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STABILITY OF VISCOUS SHOCK WAVES FOR THE ONE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DENSITY-DEPENDENT VISCOSITY*

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Abstract We study the large-time behavior toward viscous shock waves to the Cauchy problem of the one-dimensional compressible isentropic Navier-Stokes equations with density-dependent viscosity. The nonlinear stability of the viscous shock waves is shown for certain class of large initial perturbation with integral zero which can allow the initial density to have large oscillation. Our analysis relies upon the technique developed by Kanel′ and the continuation argument.

Key words viscous shock waves; density-dependent viscosity; one-dimensional compressible Navier-Stokes equations; nonlinear stability; large density oscillation

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1 Introduction

We consider the large time behavior of global solutions to Cauchy problem of the one-dimensional compressible isentropic Navier-Stokes equations with density-dependent viscosity in Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \Big(\mu(v)\dfrac{u_x}{v}\Big)_x \end{cases} \tag{1.1}$$

with prescribed initial conditions

$$(v,u)|_{t=0} = (v_0,u_0)\,, \quad \lim_{x\to\pm\infty}(v_0,u_0)(x) = (v_\pm,u_\pm)\,, \tag{1.2}$$

here $t>0$ is the time variable, $x\in\mathbb{R}$ is the Lagrangian spatial variable, and $v_\pm>0$, $u_\pm$ are given constants. The primary dependent variables are the specific volume v and the velocity u. Throughout this manuscript, the pressure p and the viscosity coefficient μ are given by

$$p(v) = av^{-\gamma}, \quad \mu(v) = bv^{-\kappa}, \tag{1.3}$$

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where $\gamma > 1$ represents the adiabatic exponent, $a > 0, b > 0$ and κ are the gas constants. Without loss of generality, we can assume that $a = b = 1$ in the rest of this manuscript.

Before stating the main problem studied in this manuscript, we first explain our motivation to study the one-dimensional compressible Navier-Stokes equations (1.1) satisfying relations (1.3). According to the study on the kinetic theory of dilute gases, if one derives the one-dimensional compressible Navier-Stokes equations from the Boltzmann equation with slab symmetry through the Chapman-Enskog expansion (see Chapman-Cowling [1]), one can deduce that the five thermodynamical variables, i.e., the density $\rho = v^{-1}$, the temperature θ, the internal energy e, the entropy s, and the pressure p, satisfy the equations of the state of the ideal polytropic gases

$$p = \frac{R\theta}{v} = \tilde{b}v^{-l}e^{s/c_v}, \quad e = c_v\theta \tag{1.4}$$

for some positive constants $l > 1$, $R > 0$, $\tilde{b} > 0$, $c_v > 0$ and the viscosity coefficient μ together with the heat conductivity coefficient $\overline{\kappa}$ are no longer positive constants but depend on the temperature. In fact for the cutoff inverse power force model (cf. [14]), the interacting potential between molecules is proportional to r^{1-s} where r denotes the distance between molecules and $s > 5$ is a constant and in such a case, one can deduce by employing the properties of the Burnett functions that the viscosity coefficient μ and the heat conductivity coefficient $\overline{\kappa}$ satisfy

$$\mu(\theta) \propto \theta^{\frac{1}{2}+\frac{2}{s-1}}, \quad \overline{\kappa}(\theta) \propto \theta^{\frac{1}{2}+\frac{2}{s-1}}. \tag{1.5}$$

Note that as $s \to +\infty$, the cutoff inverse power force model is then reduced to the hard sphere model, while the Maxwell molecule model corresponds to the case of $s = 5$.

For isentropic polytropic flows, the pressure p satisfies $p = \tilde{a}\rho^\gamma$ for some positive constants $\tilde{a} > 0, \gamma \geq 1$. Such a fact together with (1.4) imply

$$\theta = \frac{\tilde{a}}{R}\rho^{\gamma-1}. \tag{1.6}$$

Thus for isentropic polytropic flows, one can get from (1.5) and (1.6) that the dependence of the viscosity coefficient μ on θ can be transferred into the dependence of μ on the density as

$$\mu(\rho) \propto \rho^{\frac{(s+3)(\gamma-1)}{2(s-1)}}, \quad \gamma > 1, \quad s > 5, \tag{1.7}$$

which is nothing but (1.3) with $\kappa = \frac{(s+3)(\gamma-1)}{2(s-1)}$. It is worth to pointing out that although the fact that $s > 5$ from physical consideration implies that $\frac{\gamma-1}{2} < \kappa \leq \gamma - 1$, to illustrate the range of the parameters γ and κ to which our argument can be applied, we will deal with the case when (1.3) hold with the constant κ being independent of γ in the rest of the paper.

The problem we want to study is on the time-asymptotically nonlinear stability of viscous shock waves for the Cauchy problem (1.1)–(1.2). Recall that a viscous shock wave of (1.1) connecting (v_-, u_-) and (v_+, u_+) is a traveling wave solution $(v, u)(t, x) \equiv (V, U)(x - st)$ of (1.1) satisfying

$$(V, U)(-\infty) = (v_l, u_l), \quad (V, U)(+\infty) = (v_r, u_r), \tag{1.8}$$

where s is the shock speed and (v_l, u_l) and (v_r, u_r) are the given far-field states satisfying $(v_+, u_+) \in S_1S_2(v_-, u_-)$, where

$$S_1S_2(v_-, u_-) := \{(v, u) : \ u < u_- - (v - v_-)s_i(v, v_-), \ i = 1, 2\}$$

with the speed $s_i(v, v_-) = (-1)^i\sqrt{(p(v) - p(v_l))/(v_l - v)}$.

Under the above assumptions imposed on the far-fields $(v_\pm, u_\pm)$ of the initial data, following the standard arguments in [16], we can find a unique $(\bar{v}, \bar{u}) \in S_1(v_-, u_-)$ such that $(v_+, u_+) \in S_2(\bar{v}, \bar{u})$, where

$$S_i(v_l, u_l) := \{(v, u);\ u = u_l - (v - v_-)s_i(v, v_l),\ u < u_l\}$$

is the i-shock curve passing through (v_l, u_l). It is easy to show that the system (1.1) admits a 1-viscous shock wave $(V_1, U_1)(x - s_1 t)$ connecting (v_-, u_-) with $(\bar{v}, \bar{u})$ and a 2-viscous shock wave $(V_2, U_2)(x - s_2 t)$ connecting $(\bar{v}, \bar{u})$ with (v_+, u_+), and both of them are unique up to a shift, where $s_1 = s_1(v_-, \bar{v}) < 0$ and $s_2 = s_2(v_+, \bar{v}) > 0$. It is expected that the large-time behavior of global solutions of the Cauchy problem (1.1)–(1.2) is described by the superposition of the shifted 1-viscous shock wave $(V_1, U_1)(x - s_1 t + \alpha_1)$ and the shifted 2-viscous shock wave $(V_2, U_2)(x - s_2 t + \alpha_2)$:

$$(V, U)(t, x; \alpha_1, \alpha_2) := (V_1, U_1)(x - s_1 t + \alpha_1) + (V_2, U_2)(x - s_2 t + \alpha_2) - (\bar{v}, \bar{u}), \tag{1.9}$$

where the shifts α_1 and α_2 are given by

$$\alpha_1 = \frac{s_2 A + B}{\delta_1(s_1 - s_2)}, \quad \alpha_2 = \frac{s_1 A + B}{\delta_2(s_1 - s_2)} \tag{1.10}$$

with

$$A = \int_{\mathbb{R}} [v_0(x) - V_1(x) - V_2(x) + \bar{v}]\,\mathrm{d}x < +\infty, \tag{1.11}$$

$$B = \int_{\mathbb{R}} [u_0(x) - U_1(x) - U_2(x) + \bar{u}]\,\mathrm{d}x < +\infty. \tag{1.12}$$

Before stating our main result, we first recall some previous results closely related. For the case when the viscosity coefficient $\mu(v)$ is a constant, stability of viscous shock wave for small initial perturbation with "zero mass" condition was proved in Kawashima-Matsumura [8] for small-amplitude profile and in Matsumura-Nishihara [13] where the corresponding assumption imposed on the amplitude of the viscous shock profile is relaxed to the assumption that

$$2(\gamma - 1)\left(1 + \frac{v_r^{-\gamma} - v_l^{-\gamma}}{\gamma(v_r - v_l)v_l^{-\gamma-1}}\right) < (\gamma - 1) + \left[\frac{v_l^{-\gamma} - v_r^{-\gamma}}{\gamma(v_r - v_l)v_l^{-\gamma-1}}\right]^2. \tag{1.13}$$

Notice that

$$\frac{v_r^{-\gamma} - v_l^{-\gamma}}{\gamma(v_r - v_l)v_l^{-\gamma-1}} = -1 + \frac{\gamma + 1}{2v_l}(v_r - v_l) + O(1)|v_r - v_l|^2,$$

although for general $\gamma > 1$, especially for the case when γ is sufficiently large, assumption (1.13) holds only when the strength of the viscous shock profile is sufficiently small, it does hold for any v_l and v_r if $\gamma \to 1$. Later, there appeared many works treating the case when the initial perturbation is not of zero mass. In particular, the asymptotic stability for small-amplitude viscous shock wave of (1.1) and related physical systems was studied in Mascia-Zumbrun [10] and Liu-Zeng [9] with small initial perturbation. As for the asymptotic stability of viscous shock wave with large initial perturbation, it was a long-standing challenging open problem, except for the partial result obtained in [17] where viscous shock waves were shown to be time-asymptotically stable for a certain class of large initial perturbation.

For the case when the viscosity coefficient $\mu(v)$ is assumed to satisfy (1.3), there is a huge literature on mathematical studies of the compressible Navier-Stokes equations with density-dependent viscosity with various initial and boundary conditions. We here just mention some works on the large-time behavior of the solutions. Jiu-Wang-Xin [5, 6] proved the time-asymptotic stability of rarefaction waves to the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity for general initial data which may contain the vacuum. As for the stability of viscous shock wave, Matsumura-Wang [14] showed that any viscous shock wave of the system (1.1)–(1.3) with $\kappa \geq (\gamma-1)/2$ is asymptotically stable for small initial perturbations with "zero mass" condition. For the corresponding result with large initial perturbation, to the best of our knowledge, no result was obtained. The main purpose of this manuscript is devoted to this problem and what we want to show in this paper is that the viscous shock wave of the compressible Navier-Stokes equations (1.1) is still nonlinear stable for certain class of large initial perturbation which satisfies the "zero mass" condition but can allow the initial density to have large oscillation.

Now we turn to state our main result. To do so, we need first to introduce some notations as in the following: the strengths of the 1-viscous shock wave and the 2-viscous shock wave are denoted by $\delta_1 := |v_- - \bar{v}|$ and $\delta_2 := |\bar{v} - v_+|$, respectively. We also set $\delta := |u_+ - u_-|$, and

$$(\phi_0, \psi_0)(x) := \int_{-\infty}^{x} (v_0(y) - V(0, y; \alpha_1, \alpha_2), u_0(y) - U(0, y; \alpha_1, \alpha_2))\,\mathrm{d}y. \tag{1.14}$$

Second, we list some assumptions on the initial data (v_0, u_0), the strengths of the viscous shock waves δ_1, δ_2, and the shifts α_1, α_2 as follows:

(H_0) there exist δ-independent constants $\ell \geq 0$ and $C_0 > 0$ such that for each $x \in \mathbb{R}$,

$$C_0^{-1}\delta^{\ell} \leq v_0(x) \leq C_0(1 + \delta^{-\ell}); \tag{1.15}$$

(H_1) $(v_+, u_+) \in S_1S_2(v_-, u_-)$ and $(\bar{v}, \bar{u}) \in S_1(v_-, u_-)$ such that $(v_+, u_+) \in S_2(\bar{v}, \bar{u})$;

(H_2) the strengths of the viscous shock waves $\delta_1,\ \delta_2$, the shifts $\alpha_1,\ \alpha_2$ defined by (1.10) and the initial data (v_0, u_0) are assumed to satisfy

$$(v_0 - V(0, \cdot; \alpha_1, \alpha_2), u_0 - U(0, \cdot; \alpha_1, \alpha_2)) \in H^1(\mathbb{R}) \cap L^1(\mathbb{R}), \tag{1.16}$$

$$(\phi_0, \psi_0) \in L^2(\mathbb{R}), \tag{1.17}$$

and for some positive constant C_1 independent of δ,

$$C_1^{-1}\delta_2 \leq \delta_1 \leq C_1\delta_2, \quad \alpha_2 - \alpha_1 \leq C_1\delta^{-1}, \quad \text{as } \delta \to 0_+; \tag{1.18}$$

(H_3) v_- and v_+ are positive constants independent of δ.

With the above preparations in hand, we are now ready to state our main result.

Theorem 1.1 Under assumptions (H_0)–(H_3), we assume further that $0 \leq \kappa < \frac{1}{2}$, $\gamma > 1$ and

$$\|(\phi_0, \psi_0)\|_{H^1(\mathbb{R})} \leq C_2\delta^{\alpha}, \quad \|\phi_{0xx}\|_{L^2(\mathbb{R})} \leq C_2(1 + \delta^{-\beta}) \tag{1.19}$$

hold for some δ-independent positive constants C_2, α and β. If the parameters $\ell,\ \alpha$ and β are assumed to satisfy

$$\begin{cases} (3\gamma + 5\kappa + 5)\ell < \min\{2, \alpha\}, \\ \min\left\{\alpha - \dfrac{\gamma+1}{2}\ell, \dfrac{1}{4}\right\} \leq \ell(1+\kappa) + \beta, \\ \beta + \ell(1+\kappa) < \dfrac{4\gamma^2 + 3\gamma(1+2\kappa) + (1+4\kappa)(1-2\kappa)}{4\gamma^2 + 2\gamma(1+4\kappa) + 2(1+\kappa)(1-2\kappa)} \min\left\{\alpha - \dfrac{\gamma+1}{2}\ell, \dfrac{1}{4}\right\}, \end{cases} \tag{1.20}$$

then there exists a suitably small $\delta_0 > 0$ such that if $0 < \delta \le \delta_0$, the Cauchy problem (1.1)–(1.2) has a unique solution (v, u) satisfying

$$\begin{gathered}(v(t,x) - V(t,x;\alpha_1,\alpha_2), u(t,x) - U(t,x;\alpha_1,\alpha_2)) \in C\left([0,\infty); H^1(\mathbb{R})\right),\\ v(t,x) - V(t,x;\alpha_1,\alpha_2) \in L^2\left(0,\infty; H^1(\mathbb{R})\right),\\ u(t,x) - U(t,x;\alpha_1,\alpha_2) \in L^2\left(0,\infty; H^2(\mathbb{R})\right)\end{gathered}$$

and

$$C_3^{-1}\delta^{\frac{2}{1-2\kappa-\gamma}\left[\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell(1+\kappa))\right]} \le v(t,x) \le C_3\delta^{\frac{2}{1-2\kappa}\left[\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell(1+\kappa))\right]} \tag{1.21}$$

for some positive constant C_3 independent of δ. Furthermore, it holds that

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} |(v,u)(t,x) - (V,U)(t,x;\alpha_1,\alpha_2)| = 0. \tag{1.22}$$

Remark 1.2 It is easy to see that the set of the parameters $\alpha > 0,\ \beta > 0,\ \ell \ge 0$ which satisfy assumption (1.20) is not empty. In fact, since $0 \le \kappa < \frac12$ implies that

$$\frac{4\gamma^2 + 3\gamma(1+2\kappa) + (1+4\kappa)(1-2\kappa)}{4\gamma^2 + 2\gamma(1+4\kappa) + 2(1+\kappa)(1-2\kappa)} > 1,$$

we can choose $\ell = 0$, $0 < \alpha \le \frac14$, and

$$\alpha \le \beta < \frac{4\gamma^2 + 3\gamma(1+2\kappa) + (1+4\kappa)(1-2\kappa)}{4\gamma^2 + 2\gamma(1+4\kappa) + 2(1+\kappa)(1-2\kappa)}\alpha,$$

such that (1.20) holds.

Remark 1.3 If the parameter $\alpha,\ \beta,\ \ell$ satisfy $\min\{2\alpha - (\gamma+1)\ell, 1/2\} < 2(\beta + \ell(1+\kappa))$, then for $\delta > 0$ sufficiently small, we can deduce from (1.21) that for each fixed $t \ge 0$, $\operatorname{Osc} v(t) := \sup\limits_{x\in\mathbb{R}} v(t,x) - \inf\limits_{x\in\mathbb{R}} v(t,x)$, the oscillation of $v(t,x)$, can be large in our result. We must point out, however, that the H^1-norm of the initial perturbation together with the strength of the viscous shock wave are assumed to be sufficiently small in our analysis. It would be a very interesting problem to show that the viscous shock wave to the compressible Navier-Stokes equations (1.1) is nonlinear stable under general large initial perturbation or even the time-asymptotically nonlinear stable of large-amplitude viscous shock waves for a certain class of large initial perturbation like this manuscript. Note that the nonlinear stability of large-amplitude viscous shock waves to the compressible Navier-Stokes equations with constant viscosity coefficients under small initial perturbations is treated in [19].

Before concluding this section, we outline the main idea used in this paper. For general $\gamma > 1$, the argument employed in [8, 13] relies heavily on the smallness of both δ_i $(i = 1, 2)$ and the $H^2(\mathbb{R})$-norm of the initial perturbation. One of the key points in such an argument is that, based on the a priori assumption that the $H^2(\mathbb{R})$-norm of the perturbation is sufficiently small, one can deduce a uniform lower and upper positive bounds on the specific volume $v(t,x)$. With such a bound on $v(t,x)$ in hand, one can thus deduce certain a priori $H^2(\mathbb{R})$ energy-type estimates on the perturbations in terms of the initial perturbation (ϕ_0, ψ_0) provided that the strengths of the viscous shock waves are suitably small. The combination of the above analysis with the standard continuation argument yields the local stability of weak viscous shock waves for the one-dimensional compressible Navier-Stokes equations with constant viscosity.

As in [3, 11, 15] where the global stability of rarefaction waves for the one-dimensional compressible Navier-Stokes equations with constant viscosity was investigated, the main difficulty for deriving the global stability of viscous shock waves is to deduce the uniform lower and upper bounds on the specific volume v under large initial perturbation. In this paper, we use the smallness of the strengths of viscous shock waves and the $H^1(\mathbb{R})$-norm of the initial perturbation to control the possible growth of the solutions caused by the nonlinearity of the system itself, and then to derive the desired uniform lower and upper bounds on the specific volume v. It is worth pointing out that the argument developed by Kanel$'$ in [7] plays an essential role in our analysis.

The layout of this paper is as follows. After listing some notations in the rest of this section, we will state some properties of the viscous shock waves and reformulate the problem in Section 2, while Section 3 is devoted to the proof of Theorem 1.1.

Notations Throughout this paper, c and C are used to denote various generic positive constants which are independent of δ, the strength of the viscous shock wave. We will use $A \lesssim B$ ($B \gtrsim A$) if $A \leq CB$ for some positive constant C. The notation $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$. For function spaces, $L^q(\Omega)\,(1 \leq q \leq \infty)$ denotes the usual Lebesgue space on $\Omega \subset \mathbb{R}$ with norm $\|\cdot\|_{L^q(\Omega)}$, while $H^q(\Omega)$ denotes the usual Sobolev space in the L^2 sense with norm $\|\cdot\|_{H^q(\Omega)}$. To simplify the presentation, we use $\|\cdot\|$ and $\|\cdot\|_q$ to denote $\|\cdot\|_{L^2(\mathbb{R})}$ and $\|\cdot\|_{H^q(\mathbb{R})}$, respectively. The notation $(V,U)(t,x)$ will be used to denote $(V,U)(t,x;\alpha_1,\alpha_2)$ in the rest of this manuscript.

2 Preliminaries

We collect some basic properties of the viscous shock waves $(V_i,U_i)(t,x)$ $(i=1,2)$ and their superposition $(V,U)(t,x)$.

We first state the existence of the viscous shock waves $(V_i(x-s_it),U_i(x-s_it))$ $(i=1,2)$ together with their decay estimates as $x-s_it \to \pm\infty$. By using the assumption (H_3), a similar proof used in [4] leads to the following lemma. We omit its proof here.

Lemma 2.1 Assume that assumptions (H_0)–(H_3) hold, then (1.1) admits a viscous shock wave $(V_1,U_1)(x-s_2t)$ of the first family connecting (v_-,u_-) with $(\bar v,\bar u)$ with speed s_1 and a viscous shock wave $(V_2,U_2)(x-s_2t)$ of the second family connecting $(\bar v,\bar u)$ with (v_+,u_+) with speed s_2, and both of them are unique up to a shift. Moreover, there exist positive constants c which depends only on v_- and v_+, such that, for $i=1,2$,

$$
\begin{aligned}
|(V_1(\xi)-\bar v, U_1(\xi)-\bar u)| &\lesssim \delta_1 \mathrm{e}^{-c\delta_1|\xi|}, && \forall\ \xi>0,\\
|(V_2(\xi)-\bar v, U_2(\xi)-\bar u)| &\lesssim \delta_2 \mathrm{e}^{-c\delta_2|\xi|}, && \forall\ \xi<0,\\
|V_1'(\xi)| &\lesssim |V_1(\xi)-v_-|\,|V_1(\xi)-\bar v|, && \forall\ \xi\in\mathbb{R},\\
|V_2'(\xi)| &\lesssim |V_2(\xi)-\bar v|\,|V_2(\xi)-v_+|, && \forall\ \xi\in\mathbb{R},\\
U_i'(\xi) &< 0, && \forall\ \xi\in\mathbb{R},\\
|(U_i'(\xi),V_i''(\xi),U_i''(\xi))| &\lesssim |V_i'(\xi)| \lesssim \delta_i^2 \mathrm{e}^{-c\delta_i|\xi|}, && \forall\ \xi\in\mathbb{R}.
\end{aligned}
\tag{2.1}
$$

Note that $(V_i, U_i)(x - s_i t + \alpha_i)$ $(i = 1, 2)$ are exact solutions of the compressible Navier-Stokes equation (1.1), while their superposition $(V, U)(t, x)$ satisfies

$$\begin{cases} V_t - U_x = 0, \\ U_t + p(V)_x = \left(\mu(V)\dfrac{U_x}{V}\right)_x - g_x, \end{cases} \quad t > 0,\ x \in \mathbb{R}, \tag{2.2}$$

where

$$g = \mu(V)\frac{U_x}{V} - \mu(V_1)\frac{U_{1x}}{V_1} - \mu(V_2)\frac{U_{2x}}{V_2} - p(V) + p(V_1) + p(V_2) - p(\bar{v}). \tag{2.3}$$

The following lemma is concerned with some estimates on $g(t, x)$, which will play an important role in performing the energy estimates. It follows essentially from the argument in [17]. Again, we omit its proof for brevity.

Lemma 2.2 Under assumption (1.18), we have

$$\int_0^\infty \|g(t)\| \mathrm{d}t \lesssim \delta^{\frac{1}{2}}, \quad \int_0^\infty (\|g_x(t)\| + \|g_{xx}(t)\|)\, \mathrm{d}t \lesssim \delta^{\frac{3}{2}}. \tag{2.4}$$

We define $(\phi, \psi)(t, x)$ by

$$(\phi, \psi)(t, x) := \int_{-\infty}^x (v(t, y) - V(t, y; \alpha_1, \alpha_2), u(t, y) - U(t, y; \alpha_1, \alpha_2))\, dy, \tag{2.5}$$

and reformulate the original problem from (1.1) and (2.2) as

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + p(v) - p(V) = \left(\mu(v)\dfrac{u_x}{v} - \mu(V)\dfrac{U_x}{V}\right) + g, \\ (\phi, \psi)|_{t=0} = (\phi_0, \psi_0). \end{cases} \tag{2.6}$$

We then define the set of functions in which we find the solutions

$$X_{m,M}(0, T) = \left\{ (\phi(t, x), \psi(t, x)) \left| \begin{array}{c} (\phi(t, x), \psi(t, x)) \in C\left([0, T]; H^2(\mathbb{R})\right), \\ \phi_x(t, x) \in L^2\left(0, T; H^1(\mathbb{R})\right), \\ \psi_x(t, x) \in L^2\left(0, T; H^2(\mathbb{R})\right), \\ m \le V(t, x) + \phi_x(t, x) \le M \end{array} \right. \right\},$$

and the local solvability of the Cauchy problem (2.6) in such a set can be stated as in the following proposition.

Proposition 2.3 Let (ϕ_0, ψ_0) be in $H^2(\mathbb{R})$ satisfying $\|(\phi_0, \psi_0)\|_2 \le M_0$ and assume that $m \le V(0, x) + \phi_{0x}(x) \le M$ holds for each $x \in \mathbb{R}$, then there exists $t_0 > 0$ depending only on m, M and M_0 such that (2.6) has a unique solution $(\phi, \psi)(t, x) \in X_{m/2,2M}(0, t_0)$ which satisfies for each $0 \le t \le t_0$ that

$$\|\psi(t)\| \le 2\|\psi_0\|, \quad \|\psi_x(t)\| \le 2\|\psi_{0x}\|, \quad \|(\phi, \psi)(t)\|_2 \le 2M_0. \tag{2.7}$$

3 Proof of Theorem 1.1

In this section we first deduce some a priori estimates on the solution $(\phi, \psi) \in X_{1/m,M}(0, T)$ to the problem (2.6), and then prove Theorem 1.1 by using the continuation argument. We will use c and C to denote some generic positive constants independent of T, m, M and δ. Besides, we will often use the notation $(v,u)=(V+\phi_x,U+\psi_x)$, though the unknown functions are ϕ and ψ. Moreover, we denote here $N_\psi(T):=\sup_{[0,T]}\|\psi(t)\|_{L^\infty}$, or by N_ψ for simplicity. Without loss of generality, we can assume that $m\ge 1$ and $M\ge 1$.

Our first lemma is concerned with the basic energy estimate, which is stated in the following lemma.

Lemma 3.1 Under the assumptions in Theorem 1.1, there exists a sufficiently small positive constant δ_1 independent of δ such that if $0<\delta\le\delta_1$, then it holds for each $0\le t\le T$ that

$$\begin{aligned}&\|(\phi,\psi)(t)\|^2+\int_0^t\int_{\mathbb{R}}\left(|V_t|\psi^2+\psi_x^2\right)\mathrm{d}x\mathrm{d}\tau\\ \lesssim{}&\|(\phi_0,\psi_0)\|^2+\delta^{\frac12}+\int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v^{\kappa+1}}\mathrm{d}x\mathrm{d}\tau+C_1(m,M,\delta,N_\psi)\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau,\end{aligned}\tag{3.1}$$

where

$$C_1(m,M,\delta,N_\psi)=N_\psi m^{\gamma+2}+N_\psi^2M^{2\kappa}m^{\kappa+1}+M^{2\kappa}m^{2(\kappa+1)}\delta^2.\tag{3.2}$$

Proof First, $(2.6)_2$ (second equation of (2.6)) can be rewritten as

$$\begin{aligned}&\psi_t+p'(V)\phi_x-\mu(V)\frac{\psi_{xx}}{V}+(p(v)-p(V)-p'(V)\phi_x)\\ ={}&\left[\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right]\psi_{xx}+\left[\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right]U_x+g.\end{aligned}\tag{3.3}$$

Multiply $(2.6)_1$ by ϕ and (3.3) by $-p'(V)^{-1}\psi$ to find

$$\begin{aligned}&\left[\frac12\phi^2-\frac{\psi^2}{2p'(V)}\right]_t-\frac{p''(V)V_t\psi^2}{2p'(V)^2}-\frac{\mu(V)\psi_x^2}{Vp'(V)}-\left[\phi\psi-\frac{\mu(V)\psi_x\psi}{Vp'(V)}\right]_x\\ ={}&\left(\frac{\mu(V)}{Vp'(V)}\right)'V_x\psi\psi_x+(p(v)-p(V)-p'(V)\phi_x)\frac{\psi}{p'(V)}\\ &-\frac{\psi}{p'(V)}\left[\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right]\psi_{xx}-\frac{\psi}{p'(V)}\left[\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right]U_x-\frac{g\psi}{p'(V)}.\end{aligned}$$

Since v_- and v_+ are independent of δ and δ is assumed to be sufficiently small, we can deduce that $V(t,x)$ can be bounded from both below and above by some positive constants independent of δ. Integrating the above identity with respect to t and x over $[0,t]\times\mathbb{R}$ yields

$$\begin{aligned}&\|(\phi,\psi)(t)\|^2+\int_0^t\int_{\mathbb{R}}\left[|V_t|\psi^2+\psi_x^2\right]\\ \lesssim{}&\|(\phi_0,\psi_0)\|^2+\underbrace{\int_0^t\int_{\mathbb{R}}|V_x\psi\psi_x|}_{I_1}+\underbrace{\int_0^t\int_{\mathbb{R}}|(p(v)-p(V)-p'(V)\phi_x)\psi|}_{I_2}\\ &+\underbrace{\int_0^t\int_{\mathbb{R}}\left|\left[\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right]\psi\psi_{xx}\right|}_{I_3}+\underbrace{\int_0^t\int_{\mathbb{R}}\left|\left[\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right]\psi U_x\right|}_{I_4}+\underbrace{\int_0^t\int_{\mathbb{R}}|\psi g|}_{I_5}.\end{aligned}\tag{3.4}$$

Straightforward calculation leads to

$$|p(v)-p(V)-p'(V)\phi_x|\lesssim m^{\gamma+2}\phi_x^2,\quad\left|\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right|\lesssim\frac{M^\kappa|\phi_x|}{v^{\kappa+1}V^{\kappa+1}}.\tag{3.5}$$

Noting that $C\delta^2 \geq -V_t = -U_x > 0$ and using Cauchy's and Hölder's inequalities, we derive from (3.5) and (2.4) that for each $\epsilon > 0$,

$$
\begin{aligned}
I_1 &\leq \epsilon \int_0^t \int_{\mathbb{R}} \psi_x^2 \mathrm{d}x\mathrm{d}\tau + C(\epsilon) \int_0^t \int_{\mathbb{R}} V_x^2 \psi^2 \mathrm{d}x\mathrm{d}\tau, \\
I_2 &\lesssim N_\psi m^{\gamma+2} \int_0^t \int_{\mathbb{R}} \phi_x^2 \mathrm{d}x\mathrm{d}\tau, \\
I_3 &\lesssim \int_0^t \int_{\mathbb{R}} \frac{\psi_{xx}^2}{v^{\kappa+1}} \mathrm{d}x\mathrm{d}\tau + N_\psi^2 M^{2\kappa} m^{\kappa+1} \int_0^t \int_{\mathbb{R}} \phi_x^2 \mathrm{d}x\mathrm{d}\tau, \\
I_4 &\lesssim \epsilon \int_0^t \int_{\mathbb{R}} |U_x| \psi^2 \mathrm{d}x\mathrm{d}\tau + M^{2\kappa} \delta^2 m^{2\kappa+2} \int_0^t \int_{\mathbb{R}} \psi_x^2 \mathrm{d}x\mathrm{d}\tau, \\
I_5 &\lesssim \int_0^t \|\psi(\tau)\| \, \|g(\tau)\| \mathrm{d}\tau \lesssim \delta^{\frac{1}{2}} + \int_0^t \|\psi(\tau)\|^2 \, \|g(\tau)\| \mathrm{d}\tau.
\end{aligned}
$$

Estimate (3.1) can be proved by substituting the above estimates on I_j $(j = 1, \cdots, 5)$ into (3.4) and employing the Gronwall inequality. This completes the proof of Lemma 3.1. □

Lemma 3.2 Under the assumptions in Theorem 1.1, if δ is suitably small, then it holds for each $0 \leq t \leq T$ that

$$
\begin{aligned}
&\left\| \left(\phi, \psi, \psi_x, \sqrt{\Phi}, M^{-\frac{\gamma+1}{2}} \phi_x \right)(t) \right\|^2 + \int_0^t \int_{\mathbb{R}} \left[\psi_x^2 + \frac{\psi_{xx}^2}{v^{\kappa+1}} \right] \mathrm{d}x\mathrm{d}\tau \\
\lesssim &\left\| \left(\phi_0, \psi_0, \sqrt{\Phi_0}, \psi_{0x} \right) \right\|^2 + \delta^{\frac{1}{2}} + C_2(m, M, \delta, N_\psi) \int_0^t \|\phi_x(\tau)\|^2 \mathrm{d}\tau,
\end{aligned} \tag{3.6}
$$

where $\Phi_0 = \Phi|_{t=0}$, and

$$
\Phi = \Phi(v, V) = p(V)(v - V) - \int_V^v p(\eta) \mathrm{d}\eta, \tag{3.7}
$$

$$
C_2(m, M, \delta, N_\psi) = N_\psi m^{\gamma+2} + N_\psi^2 M^{2\kappa} m^{\kappa+1} + M^{2\kappa} m^{2(\kappa+1)} \delta^2 + m^{\gamma+2} \delta^2 + M^{2\kappa} m^{\kappa+1} \delta^4. \tag{3.8}
$$

Proof Multiplying $\partial_x(2.6)_1$ by $p(V) - p(v)$ and $\partial_x(2.6)_2$ by ψ_x, we obtain the following identity

$$
\begin{aligned}
&\left[\Phi + \frac{1}{2} \psi_x^2 \right]_t + \mu(v) \frac{\psi_{xx}^2}{v} + \left[\psi_x \left(p(v) - p(V) - \mu(v) \frac{\psi_{xx}}{v} - \left(\frac{\mu(v)}{v} - \frac{\mu(V)}{V} \right) U_x \right) \right]_x \\
= &- \left[\frac{\mu(v)}{v} - \frac{\mu(V)}{V} \right] U_x \psi_{xx} + \psi_x g_x - V_t \left(p(v) - p(V) - p'(V) \phi_x \right).
\end{aligned}
$$

Integrating this last identity over $[0, t] \times \mathbb{R}$ yields

$$
\begin{aligned}
&\left\| \left(\sqrt{\Phi}, \psi_x \right)(t) \right\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\psi_{xx}^2}{v^{\kappa+1}} \mathrm{d}x\mathrm{d}\tau \\
\lesssim &\left\| \left(\sqrt{\Phi_0}, \psi_{0x} \right) \right\|^2 + \underbrace{\int_0^t \int_{\mathbb{R}} \left| \left[\frac{\mu(v)}{v} - \frac{\mu(V)}{V} \right] U_x \psi_{xx} \right| \mathrm{d}x\mathrm{d}\tau}_{I_6} \\
&+ \underbrace{\int_0^t \int_{\mathbb{R}} |\psi_x| |g_x| \mathrm{d}x\mathrm{d}\tau}_{I_7} + \int_0^t \int_{\mathbb{R}} \left| V_t \left(p(v) - p(V) - p'(V) \phi_x \right) \right| \mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{3.9}
$$

Since

$$
v^{\kappa+1} - V^{\kappa+1} = (\kappa + 1) \int_0^1 \left(\theta v + (1 - \theta) V \right)^\kappa \mathrm{d}\theta \phi_x \lesssim M^\kappa |\phi_x|,
$$

we apply Cauchy's inequality to I_6 to find

$$I_6=\int_0^t\int_{\mathbb{R}}\left|\frac{\left(V^{\kappa+1}-v^{\kappa+1}\right)}{v^{\kappa+1}V^{\kappa+1}}U_x\psi_{xx}\right|\leq\epsilon\int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v^{\kappa+1}}+C(\epsilon)M^{2\kappa}m^{\kappa+1}\delta^4\int_0^t\int_{\mathbb{R}}\phi_x^2. \tag{3.10}$$

If we apply Hölder's inequality to I_7 and use (2.4), we can deduce

$$I_7\leq\int_0^t\|g_x(\tau)\|\|\psi_x(\tau)\|^2\mathrm{d}\tau+\int_0^t\|g_x(\tau)\|\mathrm{d}\tau\lesssim\int_0^t\|g_x(\tau)\|\|\psi_x(\tau)\|^2\mathrm{d}\tau+\delta^{\frac{3}{2}}. \tag{3.11}$$

Plugging (3.10), (3.11) and (3.5) into (3.9) and noting that

$$\Phi(v,V)=-\phi_x^2\int_0^1\int_0^1\theta_1p'((1-\theta_2\theta_1)V+\theta_2\theta_1v)\mathrm{d}\theta_1\mathrm{d}\theta_2\gtrsim M^{-\gamma-1}\phi_x^2, \tag{3.12}$$

we have

$$\begin{aligned}&\left\|\left(\sqrt{\Phi},\psi_x,M^{-\frac{\gamma+1}{2}}\phi_x\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v^{\kappa+1}}\mathrm{d}x\mathrm{d}\tau\\ \lesssim&\left\|\left(\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2+\left[m^{\gamma+2}\delta^2+M^{2\kappa}m^{\kappa+1}\delta^4\right]\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau+\delta^{\frac{3}{2}},\end{aligned}$$

which combined with (3.1) gives (3.6). The proof of the lemma is completed. □

We next make the estimate on the last term of (3.6), which is stated in the following lemma.

Lemma 3.3 Under the assumptions in Theorem 1.1, a δ-independent positive constant δ_2 exists such that if

$$m^{\gamma+3\kappa+3}M^{2(\gamma+\kappa+1)}\left(N_\psi+\delta^2\right)\leq\delta_2, \tag{3.13}$$

then we have for each $0\leq t\leq T$ that

$$\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau\lesssim m^{\kappa+1}M^{2\gamma+2}\left[\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2+\delta^{\frac{1}{2}}\right] \tag{3.14}$$

and

$$\begin{aligned}&\left\|\left(\phi,\psi,\psi_x,\sqrt{\Phi},M^{-\frac{\gamma+1}{2}}\phi_x\right)(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\left[\psi_x^2+\frac{\psi_{xx}^2}{v^{\kappa+1}}\right]\mathrm{d}x\mathrm{d}\tau\\ \lesssim&\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2+\delta^{\frac{1}{2}}.\end{aligned} \tag{3.15}$$

Proof Multiplying $(2.6)_2$ with ϕ_x implies

$$\begin{aligned}&(\phi_x\psi)_t+\psi_x^2-(\psi\phi_t)_x-\frac{\mu(v)\psi_{xx}\phi_x}{v}-g\phi_x\\ =&-\left(p(v)-p(V)\right)\phi_x+\left(\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right)U_x\phi_x\\ \geq&-\int_0^1p'(V+\theta\phi_x)\mathrm{d}\theta\phi_x^2\gtrsim M^{-\gamma-1}\phi_x^2,\end{aligned} \tag{3.16}$$

where we have used the fact that

$$\left(\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right)U_x\phi_x=-\frac{(\kappa+1)\int_0^1\left(\theta v+(1-\theta)V\right)^\kappa\mathrm{d}\theta\phi_x^2}{v^{\kappa+1}V^{\kappa+1}}U_x\geq0.$$

Integrating (3.16) over $[0,t]\times\mathbb{R}$, we have from Cauchy's and Hölder's inequalities that

$$M^{-\gamma-1}\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau\lesssim\|\phi_x(t)\|\|\psi(t)\|+\|\phi_{0x}\|\|\psi_0\|+\int_0^t\|\psi_x(\tau)\|^2\mathrm{d}\tau$$

$$+m^{\kappa+1}M^{\gamma+1}\int_0^t\int_{\mathbb{R}}\frac{\psi_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau+\sup_{0\le\tau\le t}\|\phi_x(\tau)\|\int_0^t\|g(\tau)\|\mathrm{d}\tau,$$

which combined with (2.4) and (3.6) implies

$$\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau\lesssim m^{\kappa+1}M^{2\gamma+2}\left[\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|^2+\delta^{\frac{1}{2}}\right]$$
$$+m^{\gamma+3\kappa+3}M^{2(\gamma+\kappa+1)}\left(N_\psi+\delta^2\right)\int_0^t\|\phi_x(\tau)\|^2\mathrm{d}\tau. \tag{3.17}$$

Noting the simple fact that

$$C_2(m,M,\delta,N_\psi)m^{\kappa+1}M^{2\gamma+2}\lesssim m^{\gamma+3\kappa+3}M^{2(\gamma+\kappa+1)}\left(N_\psi+\delta^2\right),$$

we can prove (3.14)–(3.15) and complete the proof of the lemma. □

To deduce a lower bound and an upper bound on $v(t,x)$, as in [12], we set $\tilde{v}:=v/V$ and make the estimate on $\tilde{v}_x$ in the following lemma.

Lemma 3.4 Under the assumptions in Theorem 1.1, if δ is suitably small such that (3.13) holds, then it follows that

$$\left\|\mu(v)\frac{\tilde{v}_x}{\tilde{v}}(t)\right\|^2+\int_0^t\int_{\mathbb{R}}\frac{\tilde{v}_x^2}{v^{\gamma+\kappa+2}}\mathrm{d}x\mathrm{d}\tau\lesssim\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\phi_{0x},\mu(v_0)\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right)\right\|^2+\delta^{\frac{1}{2}}. \tag{3.18}$$

Proof Since

$$\left[\mu(v)\frac{u_x}{v}-\mu(V)\frac{U_x}{V}\right]_x=\left(\mu(v)\frac{\tilde{v}_x}{\tilde{v}}\right)_t+\left[(\mu(v)-\mu(V))\frac{V_x}{V}\right]_t, \tag{3.19}$$

we differentiate $(2.6)_2$ and have

$$\left[\mu(v)\frac{\tilde{v}_x}{\tilde{v}}-\psi_x\right]_t-(p(v)-p(V))_x=-\left[(\mu(v)-\mu(V))\frac{V_x}{V}\right]_t-g_x. \tag{3.20}$$

Multiplying (3.20) with $\mu(v)\tilde{v}_x/\tilde{v}$ leads to

$$\left[\frac{1}{2}\left(\mu(v)\frac{\tilde{v}_x}{\tilde{v}}\right)^2-\mu(v)\frac{\tilde{v}_x}{\tilde{v}}\psi_x\right]_t-\mu(v)\frac{\tilde{v}_x}{\tilde{v}}(p(v)-p(V))_x$$
$$=-\left(\mu(v)\frac{\tilde{v}_x}{\tilde{v}}\right)_t\psi_x-\mu(v)\frac{\tilde{v}_x}{\tilde{v}}\left[(\mu(v)-\mu(V))\frac{V_x}{V}\right]_t-\mu(v)\frac{\tilde{v}_x}{\tilde{v}}g_x,$$

which combined with (3.19) and the identity

$$-\mu(v)\frac{\tilde{v}_x}{\tilde{v}}(p(v)-p(V))_x=-\frac{p'(v)\mu(v)V^2}{v}\tilde{v}_x^2-\mu(v)\frac{V_x\tilde{v}_x}{v}\left(p'(v)v-p'(V)V\right)$$

implies

$$\left[\frac{1}{2}\left(\mu(v)\frac{\tilde{v}_x}{\tilde{v}}\right)^2-\mu(v)\frac{\tilde{v}_x}{\tilde{v}}\psi_x\right]_t-\frac{p'(v)\mu(v)V^2}{v}\tilde{v}_x^2-\mu(v)\frac{V_x\tilde{v}_x}{v}\left(p'(v)v-p'(V)V\right)$$
$$=-\left[\psi_x\left(\mu(v)\frac{u_x}{v}-\mu(V)\frac{U_x}{V}\right)\right]_x+\mu(v)\frac{\psi_{xx}^2}{v}+\psi_{xx}\left[\frac{\mu(v)}{v}-\frac{\mu(V)}{V}\right]U_x$$
$$+\left[\psi_x-\mu(v)\frac{\tilde{v}_x}{\tilde{v}}\right]\left[(\mu(v)-\mu(V))\frac{V_x}{V}\right]_t-\mu(v)\frac{\tilde{v}_x}{\tilde{v}}g_x. \tag{3.21}$$

For each $\epsilon>0$,

$$\mu(v)\frac{V_x\tilde{v}_x}{v}\left(p'(v)v-p'(V)V\right)\le\epsilon\frac{|p'(v)|\,\mu(v)}{v}\tilde{v}_x^2+C(\epsilon)\frac{V_x^2\mu(v)}{v|p'(v)|}\left|p'(v)v-p'(V)V\right|^2$$

$$\leq \epsilon \frac{|p'(v)|\,\mu(v)}{v}\tilde{v}_x^2 + C(\epsilon) V_x^2 M^{\gamma-\kappa} m^{2\gamma+2}\phi_x^2. \tag{3.22}$$

Estimates (3.5) and (2.1) give

$$\psi_{xx}\left[\frac{\mu(v)}{v} - \frac{\mu(V)}{V}\right] U_x \lesssim |\psi_{xx} U_x| \frac{M^\kappa |\phi_x|}{v^{\kappa+1}} \lesssim \frac{\psi_{xx}^2}{v^{\kappa+1}} + \delta^4 M^{2\kappa} m^{\kappa+1}\phi_x^2. \tag{3.23}$$

Apply (2.1) to infer

$$\begin{aligned}
&\left|\left[(\mu(v) - \mu(V))\,\frac{V_x}{V}\right]_t\right| \\
=& \left|(\mu(v) - \mu(V))\left(\frac{U_x}{V}\right)_x + \left(\mu'(v)\psi_{xx} + (\mu'(v) - \mu'(V))\, U_x\right)\frac{V_x}{V}\right| \\
\lesssim& \, \delta^2 m^{\kappa+1}|\phi_x| + \delta^2 \frac{|\psi_{xx}|}{v^{\kappa+1}} + \delta^4 m^{\kappa+2}|\phi_x|.
\end{aligned} \tag{3.24}$$

Then we apply Cauchy's inequality and (3.24) to have

$$\left|\psi_x\left[(\mu(v) - \mu(V))\,\frac{V_x}{V}\right]_t\right| \lesssim \psi_x^2 + \delta^4\left[m^{2\kappa+4}\phi_x^2 + m^{\kappa+1}\frac{\psi_{xx}^2}{v^{\kappa+1}}\right] \tag{3.25}$$

and

$$\begin{aligned}
&\left|\mu(v)\frac{\tilde{v}_x}{\tilde{v}}\left[(\mu(v) - \mu(V))\,\frac{V_x}{V}\right]_t\right| \\
\lesssim& \,\epsilon\frac{|p'(v)|\,\mu(v)}{v}\tilde{v}_x^2 + C(\epsilon) M^{\gamma-\kappa}\delta^4\left[m^{2\kappa+4}\phi_x^2 + m^{\kappa+1}\frac{\psi_{xx}^2}{v^{\kappa+1}}\right].
\end{aligned} \tag{3.26}$$

Integrating (3.21) over $[0,t]\times\mathbb{R}$ and using (3.22)–(3.26), (3.14)–(3.15), and (3.13), we can obtain (3.18) by employing Gronwall's inequality and (2.4). This completes the proof of the lemma.

□

The following lemma concerns the positive lower and upper bounds on v in terms of the initial perturbation.

Lemma 3.5 Under the assumptions in Theorem 1.1, if we assume that (3.13) holds, then we have for each $(t,x)\in[0,T]\times\mathbb{R}$ that

$$B_0^{\frac{2}{1-\gamma-2\kappa}} \lesssim v(t,x) \lesssim B_0^{\frac{2}{1-2\kappa}} \tag{3.27}$$

with

$$B_0 = \left[\left\|\left(\phi_0, \psi_0, \sqrt{\Phi_0}, \psi_{0x}\right)\right\| + \delta^{\frac{1}{4}}\right]\left[\left\|\left(\phi_0, \psi_0, \sqrt{\Phi_0}, \psi_{0x}, \mu(v_0)\frac{\tilde{v}_{0x}}{\tilde{v}_0}\right)\right\| + \delta^{\frac{1}{4}}\right]. \tag{3.28}$$

Proof Rewrite $\Phi(v,V)$ as

$$\Phi(v,V) = V^{-\gamma+1}\tilde{\Phi}(\tilde{v}), \quad \tilde{\Phi}(\tilde{v}) = \tilde{v} - 1 + \frac{1}{\gamma-1}\left(\tilde{v}^{-\gamma+1} - 1\right),$$

and note that

$$\tilde{\Phi}(z) \sim \begin{cases} z, & z\to+\infty, \\ z^{-\gamma+1}, & z\to 0^+. \end{cases}$$

In order to apply Kanel′s method [7], we construct $\Psi(\tilde{v})$ as

$$\Psi(\tilde{v}) := \int_1^{\tilde{v}} \sqrt{\tilde{\Phi}(z)}\frac{\mu(z)}{z}\mathrm{d}z.$$

From the constitutive relations (1.3), we have

$$\Psi(\tilde v)\sim\begin{cases}\tilde v^{\frac12-\kappa} & \tilde v\to+\infty,\\ \tilde v^{\frac{1-\gamma}{2}-\kappa} & \tilde v\to0^+,\end{cases}$$

which implies that

$$|\Psi(\tilde v)|\gtrsim \tilde v^{\frac12-\kappa}+\tilde v^{\frac{1-\gamma}{2}-\kappa}-C \tag{3.29}$$

holds for some uniform constant $C>0$.

On the other hand, we have

$$\begin{aligned}|\Psi(\tilde v(t,x))|&=\left|\int_{-\infty}^x\frac{\partial\Psi(\tilde v)}{\partial y}(t,y)\mathrm{d}y\right|\\ &\le\int_{\mathbb R}\sqrt{\tilde\Phi(\tilde v)}\left|\mu(\tilde v)\frac{\tilde v_x}{\tilde v}\right|(t,x)\mathrm{d}x\\ &\le\left\|\sqrt{\tilde\Phi(\tilde v)}(t)\right\|\left\|\mu(\tilde v)\frac{\tilde v_x}{\tilde v}(t)\right\|.\end{aligned} \tag{3.30}$$

Noting that $\min\{v_-,\bar v,v_+\}\le V(t,x)\le\max\{v_-,\bar v,v_+\}$, and $v_\pm$, $u_\pm$ are assumed to be independent of δ, we can easily deduce (3.27) from (3.29), (3.30), (3.15) and (3.18). This completes the proof of the lemma. □

For the estimates for the second order derivatives of (ϕ,ψ). since

$$\frac{\tilde v_x}{\tilde v}=\frac{\phi_{xx}}{v}-\frac{V_x\phi_x}{Vv}, \tag{3.31}$$

we have from Lemmas 3.3–3.5 that

$$\|\phi(t)\|_2^2+\|\psi(t)\|_1^2+\int_0^t\|(\phi_x,\psi_x)(\tau)\|_1^2\mathrm{d}\tau\le C\left(\delta,\|(\phi_0,\psi_0)\|_2\right). \tag{3.32}$$

As for the estimate on $\|\psi_{xx}(t)\|$, we multiply $\partial_x(2.6)_2$ by ψ_{xxx}, integrate the resulting identity over $[0,t]\times\mathbb R$, and use the Sobolev's, Young's and Gronwall's inequalities to discover

$$\|\psi_{xx}(t)\|^2+\int_0^t\|\psi_{xxx}(\tau)\|^2\mathrm{d}\tau\le C\left(\delta,\|(\phi_0,\psi_0)\|_2\right), \tag{3.33}$$

which combined with (3.32) yields the following lemma.

Lemma 3.6 If δ is suitably small such that (3.13) holds, then we have for each $0\le t\le T$ that

$$\|(\phi,\psi)(t)\|_2^2+\int_0^t\left(\|\phi_x(\tau)\|_1^2+\|\psi_x(\tau)\|_2^2\right)\mathrm{d}\tau\le C\left(\delta,\|(\phi_0,\psi_0)\|_2\right). \tag{3.34}$$

Proof of Theorem 1.1 Since $\Phi_0(x)\lesssim\left(|V(0,x)|^{-\gamma-1}+|v(0,x)|^{-\gamma-1}\right)\phi_{0x}^2$, we get from (2.1) and the assumptions (H_0), (H_3) that

$$\left\|\sqrt{\Phi_0}\right\|\lesssim(1+\delta^{-(\gamma+1)\ell/2})\|\phi_{0x}\|,\quad \left\|\mu(v_0)\frac{\tilde v_{0x}}{\tilde v_0}\right\|\lesssim\delta^{-\ell(\kappa+1)}(\|\phi_{0xx}\|+\delta^2\|\phi_{0x}\|).$$

Hence, if (1.19)and (1.20) hold, then we have for $0<\delta<1$,

$$\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x}\right)\right\|+\delta^{\frac14}\lesssim\delta^{\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}},$$

$$\left\|\left(\phi_0,\psi_0,\sqrt{\Phi_0},\psi_{0x},\mu(v_0)\frac{\tilde v_{0x}}{\tilde v_0}\right)\right\|+\delta^{\frac14}\lesssim\delta^{-\ell(\kappa+1)-\beta}.$$

According to Proposition 2.3, there is a positive constant t_1, which depends only on δ and $\|(\phi_0,\psi_0)\|_2$, such that the Cauchy problem (2.6) admits a unique solution $(\phi(t,x),\psi(t,x))\in X_{m_0,M_0}(0,t_1)$ with $m_0=2^{-1}C_1^{-1}\delta^{\ell}$ and $M_0=2C_1(1+\delta^{-\ell})$, which satisfies (2.7) for each $0\le t\le t_1$. Hence we have from (1.19) and Sobolev's inequality that

$$N_\psi(t_1)=\sup_{[0,t_1]}\|\psi(t)\|_{L^\infty(\mathbb{R})}\le\sup_{[0,t_1]}\|\psi(t)\|^{\frac12}\|\psi_x(t)\|^{\frac12}\le 2C_2\delta^{\alpha}.$$

Consequently,

$$m_0^{-\gamma-3\kappa-3}M_0^{2(\gamma+\kappa+1)}\left(N_\psi(t_1)+\delta^2\right)\le C\delta^{\min\{\alpha,2\}-(3\gamma+5\kappa+5)\ell}.$$

Thus if $(1.20)_1$ holds, we can choose a sufficiently small constant $\delta_1<1$ such that if $0<\delta\le\delta_1$, the assumptions imposed in Lemmas 3.1–3.6 hold with $T=t_1$, $m=m_0^{-1}$ and $M=M_0$. Thus we have from (3.27) that for each $0\le t\le t_1$,

$$C_4^{-1}\delta^{\frac{2}{1-\gamma-2\kappa}\left[\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell(\kappa+1))\right]}\le v(t,x)\le C_4\delta^{\frac{2}{1-2\kappa}\left[\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell(\kappa+1))\right]}. \tag{3.35}$$

From (3.15), we can have for each $0\le t\le t_1$ that

$$\|\psi(t)\|_1\le C_5\delta^{\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}}. \tag{3.36}$$

Next if we take $(\phi(t_1,x),\psi(t_1,x))$ as the initial data, we can deduce by employing Proposition 2.3 again that the unique local solution $(\phi(t,x),\psi(t,x))$ constructed above can be extended to the time internal $[0,t_1+t_2]$ and satisfies

$$\|\psi(t)\|_{L^\infty(\mathbb{R})}\le\|\psi(t)\|_1\le 2\|\psi(t_1)\|_1\le 2C_5\delta^{\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}}$$

and

$$2^{-1}C_4^{-1}\delta^{\frac{2}{1-\gamma-2\kappa}\left[\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell(\kappa+1))\right]}\le v(t,x)\le 2C_4\delta^{\frac{2}{1-2\kappa}\left[\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell(\kappa+1))\right]}$$

for each $t_1\le t\le t_1+t_2$. Thus,

$$N_\psi(t_1+t_2)\le\max\left\{N_\psi(t_1),2C_5\delta^{\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}}\right\}\le C_6\delta^{\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}}.$$

Set

$$\begin{aligned}m_1&=2^{-1}C_4^{-1}\delta^{\frac{2}{1-\gamma-2\kappa}\left[\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell(\kappa+1))\right]},\\ M_1&=2C_4\delta^{\frac{2}{1-2\kappa}\left[\min\left\{\alpha-\frac{\gamma+1}{2}\ell,\frac14\right\}-(\beta+\ell(\kappa+1))\right]}.\end{aligned}$$

Then one can easily deduce that if the parameters $\alpha>0$, β and ℓ satisfy $(1.20)_3$, then there exists a sufficiently small $\delta_2>0$ such that if $0<\delta\le\delta_2$, the assumptions listed in Lemmas 3.1–3.6 are satisfied with $T=t_1+t_2$, $m=m_1^{-1}$ and $M=M_1$. Consequently, (3.35), (3.36) and (3.34) hold for each $0\le t\le t_1+t_2$. If we take $(\phi(t_1+t_2,x),\psi(t_1+t_2,x))$ as the initial data and employ Proposition 2.3 again, we can then extend the above solution $(\phi(t,x),\psi(t,x))$ to the time step $t=t_1+2t_2$. Repeating the above procedure, we thus extend $(\phi(t,x),\psi(t,x))$ step by step to the unique global solution and (3.35), (3.36) and (3.34) hold for all $t\ge0$. The proof of Theorem 1.1 is completed. □

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