^{2(1-n)} (r^{n-1} u)_x + 2(1-n) r^{1-2n} v u,$$

we have

$$\int_0^t \int_{\mathbb{R}_+} |(r^{n-1}u)_x v_t| \lesssim \|\theta\|_\infty \int_0^t \int_{\mathbb{R}_+} \left[\frac{r^{2(1-n)} \left|(r^{n-1}u)_x\right|^2}{v\theta} + \frac{vu^2}{r^2\theta} \right] \lesssim \|\theta\|_\infty. \tag{3.8}$$

Plug (3.6)–(3.8) into (3.5), apply Gronwall's inequality and use (3.1) to obtain (3.2). □

Lemma 3.3. *Under the assumptions listed in Theorem 1.1, we have*

$$\sup_{t\in[0,T]} \|u_x(t)\|^2 + \int_0^T \|r^{n-1}u_{xx}\|^2 \lesssim 1 + \|\theta\|_\infty^{l_2}, \tag{3.9}$$

$$\|u\|_\infty \lesssim 1 + C\|\theta\|_\infty^{l_2/4}, \tag{3.10}$$

$$\int_0^T \|r^{n-1}u_x\|^2_{L^\infty(\mathbb{R}_+)} \mathrm{d}s \lesssim 1 + \|\theta\|_\infty^{l_2}, \tag{3.11}$$

where

$$l_2 = \max\left\{2l_1 + 1, (8-b)_+\right\}. \tag{3.12}$$

Proof. Multiplying (1.12b) by u_{xx} yields

$$\left(\frac{u_x^2}{2}\right)_t + \frac{\alpha r^{2(n-1)}u_{xx}^2}{v} - \alpha u_{xx}\left[\frac{r^{2(n-1)}u_x v_x}{v^2} + (n-1)\frac{uv}{r^2} - 2(n-1)r^{n-2}u_x\right]$$
$$= (u_x u_t)_x + R u_{xx} r^{n-1}\left(\frac{\theta_x}{v} - \frac{\theta v_x}{v^2}\right) + u_{xx} r^{n-1}\cdot\frac{4a}{3}\theta^3\theta_x.$$

Integrating this last identity over $[0,t]\times\mathbb{R}_+$, we get from (2.3) and (3.2) that

$$\begin{aligned}
&\|u_x(t)\|^2 + \int_0^t \|r^{n-1}u_{xx}(s)\|^2 \mathrm{d}s \\
&\lesssim 1 + \int_0^t \int_{\mathbb{R}_+} \left[r^{2(n-1)}u_x^2 v_x^2 + \frac{u^2}{r^{2(n+1)}} + \frac{u_x^2}{r^2} + \theta_x^2 + \theta^2 v_x^2 + \theta^6\theta_x^2\right] \\
&\lesssim 1 + \left[1 + \|\theta\|_\infty^{l_1}\right]\left[\int_0^t \left\|r^{n-1}u_x\right\|^2_{L^\infty(\mathbb{R}_+)} \mathrm{d}s + \|\theta\|_\infty\right] + \|\theta\|_\infty^{(8-b)_+}.
\end{aligned} \tag{3.13}$$

Noticing

$$\left(r^{n-1}u_x\right)_x = (n-1)r^{-1}vu_x + r^{n-1}u_{xx},$$

we have

$$\int_0^t \left\|r^{n-1}u_x\right\|^2_{L^\infty(\mathbb{R}_+)} \mathrm{d}s \lesssim \int_0^t \left\|r^{n-1}u_x\right\| \left\|\left(r^{n-1}u_x\right)_x\right\| \mathrm{d}s$$

$$\lesssim \epsilon \int_0^t \|r^{n-1}u_{xx}\|^2 + \frac{C}{\epsilon}\int_0^t \|r^{n-1}u_x\|^2 \lesssim \epsilon \int_0^t \|r^{n-1}u_{xx}\|^2 + \frac{C}{\epsilon}\|\theta\|_\infty, \tag{3.14}$$

for any positive ϵ. Plugging (3.14) with $\epsilon := (1 + \|\theta\|_\infty^{l_1})^{-1}$ to (3.13) yields (3.9).

Estimates (3.10) and (3.11) follow immediately from (2.3), (3.9), and (3.14). □

Lemma 3.4. *Under the assumptions listed in Theorem 1.1, we have for all* $0 \le t \le T$ *that*

$$\|r^{n-1}v_x(t)\|^2 + \int_0^t \|r^{n-1}\sqrt{\theta}v_x(s)\|^2 \mathrm{d}s \lesssim 1 + \|\theta\|_\infty^{l_3}, \tag{3.15}$$

where

$$l_3 = \max\{1, l_2/2, (7-b)_+\}. \tag{3.16}$$

Proof. In view of (1.12a)–(1.12b), we derive

$$\alpha\left(\frac{r^{n-1}v_x}{v}\right)_t + \frac{Rr^{n-1}\theta v_x}{v^2} = u_t - \alpha(n-1)r^{n-2}u\frac{v_x}{v} + \frac{Rr^{n-1}\theta_x}{v} + \frac{4a}{3}r^{n-1}\theta^3\theta_x.$$

Multiplying this identity by $r^{n-1}v_x/v$, we utilize Cauchy's inequality, (1.2), and (2.3) to deduce

$$\|r^{n-1}v_x(t)\|^2 + \int_0^t \|r^{n-1}\sqrt{\theta}v_x(s)\|^2 \mathrm{d}s$$

$$\lesssim 1 + \int_0^t\int_{\mathbb{R}_+} \left[r^{n-2}u^2|v_x| + \frac{|(r^{n-1}u)_x|^2}{v} + r^{2n-3}|u|v_x^2 + \frac{\kappa r^{2(n-1)}\theta_x^2}{v\theta^2}\cdot\frac{\theta+\theta^7}{1+\theta^b}\right]$$

$$\lesssim 1 + \|\theta\|^{(7-b)_+} + \int_0^t\int_{\mathbb{R}_+} \left[r^{n-2}u^2|v_x| + \frac{|(r^{n-1}u)_x|^2}{v}\right] + \int_0^t \left\|\frac{u}{r}\right\|^2_{L^\infty(\mathbb{R}_+)} \|r^{n-1}v_x\|^2. \tag{3.17}$$

From (2.3) and (3.10), one has

$$\int_0^t \int_{\mathbb{R}_+} \left[r^{n-2} u^2 |v_x| + \frac{|(r^{n-1}u)_x|^2}{v} \right]$$
$$\lesssim \epsilon \int_0^t \|r^{n-1}\sqrt{\theta} v_x(s)\|^2 \mathrm{d}s + \frac{C}{\epsilon} \int_0^t \|u\|^2_{L^\infty(\mathbb{R}_+)} \int_{\mathbb{R}_+} \frac{u^2 v}{r^2\theta} + \|\theta\|_\infty \int_0^t \int_{\mathbb{R}_+} \frac{|(r^{n-1}u)_x|^2}{v\theta}$$
$$\lesssim \epsilon \int_0^t \|r^{n-1}\sqrt{\theta} v_x(s)\|^2 \mathrm{d}s + \frac{C}{\epsilon}(1 + \|\theta\|_\infty^{l_2/2}) + \|\theta\|_\infty.$$

Plug the last estimate into (3.17), employ Gronwall's inequality, and use (3.1) to obtain (3.15). □

Lemma 3.5. *Under the assumptions listed in Theorem 1.1, we have for all* $0 \le t \le T$ *that*

$$\|r^{n-1}u_x(t)\|^2 + \int_0^t \|r^{2(n-1)}u_{xx}(s)\|^2 \mathrm{d}s \lesssim 1 + \|\theta\|_\infty^{l_2+l_3}. \tag{3.18}$$

Proof. Multiplying (1.12b) by $r^{2n-2}u_{xx}$, we obtain

$$\partial_t \left(\frac{r^{2n-2}u_x^2}{2} \right) - \left(r^{2n-2}u_t u_x \right)_x + \frac{\alpha r^{4n-4}u_{xx}^2}{v} - (n-1)r^{2n-3}uu_x^2 + 2(n-1)vr^{n-2}u_x u_t$$
$$= r^{3n-3}u_{xx}\left[P_x - 2\alpha(n-1)r^{-1}u_x + \alpha(n-1)r^{-n-1}uv + \frac{\alpha r^{n-1}u_x v_x}{v^2} \right]. \tag{3.19}$$

Integrating the above identity over $[0,t) \times \mathbb{R}_+$ yields

$$\|r^{n-1}u_x(t)\|^2 + \int_0^t \|r^{2(n-1)}u_{xx}(s)\|^2 \mathrm{d}s \lesssim 1 + \sum_{j=1}^3 \mathcal{H}_j, \tag{3.20}$$

where $\mathcal{H}_j$ $(j = 1, 2, 3)$ are defined and estimated below. In view of (2.3), (3.10) and (3.11), we have

$$\mathcal{H}_1 := \int_0^t \int_{\mathbb{R}_+} \left[r^{2n-4}u_x^2 + r^{-4}u^2 + r^{4(n-1)}u_x^2 v_x^2 + r^{2n-3}|u|u_x^2 \right]$$
$$\lesssim \|\theta\|_\infty + \int_0^t \|r^{n-1}u_x\|^2_{L^\infty(\mathbb{R}_+)} \|r^{n-1}v_x\|^2 + \|\theta\|_\infty \|u\|_\infty \int_0^t \int_{\mathbb{R}_+} \frac{r^{2(n-1)}u_x^2}{\theta}$$
$$\lesssim 1 + \|\theta\|_\infty^{l_2+l_3}. \tag{3.21}$$

Then it follows from (2.3), (2.12), and (3.15) that

$$\begin{aligned}\mathcal{H}_2 &:= \int_0^t \int_{\mathbb{R}_+} r^{2(n-1)} P_x^2 \\ &\lesssim \int_0^t \int_{\mathbb{R}_+} r^{2(n-1)} \left(\theta_x^2 + \theta^2 v_x^2 + \theta^6 \theta_x^2\right) \\ &\lesssim \int_0^t \int_{\mathbb{R}_+} \frac{\kappa r^{2(n-1)}\theta_x^2}{v\theta^2} \cdot \frac{\theta^2+\theta^8}{1+\theta^b} + \|\theta\|_\infty \int_0^t \int_{\mathbb{R}_+} r^{2(n-1)}\theta v_x^2 \\ &\lesssim 1 + \|\theta\|_\infty^{l_3+1} + \|\theta\|_\infty^{(8-b)_+}. \end{aligned} \tag{3.22}$$

For the last term

$$\mathcal{H}_3 := \left| \int_0^t \int_{\mathbb{R}_+} v r^{n-2} u_x u_t \right|,$$

we obtain from (1.11), (1.12b), (2.3) that

$$\begin{aligned}\mathcal{H}_3 &\lesssim \int_0^t \int_{\mathbb{R}_+} \left[r^{n-4}|uu_x| + r^{2(n-2)}u_x^2 + r^{3n-4}|u_x u_{xx}| + r^{3n-4}|v_x|u_x^2 + r^{2n-3}|u_x P_x| \right] \\ &\lesssim \epsilon \int_0^t \|r^{2(n-1)}u_{xx}\|^2 + C(\epsilon)\|\theta\|_\infty + \underbrace{\int_0^t \int_{\mathbb{R}_+} \left[r^{3n-4}|v_x|u_x^2 + r^{2n-3}|u_x P_x| \right]}_{\mathcal{H}_{3a}}. \end{aligned} \tag{3.23}$$

By virtue of (2.3), (3.11), and (3.15), we obtain

$$\begin{aligned}\mathcal{H}_{3a} &\lesssim \|\theta\|_\infty + \int_0^t \|r^{n-1}u_x\|_{L^\infty(\mathbb{R}_+)}^2 \|r^{n-1}v_x\|^2 + \int_0^t \int_{\mathbb{R}_+} r^{2n-3}\,|u_x|\left(|\theta_x| + \theta\,|v_x| + \theta^3\,|\theta_x|\right) \\ &\lesssim 1 + \|\theta\|_\infty^{l_2+l_3} + \int_0^t \int_{\mathbb{R}_+} \frac{\kappa r^{2n-2}\theta_x^2}{v\theta^2}\frac{\theta^2+\theta^8}{1+\theta^b} + \int_0^t \int_{\mathbb{R}_+} r^{2n-2}\theta^2 v_x^2 \lesssim 1 + \|\theta\|_\infty^{l_2+l_3}. \end{aligned} \tag{3.24}$$

Inserting estimates (3.21)–(3.24) into (3.20) yields (3.18). □

4. Upper bound of the temperature

In this section we shall derive an estimate on the upper bound of θ. To this end, we set

$$X(t) := \int_0^t \int_{\mathbb{R}_+} \left(1+\theta^{b+3}(s,x)\right) \theta_t^2(s,x) \mathrm{d}x \mathrm{d}s, \tag{4.1}$$

$$Y(t) := \sup_{s\in(0,t)} \int_{\mathbb{R}_+} r^{2n-2} \left(1+\theta^{2b}(s,x)\right) \theta_x^2(s,x) \mathrm{d}x, \tag{4.2}$$

$$Z(t) := \sup_{s\in(0,t)} \int_{\mathbb{R}_+} u_{xx}^2(s,x) \mathrm{d}x, \tag{4.3}$$

and then try to deduce certain estimates among them by employing the special structure of system (1.12).

Lemma 4.1. *Under the assumptions listed in Theorem 1.1, we have for all* $0 \le t \le T$ *that*

$$\|\theta(t)\|_{L^\infty(\mathbb{R}_+)} \lesssim 1 + Y(t)^{\frac{1}{2b+6}}, \tag{4.4}$$

$$\sup_{s\in(0,t)} \|u_x(s)\|^2 \lesssim 1 + Z(t)^{\frac{1}{2}}, \quad \|u_x(t)\|_{L^\infty(\mathbb{R}_+)} \lesssim 1 + Z(t)^{\frac{3}{8}}. \tag{4.5}$$

Proof. We assume that $x \in [k, k+1]$ for some $k \in \mathbb{N}$ and $x \ge b_k(t)$. Then

$$\begin{aligned}
(\theta(t,x)-1)^{2b+6} &= (\theta(t,b_k(t))-1)^{2b+6} + \int_{b_k(t)}^{x} (2b+6)(\theta(t,y)-1)^{2b+5} \theta_x(t,y) \mathrm{d}y \\
&\lesssim 1 + \|\theta(t)-1\|_{L^\infty(\mathbb{R}_+)}^{b+3} \left[\int_{\mathbb{R}_+} (\theta-1)^4 \mathrm{d}x\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}_+} (\theta-1)^{2b} \theta_x^2 r^{2n-2} \mathrm{d}x\right]^{\frac{1}{2}} \\
&\lesssim 1 + C\|\theta(t)-1\|_{L^\infty(\mathbb{R}_+)}^{b+3} Y^{\frac{1}{2}}(t),
\end{aligned}$$

from which we can deduce (4.4) by using Cauchy's inequality. Estimates (4.5) follow directly by applying Gagliardo–Nirenberg and Sobolev inequalities. □

Our next result shows that $X(t)$ and $Y(t)$ can be controlled by $Z(t)$.

Lemma 4.2. *Under the assumptions listed in Theorem 1.1, we have for* $0 \le t \le T$ *that*

$$X(t) + Y(t) \lesssim 1 + Z(t)^{\lambda_1}. \tag{4.6}$$

Here λ_1 is given by

$$\lambda_1 = \max\left\{\frac{b+3}{4(b+4)}, \frac{3(b+3)}{2(3b+9-2l_3)}\right\}. \tag{4.7}$$

Proof. As in Kawohl [15], we set

$$K(v,\theta) = \int_0^\theta \frac{\kappa(v,\xi)}{v}\mathrm{d}\xi = \frac{\kappa_1\theta}{v} + \frac{\kappa_2\theta^{b+1}}{b+1}. \tag{4.8}$$

Thanks to (2.12), we have

$$K_t = K_v(v,\theta)(r^{n-1}u)_x + \frac{\kappa\theta_t}{v}, \tag{4.9}$$

$$K_{xt} = \left(\frac{\kappa\theta_x}{v}\right)_t + K_v(r^{n-1}u)_{xx} + K_{vv}v_x(r^{n-1}u)_x + \left(\frac{\kappa}{v}\right)_v v_x\theta_t, \tag{4.10}$$

$$|K_v(v,\theta)| + |K_{vv}(v,\theta)| \lesssim \theta. \tag{4.11}$$

We can rewrite (1.12c) as

$$e_\theta\theta_t + \theta P_\theta(r^{n-1}u)_x - \frac{\alpha\left|(r^{n-1}u)_x\right|^2}{v} = \left(\frac{r^{2n-2}\kappa\theta_x}{v}\right)_x - 2\mu(n-1)\left(r^{n-2}u^2\right)_x + \lambda\phi z. \tag{4.12}$$

Multiplying (4.12) by K_t and integrating the resulting identity over $[0,t]\times\mathbb{R}_+$ yield

$$\int_0^t\int_{\mathbb{R}_+}\left[e_\theta(v,\theta)\theta_t + \theta P_\theta(v,\theta)(r^{n-1}u)_x - \frac{\alpha\left|(r^{n-1}u)_x\right|^2}{v}\right]K_t + \int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_x}{v}K_{tx} + 2\mu(n-1)\int_0^t\int_{\mathbb{R}_+}\left(r^{n-2}u^2\right)_x K_t = \int_0^t\int_{\mathbb{R}_+}\lambda\phi z K_t. \tag{4.13}$$

Combining (4.8)–(4.13), we have

$$\int_0^t\int_{\mathbb{R}_+}\frac{e_\theta(v,\theta)\kappa(v,\theta)\theta_t^2}{v} + \int_0^t\int_{\mathbb{R}_+}\frac{\kappa(v,\theta)\theta_x}{v}\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_t$$

$$\lesssim 1 + \underbrace{\left|\int_0^t\int_{\mathbb{R}_+} e_\theta(v,\theta)\theta_t K_v(v,\theta)(r^{n-1}u)_x\right|}_{I_1} + \underbrace{\left|\int_0^t\int_{\mathbb{R}_+}\theta p_\theta(v,\theta)\left|(r^{n-1}u)_x\right|^2 K_v(v,\theta)\right|}_{I_2}$$

$$
\begin{aligned}
&+\underbrace{\left|\int_0^t\int_{\mathbb{R}_+}\frac{\theta p_\theta(v,\theta)\kappa(v,\theta)(r^{n-1}u)_x\theta_t}{v}\right|}_{I_3}+\underbrace{\left|\int_0^t\int_{\mathbb{R}_+}\frac{\alpha\left|(r^{n-1}u)_x\right|^2K_t(v,\theta)}{v}\right|}_{I_4}\\
&+\underbrace{\left|\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_x}{v}\Big(K_{vv}(v,\theta)v_x(r^{n-1}u)_x+K_v(v,\theta)(r^{n-1}u)_{xx}\Big)\right|}_{I_5}\\
&+\underbrace{\left|\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_x}{v}\left(\frac{\kappa(v,\theta)}{v}\right)_v v_x\theta_t\right|}_{I_6}+\underbrace{\left|\int_0^t\int_{\mathbb{R}_+}\lambda\phi zK_v(v,\theta)(r^{n-1}u)_x\right|}_{I_7}\\
&+\underbrace{\left|\int_0^t\int_{\mathbb{R}_+}\frac{\lambda\phi z\kappa(v,\theta)\theta_t}{v}\right|}_{I_8}+\underbrace{\left|\int_0^t\int_{\mathbb{R}_+}\Big(r^{n-2}u^2\Big)_x\left(K_v(v,\theta)(r^{n-1}u)_x+\frac{\kappa(v,\theta)\theta_t}{v}\right)\right|}_{I_9}. \qquad (4.14)
\end{aligned}
$$

We now turn to control I_k $(k=1,\dots,9)$ term by term. To do so, we first have

$$
\int_0^t\int_{\mathbb{R}_+}\frac{e_\theta(v,\theta)\kappa(v,\theta)\theta_t^2}{v}\gtrsim\int_0^t\int_{\mathbb{R}_+}\Big(1+\theta^3\Big)\Big(1+\theta^b\Big)\theta_t^2\gtrsim X(t), \qquad (4.15)
$$

and

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_x}{v}\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_t\\
&=\frac{1}{2}\int_{\mathbb{R}_+}r^{2n-2}\left(\frac{\kappa\theta_x}{v}\right)^2\mathrm{d}x-C-(n-1)\int_0^t\int_{\mathbb{R}_+}\left(\frac{\kappa\theta_x}{v}\right)^2r^{2n-3}u\\
&\gtrsim Y(t)-C-C\int_0^t\int_{\mathbb{R}_+}\left(\frac{\kappa\theta_x}{v}\right)^2r^{2n-3}u. \qquad (4.16)
\end{aligned}
$$

Furthermore, we obtain

$$
\left|\int_0^t\int_{\mathbb{R}_+}\left(\frac{\kappa\theta_x}{v}\right)^2r^{2n-3}u\right|\lesssim\int_0^t\|u\|_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+}\Big(1+\theta^b\Big)\frac{\kappa\left(r^{n-1}\theta_x\right)^2}{v^2\theta^2}\frac{\theta^2}{r}
$$

$$\lesssim \left(1+\|\theta\|_\infty^{b+2}\right)\int_0^t \|u\|^{\frac12}\|u_x\|^{\frac12}\int_{\mathbb{R}_+}\frac{\kappa\left(r^{n-1}\theta_x\right)^2}{v^2\theta^2}$$

$$\lesssim \left(1+Y(t)^{\frac{b+2}{2b+6}}\right)\left(1+Z(t)^{\frac18}\right)\lesssim C(\epsilon)\left(1+Z(t)^{\frac{b+3}{4(b+4)}}\right)+\epsilon Y(t). \tag{4.17}$$

On the other hand, it follows from Cauchy's inequality and (2.3) that

$$I_1 \lesssim \int_0^t\int_{\mathbb{R}_+}(1+\theta)^4\left|\theta_t(r^{n-1}u)_x\right|$$

$$\le \epsilon X(t)+C(\epsilon)\left(1+\|\theta\|_\infty^{(6-b)_+}\right)\int_0^t\int_{\mathbb{R}_+}\frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}\cdot v$$

$$\le \epsilon X(t)+C(\epsilon)\left(1+Y(t)^{\frac{(6-b)_+}{2b+6}}\right)\le \epsilon(X(t)+Y(t))+C(\epsilon), \tag{4.18}$$

$$I_2 \lesssim \int_0^t\int_{\mathbb{R}_+}(1+\theta)^5\left|(r^{n-1}u)_x\right|^2$$

$$\lesssim \int_0^t\int_{\mathbb{R}_+}(1+\theta)^6\cdot\frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}\lesssim 1+\|\theta\|_\infty^6\lesssim C(\epsilon)+\epsilon Y(t), \tag{4.19}$$

and

$$I_3 \lesssim \int_0^t\int_{\mathbb{R}_+}(1+\theta)^{b+4}\left|(r^{n-1}u)_x\theta_t\right|$$

$$\le \epsilon X(t)+C(\epsilon)\left(1+\|\theta\|_\infty^{b+6}\right)\int_0^t\int_{\mathbb{R}_+}\frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}$$

$$\le \epsilon X(t)+C(\epsilon)Y(t)^{\frac{b+6}{2b+6}}\le \epsilon\left(X(t)+Y(t)\right)+C(\epsilon). \tag{4.20}$$

As for the term I_4, it is easy to see that

$$I_4=\left|\int_0^t\int_{\mathbb{R}_+}\frac{\alpha\left|(r^{n-1}u)_x\right|^2}{v}\left(K_v(r^{n-1}u)_x+\frac{\kappa(v,\theta)\theta_t}{v}\right)\right|$$

$$\lesssim \int_0^t\int_{\mathbb{R}_+}\left|(r^{n-1}u)_x\right|^3\theta+\int_0^t\int_{\mathbb{R}_+}\left|(r^{n-1}u)_x\right|^2\left(1+\theta^b\right)|\theta_t| =: I_{13}^a+I_{13}^b. \tag{4.21}$$

We deduce from (3.1), (3.11), (3.18), and the fact $2l_2 + l_3 < b + 9$ (since $b > 19/4$) that

$$\begin{aligned}
\left|I_4^a\right| &\lesssim \int_0^t \int_{\mathbb{R}_+} |(r^{n-1}u)_x|^2\theta^2 + \int_0^t \int_{\mathbb{R}_+} \left|(r^{n-1}u)_x\right|^4 \\
&\lesssim \|\theta\|_\infty^3 + \int_0^t \int_{\mathbb{R}_+} \left(r^{-4}u^4 + r^{4n-4}u_x^4\right) \\
&\lesssim 1 + Y(t)^{\frac{3}{2b+6}} + \int_0^t \left\|\frac{u}{r}\right\|_{L^\infty(\mathbb{R}_+)}^2 \|u\|^2 + \int_0^t \|r^{n-1}u_x\|_{L^\infty(\mathbb{R}_+)}^2 \|r^{n-1}u_x\|^2 \\
&\lesssim 1 + Y(t)^{\frac{3}{2b+6}} + Y(t)^{\frac{2l_2+l_3}{2b+6}} \le \epsilon Y(t) + C(\epsilon)
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
\left|I_4^b\right| &\le \epsilon X(t) + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \left(1+\theta^{b-3}\right)(r^{n-1}u)_x^4 \\
&\le \epsilon X(t) + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \left(1+\theta^{b-3}\right)\left(\frac{u^4}{r^4} + r^{4n-4}u_x^4\right) \\
&\le \epsilon X(t) + C(\epsilon)\left(1+\|\theta\|_\infty^{b-3}\right)\left(1 + \int_0^t \|r^{n-1}u_x\|_{L^\infty(\mathbb{R}_+)}^2 \|r^{n-1}u_x\|^2\right) \\
&\le \epsilon X(t) + C(\epsilon)\left(1+\|\theta\|_\infty^{2l_2+l_3+b-3}\right) \le \epsilon\left(X(t)+Y(t)\right) + C(\epsilon).
\end{aligned} \tag{4.23}$$

Thus the combination of (4.21)–(4.23) and the assumption $b > 19/4$ yields

$$I_4 \le \epsilon\left(X(t)+Y(t)\right) + C(\epsilon). \tag{4.24}$$

Now for the term I_5, we have from (4.11) that

$$I_5 \lesssim \int_0^t \int_{\mathbb{R}_+} r^{2n-2}\left(1+\theta^{b+1}\right)\left[\left|\theta_x v_x (r^{n-1}u)_x\right| + \left|\theta_x (r^{n-1}u)_{xx}\right|\right] =: I_5^a + I_5^b, \tag{4.25}$$

then we can obtain from (2.3), (3.1), (3.9), (3.11), (3.15) and the fact that

$$(r^{n-1}u)_{xx} = 2(n-1)r^{-1}vu_x + (n-1)r^{-1}v_x u - (n-1)r^{-1-n}v^2u + r^{n-1}u_{xx},$$

that

$$I_5^a \lesssim 1 \int_0^t \int_{\mathbb{R}_+} \frac{\kappa r^{2n-2}\theta_x^2}{v\theta^2} \cdot \frac{1+\theta^{2b+4}}{1+\theta^b} + C \int_0^t \int_{\mathbb{R}_+} r^{2n-2} v_x^2 \left|(r^{n-1}u)_x\right|^2$$

$$\lesssim 1 + C\|\theta\|_\infty^{b+4} + C \int_0^t \int_{\mathbb{R}_+} r^{2n-2} v_x^2 \left(\frac{u^2}{r^2} + r^{2n-2}u_x^2\right)$$

$$\lesssim 1 + CY(t)^{\frac{b+4}{2b+6}} + \int_0^t \|r^{n-1}v_x\|^2 \left[\left\|\frac{u}{r}\right\|_{L^\infty(\mathbb{R}_+)}^2 + \left\|r^{n-1}u_x\right\|_{L^\infty(\mathbb{R}_+)}^2\right]$$

$$\lesssim 1 + CY(t)^{\frac{b+4}{2b+6}} + CY(t)^{\frac{l_2+l_3}{2b+6}} \le \epsilon Y(t) + C(\epsilon), \tag{4.26}$$

and

$$I_5^b \lesssim 1 \int_0^t \int_{\mathbb{R}_+} \frac{\kappa r^{2n-2}\theta_x^2}{v\theta^2} \cdot \frac{1+\theta^{2b+4}}{1+\theta^b} + \int_0^t \int_{\mathbb{R}_+} \frac{r^{2n-2}\left|(r^{n-1}u)_{xx}\right|^2}{v}$$

$$\lesssim 1 + \|\theta\|_\infty^{b+4} + \int_0^t \int_{\mathbb{R}_+} \left(r^{2n-4}u_x^2 + r^{2n-4}u^2v_x^2 + \frac{u^2}{r^4} + r^{4n-4}u_{xx}^2\right)$$

$$\lesssim 1 + Y(t)^{\frac{b+4}{2b+6}} + \int_0^t \int_{\mathbb{R}_+} \left(\frac{r^{2n-2}u_x^2}{v\theta} \cdot \frac{v\theta}{r^2} + \frac{vu^2}{r^2\theta} \cdot \frac{\theta}{vr^2}\right)$$

$$+ \int_0^t \left\|\frac{u}{r}\right\|_{L^\infty(\mathbb{R}_+)}^2 \int_{\mathbb{R}_+} r^{2n-2}v_x^2 + \int_0^t \int_{\mathbb{R}_+} r^{4n-4}u_{xx}^2$$

$$\lesssim 1 + Y(t)^{\frac{b+4}{2b+6}} + Y(t)^{\frac{l_2+l_3}{2b+6}} \le \epsilon Y(t) + C(\epsilon). \tag{4.27}$$

Combining (4.25)–(4.27) together, we get

$$I_5 \le \epsilon Y(t) + C(\epsilon). \tag{4.28}$$

As to the term I_6, it is easy to see that

$$I_6 \lesssim \int_0^t \int_{\mathbb{R}_+} r^{2n-2}\left(1+\theta^b\right)|\theta_x\theta_t v_x| \le \epsilon X(t) + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} r^{4n-4}\left(1+\theta^{b-3}\right)\theta_x^2 v_x^2$$

$$\le \epsilon X(t) + C(\epsilon) \underbrace{\int_0^t \left\|\frac{r^{n-1}\kappa(v,\theta)\theta_x}{v}\right\|_{L^\infty(\mathbb{R}_+)}^2 \left\|r^{n-1}v_x\right\|^2 \mathrm{d}s}_{I_6^a}. \tag{4.29}$$

We can deduce from (4.12) that

$$\left(\frac{r^{n-1}\kappa\theta_x}{v}\right)_x = \frac{e_\theta\theta_t}{r^{n-1}} + \frac{\theta P_\theta (r^{n-1}u)_x}{r^{n-1}} - \frac{\alpha\left|(r^{n-1}u)_x\right|^2}{vr^{n-1}} - (n-1)r^{-1}\kappa\theta_x + \frac{2\mu(n-1)\left(r^{n-2}u^2\right)_x}{r^{n-1}} - \frac{\lambda\phi z}{r^{n-1}}. \tag{4.30}$$

Thus we conclude from (4.30) that

$$\begin{aligned} I_6^a &\lesssim \left(1+\|\theta\|_\infty^{l_3}\right)\int_0^t\int_{\mathbb{R}_+}\left|\frac{r^{n-1}\kappa(v,\theta)\theta_x}{v}\right|\left|\left(\frac{r^{n-1}\kappa\theta_x}{v}\right)_x\right| \\ &\lesssim \left(1+\|\theta\|_\infty^{l_3}\right)\left(\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa\theta_x^2}{v\theta^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_{\mathbb{R}_+}(1+\theta)^{b+2}\left|\left(\frac{r^{n-1}\kappa(v,\theta)\theta_x}{v}\right)_x\right|^2\right)^{\frac{1}{2}} \\ &\lesssim \left(1+\|\theta\|_\infty^{l_3}\right)\left(\int_0^t\int_{\mathbb{R}_+}(1+\theta)^{b+2}\left(e_\theta^2\theta_t^2+\theta^2P_\theta^2u_x^2+u_x^4+\phi^2z^2\right)\right)^{\frac{1}{2}} \\ &\lesssim \left(1+Y(t)^{\frac{l_3}{2b+6}}\right)\left(\int_0^t\int_{\mathbb{R}_+}(1+\theta)^{b+2}\left(\frac{e_\theta^2\theta_t^2}{r^{2(n-1)}}+\frac{\theta^2P_\theta^2\left|(r^{n-1}u)_x\right|^2}{r^{2(n-1)}}+\frac{\left|(r^{n-1}u)_x\right|^4}{r^{2(n-1)}}\right.\right. \\ &\quad \left.\left.+\, r^{-2}\kappa^2\theta_x^2+\frac{\left|\left(r^{n-2}u^2\right)_x\right|^2}{r^{2(n-1)}}+\frac{\phi^2z^2}{r^{2(n-1)}}\right)\right)^{\frac{1}{2}}. \end{aligned} \tag{4.31}$$

For the last terms on the right hand side of (4.31), one can deduce from (3.1) that

$$\int_0^t\int_{\mathbb{R}_+}\frac{(1+\theta)^{b+2}e_\theta^2\theta_t^2}{r^{2(n-1)}} \lesssim \int_0^t\int_{\mathbb{R}_+}(1+\theta)^{b+8}\theta_t^2 \lesssim \left(1+Y(t)^{\frac{5}{2b+6}}\right)X(t), \tag{4.32}$$

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}_+}\frac{(1+\theta)^{b+2}\theta^2P_\theta^2|(r^{n-1}u)_x|^2}{r^{2(n-1)}} \\ &\lesssim \int_0^t\int_{\mathbb{R}_+}\frac{(1+\theta)^{b+11}\left|(r^{n-1}u)_x\right|^2}{v\theta} \lesssim 1+\|\theta\|_\infty^{b+11} \lesssim 1+Y(t)^{\frac{b+11}{2b+6}}, \end{aligned} \tag{4.33}$$

$$\int_0^t\int_{\mathbb{R}_+}\frac{(1+\theta)^{b+2}\left|(r^{n-1}u)_x\right|^4}{r^{2(n-1)}} \lesssim \int_0^t\int_{\mathbb{R}_+}\frac{(1+\theta)^{b+2}\left(r^{-4}u^4+r^{4(n-1)}u_x^4\right)}{r^{2(n-1)}}$$

$$\lesssim \left(1+\|\theta\|_\infty^{b+2}\right)\int_0^t \left[\left\|\frac{u}{r}\right\|^2_{L^\infty(\mathbb{R}_+)}\|u\|^2+\|\theta\|_\infty\|u_x\|^2_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+}\frac{r^{2(n-1)}u_x^2}{v\theta}\right]$$
$$\lesssim 1\left(1+Y(t)^{\frac{1}{2}}\right)\left(1+Z(t)^{\frac{3}{4}}\right), \tag{4.34}$$

and

$$\int_0^t\int_{\mathbb{R}_+}(1+\theta)^{b+2}r^{-2}\kappa^2\theta_x^2 \lesssim 1+\|\theta\|_\infty^{2b+4}\lesssim 1+Y(t)^{\frac{2b+4}{2b+6}}. \tag{4.35}$$

Moreover, from (2.2) and the fact that

$$\left(r^{n-2}u^2\right)_x=(n-2)r^{n-3}r_xu^2+2r^{n-2}uu_x=(n-2)r^{-2}vu^2+2r^{n-2}uu_x, \tag{4.36}$$

we have

$$\int_0^t\int_{\mathbb{R}_+}\frac{(1+\theta)^{b+2}\left|\left(r^{n-2}u^2\right)_x\right|^2}{r^{2(n-1)}}\lesssim\int_0^t\int_{\mathbb{R}_+}\left[\frac{(1+\theta)^{b+2}u^4}{r^{2(n-1)+4}}+\frac{r^{2(n-2)}(1+\theta)^{b+2}u^2u_x^2}{r^{2(n-1)}}\right]$$
$$\lesssim\left(1+\|\theta\|_\infty^{b+2}\right)\int_0^t\int_{\mathbb{R}_+}\frac{u^4}{r^2}+\left(1+\|\theta\|_\infty^{b+3}\right)\int_0^t\|u_x\|^2_{L^\infty(\mathbb{R}_+)}\int\frac{u^2}{r^2\theta}$$
$$\lesssim\left(1+Y(t)^{\frac{1}{2}}\right)\left(1+Z(t)^{\frac{3}{4}}\right) \tag{4.37}$$

and

$$\int_0^t\int_{\mathbb{R}_+}\frac{(1+\theta)^{b+2}\phi^2z^2}{r^{2(n-1)}}\lesssim\left(1+\|\theta\|_\infty^{b+2+\beta}\right)\int_0^t\int_{\mathbb{R}_+}\phi z^2\lesssim 1+Y(t)^{\frac{b+2+\beta}{2b+6}}. \tag{4.38}$$

Consequently, we obtain by combining the estimates (4.31)–(4.38) that

$$\int_0^t\left\|\frac{r^{n-1}\kappa(v,\theta)\theta_x}{v}\right\|^2_{L^\infty(\mathbb{R}_+)}\left\|r^{n-1}v_x\right\|^2\mathrm{d}s\lesssim\left(1+Y(t)^{\frac{l_3}{2b+6}}\right)\left[\left(1+Y(t)^{\frac{5}{2b+6}}\right)X(t)\right.$$
$$\left.+Y(t)^{\frac{b+11}{2b+6}}+\left(1+Y(t)^{\frac{1}{2}}\right)\left(1+Z(t)^{\frac{3}{4}}\right)+Y(t)^{\frac{2b+4}{2b+6}}+Y(t)^{\frac{b+2+\beta}{2b+6}}\right]^{\frac{1}{2}}$$
$$\le\epsilon\left(X(t)+Y(t)\right)+C(\epsilon)\left(1+Z(t)^{\frac{3(b+3)}{2(3b+9-2l_3)}}\right). \tag{4.39}$$

Combining (4.29)–(4.39), we can get the following estimate on I_6

$$|I_6| \le \epsilon\,(X(t)+Y(t)) + C(\epsilon)\left(1+Z(t)^{\frac{3(b+3)}{2(3b+9-2l_3)}}\right). \tag{4.40}$$

For I_7 and I_8, we get from (2.2) and assumptions $b > 19/4$ and $0 \le \beta < b+9$ that

$$\begin{aligned} I_7 &= \left|\int_0^t\int_{\mathbb{R}_+} \lambda\phi z K_v(v,\theta)(r^{n-1}u)_x\right| \\ &\lesssim \left(1+\|\theta\|_\infty^{\frac{\beta+3}{2}}\right)\left(\int_0^t\int_{\mathbb{R}_+}\phi z^2\right)^{\frac{1}{2}}\left(\int_0^t\int_{\mathbb{R}_+}\frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}\right)^{\frac{1}{2}} \\ &\lesssim 1+Y(t)^{\frac{\beta+3}{4b+12}} \le \epsilon Y(t)+C(\epsilon), \end{aligned} \tag{4.41}$$

$$\begin{aligned} I_8 &= \left|\int_0^t\int_{\mathbb{R}_+}\frac{\lambda\phi z\kappa\,(v,\theta)\,\theta_t}{v}\right| \le \epsilon X(t)+C(\epsilon)\int_0^t\int_{\mathbb{R}_+}\left(1+\theta^{b+\beta-3}\right)\phi z^2 \\ &\le \epsilon X(t)+C(\epsilon)\left(1+\|\theta\|_\infty^{b+\beta-3}\right)\int_0^t\int_{\mathbb{R}_+}\phi z^2 \le \epsilon(X(t)+Y(t))+C(\epsilon). \end{aligned} \tag{4.42}$$

Finally, for the term I_9, one can infer from (4.36) that

$$\begin{aligned} I_9 &\lesssim \int_0^t\int_{\mathbb{R}_+}\left(r^{-2}u^2+r^{n-2}\,|uu_x|\right)\left[\theta\left|(r^{n-1}u)_x\right|+\left(1+\theta^b\right)|\theta_t|\right] \\ &\lesssim \int_0^t\int_{\mathbb{R}_+}\Big[r^{-2}u^2\theta\left|(r^{n-1}u)_x\right|+r^{-2}u^2\left(1+\theta^b\right)|\theta_t|+r^{n-2}\left(1+\theta^b\right)|uu_x\theta_t| \\ &\quad + r^{n-2}\theta\left|uu_x(r^{n-1}u)_x\right|\Big] := I_{18}^1+I_{18}^2+I_{18}^3+I_{18}^4. \end{aligned} \tag{4.43}$$

We estimate I_9^j $(j=1,2,3,4)$ term by term as follows. First, we have from (3.1) that

$$\begin{aligned} I_9^1 &\lesssim \int_0^t\int_{\mathbb{R}_+}\frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}\cdot\theta^3+\int_0^t\int_{\mathbb{R}_+}\frac{u^4}{r^4} \\ &\lesssim \|\theta\|_\infty^3+\int_0^t\left\|\frac{u}{r}\right\|^2_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+}u^2 \lesssim 1+Y(t)^{\frac{3}{2b+6}} \le \epsilon Y(t)+C(\epsilon), \end{aligned} \tag{4.44}$$

$$I_9^2 \leq \epsilon X(t) + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} r^{-4}u^4 \left(1+\theta^{b-3}\right)$$
$$\leq \epsilon X(t) + C(\epsilon)\left(1+\|\theta\|_\infty^{b-3}\right) \leq \epsilon\left(X(t)+Y(t)\right)+C(\epsilon). \tag{4.45}$$

As for the term I_9^3, we have from (3.1) and (3.18) that

$$I_9^3 \leq \epsilon X(t) + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \frac{r^{2(n-1)}u^2u_x^2\left(1+\theta^{b-3}\right)}{r^2}$$
$$\leq \epsilon X(t) + C(\epsilon)\left(1+\|\theta\|_\infty^{b-3}\right)\int_0^t \left\|\frac{u}{r}\right\|^2_{L^\infty(\mathbb{R}_+)} \int_{\mathbb{R}_+} r^{2(n-1)}u_x^2$$
$$\leq \epsilon X(t) + C(\epsilon)\left(1+\|\theta\|_\infty^{b-3+l_2+l_3}\right) \leq \epsilon\left(X(t)+Y(t)\right)+C(\epsilon). \tag{4.46}$$

For I_9^4, we can get from (2.3), (3.1) and the fact that $l_2 < 2b+6$ since $b > 19/4$ that

$$I_9^4 \lesssim \int_0^t \int_{\mathbb{R}_+} \frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}\cdot\theta^3 + \int_0^t \int_{\mathbb{R}_+} r^{2n-4}u^2u_x^2$$
$$\lesssim \|\theta\|_\infty^3 + \int_0^t \left\|r^{n-1}u_x\right\|^2_{L^\infty(\mathbb{R}_+)} \int_{\mathbb{R}_+} u^2$$
$$\lesssim 1 + Y(t)^{\frac{3}{2b+6}} + Y(t)^{\frac{l_2}{2b+6}} \leq \epsilon Y(t) + C(\epsilon). \tag{4.47}$$

Thus it follows from (4.43)–(4.47) that

$$I_9 \leq \epsilon\left(X(t)+Y(t)\right)+C(\epsilon). \tag{4.48}$$

With the above estimates in hand, if we define λ_1 as in (4.7), by combining all the above estimates and choosing $\epsilon > 0$ small enough, we can deduce estimate (4.6). Since $b > 19/4$, one gets from (3.3), (3.12), (3.16) and (4.7) that $0 < \lambda_1 < 1$. □

Our next result in this section is to show that $Z(t)$ can be bounded by $X(t)$ and $Y(t)$.

Lemma 4.3. *Under the assumptions listed in Theorem 1.1, we have for all* $0 \leq t \leq T$ *that*

$$Z(t) \lesssim 1 + X(t) + Y(t) + Z(t)^{\lambda_2}. \tag{4.49}$$

Here λ_2 is given by

$$\lambda_2 = \max\left\{\frac{b+3}{2(2b+5)}, \frac{3(b+3)}{2(2b+6-l_1)}\right\}. \tag{4.50}$$

It is easy to see that $\lambda_2 \in (0,1)$ provided that $b > \frac{19}{4}$.

Proof. Differentiating (1.12b) with respect to t and multiplying it by u_t, we obtain

$$\left(\frac{u_t^2}{2}\right)_t = \left\{\left[r^{n-1}\left(\alpha\frac{(r^{n-1}u)_x}{v} - P\right)\right]_t u_t\right\}_x - \left[r^{n-1}\left(\alpha\frac{(r^{n-1}u)_x}{v} - P\right)\right]_t u_{tx}$$
$$- \left[\left(r^{n-1}\right)_x\left(\alpha\frac{(r^{n-1}u)_x}{v} - P\right)\right]_t u_t.$$

Then integrating the above identity over $[0,t]\times\mathbb{R}_+$ yields

$$\frac{\|u_t(t)\|^2}{2} + \int_0^t\int_{\mathbb{R}_+}\frac{\alpha r^{2(n-1)}u_{tx}^2}{v} + \alpha(n-1)^2\int_0^t\int_{\mathbb{R}_+}\frac{vu_t^2}{r^2}$$
$$\leq C + \underbrace{C\int_0^t\int_{\mathbb{R}_+} r^{n-3}u^2|u_{tx}|}_{I_{10}} \underbrace{-2\alpha(n-1)\int_0^t\int_{\mathbb{R}_+} r^{n-2}u_tu_{tx}}_{I_{11}} + \underbrace{C\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-3}|uu_xu_{tx}|}{v}}_{I_{12}}$$
$$+ \underbrace{\alpha\int_0^t\int_{\mathbb{R}_+}\frac{r^{2(n-1)}u_xu_{tx}v_t}{v^2}}_{I_{13}} + \underbrace{\int_0^t\int_{\mathbb{R}_+} r^{n-1}P_tu_{tx}}_{I_{14}} + \underbrace{(n-1)\int_0^t\int_{\mathbb{R}_+} r^{n-2}Puu_{tx}}_{I_{15}}$$
$$+ \underbrace{C\int_0^t\int_{\mathbb{R}_+}\frac{|vu^2u_t|}{r^3}}_{I_{16}} + \underbrace{C\int_0^t\int_{\mathbb{R}_+} r^{-2}|uu_tv_t|}_{I_{17}} + \underbrace{C\int_0^t\int_{\mathbb{R}_+} r^{n-3}|uu_xu_t|}_{I_{18}}$$
$$+ \underbrace{C\int_0^t\int_{\mathbb{R}_+} r^{-1}P|u_tv_t|}_{I_{19}} + \underbrace{C\int_0^t\int_{\mathbb{R}_+} r^{-2}|vuPu_t|}_{I_{20}} + \underbrace{C\int_0^t\int_{\mathbb{R}_+} vr^{-1}|u_tP_t|}_{I_{21}}. \tag{4.51}$$

Now we turn to estimate I_k ($k = 10, 11, \cdots, 21$) term by term. To this end, we compute from (2.3) and (3.1) that

$$I_{10} \leq \epsilon\int_0^t\int_{\mathbb{R}_+}\frac{\alpha r^{2(n-1)}u_{tx}^2}{v} + C(\epsilon)\int_0^t\left\|\frac{u}{r}\right\|_{L^\infty(\mathbb{R}_+)}^2\|u\|^2 \leq \epsilon\int_0^t\int_{\mathbb{R}_+}\frac{\alpha r^{2(n-1)}u_{tx}^2}{v} + C(\epsilon), \tag{4.52}$$

and

$$I_{11} = \alpha(n-1)\int_0^t\int_{\mathbb{R}_+}\left(r^{n-2}\right)_x u_t^2 = \alpha(n-1)(n-2)\int_0^t\int_{\mathbb{R}_+} r^{-2}vu_t^2. \tag{4.53}$$

Since

$$I_{12} \le \epsilon\int_0^t\int_{\mathbb{R}_+}\frac{\alpha r^{2(n-1)}u_{tx}^2}{v} + C(\epsilon)\int_0^t\int_{\mathbb{R}_+} r^{2(n-1)-2}u^2u_x^2, \tag{4.54}$$

and

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+} r^{2(n-1)-2}u^2u_x^2 \lesssim \int_0^t\int_{\mathbb{R}_+}\frac{r^{2(n-1)}u_x^2}{v\theta}\cdot\frac{u^2\theta}{r^2}\\
&\lesssim \|\theta\|_\infty\int_0^t \|u\|^2_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+}\frac{r^{2(n-1)}u_x^2}{v\theta} \lesssim \|\theta\|_\infty\int_0^t \|u\|\|u_x\|\int_{\mathbb{R}_+}\frac{r^{2(n-1)}u_x^2}{v\theta}\\
&\lesssim \left(1+Y^{\frac{1}{2b+6}}(t)\right)\left(1+Z^{\frac{1}{4}}(t)\right) \lesssim \left(1+Y(t)+Z^{\frac{b+3}{2(2b+5)}}(t)\right),
\end{aligned}\tag{4.55}$$

we can deduce that

$$I_{12} \le \epsilon\int_0^t\int_{\mathbb{R}_+}\frac{\alpha r^{2(n-1)}u_{tx}^2}{v} + C(\epsilon)\left(1+Y(t)+Z^{\frac{b+3}{2(2b+5)}}(t)\right). \tag{4.56}$$

For the term I_{13}, we get

$$I_{13} \le \epsilon\int_0^t\int_{\mathbb{R}_+}\frac{\alpha r^{2(n-1)}u_{tx}^2}{v} + C(\epsilon)\int_0^t\int_{\mathbb{R}_+}\frac{r^{2(n-1)}u_x^2(r^{n-1}u)_x^2}{v^2}. \tag{4.57}$$

Since

$$\int_0^t\int_{\mathbb{R}_+}\frac{r^{2(n-1)}u_x^2\left|(r^{n-1}u)_x\right|^2}{v^2} \lesssim \int_0^t\int_{\mathbb{R}_+}\left(\frac{r^{2(n-1)}u_x^2u^2}{r^2} + r^{4(n-1)}u_x^4\right), \tag{4.58}$$

and

$$\int_0^t \int_{\mathbb{R}_+} r^{4(n-1)} u_x^4 \lesssim \int_0^t \left\| r^{n-1} u_x \right\|_{L^\infty(\mathbb{R}_+)}^2 \int_{\mathbb{R}_+} r^{2(n-1)} u_x^2$$

$$\lesssim 1 + \|\theta\|_\infty^{2l_2+l_3} \lesssim 1 + Y^{\frac{2l_2+l_3}{2b+6}} \lesssim 1 + Y(t), \tag{4.59}$$

we combine (4.55), (4.58)–(4.59) to find

$$\int_0^t \int_{\mathbb{R}_+} \frac{r^{2(n-1)} u_x^2 \left|(r^{n-1}u)_x\right|^2}{v^2} \lesssim 1 + Y(t) + Z^{\frac{b+3}{2(2b+5)}}(t). \tag{4.60}$$

Consequently, we can conclude from (4.57)–(4.60) that

$$I_{13} \le \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{\alpha r^{2(n-1)} u_{tx}^2}{v} + C(\epsilon)\left(1 + Y(t) + Z^{\frac{b+3}{2(2b+5)}}(t)\right). \tag{4.61}$$

Finally for I_k ($k = 14, 15, \cdots, 21$), it follows from (1.2) and (2.12) that

$$\begin{aligned} I_{14} &\le \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{\alpha r^{2(n-1)} u_{tx}^2}{v} + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \left[\left(1+\theta^6\right)\theta_t^2 + \theta^2 \left|(r^{n-1}u)_x\right|^2\right] \\ &\le \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{\alpha r^{2(n-1)} u_{tx}^2}{v} + C(\epsilon)\left(X(t) + \int_0^t \int_{\mathbb{R}_+} \frac{\left|(r^{n-1}u)_x\right|^2}{v\theta} \cdot \theta^3\right) \\ &\le \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{\alpha r^{2(n-1)} u_{tx}^2}{v} + C(\epsilon)\left(1 + X(t) + Y(t)\right), \end{aligned} \tag{4.62}$$

$$\begin{aligned} I_{15} &\le \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{\alpha r^{2(n-1)} u_{tx}^2}{v} + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \frac{u^2}{r^2\theta} \cdot \theta\left(\theta^2 + \theta^8\right) \\ &\le \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{\alpha r^{2(n-1)} u_{tx}^2}{v} + C(\epsilon)\left(1 + Y(t)\right), \end{aligned} \tag{4.63}$$

$$I_{16} \le \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \frac{u^4}{r^4} \le \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon), \tag{4.64}$$

$$I_{17} \leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon)\|\theta\|_\infty \int_0^t \|u\|_{L^\infty(\mathbb{R}_+)}^2 \int_{\mathbb{R}_+} \frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}$$

$$\leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon)\|\theta\|_\infty \int_0^t \|u\|\|u_x\| \int_{\mathbb{R}_+} \frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}$$

$$\leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon)\left(1 + Y(t) + Z^{\frac{b+3}{2(2b+5)}}(t)\right), \tag{4.65}$$

$$I_{18} \leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \frac{r^{2(n-1)} u_x^2}{v\theta} \cdot \frac{v\theta u^2}{r^2}$$

$$\leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon)\|\theta\|_\infty \int_0^t \|u\|_{L^\infty(\mathbb{R}_+)}^2 \int_{\mathbb{R}_+} \frac{r^{2(n-1)} u_x^2}{v\theta}$$

$$\leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon)\left(1 + Y(t) + Z^{\frac{b+3}{2(2b+5)}}(t)\right), \tag{4.66}$$

$$I_{19} \leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \frac{\left|(r^{n-1}u)_x\right|^2}{v\theta} \cdot \theta\left(\theta^2 + \theta^8\right)$$

$$\leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon)\left(1 + \|\theta\|_\infty^9\right)$$

$$\leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon)\left(1 + Y(t)\right), \tag{4.67}$$

$$I_{20} \leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \frac{u^2}{r^2\theta} \cdot \theta\left(\theta^2 + \theta^8\right)$$

$$\leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon)\left(1 + Y(t)\right), \tag{4.68}$$

and

$$I_{21} \leq \epsilon \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} + C(\epsilon)\left(1 + X(t) + Y(t)\right). \tag{4.69}$$

Combining estimates (4.51)–(4.69) and choosing $\epsilon > 0$ small enough, we arrive at

$$\|u_t(t)\|^2 + \int_0^t \int_{\mathbb{R}_+} \frac{r^{2(n-1)} u_{tx}^2}{v} + \int_0^t \int_{\mathbb{R}_+} \frac{v u_t^2}{r^2} \lesssim 1 + X(t) + Y(t) + Z^{\frac{b+3}{2(2b+5)}}(t). \tag{4.70}$$

To yield an estimate on $\|u_{xx}(t)\|$, $(1.12)_2$ tells us that

$$u_{xx} = \frac{u_t v}{\alpha r^{2(n-1)}} + \frac{(n-1)v^2 u}{r^{2n}} - \frac{2(n-1)v u_x}{r^n} + v^{-1} u_x v_x + \frac{P_x v}{\alpha r^{n-1}}. \tag{4.71}$$

Thus one gets from (4.71) that

$$\begin{aligned}
\|u_{xx}(t)\|^2 &\lesssim \int_{\mathbb{R}_+} \left(\frac{u_t^2}{r^{4(n-1)}} + \frac{u^2}{r^{4n}} + \frac{u_x^2}{r^{2n}} + u_x^2 v_x^2 + \frac{P_x^2}{r^{2(n-1)}} \right) \mathrm{d}x \\
&\lesssim 1 + \int_{\mathbb{R}_+} \left(u_t^2 + P_x^2 + u_x^2 + u_x^2 v_x^2 \right) \mathrm{d}x \\
&\lesssim 1 + \|u_t(t)\|^2 + \int_{\mathbb{R}_+} \left(1 + \theta^6\right) \theta_x^2 \mathrm{d}x + \int_{\mathbb{R}_+} \left(\theta^2 + u_x^2\right) v_x^2 \mathrm{d}x + \int_{\mathbb{R}_+} u_x^2 \mathrm{d}x \\
&\lesssim 1 + X(t) + Y(t) + Z^{\frac{b+3}{2(2b+5)}}(t) + \|\theta\|_\infty^2 \|v_x(t)\|^2 + \|u_x\|_\infty^2 \|v_x(t)\|^2 + \|u_x\|^2,
\end{aligned} \tag{4.72}$$

while the last three terms in the right hand side of (4.72) can be bounded as follows:

$$\|\theta\|_\infty^2 \|v_x(t)\|^2 \lesssim 1 + \|\theta\|_\infty^{2+l_1} \lesssim 1 + Y(t)^{\frac{2+l_1}{2b+6}} \lesssim 1 + Y(t), \tag{4.73}$$

$$\begin{aligned}
\|u_x\|_\infty^2 \|v_x(t)\|^2 &\lesssim \left(1 + \|\theta\|_\infty^{l_1}\right)\left(1 + Z(t)^{\frac{3}{4}}\right) \\
&\lesssim \left(1 + Z(t)^{\frac{3}{4}} + Y(t)^{\frac{l_1}{2b+6}} + Y(t)^{\frac{l_1}{2b+6}} Z(t)^{\frac{3}{4}}\right) \lesssim 1 + Y(t) + Z(t)^{\frac{3(b+3)}{2(2b+6-l_1)}},
\end{aligned} \tag{4.74}$$

$$\|u_x\|^2 \lesssim 1 + \|\theta\|_\infty^{l_2} \lesssim 1 + Y(t)^{\frac{l_2}{2b+6}} \lesssim 1 + Y(t). \tag{4.75}$$

Thus it follows from (4.72)–(4.75) that

$$\|u_{xx}(t)\|^2 \lesssim 1 + X(t) + Y(t) + Z(t)^{\lambda_2}, \tag{4.76}$$

where λ_2 is given by (4.50). Having obtained (4.76), we can get estimate (4.49) by using the definition of $Z(t)$. □

Combining Lemmas 4.1–4.3, we can deduce that $Y(t) \lesssim 1$. Thus we can get the desired upper bounds on $\theta(t,x)$ from (4.4). In fact, we can deduce from Lemmas 2.1 and 4.3 that:

Lemma 4.4. *Under the assumptions listed in Theorem 1.1, there exists a positive constant* $\overline{\Theta}$*, which depends only on the constants* μ*,* λ_1*,* λ*,* K*,* A*,* d*,* R*,* c_v*,* a*,* κ_1*,* κ_2*,* n *and the initial data* $(v_0(x), u_0(x), \theta_0(x), z_0(x))$*, such that*

$$\theta(t,x) \le \overline{\Theta} \quad \forall\, (t,x) \in [0,T] \times \mathbb{R}_+. \tag{4.77}$$

Moreover, we have for $0 \le t \le T$ *that*

$$\begin{aligned}&\|(v-1,u,\theta-1,z)(t)\|^2 + \left\|r^{n-1}(v_x,u_x,\theta_x)(t)\right\|^2 + \|u_{xx}(t)\|^2 + \|z(t)\|_{L^1(\mathbb{R}_+)}\\ &+\int_0^t \left\|\left(r^{2n-2}u_{xx}, r^{n-1}\sqrt{\theta}v_x, r^{n-1}u_{tx}, r^{n-1}u_x, r^{n-1}\theta_x, \theta_t, r^{n-1}z_x\right)(s)\right\|^2 \mathrm{d}s \lesssim 1\end{aligned} \tag{4.78}$$

and

$$\|u_x(t)\|_{L^\infty(\mathbb{R}_+)} \lesssim 1, \quad \|u(t)\|_{L^\infty(\mathbb{R}_+)} \lesssim 1, \quad \int_0^t \left\|r^{n-1}u_x\right\|^2_{L^\infty(\mathbb{R}_+)} \mathrm{d}s \lesssim 1. \tag{4.79}$$

Lemma 4.5. *Under the assumptions listed in Theorem 1.1, we have for* $0 \le t \le T$ *that*

$$\|\theta_x(t)\|^2 + \int_0^t \left\|r^{n-1}\theta_{xx}(s)\right\|^2 \mathrm{d}s \lesssim 1, \tag{4.80}$$

$$\int_0^t \left\|r^{n-1}\theta_x(s)\right\|^2_{L^\infty(\mathbb{R}_+)} \mathrm{d}s \lesssim 1. \tag{4.81}$$

Proof. Multiplying (4.12) by θ_{xx}/e_θ, one has

$$\begin{aligned}&\frac{1}{2}\left(\theta_x^2\right)_t - (\theta_t\theta_x)_x + \frac{r^{2n-2}\kappa\theta_{xx}^2}{ve_\theta}\\ &= \frac{\theta P_\theta (r^{n-1}u)_x\theta_{xx}}{e_\theta} - \frac{\alpha (r^{n-1}u)_x^2\theta_{xx}}{ve_\theta} - \frac{(2n-2)r^{2n-2}\kappa\theta_x\theta_{xx}}{e_\theta} - \frac{r^{2n-2}\kappa_v v_x\theta_x\theta_{xx}}{ve_\theta}\\ &\quad - \frac{r^{2n-2}\kappa_\theta\theta_x^2\theta_{xx}}{ve_\theta} + \frac{r^{2n-2}\kappa v_x\theta_x\theta_{xx}}{v^2e_\theta} + \frac{2\mu(n-1)\left(r^{n-2}u^2\right)_x\theta_{xx}}{e_\theta} - \frac{\lambda\phi z\theta_{xx}}{e_\theta}.\end{aligned} \tag{4.82}$$

Integrating the above identity with respect to x over $\mathbb{R}_+$, one has

$$\frac{1}{2}\frac{d}{dt}\|\theta_x(t)\|^2 + \int_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)} \le \underbrace{\left|\int_{\mathbb{R}_+} \frac{\theta P_\theta(v,\theta)(r^{n-1}u)_x\theta_{xx}}{e_\theta(v,\theta)}\right|}_{I_{22}} + \underbrace{\left|\int_{\mathbb{R}_+} \frac{\alpha\left|(r^{n-1}u)_x\right|^2\theta_{xx}}{ve_\theta(v,\theta)}\right|}_{I_{23}}$$

$$
+\underbrace{\left|\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa_v(v,\theta)v_x\theta_x\theta_{xx}}{ve_\theta(v,\theta)}\right|}_{I_{24}}+\underbrace{\left|\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa_\theta(v,\theta)\theta_x^2\theta_{xx}}{ve_\theta(v,\theta)}\right|}_{I_{25}}+\underbrace{\left|\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_x v_x\theta_{xx}}{v^2e_\theta(v,\theta)}\right|}_{I_{26}}
$$

$$
+\underbrace{\left|\int_{\mathbb{R}_+}\frac{\lambda\phi z\theta_{xx}}{e_\theta(v,\theta)}\right|}_{I_{27}}+\underbrace{\left|\int_{\mathbb{R}_+}\frac{(2n-2)r^{2n-2}\kappa\theta_x\theta_{xx}}{e_\theta(v,\theta)}\right|}_{I_{28}}+\underbrace{\left|\int_{\mathbb{R}_+}\frac{2\mu(n-1)\left(r^{n-2}u^2\right)_x\theta_{xx}}{e_\theta(v,\theta)}\right|}_{I_{29}}. \tag{4.83}
$$

Now we turn to estimate the terms I_k $(k=22,\dots,29)$. To begin with, we have

$$
\begin{aligned}
I_{22} &\le \epsilon\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x+C(\epsilon)\int_{\mathbb{R}_+}\frac{\left|(r^{n-1}u)_x\right|^2}{v\theta}\cdot\frac{\theta^3P_\theta^2(v,\theta)}{\kappa(v,\theta)r^{2n-2}e_\theta(v,\theta)}\mathrm{d}x\\
&\le \epsilon\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x+C(\epsilon)V(t).
\end{aligned}\tag{4.84}
$$

Estimates (4.77) and (4.79) imply

$$
\begin{aligned}
I_{23} &\le \epsilon\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x+C(\epsilon)\int_{\mathbb{R}_+}\frac{\left|(r^{n-1}u)_x\right|^4}{r^{2n-2}\kappa(v,\theta)e_\theta(v,\theta)}\mathrm{d}x\\
&\le \epsilon\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x+C(\epsilon)\int_{\mathbb{R}_+}\frac{u^4+r^{4(n-1)}u_x^4}{r^{2n-2}\kappa(v,\theta)e_\theta(v,\theta)}\mathrm{d}x\\
&\le \epsilon\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x+C(\epsilon)\left(\left\|\frac{u}{r}\right\|_{L^\infty(\mathbb{R}_+)}^2+V(t)\right).
\end{aligned}\tag{4.85}
$$

By virtue of Sobolev's inequality and Lemma 4.4, we infer

$$
\begin{aligned}
I_{24} &\le \epsilon\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x+C(\epsilon)\int_{\mathbb{R}_+}r^{2n-2}v_x^2\cdot\frac{\kappa_v^2(v,\theta)\theta_x^2}{e_\theta(v,\theta)\kappa(v,\theta)}\mathrm{d}x\\
&\le 2\epsilon\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x+C(\epsilon)\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_x^2}{v\theta^2}\cdot\frac{\theta^2e_\theta}{r^{4n-4}\kappa^2}\mathrm{d}x\\
&\le 2\epsilon\int_{\mathbb{R}_+}\frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x+C(\epsilon)V(t),
\end{aligned}\tag{4.86}
$$

$$
\begin{aligned}
I_{25} &\le \epsilon \int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x + C(\epsilon)\int \frac{r^{2n-2}\theta_x^4\kappa_\theta^2(v,\theta)}{\kappa(v,\theta)e_\theta(v,\theta)}\mathrm{d}x \\
&\le \epsilon \int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x + C(\epsilon)\,\|\theta_x(t)\|\,\|\theta_{xx}(t)\| \int\limits_{\mathbb{R}_+} r^{2n-2}\theta_x^2\mathrm{d}x \\
&\le 2\epsilon \int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x + C(\epsilon)V(t),
\end{aligned}
\tag{4.87}
$$

and

$$
\begin{aligned}
I_{26} &\le \epsilon \int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x + C(\epsilon)\int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_x^2 v_x^2}{e_\theta(v,\theta)}\mathrm{d}x \\
&\le \epsilon \int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x + C(\epsilon)\,\|\theta_x\|_{L^\infty(\mathbb{R}_+)}^2 \int\limits_{\mathbb{R}_+} r^{2n-2}v_x^2\mathrm{d}x \\
&\le 2\epsilon \int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x + C(\epsilon)V(t).
\end{aligned}
\tag{4.88}
$$

It follows from (1.5), (4.77) and (4.78) that

$$
I_{27} \le \epsilon \int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x + C(\epsilon)\int\limits_{\mathbb{R}_+} \phi z^2\mathrm{d}x,
\tag{4.89}
$$

$$
I_{28} \le \epsilon \int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x + C(\epsilon)V(t),
\tag{4.90}
$$

and

$$
\begin{aligned}
I_{29} &\lesssim \int\limits_{\mathbb{R}_+} \frac{\left(r^{-2}u^2 + r^{n-2}\,|uu_x|\right)\theta_{xx}}{e_\theta(v,\theta)}\mathrm{d}x \\
&\le \epsilon \int\limits_{\mathbb{R}_+} \frac{r^{2n-2}\kappa(v,\theta)\theta_{xx}^2}{ve_\theta(v,\theta)}\mathrm{d}x + C(\epsilon)\left(\left\|\frac{u}{r}\right\|_{L^\infty(\mathbb{R}_+)}^2 + V(t)\right).
\end{aligned}
\tag{4.91}
$$

Combining (4.83)–(4.91) and by choosing $\epsilon > 0$ small enough, we arrive at

$$
\frac{d}{dt}\|\theta_x(t)\|^2 + \left\|r^{n-1}\theta_{xx}(t)\right\|^2 \lesssim V(t) + \left\|\frac{u}{r}\right\|_{L^\infty(\mathbb{R}_+)}^2 + \int\limits_{\mathbb{R}_+} \phi(t,x)z^2(t,x)\mathrm{d}x.
\tag{4.92}
$$

Integrating the inequality (4.92) over $(0,t)$ and using estimates (2.2), (2.3) and (3.1) yield (4.80).

In view of (2.3), Lemma 4.4, (4.80) and Sobolev's inequality, we find

$$\int_0^t \left\| r^{n-1}\theta_x(s) \right\|_{L^\infty(\mathbb{R}_+)}^2 \mathrm{d}s \lesssim \int_0^t \left\| r^{n-1}\theta_x \right\| \left\| \left(r^{n-1}\theta_x \right)_x \right\| \mathrm{d}s$$

$$\lesssim \int_0^t \left\| r^{n-1}\theta_x \right\|^2 \mathrm{d}s + \int_0^t \left\| \left(r^{n-1}\theta_x \right)_x \right\|^2 \mathrm{d}s$$

$$\lesssim 1 + \int_0^t \int_{\mathbb{R}_+} \left(\frac{\theta_x^2}{r^2} + r^{2(n-1)}\theta_{xx}^2 \right) \lesssim 1 + \int_0^t \int_{\mathbb{R}_+} \frac{\kappa r^{2(n-1)}\theta_x^2}{v\theta^2} \cdot \theta^2 \lesssim 1.$$

This completes the proof of this Lemma. □

Lemma 4.6. *Let the assumptions listed in Theorem 1.1 hold. Then for any* $0 \le t \le T$*,*

$$\left\| r^{n-1} z_x(t) \right\|^2 + \int_0^t \| z_t(s) \|^2 \mathrm{d}s \lesssim 1 \tag{4.93}$$

$$\| z_x(t) \|^2 + \int_0^t \left\| r^{n-1} z_{xx}(s) \right\|^2 \mathrm{d}s \lesssim 1. \tag{4.94}$$

Proof. To prove (4.93), we multiply equation (1.12d) by z_t to get

$$\left(\frac{d r^{2n-2} z_x^2}{2v^2} \right)_t + z_t^2 = d \left(\frac{r^{2n-2} z_x z_t}{v^2} \right)_x + \frac{d(n-1) r^{2n-3} u z_x^2}{v^2} - \frac{d r^{2n-2} (r^{n-1}u)_x z_x^2}{v^3} - \phi z z_t.$$

Integrating this last identity with respect to t and x, we have

$$\int_{\mathbb{R}_+} \frac{d r^{2n-2} z_x^2}{2v^2} \mathrm{d}x + \int_0^t \int_{\mathbb{R}_+} z_t^2 = \int_{\mathbb{R}_+} \frac{d r^{2n-2} z_x^2}{2v^2}(0,x) \mathrm{d}x$$

$$+ \underbrace{\int_0^t \int_{\mathbb{R}_+} \frac{d(n-1) r^{2n-3} u z_x^2}{v^2}}_{I_{30}} - \underbrace{\int_0^t \int_{\mathbb{R}_+} \frac{d r^{2n-2} (r^{n-1}u)_x z_x^2}{v^3}}_{I_{31}} - \underbrace{\int_0^t \int_{\mathbb{R}_+} \phi z z_t}_{I_{32}}. \tag{4.95}$$

To control I_k $(k = 30, 31, 32)$, we can get from (2.2) and (4.79) that

$$|I_{30}| \lesssim \int_0^t \int_{\mathbb{R}_+} \frac{r^{2n-2} z_x^2}{v^2} \lesssim 1, \tag{4.96}$$

and

$$|I_{31}| \lesssim \int_0^t \int_{\mathbb{R}_+} \frac{r^{2n-2} z_x^2}{v^2} \left(r^{-1} |u| + r^{n-1} |u_x| \right)$$
$$\lesssim 1 + \int_0^t \left\| r^{n-1} u_x \right\|_{L^\infty(\mathbb{R}_+)}^2 \int_{\mathbb{R}_+} \frac{r^{2n-2} z_x^2}{v^2}. \tag{4.97}$$

As for the term I_{32}, we get from (2.2) that

$$|I_{32}| \le \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} z_t^2 + C \int_0^t \int_{\mathbb{R}_+} \phi^2 z^2 \le \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} z_t^2 + C. \tag{4.98}$$

Substituting (4.96)–(4.98) into (4.95) and using Gronwall's inequality yield (4.93).

Multiply (1.12d) by z_{xx} to infer

$$\left(\frac{z_x^2}{2} \right)_t - (z_t z_x)_x + \frac{d r^{2n-2} z_{xx}^2}{v^2} = -\frac{(2n-2) d r^{n-2} z_x z_{xx}}{v^2} + \frac{2 d r^{2n-2} z_x v_x z_{xx}}{v^3} + \phi z z_{xx}.$$

Integrating the above identity over $[0, t) \times \mathbb{R}_+$, we arrive at

$$\|z_x(t)\|^2 + \int_0^t \left\| r^{n-1} z_{xx}(s) \right\|^2 \mathrm{d}s$$
$$\lesssim 1 + \underbrace{\int_0^t \int_{\mathbb{R}_+} r^{n-2} |z_x z_{xx}|}_{I_{33}} + \underbrace{\int_0^t \int_{\mathbb{R}_+} r^{2n-2} |z_x v_x z_{xx}|}_{I_{34}} + \underbrace{\int_0^t \int_{\mathbb{R}_+} \phi z |z_{xx}|}_{I_{35}}. \tag{4.99}$$

The terms I_k $(42 \le k \le 44)$ can be estimated by employing Cauchy's and Sobolev's inequalities, Lemmas 2.1 and 4.4 as follows:

$$|I_{33}| \le \frac{1}{8} \int_0^t \left\| r^{n-1} z_{xx}(s) \right\|^2 \mathrm{d}s + C \int_0^t \int_{\mathbb{R}_+} r^{-2} z_x^2 \le \frac{1}{8} \int_0^t \left\| r^{n-1} z_{xx}(s) \right\|^2 \mathrm{d}s + C, \tag{4.100}$$

$$|I_{34}| \le \frac{1}{8} \int_0^t \left\| r^{n-1} z_{xx}(s) \right\|^2 \mathrm{d}s + C \int_0^t \int_{\mathbb{R}_+} r^{2n-2} v_x^2 z_x^2$$
$$\le \frac{1}{8} \int_0^t \left\| r^{n-1} z_{xx}(s) \right\|^2 \mathrm{d}s + C \int_0^t \|z_x\| \|z_{xx}\| \mathrm{d}s \le \frac{1}{4} \int_0^t \left\| r^{n-1} z_{xx}(s) \right\|^2 \mathrm{d}s + C, \tag{4.101}$$

and

$$|I_{35}| \leq \frac{1}{8}\int_0^t \left\|r^{n-1}z_{xx}(s)\right\|^2 \mathrm{d}s + C\int_0^t\int_{\mathbb{R}_+} \phi^2 z^2$$

$$\leq \frac{1}{8}\int_0^t \left\|r^{n-1}z_{xx}(s)\right\|^2 \mathrm{d}s + C\|\theta\|_\infty^\beta \int_0^t\int_{\mathbb{R}_+} \phi z^2 \mathrm{d}s \leq \frac{1}{8}\int_0^t \left\|r^{n-1}z_{xx}(s)\right\|^2 \mathrm{d}s + C. \quad (4.102)$$

Then (4.94) follows from (4.99)–(4.102). □

Lemma 4.7. *Under the assumptions listed in Theorem 1.1, we get for any* $0 \leq t \leq T$ *that*

$$\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2}\mathrm{d}x + \int_0^t\int_{\mathbb{R}_+} \frac{Rr^{2n-2}\theta v_{xx}^2}{v^3} \lesssim 1 \quad (4.103)$$

$$\int_0^t \left\|r^{n-1}v_x(s)\right\|_{L^\infty(\mathbb{R}_+)}^2 \mathrm{d}s \lesssim 1. \quad (4.104)$$

Proof. Differentiate (3.4) with respect to x and multiply it by $r^{n-1}v_{xx}/v$ to obtain

$$\begin{aligned}
&\frac{\alpha}{2}\left[\frac{r^{2n-2}v_{xx}^2}{v^2}\right]_t + \frac{Rr^{2n-2}\theta v_{xx}^2}{v^3} = \frac{r^{n-1}u_{tx}v_{xx}}{v} \\
&- \alpha(n-1)r^{n-2}v_{xx}\left(\frac{(r^{n-1}u)_x}{v}\right)_x + \frac{\alpha(n-1)r^{2n-3}uv_{xx}^2}{v^2} + \frac{\alpha r^{2n-2}v_{xx}}{v}\left(\frac{v_x^2}{v^2}\right)_t \\
&+ (n-1)r^{n-2}v_{xx}\left(\frac{R\theta_x}{v} - \frac{R\theta v_x}{v^2} + \frac{4}{3}a\theta^3\theta_x\right) + \frac{Rr^{2n-2}v_{xx}\theta_{xx}}{v^2} \\
&- \frac{2Rr^{2n-2}v_{xx}v_x\theta_x}{v^3} + \frac{2Rr^{2n-2}\theta v_{xx}v_x^2}{v^4} + \frac{4ar^{2n-2}\theta^2 v_{xx}\theta_x^2}{v} + \frac{4a\theta^3 r^{2n-2}v_{xx}\theta_{xx}}{3v}.
\end{aligned}$$

Integrating this last identity over $[0,t]\times\mathbb{R}_+$ implies

$$\begin{aligned}
&\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2}\mathrm{d}x + \int_0^t\int_{\mathbb{R}_+} \frac{Rr^{2n-2}\theta v_{xx}^2}{v^3} \lesssim 1 + \underbrace{\int_0^t\int_{\mathbb{R}_+} \frac{r^{n-1}|u_{tx}v_{xx}|}{v}}_{I_{36}} \\
&+ \underbrace{\int_0^t\int_{\mathbb{R}_+} r^{n-2}\left|v_{xx}\left(\frac{(r^{n-1}u)_x}{v}\right)_x\right|}_{I_{37}} + \underbrace{\int_0^t\int_{\mathbb{R}_+} \frac{r^{2n-3}|u|v_{xx}^2}{v^2}}_{I_{38}}
\end{aligned}$$

$$+\underbrace{\int_0^t\int_{\mathbb{R}_+}\left|\frac{r^{2n-2}v_{xx}}{v}\left(\frac{v_x^2}{v^2}\right)_t\right|}_{I_{39}}+\underbrace{\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}|v_{xx}\theta_{xx}|}{v^2}}_{I_{40}}$$

$$+\underbrace{\int_0^t\int_{\mathbb{R}_+}r^{n-2}\left|v_{xx}\left(\frac{R\theta_x}{v}-\frac{R\theta v_x}{v^2}+\frac{4}{3}a\theta^3\theta_x\right)\right|}_{I_{41}}+\underbrace{\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}|v_{xx}\theta_x v_x|}{v^3}}_{I_{42}}$$

$$+\underbrace{\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}\theta v_x^2|v_{xx}|}{v^4}}_{I_{43}}+\underbrace{\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}\theta^2\theta_x^2|v_{xx}|}{v}}_{I_{44}}+\underbrace{\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}\theta^3|v_{xx}\theta_{xx}|}{v}}_{I_{45}}. \tag{4.105}$$

Now we turn to estimate terms I_j $(j=36,\dots,45)$. For this purpose, we get first from (4.70), Lemma 4.4 and (2.9) with $m=1/4$ that

$$\begin{aligned}I_{36}&\le\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{20v^3}+C\int_0^t V(s)\int_{\mathbb{R}_+}\frac{r^{2n-2}v_{xx}^2}{v^2}+C\int_0^t\int_{\mathbb{R}_+}\frac{r^{2n-2}u_{tx}^2}{v}\\&\le\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{20v^3}+C\int_0^t V(s)\int_{\mathbb{R}_+}\frac{r^{2n-2}v_{xx}^2}{v^2}+C.\end{aligned}\tag{4.106}$$

Next, due to the fact that

$$\left(\frac{(r^{n-1}u)_x}{v}\right)_x=\frac{2(n-1)u_x}{r}-\frac{(n-1)uv}{r^{n+1}}-\frac{r^{n-1}u_xv_x}{v^2}+\frac{r^{n-1}u_{xx}}{v},$$

we can obtain from (2.3), (2.9), (3.2), (3.9), (3.11) and (4.77) that

$$\begin{aligned}I_{37}&\le C\int_0^t\int_{\mathbb{R}_+}\left(\frac{v|uv_{xx}|}{r^3}+r^{n-3}|u_xv_{xx}|+\frac{r^{2n-3}|v_{xx}v_xu_x|}{v^2}+\frac{r^{2n-3}|v_{xx}u_{xx}|}{v}\right)\\&\le\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{20v^3}+C\int_0^t V(s)\int_{\mathbb{R}_+}\frac{r^{2n-2}v_{xx}^2}{v^2}\\&\quad+C\int_0^t\int_{\mathbb{R}_+}\left(\frac{vu^2}{r^2\theta}+\frac{r^{2(n-1)}u_x^2}{v\theta}+r^{2n-2}u_x^2v_x^2+r^{2n-2}u_{xx}^2\right)\end{aligned}$$

$$\leq \int_0^t \int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{20v^3} + C + C\int_0^t \left[V(s)\|r^{n-1}v_{xx}\|^2 + \left\| r^{n-1}u_x \right\|_{L^\infty(\mathbb{R}_+)}^2 \|v_x\|^2 \right]$$

$$\leq \int_0^t \int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{20v^3} + C\int_0^t V(s)\|r^{n-1}v_{xx}\|^2 + C. \tag{4.107}$$

And

$$I_{38} \leq \int_0^t \int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{20v^3} + C\int_0^t V(s)\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2} + C\int_0^t \int_{\mathbb{R}_+} \frac{r^{2n-4}u^2v_{xx}^2}{v^2}$$

$$\leq \int_0^t \int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{20v^3} + C\int_0^t \left(V(s) + \left\| \frac{u}{r} \right\|_{L^\infty(\mathbb{R}_+)}^2 \right) \int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2}. \tag{4.108}$$

For the term I_{39}, since

$$\left(\frac{v_x^2}{v^2} \right)_t = \frac{2v_x}{v} \cdot \frac{v_{tx}v - v_x v_t}{v^2} = \frac{2v_x(r^{n-1}u)_{xx}}{v^2} - \frac{2v_x^2(r^{n-1}u)_x}{v^2},$$

then we have

$$I_{39} \lesssim \underbrace{\int_0^t \int_{\mathbb{R}_+} \frac{r^{2n-2}\left| v_x^2 v_{xx}(r^{n-1}u)_x \right|}{v^4}}_{I_{39}^1} + \underbrace{\int_0^t \int_{\mathbb{R}_+} \frac{r^{2n-2}\left| v_x v_{xx}(r^{n-1}u)_{xx} \right|}{v^3}}_{I_{39}^2}. \tag{4.109}$$

As for the term I_{39}^1, we can deduce that

$$I_{39}^1 \leq \int_0^t \int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{80v^3} + C\int_0^t \left[V(s)\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2} + \int_{\mathbb{R}_+} r^{2n-2}v_x^4 \left[(r^{n-1}u)_x \right]^2 \right]$$

$$\leq \int_0^t \int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{80v^3} + C\int_0^t \left[V(s)\|r^{n-1}v_{xx}\|^2 + \|v_x\|_{L^\infty(\mathbb{R}_+)}^2 \left\| (r^{n-1}u)_x \right\|_{L^\infty(\mathbb{R}_+)}^2 \right]$$

$$\leq \int_0^t \int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{40v^3} + C\int_0^t \left[V(s)\|r^{n-1}v_{xx}\|^2 + \|(r^{n-1}u)_{xx}\|^2 \right]. \tag{4.110}$$

Here we have used (4.78) and the fact that

$$\left\|(r^{n-1}u)_x\right\|^2 \lesssim \int_{\mathbb{R}_+} \left(\frac{u^2}{r^2} + r^{2n-2}u_x^2\right) \mathrm{d}x \lesssim 1. \tag{4.111}$$

On the other hand, we can infer from (2.3), (3.1), (4.77) and (4.78) that

$$\begin{aligned}
\int_0^t \|(r^{n-1}u)_{xx}\|^2 &\lesssim \int_0^t \int_{\mathbb{R}_+} \left(\frac{u^2}{r^{2n+2}} + \frac{u^2 v_x^2}{r^2} + \frac{u_x^2}{r^2} + r^{2n-2}u_{xx}^2\right) \\
&\lesssim 1 + \int_0^t \int_{\mathbb{R}_+} \frac{u^2}{r^2\theta} \cdot \theta + \int_0^t \left\|\frac{u}{r}\right\|^2_{L^\infty(\mathbb{R}_+)} \int_{\mathbb{R}_+} v_x^2 + \int_0^t \int_{\mathbb{R}_+} \frac{r^{2n-2}u_x^2}{v\theta}\theta \lesssim 1.
\end{aligned} \tag{4.112}$$

The combination of (4.110) and (4.112) shows that

$$I_{39}^1 \le \int_0^t \int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{40v^3} + C\int_0^t V(s)\|r^{n-1}v_{xx}\|^2 + C. \tag{4.113}$$

Moreover, it follows from (2.9), (3.1), (4.78), (4.79) that

$$\begin{aligned}
I_{39}^2 &\le \int_0^t \int_{\mathbb{R}_+} \frac{r^{2(n-1)}\theta v_{xx}^2}{80v^3} + C\int_0^t \left[V(s)\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2} + \int_{\mathbb{R}_+} r^{2n-2}v_x^2\left[(r^{n-1}u)_{xx}\right]^2\right] \\
&\le \int_0^t \int_{\mathbb{R}_+} \frac{r^{2(n-1)}\theta v_{xx}^2}{80v^3} + C\int_0^t V(s)\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2} \\
&\quad + C\int_0^t \int_{\mathbb{R}_+} \left(\frac{v_x^2 u^2}{r^4} + \frac{r^{2n-2}u^2 v_x^4}{r^2} + \frac{r^{2n-2}u_x^2 v_x^2}{r^2} + r^{4n-4}v_x^2 u_{xx}^2\right) \\
&\le \int_0^t \int_{\mathbb{R}_+} \frac{r^{2(n-1)}\theta v_{xx}^2}{80v^3} + C\int_0^t \left[V(s)\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2} + \left\|\frac{u}{r}\right\|^2_{L^\infty(\mathbb{R}_+)} \|v_x\|^2\right] \\
&\quad + C\int_0^t \|v_x\|^2_{L^\infty(\mathbb{R}_+)}\|(r^{n-1}v_x, r^{n-1}u_x, r^{2n-2}u_{xx})\|^2 \\
&\le C + \frac{1}{80}\int_0^t \|r^{n-1}\sqrt{\theta}v_{xx}\|^2 + C\int_0^t \left[V(s)\|r^{n-1}v_{xx}\|^2 + \|v_x\|\|v_{xx}\|(1 + \|r^{2n-2}u_{xx}\|^2)\right]
\end{aligned}$$

$$\leq C + \frac{1}{40}\int_0^t \|r^{n-1}\sqrt{\theta}v_{xx}\|^2 + C\int_0^t V(s)\|r^{n-1}v_{xx}\|^2$$
$$+ C\int_0^t \|v_x\|^2 + C\int_0^t \|r^{2n-2}u_{xx}\|^2\|r^{n-1}v_{xx}\|^2 \mathrm{d}s. \tag{4.114}$$

Furthermore, (2.9) and (4.78) tell us that

$$\int_0^t\int_{\mathbb{R}_+} v_x^2 \lesssim \int_0^t\int_{\mathbb{R}_+} \theta v_x^2 + C\int_0^t V(s)\int_{\mathbb{R}_+} v_x^2 \lesssim 1. \tag{4.115}$$

Substituting (4.113)–(4.115) into (4.109) gives

$$I_{39} \leq C + \frac{1}{40}\int_0^t \|r^{n-1}\sqrt{\theta}v_{xx}\|^2 + C\int_0^t V(s)\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2}$$
$$+ C\int_0^t \|r^{2n-2}u_{xx}\|^2\|r^{n-1}v_{xx}\|^2 \mathrm{d}s. \tag{4.116}$$

For I_{40}, by making use of (2.9) and (4.80), we find

$$I_{40} \leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3} + C\int_0^t V(s)\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2} + C\int_0^t\int_{\mathbb{R}_+} r^{2n-2}\theta_{xx}^2$$
$$\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3} + C\int_0^t V(s)\int_{\mathbb{R}_+} \frac{r^{2n-2}v_{xx}^2}{v^2} + C. \tag{4.117}$$

As for the term I_{41}, we have from (2.3), (4.77) and (4.78) that

$$I_{41} \leq C\int_0^t\int_{\mathbb{R}_+} \left(\frac{r^{n-2}|v_{xx}\theta_x|}{v} + \frac{r^{n-2}\theta|v_{xx}v_x|}{v^2} + \theta^3|v_{xx}\theta_x|\right)$$
$$\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3} + C\int_0^t\int_{\mathbb{R}_+} \left[\frac{r^{2n-2}\theta_x^2}{v\theta^2}\cdot\left(\theta+\theta^7\right) + \theta v_x^2\right]$$
$$\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+} \frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3} + C. \tag{4.118}$$

In view of (4.78) and (4.115) and similar to that of (4.114), we conclude

$$\begin{aligned}
I_{42} &\leq \frac{1}{40}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C\int_0^t V(s)\int_{\mathbb{R}_+}\frac{r^{2n-2}v_{xx}^2}{v^2}+C\int_0^t\|v_x\|_{L^\infty(\mathbb{R}_+)}^2\int_{\mathbb{R}_+}r^{2n-2}\theta_x^2\\
&\leq \frac{1}{40}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C\int_0^t V(s)\int_{\mathbb{R}_+}\frac{r^{2n-2}v_{xx}^2}{v^2}+C\int_0^t\|v_x\|\|v_{xx}\|\mathrm{d}s\\
&\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C\int_0^t V(s)\int_{\mathbb{R}_+}\frac{r^{2n-2}v_{xx}^2}{v^2}+C
\end{aligned}\tag{4.119}$$

and

$$\begin{aligned}
I_{43} &\leq \frac{1}{40}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C\int_0^t\|v_x\|_{L^\infty(\mathbb{R}_+)}^2\int_{\mathbb{R}_+}r^{2n-2}v_x^2\\
&\leq \frac{1}{40}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C\int_0^t\|v_x\|\|v_{xx}\|\mathrm{d}s\\
&\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C\int_0^t V(s)\int_{\mathbb{R}_+}\frac{r^{2n-2}v_{xx}^2}{v^2}+C.
\end{aligned}\tag{4.120}$$

For the term I_{44}, one gets from (4.78) and (4.81) that

$$\begin{aligned}
I_{44} &\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C\int_0^t\int_{\mathbb{R}_+}r^{2n-2}\theta_x^4\\
&\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C\int_0^t\left\|r^{n-1}\theta_x(s)\right\|_{L^\infty(\mathbb{R}_+)}^2\int_{\mathbb{R}_+}\theta_x^2\\
&\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C.
\end{aligned}\tag{4.121}$$

Furthermore, we conclude from (4.80) that

$$I_{45}\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C\int_0^t\int_{\mathbb{R}_+}r^{2n-2}\theta_{xx}^2\leq \frac{1}{20}\int_0^t\int_{\mathbb{R}_+}\frac{Rr^{2(n-1)}\theta v_{xx}^2}{v^3}+C.\tag{4.122}$$

Combining (4.105)–(4.122) and taking full advantage of (3.1), (4.78) and Gronwall's inequality, we can obtain (4.103).

It remains to estimate the term $\int_0^t \left\|r^{n-1}v_x(s)\right\|^2_{L^\infty(\mathbb{R}_+)} ds$. To this end, noticing that

$$\left(r^{n-1}v_x\right)_x = (n-1)r^{-1}vv_x + r^{n-1}v_{xx},$$

we have

$$\begin{aligned}
&\int_0^t \left\|r^{n-1}v_x\right\|^2_{L^\infty(\mathbb{R}_+)} ds \lesssim \int_0^t \left\|r^{n-1}v_x\right\| \left\|\left(r^{n-1}v_x\right)_x\right\| ds \\
&\lesssim \int_0^t \int_{\mathbb{R}_+} \left(r^{2n-2}v_x^2 + v_x^2 + r^{2n-2}v_{xx}^2\right) \\
&\lesssim 1 + \int_0^t \int_{\mathbb{R}_+} \left(\theta r^{2n-2}v_x^2 + \theta r^{2n-2}v_{xx}^2\right) + \int_0^t V(s)\|(r^{n-1}v_x, r^{n-1}v_{xx})\|^2 ds \lesssim 1. \qquad (4.123)
\end{aligned}$$

Here we have used (2.9), (2.12), (4.78), (4.103) and (4.115). □

5. Lower bound on temperature

We shall derive a local-in-time lower bound on the temperature $\theta(t, x)$ in the following lemma.

Lemma 5.1. *Under the assumptions listed in Theorem 1.1, for all* $0 \le s \le t \le T$ *and* $x \in \mathbb{R}_+$,

$$\theta(t,x) \ge \frac{\min_{\mathbb{R}_+} \theta(s,\cdot)}{C + C(t-s+1)\min_{\mathbb{R}_+}\theta(s,\cdot)}. \qquad (5.1)$$

Proof. By setting $h(t, x) := 1/\theta(t, x)$, we multiply (4.12) by θ^{-2} to get

$$\begin{aligned}
e_\theta h_t = &\left(\frac{r^{2n-2}\kappa h_x}{v}\right)_x - \left\{\frac{2r^{2n-2}\kappa h_x^2}{vh} + \frac{h^2}{v}\left((n-1)\delta\alpha - 2(n-2)\mu\right)\right. \\
&\times \left[\frac{uv}{r} + \frac{(\alpha\delta - 2\mu)\, r^{n-1}u_x}{(n-1)\delta\alpha - 2(n-2)\mu}\right]^2 + \frac{2\mu\,(n\delta\alpha - 2(n-1)\mu)\, r^{2n-2}u_x^2h^2}{v\left((n-1)\delta\alpha - 2(n-2)\mu\right)} \\
&\left. + \frac{\alpha\,(1-\delta)\,h^2}{v}\left[(r^{n-1}u)_x - \frac{vP_\theta}{2\alpha\,(1-\delta)\,h}\right]^2 + \lambda\phi z h^2\right\} + \frac{vP_\theta^2}{4\alpha\,(1-\delta)}, \qquad (5.2)
\end{aligned}$$

where $\alpha = 2\mu + \zeta > 0$ and

$$0 \le \frac{2(n-2)\mu}{(n-1)\alpha} < \frac{2(n-1)\mu}{n\alpha} < \delta < 1. \qquad (5.3)$$

By virtue of (4.77), one can infer

$$h_t \leq \frac{1}{e_\theta}\left(\frac{r^{2n-2}\kappa(v,\theta)h_x}{v}\right)_x + \frac{vP_\theta^2}{4\alpha(1-\delta)e_\theta} \leq \frac{1}{e_\theta}\left(\frac{r^{2n-2}\kappa(v,\theta)h_x}{v}\right)_x + C. \tag{5.4}$$

For $g(t,x) := h(t,x) - 1$, we have

$$g_t \leq \frac{1}{e_\theta}\left(\frac{r^{2n-2}\kappa(v,\theta)g_x}{v}\right)_x + C. \tag{5.5}$$

Multiply (5.5) by $(g_+)^{2p-1}$ with $g_+(t,x) := \max\{g(t,x),0\}$ to yield

$$\begin{aligned} &\|g_+\|_{L^{2p}(\mathbb{R}_+)}^{2p-1}\left(\|g_+\|_{L^{2p}(\mathbb{R}_+)}\right)_t \\ &\quad\lesssim \int\limits_{\mathbb{R}_+}(g_+)^{2p-1}\,\mathrm{d}x + \int\limits_{\mathbb{R}_+}\frac{(g_+)^{2p-1}}{e_\theta}\left(\frac{r^{2n-2}\kappa(v,\theta)g_x}{v}\right)_x\mathrm{d}x. \end{aligned}\tag{5.6}$$

And we also have

$$\begin{aligned} \frac{(g_+)^{2p-1}}{e_\theta}\left(\frac{r^{2n-2}\kappa(v,\theta)g_x}{v}\right)_x &= \left(\frac{r^{2n-2}\kappa(v,\theta)(g_+)^{2p-1}g_x}{ve_\theta}\right)_x \\ &+ \frac{r^{2n-2}\kappa(v,\theta)(g_+)^{2p-1}(g_+)_x(e_\theta)_x}{ve_\theta^2} - \frac{(2p-1)r^{2n-2}\kappa(v,\theta)(g_+)^{2p-2}\left[(g_+)_x\right]^2}{ve_\theta}. \end{aligned}\tag{5.7}$$

It is easy to see

$$\begin{aligned} &\frac{r^{2n-2}\kappa(v,\theta)(g_+)^{2p-1}(g_+)_x(e_\theta)_x}{ve_\theta^2} \\ &\leq \frac{(2p-1)r^{2n-2}\kappa(v,\theta)(g_+)^{2p-2}\left[(g_+)_x\right]^2}{ve_\theta} + \frac{C}{2p-1}\frac{r^{2n-2}\kappa(v,\theta)(g_+)^{2p}\left((e_\theta)_x\right)^2}{ve_\theta^3}. \end{aligned}\tag{5.8}$$

Combining (5.6)–(5.8) and making use of Hölder's inequality yield

$$\begin{aligned} \|g_+(t)\|_{L^{2p}(\mathbb{R}_+)}^{2p-1}\left(\|g_+(t)\|_{L^{2p}(\mathbb{R}_+)}\right)_t &\leq C\|g_+(t)\|_{L^{\frac{2p}{3}}(\mathbb{R}_+)}^{\frac{1}{2}}\|g_+(t)\|_{L^{2p}(\mathbb{R}_+)}^{\frac{4p-3}{2}} \\ &+ \frac{C}{2p-1}\|g_+(t)\|_{L^{2p}(\mathbb{R}_+)}^{2p-1}\left\|\frac{r^{2n-2}\kappa(v,\theta)g_+\left((e_\theta)_x\right)^2}{ve_\theta^3}\right\|_{L^{2p}(\mathbb{R}_+)}. \end{aligned}$$

That is,

$$\left(\|g_+(t)\|_{L^{2p}(\mathbb{R}_+)}\right)_t \leq C\frac{\|g_+(t)\|_{L^{\frac{2p}{3}}(\mathbb{R}_+)}^{\frac{1}{2}}}{\|g_+(t)\|_{L^{2p}(\mathbb{R}_+)}^{\frac{1}{2}}} + \frac{C}{2p-1}\left\|\frac{r^{2n-2}\kappa(v,\theta)g_+\left((e_\theta)_x\right)^2}{ve_\theta^3}\right\|_{L^{2p}(\mathbb{R}_+)}. \tag{5.9}$$

Integrating (5.9) with respect to t over (s,t), we arrive at

$$\|g_+(t)\|_{L^{2p}(\mathbb{R}_+)} \le \|g_+(s)\|_{L^{2p}(\mathbb{R}_+)} + C\int_s^t \frac{\|g_+(\tau)\|^{\frac12}_{L^{\frac{2p}{3}}(\mathbb{R}_+)}}{\|g_+(\tau)\|^{\frac12}_{L^{2p}(\mathbb{R}_+)}}\mathrm{d}\tau + \frac{C}{2p-1}\int_s^t \left\|\frac{r^{2n-2}\kappa(v,\theta)\, g_+\left((e_\theta)_x\right)^2}{ve_\theta^3}\right\|_{L^{2p}(\mathbb{R}_+)}\mathrm{d}\tau. \tag{5.10}$$

Moreover, one can conclude from (2.12) that

$$\left((e_\theta)_x\right)^2 = \left(4a\theta^3 v_x + 12av\theta^2\theta_x\right)^2 \lesssim \theta^6 v_x^2 + \theta^4\theta_x^2. \tag{5.11}$$

If we define $\Omega_1(t) = \{x \in \mathbb{R}_+ | 0 < \theta(t,x) < 1\}$, then it follows from (2.3), (2.12), (4.77), (4.81), (4.104) and (5.11) that

$$\begin{aligned}
&\int_s^t \left\|\frac{r^{2n-2}\kappa(v,\theta)\, g_+\left((e_\theta)_x\right)^2}{ve_\theta^3}\right\|_{L^{2p}(\mathbb{R}_+)}\mathrm{d}\tau = \int_s^t \left\|\frac{r^{2n-2}\kappa(v,\theta)\, g\left((e_\theta)_x\right)^2}{ve_\theta^3}\right\|_{L^{2p}(\Omega_1(\tau))}\mathrm{d}\tau\\
&\le C\int_s^t \left\|\frac{r^{2n-2}\left(1+\theta^b\right)(1-\theta)\left(\theta^6 v_x^2+\theta^4\theta_x^2\right)}{\theta}\right\|_{L^{2p}(\Omega_1(\tau))}\mathrm{d}\tau\\
&\lesssim \int_s^t \left\|r^{2n-2}(1-\theta)\left(v_x^2+\theta_x^2\right)\right\|_{L^{2p}(\Omega_1(\tau))}\mathrm{d}\tau\\
&\lesssim \int_0^t \left(\left\|r^{n-1}v_x(\tau)\right\|^2_{L^\infty(\mathbb{R}_+)} + \left\|r^{n-1}\theta_x(\tau)\right\|^2_{L^\infty(\mathbb{R}_+)}\right)\left(\int_{\mathbb{R}_+}(1-\theta(\tau,x))^{2p}\,\mathrm{d}x\right)^{\frac{1}{2p}}\mathrm{d}\tau \lesssim 1.
\end{aligned} \tag{5.12}$$

Hence (5.10) and (5.12) give birth to

$$\|g_+(t)\|_{L^{2p}(\mathbb{R}_+)} \le \|g_+(s)\|_{L^{2p}(\mathbb{R}_+)} + C\int_s^t \frac{\|g_+(\tau)\|^{\frac12}_{L^{\frac{2p}{3}}(\mathbb{R}_+)}}{\|g_+(\tau)\|^{\frac12}_{L^{2p}(\mathbb{R}_+)}}\mathrm{d}\tau + \frac{C}{2p-1}. \tag{5.13}$$

Then we can deduce from the fact $g_+(t,x) \in L^\infty(\mathbb{R}_+)\cap L^2(\mathbb{R}_+)$ that

$$\lim_{p\to+\infty}\|g_+(t)\|_{L^p(\mathbb{R}_+)} = \|g_+(t)\|_{L^\infty(\mathbb{R}_+)}. \tag{5.14}$$

Letting $p \to +\infty$ in (5.13) and using (5.14), we have

$$\|g_+(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} \leq \|g_+(s,\cdot)\|_{L^\infty(\mathbb{R}_+)} + C(t-s). \tag{5.15}$$

Then (5.1) follows from (5.15) and the definition of $g_+(t,x)$ immediately. □

With Lemmas 2.1–5.1 in hand, we can deduce Theorem 1.1 by using the continuation argument introduced in Wang and Zhao [27] and we omit the details for brevity.

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