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Inflow problem for the one-dimensional compressible Navier–Stokes equations under large initial perturbation

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Abstract

This paper is concerned with the inflow problem for the one-dimensional compressible Navier–Stokes equations. For such a problem, Matsumura and Nishihara showed in [10] that there exists boundary layer solution to the inflow problem, and that both the boundary layer solution, the rarefaction wave, and the superposition of boundary layer solution and rarefaction wave are nonlinear stable under small initial perturbation. The main purpose of this paper is to show that similar stability results for the boundary layer solution and the supersonic rarefaction wave still hold for a class of large initial perturbation which can allow the initial density to have large oscillation. The proofs are given by an elementary energy method and the key point is to deduce the desired lower and upper bounds on the density function.

Keywords: Compressible Navier–Stokes equations; Boundary layer solution; Inflow problem; Rarefaction wave; Large initial perturbation; Large density oscillation

Contents

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1. Introduction

This paper is concerned with the large time behaviors of solutions to the inflow problem for one-dimensional compressible Navier–Stokes equations on the half line $\mathbb{R}_+ = (0, +\infty)$, which is an initial–boundary value problem in Eulerian coordinates:

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ (\rho u)_t + \big(\rho u^2 + \tilde{p}\big)_x = \mu u_{xx}, & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ (\rho, u)|_{x=0} = (\rho_-, u_-), & u_- > 0, \\ (\rho, u)(0, x) = (\rho_0, u_0)(x) \to (\rho_+, u_+), & \text{as } x \to +\infty. \end{cases} \tag{1.1}$$

Here, $\rho(>0)$, u, and $\tilde{p} = \tilde{p}(\rho) = \rho^\gamma$ with $\gamma \geq 1$ being the adiabatic exponent are, respectively, the density, the velocity, and the pressure, while the viscosity coefficient μ (>0), farfield states $\rho_\pm$ (>0) and $u_\pm$ are constants.

We assume that the initial data $(\rho_0(x), u_0(x))$ satisfy the boundary condition $(1.1)_3$ as a compatibility condition, i.e.

$$\rho_0(0) = \rho_-, \qquad u_0(0) = u_-.$$

The assumption $u_- > 0$ implies that, through the boundary $x = 0$ the fluid with the density ρ_- flows into the region $\mathbb{R}_+$, and hence the problem (1.1) is called the inflow problem. The cases of $u_- = 0$ and $u_- < 0$, the problems where the condition $\rho(t, 0) = \rho_-$ is removed, are called the impermeable wall problem and the outflow problem, respectively.

For the case of $u_- > 0$, as in [10], the inflow problem (1.1) can then be transformed to the problem in the Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, & x > s_- t, \ t > 0, \\ u_t + p(v)_x = \mu\left(\dfrac{u_x}{v}\right)_x, & x > s_- t, \ t > 0, \\ (v, u)|_{x=s_- t} = (v_-, u_-), & u_- > 0, \\ (v, u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+), & \text{as } x \to +\infty, \end{cases} \tag{1.2}$$

where

$$p(v) = v^{-\gamma}, \qquad v = \frac{1}{\rho}, \qquad v_\pm = \frac{1}{\rho_\pm}, \qquad s_- = -\frac{u_-}{v_-} < 0. \tag{1.3}$$

The characteristic speeds of the corresponding hyperbolic system of (1.2) are

$$\lambda_1 = -\sqrt{-p'(v)}, \qquad \lambda_2 = \sqrt{-p'(v)}, \tag{1.4}$$

and the sound speed $c(v)$ is defined by

$$c(v) = v\sqrt{-p'(v)} = \sqrt{\gamma}\, v^{-\frac{\gamma-1}{2}}. \tag{1.5}$$

Comparing $|u|$ with $c(v)$, we divide the phase space $\mathbb{R}_+ \times \mathbb{R}_+$ into three regions:

$$\Omega_{sub} := \{(v,u);\ |u| < c(v),\ v > 0,\ u > 0\},$$
$$\Gamma_{trans} := \{(v,u);\ |u| = c(v),\ v > 0,\ u > 0\},$$
$$\Omega_{super} := \{(v,u);\ |u| > c(v),\ v > 0,\ u > 0\},$$

which are called the subsonic, transonic and supersonic regions, respectively.

In the phase plane, we denote the curves through a point (v_1, u_1)

$$BL(v_1,u_1) := \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R}_+;\ \frac{u}{v} = \frac{u_1}{v_1}\right\}$$

and

$$R_i(v_1,u_1) := \left\{(v,u) \in \mathbb{R}_+ \times \mathbb{R}_+;\ u = u_1 - \int_{v_1}^{v} \lambda_i(s)\,\mathrm{d}s,\ u > u_1\right\} \quad (i = 1,2)$$

as the boundary layer line and the i-rarefaction wave curve, respectively.

For the precise description of the large time behaviors of solutions to the initial–boundary value problem in the half line for the one-dimensional isentropic model system $(1.1)_1$–$(1.1)_2$ of compressible viscous gas, a complete classification in terms of $(v_\pm, u_\pm)$ for the impermeable wall problem, the inflow problem, and the outflow problem is given by Matsumura in [6]. For the rigorous mathematical justification of this classification, some results have been obtained which can be summarized as in the following:

- For the impermeable wall problem, to describe its large time behaviors, it is unnecessary to introduce the boundary layer solution and the nonlinear stability of the viscous shock wave and the rarefaction wave are well-understood, cf. [7,9]. It is worth pointing out that although the nonlinear stability result for the viscous shock wave in [7] is obtained only for small initial perturbation, the corresponding result in [9] for the rarefaction wave holds for any large initial perturbation;
- For the outflow problem, Kawashima, Nishibata, and Zhu [4] and Kawashima and Zhu [5] showed that the boundary layer solution together with the superposition of the boundary layer solution and the rarefaction wave are asymptotically nonlinear stable under small initial

perturbation, while Nakamura, Nishibata and Yuge [11] investigated the convergence rate toward the boundary layer solution. Recently, Huang and Qin [2] show that not only the boundary layer solution but also the superposition of the boundary layer solution and the rarefaction wave are still stable under large initial perturbation and improve the works of [4] and [5];
- For the inflow problem (1.2), Matsumura and Nishihara [10] established the asymptotic stability of the boundary layer solution and the superposition of the boundary layer solution and the rarefaction wave when $(v_-, u_-) \in \Omega_{sub}$ together with the assumption that the initial perturbation is small. Shi [15] studied the rarefaction wave case when $(v_-, u_-) \in \Omega_{super}$ under small initial perturbation. Huang, Matsumura and Shi [1] demonstrated the stability of the viscous shock wave and the boundary layer solution to the inflow problem (1.2), also under small initial perturbation. As to the inflow problem for the full Navier–Stokes equations, the interested readers are referred to Qin and Wang [14].

It is worth pointing out that for the impermeable wall problem and the outflow problem, the corresponding stability results on the boundary layer solution, the rarefaction wave, and/or their superposition hold true even for certain class of large initial perturbation. Thus a problem of interest is how about the case for the inflow problem, that is, do similar stability results on the boundary layer solution and the rarefaction wave hold for the inflow problem? The main purpose of this paper is devoted to this problem. More precisely, what we are interested in this paper is to consider the following two cases concerning the boundary layer solution and the rarefaction wave for the inflow problem (1.1):

Case I: $(v_-, u_-) \in \Omega_{sub}$ and $(v_+, u_+) \in BL_+(v_-, u_-) \cup BL_-(v_-, u_-)$. Then the time-asymptotic state of the solutions to the inflow problem (1.1) is described by the boundary layer solution $(V, U)(x - s_- t)$ which connects (v_-, u_-) with (v_+, u_+), where

$$BL_+(v_-, u_-) := \{(v, u) \in BL(v_-, u_-);\ v_- < v \le v_*\}$$

and

$$BL_-(v_-, u_-) := \{(v, u) \in BL(v_-, u_-);\ 0 < v < v_-\}.$$

Here, (v_*, u_*) is the intersection point of $BL(v_-, u_-)$ and Γ_{trans}, i.e.,

$$\sqrt{-p'(v_*)} = \frac{u_-}{v_-}, \qquad u_* = \frac{u_-}{v_-} v_*. \tag{1.6}$$

The boundary layer solution $(V, U)(x - s_- t)$ will be explained in the next section.

Case II: $(v_-, u_-) \in \Omega_{super}$ and $(v_+, u_+) \in R_1(v_-, u_-)$ (or $(v_+, u_+) \in R_2(v_-, u_-)$). Then the time-asymptotic state of the solutions to the inflow problem (1.1) is given by the 1-rarefaction wave $(v_1^r, u_1^r)(x/t)$ (or the 2-rarefaction wave $(v_2^r, u_2^r)(x/t)$) connecting (v_-, u_-) with (v_+, u_+).

What we want to show is on the nonlinear stability of both the boundary layer solution and the rarefaction wave for a class of large initial perturbation which can allow the initial density to have large oscillation, which improve the works of Matsumura and Nishihara [10] and Shi [15]. The precise statements of our main results will be given in Theorems 1–3 below.

The present paper is organized as follows. After stating the notations, in Section 2, we introduce some properties of the boundary layer solution and the smooth rarefaction wave, and then state the main results. In Section 3, we establish a priori estimates and then prove the stability of boundary layer by making use of Kanel's technique. In Section 4, the stability of rarefaction wave under large initial perturbation will be treated by the similar method.

1.1. Notations

Throughout this paper, c and C denote some positive constants (generally large), ϵ, λ stand for some positive constants (generally small), and $C(\cdot,\cdot)$ denotes some generic positive constant depending only on the quantities listed in the parenthesis. Notice that all the constants c, C, $C(\cdot,\cdot)$, ϵ, and λ may take different values in different places. $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \le CB$ and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. For function spaces, $L^p(\mathbb{R}_+)$ $(1 \le p \le \infty)$ denotes the usual Lebesgue space on $\mathbb{R}_+$ with norm $\|\cdot\|_{L^p}$ and $H^k(\mathbb{R}_+)$ the usual Sobolev space in the L^2 sense with norm $\|\cdot\|_p$. We note $\|\cdot\| = \|\cdot\|_{L^2}$ for simplicity. Finally, we denote by $C^k(I; H^p)$ the space of k-times continuously differentiable functions on the interval I with values in $H^p(\mathbb{R}_+)$ and $L^2(I; H^p)$ the space of L^2-functions on I with values in $H^p(\mathbb{R}_+)$.

2. Preliminaries and main results

2.1. Boundary layer solution

First we recall some properties about the boundary layer solution. In [10], it is shown that if $(v_-, u_-) \in \Omega_{sub}$ and $(v_+, u_+) \in BL_+(v_-, u_-) \cup BL_-(v_-, u_-)$, the solution to (1.2) tends to a boundary layer $(V, U)(\xi) \equiv (V, U)(x - s_- t)$ which is defined by

$$\begin{cases} -s_- V_\xi - U_\xi = 0, \quad \xi > 0, \\ -s_- U_\xi + p(V)_\xi = \mu\left(\dfrac{U_\xi}{V}\right)_\xi, \\ (V, U)(0) = (v_-, u_-), \quad (V, U)(\infty) = (v_+, u_+). \end{cases} \tag{2.1}$$

The strength of the boundary layer solution $(V, U)(\xi)$ is measured by

$$\delta := |u_+ - u_-|. \tag{2.2}$$

The existence and the properties of the boundary layer solution $(V, U)(\xi)$ are given in the following lemma.

Lemma 2.1. *(Cf. [10].) If $(v_-, u_-) \in \Omega_{sub}$ and $(v_+, u_+) \in BL_+(v_-, u_-) \cup BL_-(v_-, u_-)$, there exists a unique solution $(V, U)(\xi)$ to (2.1) satisfying for $k = 0, 1, 2$,*

$$\begin{aligned} &\left|\partial_\xi^k (V - v_+, U - u_+)(\xi)\right| \le C\delta e^{-c\xi} \quad \text{if } v_+ < v_*, \\ &\left|\partial_\xi^k (V - v_+, U - u_+)(\xi)\right| \le C\delta^{1+k}(1 + \delta\xi)^{-1-k} \quad \text{if } v_+ = v_*, \end{aligned} \tag{2.3}$$

and

$$\left|(V_\xi, V_{\xi\xi}, U_{\xi\xi})\right| \le C|U_\xi|, \tag{2.4}$$

the constants c and C depending only on (v_-, u_-). *Furthermore, the boundary layer* $(V, U)(\xi)$ *is monotonic, that is,* $V_\xi \gtrless 0$ *and* $U_\xi \gtrless 0$ *if* $u_+ \gtrless u_-$.

The first aim of this paper is to show the boundary layer solution obtained in Lemma 2.1 is still stable under some large initial perturbation. Defining the perturbation $(\phi, \psi)(t, \xi)$ by

$$(\phi, \psi)(t, \xi) = (v, u)(t, \xi) - (V, U)(\xi), \tag{2.5}$$

we get from (1.2) and (2.1) that (ϕ, ψ) satisfies

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0, \quad \xi > 0,\ t > 0, \\ \psi_t - s_-\psi_\xi + \big(p(V+\phi) - p(V)\big)_\xi = \mu\left(\dfrac{U_\xi + \psi_\xi}{V+\phi} - \dfrac{U_\xi}{V}\right)_\xi, \\ (\phi, \psi)|_{\xi=0} = (0, 0), \\ (\phi, \psi)|_{t=0} = (\phi_0, \psi_0) := (v_0 - V, u_0 - U). \end{cases} \tag{2.6}$$

The solution space is

$$X_{m,M}(0, T) = \Big\{(\phi, \psi) \in C\big([0, T]; H_0^1\big);\ \phi_\xi \in L^2\big(0, T; L^2\big),\ \psi_\xi \in L^2\big(0, T; H^1\big), \\ \sup_{[0,T]\times\mathbb{R}_+} (V+\phi)(t, \xi) \le M,\ \inf_{[0,T]\times\mathbb{R}_+} (V+\phi)(t, \xi) \ge m\Big\}.$$

Then the time-local existence of the solution $(\phi, \psi)(t, \xi)$ to (2.6) is quoted in the next lemma.

Lemma 2.2. *(See [10].) Let* (ϕ_0, ψ_0) *be in* $H_0^1(\mathbb{R}_+)$. *If* $\sup_{\mathbb{R}_+}(V + \phi_0) \le M$ *and* $\inf_{\mathbb{R}_+}(V + \phi_0) \ge m$, *then there exists* $t_0 > 0$ *depending only on* m, M *and* $\|(\phi_0, \psi_0)\|_1$ *such that* (2.6) *has a unique solution* $(\phi, \psi) \in X_{m/2,2M}(0, t_0)$ *satisfying*

$$\big\|(\phi, \psi)(t)\big\|_1 \le 2\big\|(\phi_0, \psi_0)\big\|_1 \tag{2.7}$$

and

$$s_-\phi_\xi(t, 0) + \psi_\xi(t, 0) = 0 \tag{2.8}$$

for each $0 \le t \le t_0$.

Under the above preparation, we give the following stability result of the boundary layer solution $(V, U)(\xi)$ which is increasing.

Theorem 1. *Assume that* $(v_-, u_-) \in \Omega_{sub}$ *and* $(v_+, u_+) \in BL_+(v_-, u_-)$. *Let* $(\phi_0, \psi_0) \in H_0^1(\mathbb{R}_+)$ *satisfy*

$$\big\|(\phi_0, \psi_0)\big\| \le C\epsilon^\alpha, \qquad \big\|(\phi_{0\xi}, \psi_{0\xi})\big\| \le C\big(\epsilon^{-\beta} + 1\big), \quad \text{and} \quad C^{-1}\epsilon^l \le V + \phi_0 \le C\epsilon^{-l}, \tag{2.9}$$

where C is a positive constant independent of ϵ. If the indices $l \geq 0$, α and β satisfy

$$\begin{cases} \alpha > \dfrac{\gamma+2}{2}l, \quad \alpha+\beta > (\gamma+1)l, \quad \beta \geq \dfrac{\gamma-1}{2}l, \\ \alpha-\beta \leq \dfrac{\gamma+3}{2}l, \\ \beta-\alpha < \min\left\{\dfrac{2(\gamma-1)}{\gamma+1}\alpha - \dfrac{3\gamma^2+4\gamma+1}{2\gamma+2}l, \dfrac{3(\gamma-1)}{\gamma^2+1}\alpha - \dfrac{\gamma^2+6\gamma+1}{2(\gamma^2+1)}\gamma l\right\}, \\ \beta-\alpha < \min\left\{\alpha-(\gamma+2)l, \dfrac{2\alpha}{\gamma} - \dfrac{\gamma^2+5\gamma+2}{2\gamma}l\right\}, \end{cases} \tag{2.10}$$

then there exists a suitably small $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$, (1.2) has a unique solution (v,u) satisfying $(v-V, u-U) \in C([0,\infty); H_0^1)$, where (V,U) is defined by (2.1).

Furthermore, it holds

$$\sup_{x \geq s_- t} \left|(v,u)(t,x) - (V,U)(x-s_- t)\right| \to 0, \quad \text{as } t \to \infty. \tag{2.11}$$

Remark 2.1. Several remarks concerning Theorem 1 are listed below:

- Here in Theorem 1 the strength of the boundary layer solution is not assumed to be small and thus we can show the nonlinear stability of strong increasing boundary layer solution. Since $l = 0$ and $0 < \alpha \leq \beta < \alpha + \min\{1, \frac{2}{\gamma}, \frac{2(\gamma-1)}{\gamma+1}, \frac{3(\gamma-1)}{\gamma^2+1}\}\alpha$ imply (2.10) and in this case the oscillation of the initial density can be large.
- It is easy to construct some initial perturbation $(\phi_0(x), \psi_0(x))$ satisfying the conditions listed in Theorem 1. In fact for each function $(f(x), g(x)) \in H^1(\mathbb{R}_+)$ and each α, β satisfying the conditions listed in Theorem 1, if we set

$$\phi(x) = \epsilon^{\frac{\alpha+\beta}{2}} f\left(\epsilon^{\beta-\alpha}x\right), \qquad \psi(x) = \epsilon^{\frac{\alpha+\beta}{2}} g\left(\epsilon^{\beta-\alpha}x\right),$$

  one can verify that such a $(\phi(x), \psi(x))$ satisfies all the conditions listed in Theorem 1.

For the case when the boundary layer solution is decreasing, we have

Theorem 2. *Assume that $(v_-, u_-) \in \Omega_{sub}$ and $(v_+, u_+) \in BL_-(v_-, u_-)$. Let $(\phi_0, \psi_0) \in H_0^1(\mathbb{R}_+)$ satisfy*

$$\left\|(\phi_0, \psi_0)\right\| \leq C\delta^\alpha, \qquad \left\|(\phi_{0\xi}, \psi_{0\xi})\right\| \leq C\left(\delta^{-\beta}+1\right) \quad \text{and} \quad C^{-1}\delta^l \leq V + \phi_0 \leq C\delta^{-l}, \tag{2.12}$$

where C is a positive constant independent of δ. If the indices $l \geq 0$, α and β satisfy

$$
\begin{cases}
\alpha > \dfrac{\gamma+2}{2}l, \quad \gamma l < \alpha+\beta < \dfrac{1}{2} - \dfrac{\gamma+5}{2}l, \\
\alpha-\beta \le \dfrac{\gamma+3}{2}l, \\
\beta-\alpha < \min\left\{\alpha-(\gamma+2)l, \dfrac{(\gamma-1)(1-4\alpha)}{2\gamma^2+6\gamma-2} - \dfrac{\gamma^3+4\gamma^2+8\gamma-1}{2\gamma^2+6\gamma-2}l\right\}
\end{cases}
\tag{2.13}
$$

then there exists $\delta_0 > 0$ suitably small such that if $\delta \le \delta_0$, (1.2) has a unique solution (v,u) satisfying $(v-V, u-U) \in C([0,\infty); H_0^1)$ and (2.11), where (V,U) is defined by (2.1).

Remark 2.2. $l = 0$, $\alpha+\beta < \frac{1}{2}$ and $0 < \alpha \le \beta < \alpha + \min\{\alpha, \frac{(\gamma-1)(1-4\alpha)}{2\gamma^2+6\gamma-2}\}$ imply (2.13) and also in such a case the oscillation of the initial density can be large. We also mention that a similar result was also obtained by Qin in [13] for the case when the L^2-norm of the first order derivative of the perturbation (ϕ,ψ) is independent of ϵ or the strength δ of the boundary layer solutions.

2.2. Rarefaction wave

We only consider the case when $(v_-,u_-) \in \Omega_{super}$ and $(v_+,u_+) \in R_1(v_-,u_-)$ because the study of $(v_+,u_+) \in R_2(v_-,u_-)$ is similar to that of $(v_+,u_+) \in R_1(v_-,u_-)$. Then the solution to the inflow problem (1.2) is expected to tend to the 1-rarefaction wave connecting (v_-,u_-) and (v_+,u_+).

Since the rarefaction wave is only Lipschitz continuous, we shall construct a smooth approximation for the rarefaction wave as follows. First consider the Riemann problem for Burgers' equation:

$$
\begin{cases}
w_t + w w_x = 0, \\
w(0,x) = w_0^R(x) = \begin{cases} w_-, & x<0, \\ w_+, & x>0, \end{cases}
\end{cases}
\tag{2.14}
$$

where $w_\pm = \lambda_1(v_\pm)$. It is obvious that $w_- < w_+$. Then it is well known that (2.14) has a continuous weak solution $w^r(x/t)$ given by

$$
w^r(x/t) = \begin{cases} w_-, & x < w_- t, \\ x/t, & w_- t \le x \le w_+ t, \\ w_+, & x > w_+ t. \end{cases}
\tag{2.15}
$$

Define $(v^r,u^r)(x/t)$ by

$$
\begin{cases}
v^r = \lambda_1^{-1}(w^r), \\
u^r = u_- - \displaystyle\int_{v_-}^{v^r} \lambda_1(s)\mathrm{d}s.
\end{cases}
\tag{2.16}
$$

Then by a simple calculation, $(v^r,u^r)(x/t)$ satisfies the following Riemann problem of Euler equations, i.e.,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0, \\ (v,u)(0,x) = \begin{cases} (v_-,u_-), & x<0, \\ (v_+,u_+), & x>0. \end{cases} \end{cases} \tag{2.17}$$

To construct the smooth approximate rarefaction wave $(\tilde{V},\tilde{U})(t,x)$, we consider the following Cauchy problem for Burgers' equation:

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = w_0(x) = \begin{cases} w_-, & x<0, \\ w_- + C_q\delta_r \int_0^{\epsilon x} y^q e^{-y}\mathrm{d}y, & x \geq 0, \end{cases} \end{cases} \tag{2.18}$$

where $\delta_r = w_+ - w_-$, $q \geq 10$ is some constant, C_q is a constant such that $C_q\delta_r \int_0^\infty y^q e^{-y}\mathrm{d}y = 1$, $\epsilon \leq 1$ is a positive constant to be determined later. Then we have

Lemma 2.3. *(See [2].) The problem (2.18) has a unique smooth solution $w(t,x)$ satisfying*

(i) $w_- \leq w(t,x) < w_+,\ w_x \geq 0$;
(ii) *For each $1 \leq p \leq \infty$, there exists a constant C, depending only on p and q such that for $t \geq 0$,*

$$\|w_x(t)\|_{L^p} \leq C\min\{\delta_r\epsilon^{1-1/p}, \delta_r^{1/p}t^{-1+1/p}\},$$
$$\|w_{xx}(t)\|_{L^p} \leq C\min\{\delta_r\epsilon^{2-1/p}, \delta_r^{1/q}t^{-1+1/q}\};$$

(iii) *when $x \leq w_-t$, $\partial_x^k(w(t,x)-w_-) = 0$ for $k=0,1,2$;*
(iv) $\sup_{x\in\mathbb{R}}|w(1+t,x) - w^r(x/t)| \to 0$, *as $t \to \infty$.*

We recall (2.16) and so define the smooth approximation $(\tilde{V},\tilde{U})(t,x)$ to $(v^r,u^r)(x/t)$ by

$$\begin{cases} \tilde{V}(t,x) = \lambda_1^{-1}\big(w(1+t,x)\big), \\ \tilde{U}(t,x) = u_- - \int_{v_-}^{\tilde{V}(t,x)} \lambda_1(s)\mathrm{d}s, \end{cases} \tag{2.19}$$

which satisfies

$$\begin{cases} \tilde{V}_t - \tilde{U}_x = 0, & x\in\mathbb{R},\ t>0, \\ \tilde{U}_t + p(\tilde{V})_x = 0. \end{cases} \tag{2.20}$$

Then, define

$$(V,U)(t,\xi) = (\tilde{V},\tilde{U})(t,x)|_{x\geq s_-t}, \quad \xi = x - s_-t, \tag{2.21}$$

and the following lemma holds.

Lemma 2.4. *$(V,U)(t,\xi)$ satisfies*

(i) $U_\xi \ge 0$;
(ii) *For each $1 \le p \le \infty$, there exists a constant C, depending only on p, q and $v_\pm$, such that for $t \ge 0$,*

$$\left\|(V_\xi, U_\xi)(t)\right\|_{L^p} \le C\min\left\{\epsilon^{1-1/p}, (1+t)^{-1+1/p}\right\}, \tag{2.22}$$

$$\left\|\left(V_{\xi\xi}, U_{\xi\xi}, (U_\xi/V)_\xi\right)(t)\right\|_{L^p} \le C\min\left\{\epsilon^{2-1/p}, (1+t)^{-1+1/q}\right\}; \tag{2.23}$$

(iii) $\sup_{\xi\in\mathbb{R}_+} |(V,U)(t,\xi) - (v^r,u^r)(\frac{\xi+s_-t}{t})| \to 0$, *as $t \to \infty$;*

and also

$$\begin{cases} V_t - s_-V_\xi - U_\xi = 0, \quad \xi > 0,\ t > 0, \\ U_t - s_-U_\xi + p(V)_\xi = 0, \\ (V,U)(t,0) = (v_-,u_-), \quad (V,U)(t,\infty) = (v_+,u_+). \end{cases} \tag{2.24}$$

Put the perturbation $(\phi,\psi)(t,\xi)$ by

$$(\phi,\psi)(t,\xi) = (v,u)(t,\xi) - (V,U)(\xi), \tag{2.25}$$

then the reformulated problem is

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0, \quad \xi > 0,\ t > 0, \\ \psi_t - s_-\psi_\xi + \left(p(V+\phi) - p(V)\right)_\xi = \mu\left(\dfrac{U_\xi + \psi_\xi}{V+\phi}\right)_\xi, \\ (\phi,\psi)|_{\xi=0} = (0,0), \\ (\phi,\psi)|_{t=0} = (\phi_0,\psi_0)(\xi) := (v_0 - V, u_0 - U)(\xi), \end{cases} \tag{2.26}$$

from (1.2) and (2.24). Then the time-local existence of the solution $(\phi,\psi)(t,\xi)$ to (2.26) is quoted in the next lemma.

Lemma 2.5. *Let (ϕ_0,ψ_0) be in $H_0^1(\mathbb{R}_+)$. If $\sup_{\mathbb{R}_+}(V+\phi_0) \le M$ and $\inf_{\mathbb{R}_+}(V+\phi_0) \ge m$, then there exists $t_0 > 0$ depending only on m, M and $\|(\phi_0,\psi_0)\|_1$ such that (2.26) has a unique solution $(\phi,\psi) \in X_{m/2,2M}(0,t_0)$ satisfying (2.7) and (2.8) for each $0 \le t \le t_0$.*

With the above result in hand, for the nonlinear stability of supersonic rarefaction wave, we can get the following theorem.

Theorem 3. *Assume that $(v_-,u_-) \in \Omega_{super}$ and $(v_+,u_+) \in R_1(v_-,u_-)$. Let $(\phi_0,\psi_0) \in H_0^1(\mathbb{R}_+)$ satisfy*

$$\left\|(\phi_0,\psi_0)\right\|_1 \le C\left(\epsilon^{-\alpha} + 1\right), \qquad u_- \ge C\epsilon^{-l_0}, \quad \text{and} \quad C^{-1}\epsilon^l \le V + \phi_0 \le C\epsilon^{-l}, \tag{2.27}$$

where C is a positive constant independent of ϵ. If the indices $l \ge 0$, α and l_0 satisfy

$$\begin{cases} l \leq \dfrac{1}{8\gamma - 2}, \\ l_0 > 2\alpha + (\gamma+1)l, \\ 4\alpha + 2(\gamma+1)l \geq \max\{l, (\gamma-1)l\}, \\ 4\alpha + 2(\gamma+1)l < \min\left\{\dfrac{\gamma-1}{6\gamma^2 - 6\gamma + 2}, \dfrac{2(\gamma-1)}{3\gamma^2+3\gamma+9}\right\}, \end{cases} \tag{2.28}$$

there exists $\epsilon_0 > 0$ *suitably small such that if* $\epsilon \leq \epsilon_0$, *(1.2) has a unique solution* (v, u) *satisfying* $(v - V, u - U) \in C([0, \infty); H_0^1)$, *where* (V, U) *is defined by (2.21). Furthermore, it holds*

$$\sup_{x \geq s_- t} \left|(v, u)(t, x) - (V, U)(t, x - s_- t)\right| \to 0, \quad \text{as } t \to \infty. \tag{2.29}$$

Remark 2.3. $l = 0$, $0 \leq \alpha < \frac{1}{4}\min\{\frac{\gamma-1}{6\gamma^2-6\gamma+2}, \frac{2(\gamma-1)}{3\gamma^2+3\gamma+9}\}$ and $l_0 > 2\alpha$ imply (2.28).

2.3. Main difficulties and ideas

To deduce the desired nonlinear stability result for the boundary layer solution, the rarefaction wave, and/or their superposition by the elementary energy method as in [9], it is sufficient to deduce certain uniform (with respect to the time variable t) energy type estimates on the solutions $(\phi(t, \xi), \psi(t, \xi))$ and the main difficulties to do so lie in the following:

- How to control the possible growth of $(\phi(t, \xi), \psi(t, \xi))$ caused by the nonlinearity of Eq. $(1.1)_1$–$(1.1)_2$?
- How to control the term

$$\int_0^t \frac{1}{v_-^2}\phi_\xi^2(\tau, 0)\mathrm{d}\tau = \int_0^t \frac{1}{u_-^2}\psi_\xi^2(\tau, 0)\mathrm{d}\tau$$

  which is due to the inflow boundary condition $(1.1)_3$?

The argument employed in [9] is to use the smallness of $N(T) := \sup_{0 \leq t \leq T} \|(\phi, \psi)(t)\|_1$ to overcome the above difficulties. One of the key points in such an argument is that, based on the a priori assumption that $N(T)$ is sufficiently small, one can deduce a uniform lower and upper positive bounds on the specific volume $v(t, \xi)$. With such a bound on $v(t, \xi)$ in hand, one can thus deduce certain a priori $H^1(\mathbb{R}_+)$ energy type estimates on $(\phi(t, \xi), \psi(t, \xi))$ in terms of the initial perturbation $(\phi_0(\xi), \psi_0(\xi))$. Then combination of the above analysis with the standard continuation argument yields the corresponding nonlinear stability result. It is worth pointing out that for the case when the strength of the underlying profile is small, for the nonlinear stability result obtained in [9], $\mathrm{Osc}\, v(t) := \sup_{\xi \in \mathbb{R}_+} v(t, \xi) - \inf_{\xi \in \mathbb{R}_+} v(t, \xi)$, the oscillation of the specific volume $v(t, \xi)$, should be sufficiently small also for all $t \in \mathbb{R}_+$.

What we are interested in this paper is to deduce the corresponding nonlinear stability results for the two cases listed in the introduction for a class of initial perturbation which can allow the initial density to have large oscillation, the argument used in [9] cannot be used any longer. Our main ideas to yield the desired nonlinear stability results are the following:

- For the nonlinear stability of the boundary layer solution listed in Case I of the introduction, our main observation is that for the case when the underlying boundary layer solution is increasing, the basic energy estimate, cf. the estimate (3.1), tells us that for each $t \in [0, T]$ the instant energy $\mathcal{E}(t) = \int_0^{+\infty} (\Phi(v(t,\xi), V(\xi)) + \frac{1}{2}\psi^2(t,\xi))\mathrm{d}\xi$ is bounded by the initial energy $\mathcal{E}(0)$ and thus one may use the smallness of $\mathcal{E}(0)$ to overcome the two difficulties mentioned above. Since our main purpose is to get a nonlinear stability result for which the oscillation of the specific volume $v(t,\xi)$ can be large, we need to deduce a precise estimates on $v(t,\xi)$ in terms of $\mathcal{E}(0)$ so that the whole analysis can be carried out. It is worth to emphasize that Kanel's argument [3] plays an important role in this step and it was to guarantee that the whole analysis to be carried out smoothly that we need to ask the parameters α, β, and l to satisfy the conditions listed in Theorem 1.
- For the case when the boundary layer solution is decreasing, the analysis can be adopted directly since in such a case the basic energy estimate is not self-contained. Even so, if the strength of the boundary layer solution is small, one can use the smallness of both the initial energy and the strength of the boundary layer solution to yield a nonlinear stability result similar to that of the case when the boundary layer solution is increasing.
- For the nonlinear stability of supersonic rarefaction wave corresponding to the Case II listed in the introduction, our main idea is to use the largeness of u_- to deal with the two difficulties mentioned above. In such a case, we do not ask the initial energy to be small and thus such a result holds for a class of large initial perturbation.

3. Stability of the boundary layer solution

3.1. *Proof of Theorem 1*

In this subsection, we first assume that $(v_-, u_-) \in \Omega_{sub}$, $(v_+, u_+) \in BL_+(v_-, u_-)$ and the problem (2.6) has a solution $(\phi, \psi) \in X_{m,M}(0, T)$ satisfying (2.8) for some $T > 0$ and each $0 \le t \le T$. We also simply write c and C as positive constants independent of T, m, M and ϵ. Recall that the notation $A \lesssim B$ is used to denote that $A \le CB$ holds uniformly for some positive constant independent of T, m, M and ϵ. Besides, we will often use the notation $(v, u) = (V + \phi, U + \psi)$ so that $m \le v \le M$, though the unknown functions are ϕ and ψ. Without loss of generality, we choose m and M such that $1/m, M \ge 1$.

Now we devote ourselves to the basic energy estimate.

Lemma 3.1. *It holds that for each* $0 \le t \le T$,

$$\left\|(\sqrt{\Phi}, \psi)(t)\right\|^2 + \int_0^t \int_0^\infty \left[\frac{\psi_\xi^2}{v} + \frac{|U_\xi \phi \psi_\xi|}{v} + |U_\xi|\big(p(v) - p(V) - p'(V)\phi\big)\right] \mathrm{d}\xi\, \mathrm{d}\tau \lesssim \left\|(\sqrt{\Phi_0}, \psi_0)\right\|^2, \tag{3.1}$$

where

$$\Phi = \Phi(v, V) = p(V)(v - V) - \int_V^v p(\eta)\mathrm{d}\eta. \tag{3.2}$$

Proof. Multiplying (2.6)$_1$ (the first equation of (2.6)) and (2.6)$_2$ by $p(V) - p(v)$ and ψ, respectively, and summing these two identities, we find a divergence from

$$\left(\Phi + \frac{1}{2}\psi^2\right)_t + \mu\frac{\psi_\xi^2}{v} - \mu\frac{U_\xi \phi \psi_\xi}{vV} + U_\xi\big(p(v) - p(V) - p'(V)\phi\big)$$
$$= \left[s_-\left(\Phi + \frac{1}{2}\psi^2\right) + \big(p(V) - p(v)\big)\psi + \mu\left(\frac{U_\xi + \psi_\xi}{V + \phi} - \frac{U_\xi}{V}\right)\psi\right]_\xi. \tag{3.3}$$

We write

$$p(v) - p(V) - p'(V)\phi = f(v, V)\phi^2. \tag{3.4}$$

Put $X = V/v > 0$ and then recall Bernoulli's inequality

$$X^{\gamma+1} = X(1 + X - 1)^\gamma \geq X + \gamma X(X - 1)$$

to find

$$Vvf(v, V) = V^{-\gamma}\frac{X^{\gamma+1} - (\gamma + 1)X + \gamma}{(X - 1)^2} \geq \gamma V^{-\gamma}. \tag{3.5}$$

Noting that $v_- \leq V \leq v_+$, we have from (2.1) that

$$0 \leq \mu U_\xi = V\big(s_-^2(V - v_+) + p(V) - p(v_+)\big) \leq Vp(V) = V^{-\gamma+1}.$$

Thus, the discriminant D of

$$\mu\frac{\psi_\xi^2}{v} - \mu\frac{U_\xi \phi \psi_\xi}{vV} + U_\xi\big(p(v) - p(V) - p'(V)\phi\big)$$

satisfies

$$D = \frac{\mu U_\xi}{V^2 vf(v, V)} - 4 \leq \frac{1}{\gamma} - 4 < 0. \tag{3.6}$$

Therefore, we integrate (3.3) over $(0, t) \times (0, \infty)$ to get (3.1). □

Next, following [8], we set $\tilde{v} := v/V$. Then we have $\Phi(v, V) = V^{-\gamma+1}\tilde{\Phi}(\tilde{v})$ with

$$\tilde{\Phi}(\tilde{v}) = \tilde{v} - 1 + \frac{1}{\gamma - 1}\big(\tilde{v}^{-\gamma+1} - 1\big). \tag{3.7}$$

Eq. (2.6)$_2$ is also written as

$$\left(\mu\frac{\tilde{v}_\xi}{\tilde{v}} - \psi\right)_t - s_-\left(\mu\frac{\tilde{v}_\xi}{\tilde{v}} - \psi\right)_\xi + \frac{\gamma \tilde{v}_\xi}{V^\gamma \tilde{v}^{\gamma+1}} = \frac{\gamma V_\xi}{V^{\gamma+1}}\big(1 - \tilde{v}^{-\gamma}\big). \tag{3.8}$$

Multiplying (3.8) by $\tilde{v}_\xi/\tilde{v}$, we discover

$$\begin{aligned}&\left[\frac{\mu}{2}\left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)^2-\psi\frac{\tilde{v}_\xi}{\tilde{v}}\right]_t+\frac{\gamma\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\\&\quad+\left[\psi\frac{\tilde{v}_t}{\tilde{v}}-\frac{\gamma V_\xi}{V^{\gamma+1}}\left(\frac{\tilde{v}^{-\gamma}-1}{\gamma}+\ln\tilde{v}\right)-\frac{\mu s_-}{2}\left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)^2\right]_\xi\\&=\frac{\psi_\xi^2}{v}-\frac{U_\xi\phi\psi_\xi}{vV}\underbrace{-\gamma\frac{VV_{\xi\xi}-(\gamma+1)V_\xi^2}{V^{\gamma+2}}\left(\frac{\tilde{v}^{-\gamma}-1}{\gamma}+\ln\tilde{v}\right)}_{I_1}.\end{aligned}\tag{3.9}$$

We utilize (2.4) and the fact that

$$p(v)-p(V)-p'(V)\phi=V^{-\gamma}\big(\tilde{v}^{-\gamma}-1+\gamma(\tilde{v}-1)\big)\tag{3.10}$$

to find

$$|I_1|\lesssim|U_\xi|\big(\tilde{v}^{-\gamma}-1+\gamma(\tilde{v}-1)\big)\lesssim|U_\xi|\big(p(v)-p(V)-p'(V)\phi\big).$$

Thus, integrating (3.9) over $(0,t)\times(0,\infty)$ yields

$$\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2+\int_0^t\int_0^\infty\frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau\lesssim\left\|\left(\sqrt{\Phi_0},\psi_0,\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right)\right\|^2+\int_0^t\left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)^2(\tau,0)\mathrm{d}\tau.\tag{3.11}$$

We have to control the final term of (3.11), $\int_0^t(\frac{\tilde{v}_\xi}{\tilde{v}})^2(\tau,0)\mathrm{d}\tau$. We note here that Matsumura and Nishihara [10] could control it under the smallness assumption that

$$N(T):=\sup_{0\le t\le T}\big\|(\phi,\psi)(t)\big\|_1\ll 1,\tag{3.12}$$

while our goal in this paper is to investigate the stability of the boundary layer solution and the rarefaction wave without such a smallness condition (3.12).

Since

$$\frac{\tilde{v}_\xi}{\tilde{v}}=\frac{\phi_\xi}{v}-\frac{V_\xi\phi}{vV},\tag{3.13}$$

we get from $\phi(\tau,0)=0$ and (2.8) that

$$\left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)^2(\tau,0)=\frac{1}{v_-^2}\phi_\xi^2(\tau,0)=\frac{1}{u_-^2}\psi_\xi^2(\tau,0),\tag{3.14}$$

which together with (3.11) implies the following lemma.

Lemma 3.2. *It holds that*

$$\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2+\int_0^t\int_0^\infty\frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau\lesssim\left\|\left(\sqrt{\Phi_0},\psi_0,\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right)\right\|^2+\int_0^t\psi_\xi^2(\tau,0)\mathrm{d}\tau. \tag{3.15}$$

We next estimate the last term of (3.15). Apply Hölder's inequality to find

$$\psi_\xi^2(\tau,0)=-2\int_0^\infty\psi_\xi\psi_{\xi\xi}(\tau,\xi)\mathrm{d}\xi\le 2M\left\|\frac{\psi_\xi}{\sqrt{v}}(\tau)\right\|\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\| \tag{3.16}$$

and

$$\int_0^t\psi_\xi^2(\tau,0)\mathrm{d}\tau\le 2M\left[\int_0^t\left\|\frac{\psi_\xi}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau\right]^{\frac{1}{2}}\left[\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau\right]^{\frac{1}{2}}. \tag{3.17}$$

Then applying Cauchy's inequality, we have from (3.17) that for each $a>1$,

$$\int_0^t\psi_\xi^2(\tau,0)\mathrm{d}\tau\le a\int_0^t\left\|\frac{\psi_\xi}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau+4M^2a^{-1}\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau. \tag{3.18}$$

Substituting (3.18) into (3.15) and using the basic energy estimate (3.1), we deduce

$$\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2+\int_0^t\int_0^\infty\frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau\lesssim\left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2+a\left\|(\sqrt{\Phi_0},\psi_0)\right\|^2+M^2a^{-1}\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau. \tag{3.19}$$

It is necessary to estimate the last term of (3.19). For this, we multiply $(2.6)_2$ by $-\psi_{\xi\xi}$ to find

$$\begin{aligned}&\left(\frac{1}{2}\psi_\xi^2\right)_t+\left(\frac{s_-}{2}\psi_\xi^2-\psi_t\psi_\xi\right)_\xi+\mu\frac{\psi_{\xi\xi}^2}{v}\\&=\underbrace{\mu\psi_{\xi\xi}\frac{V_\xi\psi_\xi}{v^2}+\mu\psi_{\xi\xi}\frac{\phi_\xi\psi_\xi}{v^2}}_{R_1}+\underbrace{\psi_{\xi\xi}\phi_\xi\left(p'(v)+\mu\frac{U_\xi}{v^2}\right)}_{R_2}\\&\quad+\underbrace{\psi_{\xi\xi}\left[\big(p'(v)-p'(V)\big)V_\xi+\mu\frac{U_{\xi\xi}\phi}{vV}-\mu\frac{U_\xi V_\xi\phi}{v^2V}-\mu\frac{U_\xi V_\xi\phi}{vV^2}\right]}_{R_3}.\end{aligned} \tag{3.20}$$

Apply Cauchy's inequality and (3.13) to find

$$R_1 \le \frac{\mu}{16}\frac{\psi_{\xi\xi}^2}{v} + CV_\xi^2\frac{\psi_\xi^2}{v^3} + 8\mu\frac{\phi_\xi^2\psi_\xi^2}{v^3} \tag{3.21}$$

and

$$R_2 \le \frac{\mu}{32}\frac{\psi_{\xi\xi}^2}{v} + Cv\big(v^{-2\gamma-2} + v^{-4}U_\xi^2\big)\big(\tilde{v}_\xi^2 + V_\xi^2\phi^2\big). \tag{3.22}$$

Noting that

$$\big|p'(v) - p'(V)\big| = \int_0^1 p''\big(\theta v + (1-\theta)V\big)\mathrm{d}\theta|\phi| \lesssim \big(v^{-\gamma-2} + 1\big)|\phi|, \tag{3.23}$$

we have

$$R_3 \le \frac{\mu}{32}\frac{\psi_{\xi\xi}^2}{v} + Cv\big[V_\xi^2\big(v^{-2\gamma-4} + 1\big) + v^{-2}U_{\xi\xi}^2 + \big(v^{-4} + v^{-2}\big)U_\xi^2V_\xi^2\big]\phi^2. \tag{3.24}$$

We now estimate the last term of (3.21). Applying Sobolev's inequality and Cauchy's inequality, we get

$$\begin{aligned}
\int_0^\infty \frac{\phi_\xi^2\psi_\xi^2}{v^3}\mathrm{d}\xi &\le \left\|\frac{\phi_\xi}{v}\right\|^2\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|_{L^\infty}^2 \\
&\le \left\|\frac{\phi_\xi}{v}\right\|^2\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|\left[\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}\right\| + \left\|\frac{\phi_\xi\psi_\xi}{2v^{3/2}}\right\| + \left\|\frac{V_\xi\psi_\xi}{2v^{3/2}}\right\|\right] \\
&\le \frac{1}{128}\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}\right\|^2 + \frac{1}{2}\left\|\frac{\phi_\xi\psi_\xi}{v^{3/2}}\right\|^2 + \frac{1}{2}\left\|\frac{V_\xi\psi_\xi}{v^{3/2}}\right\|^2 + C\left\|\frac{\phi_\xi}{v}\right\|^4\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|^2,
\end{aligned} \tag{3.25}$$

which implies

$$\int_0^\infty \frac{\phi_\xi^2\psi_\xi^2}{v^3}\mathrm{d}\xi \le \frac{1}{64}\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}\right\|^2 + \left\|\frac{V_\xi\psi_\xi}{v^{3/2}}\right\|^2 + C\left\|\frac{\phi_\xi}{v}\right\|^4\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|^2. \tag{3.26}$$

Integrating (3.21) over $(0,\infty)$ and then plugging (3.26) into this last inequality, we find

$$\int_0^\infty R_1\mathrm{d}\xi \le \frac{3\mu}{16}\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}\right\|^2 + C\left\|\frac{V_\xi\psi_\xi}{v^{3/2}}\right\|^2 + C\left\|\frac{\phi_\xi}{v}\right\|^4\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|^2. \tag{3.27}$$

Therefore, we integrate (3.20) over $(0,t)\times(0,\infty)$ and then recall the above estimates (3.22), (3.24), (3.27) and $m \le v \le M$ to conclude

$$\begin{aligned}
&\left\|\psi_\xi(t)\right\|^2+\int_0^t|s_-|\psi_\xi^2(\tau,0)\mathrm{d}\tau+\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau\\
&\quad\lesssim\|\psi_{0\xi}\|^2+\int_0^t\int_0^\infty\left(m^{1-\gamma}+M^{\gamma-1}U_\xi^2\right)\frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau\\
&\qquad+\int_0^t\int_0^\infty\left[J\phi^2+m^{-2}V_\xi^2\frac{\psi_\xi^2}{v}\right]\mathrm{d}\xi\,\mathrm{d}\tau+\int_0^t\left\|\frac{\phi_\xi}{v}\right\|^4\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|^2\mathrm{d}\tau
\end{aligned}\tag{3.28}$$

with

$$J=\left(v^{-2\gamma-3}+v\right)V_\xi^2+v^{-1}U_{\xi\xi}^2+\left(v^{-1}+v^{-3}\right)U_\xi^2V_\xi^2.\tag{3.29}$$

We use (2.4) and (3.5) to deduce

$$J\lesssim\left(v^{-2\gamma-3}+v\right)U_\xi^2\lesssim\left(m^{-2\gamma-2}+M^2\right)|U_\xi|f(v,V).\tag{3.30}$$

Hence, we obtain from (3.1) and (3.4) that

$$\int_0^t\int_0^\infty\left[J\phi^2+m^{-2}V_\xi^2\frac{\psi_\xi^2}{v}\right]\mathrm{d}\xi\,\mathrm{d}\tau\lesssim\left(m^{-2\gamma-2}+M^2\right)\left\|(\sqrt{\Phi_0},\psi_0)\right\|^2\tag{3.31}$$

and

$$\int_0^t\left\|\frac{\phi_\xi}{v}\right\|^4\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|^2\mathrm{d}\tau\lesssim\left\|(\sqrt{\Phi_0},\psi_0)\right\|^2\sup_{0\le\tau\le t}\left\|\frac{\phi_\xi}{v}(\tau)\right\|^4.\tag{3.32}$$

Plugging estimates (3.31) and (3.32) into (3.28), we have

$$\begin{aligned}
&\left\|\psi_\xi(t)\right\|^2+\int_0^t|s_-|\psi_\xi^2(\tau,0)\mathrm{d}\tau+\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau\\
&\quad\lesssim\|\psi_{0\xi}\|^2+\left(m^{1-\gamma}+M^{\gamma-1}\right)\int_0^t\int_0^\infty\frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau\\
&\qquad+\left(m^{-2\gamma-2}+M^2\right)\left\|(\sqrt{\Phi_0},\psi_0)\right\|^2+\left\|(\sqrt{\Phi_0},\psi_0)\right\|^2\sup_{0\le\tau\le t}\left\|\frac{\phi_\xi}{v}(\tau)\right\|^4.
\end{aligned}\tag{3.33}$$

Since

$$\Phi(v,V)=-\int_0^1\int_0^1\theta_1 p'\big(\theta_1\theta_2 v+(1-\theta_1\theta_2)V\big)\mathrm{d}\theta_1\mathrm{d}\theta_2\phi^2, \tag{3.34}$$

we have

$$C^{-1}M^{-\gamma-1}\phi^2\le\Phi(v,V)\le Cm^{-\gamma-1}\phi^2. \tag{3.35}$$

Then we deduce from (3.13) that

$$\left\|\frac{\phi_\xi}{v}(\tau)\right\|^2\le C\|V_\xi(\tau)\|^2_{L^\infty}m^{-2}M^{\gamma+1}\big\|\sqrt{\Phi}(\tau)\big\|^2+\left\|\frac{\tilde v_\xi}{\tilde v}(\tau)\right\|^2. \tag{3.36}$$

Hence we have from (3.1), (3.15) and (3.17) that

$$\begin{aligned}\left\|\frac{\phi_\xi}{v}(\tau)\right\|^4&\lesssim m^{-4}M^{2\gamma+2}\big\|(\sqrt{\Phi_0},\psi_0)\big\|^4+\left\|\frac{\tilde v_{0\xi}}{\tilde v_0}\right\|^4+\left[\int_0^t\psi_\xi^2(\tau,0)\mathrm{d}\tau\right]^2\\&\lesssim m^{-4}M^{2\gamma+2}\big\|(\sqrt{\Phi_0},\psi_0)\big\|^4+\left\|\frac{\tilde v_{0\xi}}{\tilde v_0}\right\|^4+M^2\big\|(\sqrt{\Phi_0},\psi_0)\big\|^2\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau.\end{aligned} \tag{3.37}$$

Substituting (3.37) into (3.33), we find a positive constant c such that if

$$M^2\big\|(\sqrt{\Phi_0},\psi_0)\big\|^4\le c, \tag{3.38}$$

then

$$\begin{aligned}&\|\psi_\xi(t)\|^2+\int_0^t|s_-|\psi_\xi^2(\tau,0)\mathrm{d}\tau+\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau\\&\quad\lesssim h(m,M,\phi_0,\psi_0)+\big(m^{1-\gamma}+M^{\gamma-1}\big)\int_0^t\int_0^\infty\frac{\tilde v_\xi^2}{v^\gamma\tilde v^2}\mathrm{d}\xi\mathrm{d}\tau,\end{aligned} \tag{3.39}$$

with

$$\begin{aligned}&h(m,M,\phi_0,\psi_0)\\&\quad=\|\psi_{0\xi}\|^2+\left[m^{-2\gamma-2}+M^2+m^{-4}M^{2\gamma+2}\big\|(\sqrt{\Phi_0},\psi_0)\big\|^4+\left\|\frac{\tilde v_{0\xi}}{\tilde v_0}\right\|^4\right]\big\|(\sqrt{\Phi_0},\psi_0)\big\|^2.\end{aligned} \tag{3.40}$$

Plugging (3.39) into (3.19) gives the following lemma.

Lemma 3.3. *There exists $c_0 > 0$ independent of T, m, M and a, such that if*

$$g_0(a, m, M, \phi_0, \psi_0) := M^2\left\|(\sqrt{\Phi_0}, \psi_0)\right\|^4 + M^2 a^{-1}\left(m^{1-\gamma} + M^{\gamma-1}\right) \le c_0, \tag{3.41}$$

it holds that for each $a > 1$ and $0 \le t \le T$,

$$\begin{aligned}
&\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2 + \int_0^t\int_0^\infty \frac{\tilde{v}_\xi^2}{v^\gamma \tilde{v}^2} \mathrm{d}\xi \mathrm{d}\tau \\
&\quad \lesssim \left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2 + a\left\|(\sqrt{\Phi_0}, \psi_0)\right\|^2 + M^2 a^{-1} h(m, M, \phi_0, \psi_0)
\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}
&\left\|\psi_\xi(t)\right\|^2 + \int_0^t |s_-|\psi_\xi^2(\tau, 0)\mathrm{d}\tau + \int_0^t \left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2 \mathrm{d}\tau \\
&\quad \lesssim h(m, M, \phi_0, \psi_0) + \left(m^{1-\gamma} + M^{\gamma-1}\right)\left[\left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2 + a\left\|(\sqrt{\Phi_0}, \psi_0)\right\|^2\right],
\end{aligned} \tag{3.43}$$

where $h(m, M, \phi_0, \psi_0)$ is given by (3.40).

Proof of Theorem 1. Without loss of generality, we assume that $\epsilon \le 1$. First, we note from (3.35), (3.13) and the initial conditions (2.9) that

$$\left\|(\sqrt{\Phi_0}, \psi_0)\right\| \le C\epsilon^{\alpha - \frac{\gamma+1}{2}l}, \qquad \left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\| \le C\epsilon^{-l}\left(\|\phi_{0\xi}\| + \|V_\xi \phi\|\right) \le C\epsilon^{-l-\beta}.$$

Since $(\phi_0, \psi_0) \in H_0^1(\mathbb{R}_+)$, we apply Lemma 2.2 to find $t_0 > 0$ such that the problem (2.6) has a unique solution $(\phi, \psi) \in X_{m_0, M_0}(0, t_0)$ with $1/m_0, M_0 \lesssim \epsilon^{-l}$. Then we find that for $a = \epsilon^{-2\alpha - 2\beta + (\gamma-1)l} > 1$,

$$g_0(a, m_0, M_0, \phi_0, \psi_0) \lesssim \epsilon^{4\alpha - 2(\gamma+2)l} + \epsilon^{2\alpha + 2\beta - 2\gamma l}.$$

Hence if $(2.10)_1$ holds, there exists $\epsilon_1 > 0$ such that (3.41) holds for each $0 < \epsilon \le \epsilon_1$. Furthermore, we have that if $(2.10)_1$ holds,

$$\begin{aligned}
h(m_0, M_0, \phi_0, \psi_0) &\lesssim \epsilon^{-2\beta} + \left[\epsilon^{-(2\gamma+2)l} + \epsilon^{4\alpha - (4\gamma+8)l} + \epsilon^{-4l-4\beta}\right]\epsilon^{2\alpha - (\gamma+1)l} \\
&\lesssim \epsilon^{-2\alpha - 4\beta + (\gamma-1)l}.
\end{aligned}$$

Clearly we conclude from Lemma 3.3 that for each $0 < \epsilon \le \epsilon_1$,

$$\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2 + \int_0^t\int_0^\infty \frac{\tilde{v}_\xi^2}{v^\gamma \tilde{v}^2} \mathrm{d}\xi \mathrm{d}\tau \lesssim \epsilon^{-2l-2\beta}.$$

To get the upper and lower bounds for the specific volume v, we follow [9] and introduce

$$\Psi(\tilde{v}) := \int_1^{\tilde{v}} \frac{\sqrt{\tilde{\Phi}(\eta)}}{\eta} \mathrm{d}\eta, \tag{3.44}$$

where $\tilde{\Phi}$ is defined by (3.7). Then we have from $\tilde{v}(t_0, \infty) = 1$ and Lemma 3.1 that

$$\left|\Psi\big(\tilde{v}(t_0,\xi)\big)\right| = \left|\int_\infty^\xi \Psi_\xi\big(\tilde{v}(t_0,\xi)\big)\mathrm{d}\xi\right| \le \left\|\sqrt{\tilde{\Phi}}(t_0)\right\| \left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t_0)\right\| \lesssim \epsilon^\theta, \tag{3.45}$$

where $\theta = \alpha - \beta - (\gamma+3)l/2$. Since $\Psi(\tilde{v}) = O(\tilde{v}^{1/2})$ as $\tilde{v} \to \infty$ and $\Psi(\tilde{v}) = O(\tilde{v}^{(1-\gamma)/2})$ as $\tilde{v} \to 0+$, (3.45) yields

$$\epsilon^{2\theta/(1-\gamma)} \lesssim v(t_0,\xi) \lesssim \epsilon^{2\theta}, \quad \forall \xi \in \mathbb{R}_+. \tag{3.46}$$

Since $(2.10)_2$ implies $2\theta/(\gamma-1) \le 0$ and $2\theta \le 0$, we exploit Lemma 2.2 again and recall (3.46) to find $t_1 > 0$ such that (2.6) has a unique solution $(\phi,\psi) \in X_{m_1,M_1}(0, t_0+t_1)$, where $m_1 \gtrsim \epsilon^{2\theta/(1-\gamma)}$ and $M_1 \lesssim \epsilon^{2\theta}$. Note that $\theta \le 0$ implies that $\epsilon^{4\theta} \ge \epsilon^{\frac{4\gamma+4}{\gamma-1}\theta}$. Thus we have

$$h\big(m_1, M_1, \phi(t_0), \psi(t_0)\big) \lesssim \epsilon^{-2\beta} + \big[\epsilon^{\frac{4\gamma+4}{\gamma-1}\theta} + \epsilon^{\frac{8\theta}{\gamma-1}+(4\gamma+4)\theta+4\alpha-2(\gamma+1)l} + \epsilon^{-4l-4\beta}\big]\epsilon^{2\alpha-(\gamma+1)l},$$

and therefore the right-hand side of (3.42) is bounded by $C\epsilon^{-2l-2\beta}$. By elementary calculations, we conclude from $(2.10)_{1,3,4}$ that

$$h\big(m_1, M_1, \phi(t_0), \psi(t_0)\big) \lesssim \epsilon^{-4\theta-2\alpha-4\beta+(\gamma-3)l}.$$

We recall $a = \epsilon^{-2\alpha-2\beta+(\gamma-1)l} > 1$ to deduce

$$g_0\big(a, m_1, M_1, \phi(t_0), \psi(t_0)\big) \lesssim \epsilon^{4\theta+4\alpha-2(\gamma+1)l} + \epsilon^{6\theta+2\alpha+2\beta-(\gamma-1)l} + \epsilon^{(2\gamma+2)\theta+2\alpha+2\beta-(\gamma-1)l}.$$

Then we have from $(2.10)_4$ that there exists $\epsilon_0 > 0$ such that (3.41) holds for each $\epsilon \le \epsilon_0$. Combining Lemma 2.2 and the continuation process, we can prove (2.1) has the global-in-time solution $(\phi,\psi) \in X_{m_1,M_1}(0,\infty)$ satisfying

$$\left\|(\phi,\psi)(t)\right\|^2 + \int_0^t \left\|\big(|U_\xi|^{1/2}\phi, \phi_\xi, \psi, \psi_\xi\big)(\tau)\right\|^2 \mathrm{d}\tau \le C, \quad \forall t \in \mathbb{R}_+, \tag{3.47}$$

the constant C depending only on ϵ. And so the asymptotic behavior of the solution (2.11) is concluded by employing Sobolev's inequality. □

3.2. Proof of Theorem 2

In this subsection, we assume that $(v_-, u_-) \in \Omega_{sub}$, $(v_+, u_+) \in BL_-(v_-, u_-)$ and (2.6) has a solution $(\phi, \psi) \in X_{m,M}(0, T)$ satisfying (2.8) for some $T > 0$ and each $0 \le t \le T$. As in the proof of Theorem 1, c and C are used to denote some positive constants independent of T, m, M and δ and the notation $A \lesssim B$ stands for that $A \le CB$ holds uniformly for some positive constant independent of T, m, M and δ. Without loss of generality, we choose m and M such that $1/m, M \ge 1$.

The basic energy estimate is stated as follows.

Lemma 3.4. *If δ is suitably small, then it holds that for each $0 \le t \le T$,*

$$\left\|(\sqrt{\Phi}, \psi)(t)\right\|^2 + \int_0^t \int_0^\infty \left[\frac{\psi_\xi^2}{v} + \frac{|U_\xi \phi \psi_\xi|}{v} + |U_\xi|\big(p(v) - p(V) - p'(V)\phi\big)\right] \mathrm{d}\xi\, \mathrm{d}\tau$$
$$\lesssim \left\|(\sqrt{\Phi_0}, \psi_0)\right\|^2 + \delta m^{-\gamma-2} \int_0^t \left\|\phi_\xi(\tau)\right\|^2 \mathrm{d}\tau, \tag{3.48}$$

where $\Phi = \Phi(v, V)$ is defined by (3.2) and $\delta = |u_+ - u_-|$ is the strength of the boundary layer (V, U).

Proof. First we recall $(v_+, u_+) \in BL_-(v_-, u_-)$ and thus we have $U_\xi < 0$. Noting that (3.5) and (2.3) hold, we conclude that if δ is suitably small, the discriminant D of

$$\mu \frac{\psi_\xi^2}{v} - \mu \frac{U_\xi \phi \psi_\xi}{vV} + |U_\xi|\big(p(v) - p(V) - p'(V)\phi\big)$$

satisfies

$$D = \frac{\mu |U_\xi|}{V^2 v f(v, V)} - 4 \le \frac{C\delta}{\gamma V^{-\gamma+1}} - 4 < 0,$$

where $f(v, V)$ is defined by (3.4). Then we integrate (3.3) over $(0, t) \times (0, \infty)$ to find that

$$\left\|(\sqrt{\Phi}, \psi)(t)\right\|^2 + \int_0^t \int_0^\infty \left[\frac{\psi_\xi^2}{v} + \frac{|U_\xi \phi \psi_\xi|}{v} + |U_\xi|\big(p(v) - p(V) - p'(V)\phi\big)\right] \mathrm{d}\xi\, \mathrm{d}\tau$$
$$\lesssim \left\|(\sqrt{\Phi_0}, \psi_0)\right\|^2 + \int_0^t \int_0^\infty |U_\xi|\big(p(v) - p(V) - p'(V)\phi\big) \mathrm{d}\xi\, \mathrm{d}\tau. \tag{3.49}$$

To estimate the last term of (3.49), we apply the idea in Nikkuni and Kawashima [12], i.e.,

$$\phi(t, \xi) = \phi(t, 0) + \int_0^\xi \phi_\xi(t, \eta) \mathrm{d}\eta \le \xi^{\frac{1}{2}} \left\|\phi_\xi(t)\right\|. \tag{3.50}$$

Since

$$p(v)-p(V)-p'(V)\phi=\int_0^1\int_0^1 p''\big(\theta_1\theta_2 v+(1-\theta_1\theta_2)V\big)\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_1\phi^2\lesssim m^{-\gamma-2}\phi^2, \quad (3.51)$$

we deduce from $|U_\xi(\xi)|\lesssim \delta e^{-c\xi}$ and (3.50) that

$$\int_0^t\int_0^\infty |U_\xi|\big(p(v)-p(V)-p'(V)\phi\big)\mathrm{d}\xi\mathrm{d}\tau\lesssim \delta m^{-\gamma-2}\int_0^t\int_0^\infty e^{-c\xi}\phi^2\mathrm{d}\xi\mathrm{d}\tau$$
$$\lesssim \delta m^{-\gamma-2}\int_0^t \big\|\phi_\xi(\tau)\big\|^2\mathrm{d}\tau. \quad (3.52)$$

We then substitute (3.52) into (3.49) to find (3.48). □

We now estimate the last term of (3.48). For this, we apply $|V_\xi(\xi)|\lesssim \delta e^{-c\xi}$ and (3.50) to conclude

$$\int_0^t\int_0^\infty V_\xi^2\phi^2\mathrm{d}\xi\mathrm{d}\tau\lesssim \delta^2\int_0^t\big\|\phi_\xi(\tau)\big\|^2\mathrm{d}\tau. \quad (3.53)$$

Then, we obtain from (3.13) that

$$\int_0^t\big\|\phi_\xi(\tau)\big\|^2\mathrm{d}\tau\lesssim M^{\gamma+2}\int_0^t\int_0^\infty \frac{\tilde v_\xi^2}{v^\gamma\tilde v^2}\mathrm{d}\xi\mathrm{d}\tau+\delta^2\int_0^t\big\|\phi_\xi(\tau)\big\|^2\mathrm{d}\tau. \quad (3.54)$$

Consequently if δ is sufficiently small,

$$\int_0^t\big\|\phi_\xi(\tau)\big\|^2\mathrm{d}\tau\lesssim M^{\gamma+2}\int_0^t\int_0^\infty \frac{\tilde v_\xi^2}{v^\gamma\tilde v^2}\mathrm{d}\xi\mathrm{d}\tau. \quad (3.55)$$

Plugging (3.55) into (3.48) gives that if δ is sufficiently small, it holds that

$$\big\|(\sqrt{\Phi},\psi)(t)\big\|^2+\int_0^t\int_0^\infty\bigg[\frac{\psi_\xi^2}{v}+\frac{|U_\xi\phi\psi_\xi|}{v}+|U_\xi|\big(p(v)-p(V)-p'(V)\phi\big)\bigg]\mathrm{d}\xi\mathrm{d}\tau$$
$$\lesssim \big\|(\sqrt{\Phi_0},\psi_0)\big\|^2+\delta m^{-\gamma-2}M^{\gamma+2}\int_0^t\int_0^\infty\frac{\tilde v_\xi^2}{v^\gamma\tilde v^2}\mathrm{d}\xi\mathrm{d}\tau. \quad (3.56)$$

Similar to proving Lemma 3.2, we obtain from (3.56) that if δ is suitably small,

$$\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2+\int_0^t\int_0^\infty \frac{\tilde{v}_\xi^2}{v^\gamma \tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau$$
$$\lesssim \left\|\left(\sqrt{\Phi_0},\psi_0,\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right)\right\|^2+\int_0^t \psi_\xi^2(\tau,0)\mathrm{d}\tau+\delta m^{-\gamma-2}M^{\gamma+2}\int_0^t\int_0^\infty \frac{\tilde{v}_\xi^2}{v^\gamma \tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau,$$

which implies that if $\delta m^{-\gamma-2}M^{\gamma+2}$ is sufficiently small,

$$\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2+\int_0^t\int_0^\infty \frac{\tilde{v}_\xi^2}{v^\gamma \tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau \lesssim \left\|\left(\sqrt{\Phi_0},\psi_0,\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right)\right\|^2+\int_0^t \psi_\xi^2(\tau,0)\mathrm{d}\tau. \tag{3.57}$$

Now we substitute (3.18) into (3.57) to conclude the following lemma.

Lemma 3.5. *There exists a constant c_1 independent of T, m, M, δ and a, such that if*

$$g_1(\delta,a,m,M):=a\delta m^{-\gamma-2}M^{\gamma+2}\le c_1, \tag{3.58}$$

then it holds for each $a>1$ and $0\le t\le T$,

$$\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2+\int_0^t\int_0^\infty \frac{\tilde{v}_\xi^2}{v^\gamma \tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau \lesssim \left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2+a\left\|(\sqrt{\Phi_0},\psi_0)\right\|^2+M^2a^{-1}\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau \tag{3.59}$$

and

$$\|(\sqrt{\Phi},\psi)(t)\|^2+\int_0^t\int_0^\infty\left[\frac{\psi_\xi^2}{v}+\frac{|U_\xi\phi\psi_\xi|}{v}+|U_\xi|\big(p(v)-p(V)-p'(V)\phi\big)\right]\mathrm{d}\xi\,\mathrm{d}\tau$$
$$\lesssim \|(\sqrt{\Phi_0},\psi_0)\|^2+\delta m^{-\gamma-2}M^{\gamma+2}\left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2+M^2a^{-1}\delta m^{-\gamma-2}M^{\gamma+2}\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau. \tag{3.60}$$

Next, we will use (3.28) to estimate the term $\int_0^t\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\|^2\mathrm{d}\tau$. We employ (3.53) and (3.55) to obtain

$$\int_0^t\int_0^\infty J\phi^2\mathrm{d}\xi\,\mathrm{d}\tau\lesssim\int_0^t\int_0^\infty\big(v^{-2\gamma-3}+v\big)V_\xi^2\phi^2\mathrm{d}\xi\,\mathrm{d}\tau\lesssim\big(m^{-2\gamma-3}+M\big)\delta^2M^{\gamma+2}\int_0^t\int_0^\infty\frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau. \tag{3.61}$$

We have from (3.56) that

$$\int_0^t\int_0^\infty m^{-2}V_\xi^2\frac{\psi_\xi^2}{v}\mathrm{d}\xi\,\mathrm{d}\tau \lesssim \delta^2 m^{-2}\big\|(\sqrt{\Phi_0},\psi_0)\big\|^2+\delta^3 m^{-\gamma-4}M^{\gamma+2}\int_0^t\int_0^\infty \frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau. \tag{3.62}$$

Substituting (3.61) and (3.62) into (3.28), we conclude that if (3.58) holds, then

$$\begin{aligned}&\big\|\psi_\xi(t)\big\|^2+\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau\\ &\quad\lesssim \big\|(\sqrt{\Phi_0},\psi_0,\psi_{0\xi})\big\|^2+m^{1-\gamma}\int_0^t\int_0^\infty\frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau+\int_0^t\left\|\frac{\phi_\xi}{v}\right\|^4\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|^2\mathrm{d}\tau.\end{aligned} \tag{3.63}$$

Then combinations of (3.59) and (3.63) give the following lemma.

Lemma 3.6. *Assume that (3.58) holds. There exists a constant c_2 independent of T, m, M, δ and a, such that if*

$$g_2(a,m,M):=a^{-1}m^{1-\gamma}M^2\le c_2, \tag{3.64}$$

then it holds for each $a>1$ and $0\le t\le T$,

$$\begin{aligned}&\big\|\psi_\xi(t)\big\|^2+\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau\\ &\quad\lesssim \|\psi_{0\xi}\|^2+m^{1-\gamma}a\big\|(\sqrt{\Phi_0},\psi_0)\big\|^2+m^{1-\gamma}\left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2+\int_0^t\left\|\frac{\phi_\xi}{v}\right\|^4\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|^2\mathrm{d}\tau.\end{aligned} \tag{3.65}$$

We now need to estimate the last term in (3.65). First substitute (3.59) and (3.60) into (3.36), and then recall (3.64) to deduce

$$\left\|\frac{\phi_\xi}{v}(\tau)\right\|^2\lesssim\left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2+a\big\|(\sqrt{\Phi_0},\psi_0)\big\|^2+M^2a^{-1}\int_0^\tau\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(s)\right\|^2\mathrm{d}s. \tag{3.66}$$

Applying the inequality

$$(c+b)^3\le 4\big(c^3+b^3\big)\quad \text{if } c,b\ge 0, \tag{3.67}$$

and noting $\delta m^{\gamma+2}M^{\gamma+2}\lesssim a^{-1}$, we discover from (3.66) and (3.60) that

$$\int_0^t \left\| \frac{\phi_\xi}{v}(\tau) \right\|^4 \left\| \frac{\psi_\xi}{\sqrt{v}}(\tau) \right\|^2 \mathrm{d}\tau$$

$$\lesssim a^{-1}\left[\left\| \frac{\tilde{v}_{0\xi}}{\tilde{v}_0} \right\|^2 + a\left\| (\sqrt{\Phi_0}, \psi_0) \right\|^2 \right]^3 + a^{-1}\left[M^2 a^{-1} \int_0^t \left\| \frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau) \right\|^2 \mathrm{d}\tau \right]^3. \tag{3.68}$$

To close the energy estimate (3.65), we will employ the following result due to Strauss [16].

Lemma 3.7. *Let $M(t)$ be a non-negative continuous function of t satisfying the inequality*

$$M(t) \le A_1 + A_2 M(t)^\kappa \tag{3.69}$$

in some interval containing 0*, where A_1 and A_2 are positive constants and $\kappa > 1$. Then there is a positive constant C such that if $M(0) \le A_1$ and*

$$A_1 A_2^{\frac{1}{\kappa-1}} < C\left(1-\kappa^{-1}\right)\kappa^{-\frac{1}{\kappa-1}}, \tag{3.70}$$

then in the same interval

$$M(t) \le \frac{A_1}{1-\kappa^{-1}}. \tag{3.71}$$

Now we write

$$M(t) = \left\| \psi_\xi(t) \right\|^2 + \int_0^t \left\| \frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau) \right\|^2 \mathrm{d}\tau, \tag{3.72}$$

and

$$h_1 = \|\psi_{0\xi}\|^2 + m^{1-\gamma} a \left\| (\sqrt{\Phi_0}, \psi_0) \right\|^2 + m^{1-\gamma} \left\| \frac{\tilde{v}_{0\xi}}{\tilde{v}_0} \right\|^2 + a^{-1}\left[\left\| \frac{\tilde{v}_{0\xi}}{\tilde{v}_0} \right\|^2 + a\left\| (\sqrt{\Phi_0}, \psi_0) \right\|^2 \right]^3. \tag{3.73}$$

Then we substitute (3.68) into (3.65) to find

$$M(t) \lesssim h_1 + a^{-4} M^6 M(t)^3. \tag{3.74}$$

Noting that $M(0) \lesssim h_1$ holds, we can close the estimates (3.65), (3.59) and (3.60) by applying Lemma 3.7.

Lemma 3.8. *Suppose that (3.58) and (3.64) hold. There is a constant c_3 independent of T, m, M, δ and a, such that if*

$$g_3(\delta, a, m, M, \phi_0, \psi_0) := h_1 a^{-2} M^3 < c_3, \tag{3.75}$$

where A_1 is defined by (3.73), then it holds that for each $a > 1$ and $0 \le t \le T$,

$$\|\psi_\xi(t)\|^2 + \int_0^t \left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2 d\tau \lesssim h_1, \tag{3.76}$$

$$\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2 + \int_0^t \int_0^\infty \frac{\tilde{v}_\xi^2}{v^\gamma \tilde{v}^2} d\xi d\tau \lesssim \left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2 + a\|(\sqrt{\Phi_0}, \psi_0)\|^2 + M^2 a^{-1} h_1 \tag{3.77}$$

and

$$\|(\sqrt{\Phi}, \psi)(t)\|^2 + \int_0^t \int_0^\infty \left[\frac{\psi_\xi^2}{v} + \frac{|U_\xi \phi \psi_\xi|}{v} + |U_\xi|\big(p(v) - p(V) - p'(V)\phi\big)\right] d\xi d\tau$$
$$\lesssim \|(\sqrt{\Phi_0}, \psi_0)\|^2 + \delta m^{-\gamma-2} M^{\gamma+2}\left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2 + M^2 a^{-1} \delta m^{-\gamma-2} M^{\gamma+2} h_1. \tag{3.78}$$

Proof of Theorem 2. Without loss of generality, we assume that $\delta \le 1$. First, we note from (3.35), (3.13) and the initial conditions (2.12) that

$$\|(\sqrt{\Phi_0}, \psi_0)\| \le C\delta^{\alpha - \frac{\gamma+1}{2}l}, \qquad \left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\| \le C\delta^{-l}\big(\|\phi_{0\xi}\| + \|V_\xi \phi\|\big) \le C\delta^{-l-\beta}.$$

Since $(\phi_0, \psi_0) \in H_0^1(\mathbb{R}_+)$, we apply Lemma 2.2 to find $t_0 > 0$ such that the problem (2.6) has a unique solution $(\phi, \psi) \in X_{m_0, M_0}(0, t_0)$ with $1/m_0, M_0 \lesssim \delta^{-l}$. Then we find that for $a = \delta^{-2\alpha-2\beta+(\gamma-1)l} > 1$,

$$g_1(\delta, a, m_0, M_0) + g_2(a, m_0, M_0) \lesssim \delta^{1-2\alpha-2\beta-(\gamma+5)l} + \delta^{2\alpha+2\beta-2\gamma l},$$

and

$$g_3(\delta, a, m_0, M_0, \phi_0, \psi_0) \lesssim \left[\delta^{-2\beta-(\gamma+1)l} + \delta^{2\alpha-4\beta-(\gamma+5)l}\right]\delta^{4\alpha+4\beta-(2\gamma+1)l}.$$

Hence if $(2.13)_1$ holds, there exists $\delta_1 > 0$ such that (3.58), (3.64) and (3.75) hold for each $\delta \le \delta_1$. Next we compute from $(2.13)_1$ that the right-hand sides of (3.77) and (3.78) are bounded by $C_1\delta^{-2\beta-2l}$ and $C_2\delta^{2\alpha-(\gamma+1)l}$, respectively. We conclude from Lemma 3.8 that

$$\left|\Psi\big(\tilde{v}(t_0, \xi)\big)\right| \le \|\sqrt{\tilde{\Phi}}(t_0)\|\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t_0)\right\| \lesssim \delta^\theta, \tag{3.79}$$

where Ψ is defined by (3.44) and $\theta = \alpha - \beta - (\gamma+3)l/2$. Hence

$$\delta^{2\theta/(1-\gamma)} \lesssim v(t_0, \xi) \lesssim \delta^{2\theta}, \quad \forall \xi \in \mathbb{R}_+. \tag{3.80}$$

Since $(2.13)_2$ implies $2\theta/(\gamma-1) \le 0$ and $2\theta \le 0$, we apply Lemma 2.2 again and recall (3.46) to find $t_1 > 0$ such that (2.6) has a unique solution $(\phi, \psi) \in X_{m_1, M_1}(0, t_0 + t_1)$, where

$m_1 \gtrsim \delta^{2\theta/(1-\gamma)}$ and $M_1 \lesssim \delta^{2\theta}$. By elementary calculations, we conclude from (2.13)$_3$ that there exists $0 < \delta_2 \le \delta_1$ such that if $\delta \le \delta_2$, then the right-hand sides of (3.77) and (3.78) are bounded by $C_1\delta^{-2\beta-2l}$ and $C_2\delta^{2\alpha-(\gamma+1)l}$, respectively. Since

$$g_1(\delta, a, m_1, M_!) + g_2(a, m_1, M_1) \lesssim \delta^{1-2\alpha-2\beta+(\gamma-1)l+\frac{2\gamma(\gamma+2)}{\gamma-1}\theta} + \delta^{2\alpha+2\beta-(\gamma-1)l+6\theta}$$

and

$$g_3\big(\delta, a, m_1, M_1, \phi(t_0), \psi(t_0)\big) \lesssim \big[\delta^{2\theta-2\beta-2l} + \delta^{2\alpha-4\beta-(\gamma+5)l}\big]\delta^{4\alpha+4\beta-(2\gamma-2)l+6\theta},$$

(2.13)$_3$ implies that there exists $0 < \delta_0 \le \delta_2$ such that (3.64) and (3.75) hold for each $\delta \le \delta_0$. Then, combining Lemma 2.2 and the continuation process, we can prove (2.1) has the global solution in time $(\phi, \psi) \in X_{m_1,M_1}(0, \infty)$ satisfying (3.47) with the constant C depending only on δ. Thus, the asymptotic behavior of the solution (2.11) is concluded by employing Sobolev's inequality. □

4. Stability of rarefaction wave

In this section, we investigate the case when $(v_-, u_-) \in \Omega_{super}$ and $(v_+, u_+) \in R_1(v_-, u_-)$, and we often use the notations same as (3.2), (3.4) and so on, though U and V are the rarefaction waves different from the boundary layer solutions. We assume that (2.26) has a solution $(\phi, \psi) \in X_{m,M}(0, T)$ satisfying (2.8) for some $T > 0$ and each $0 \le t \le T$. As before, we also simply write c and C as positive constants independent of T, m, M and ϵ and the notation $A \lesssim B$ will mean that $A \le CB$ holds uniformly for some positive constant independent of T, m, M and ϵ. Without loss of generality, we choose m and M such that $1/m, M \ge 1$.

Our first result is concerned with the basic energy estimate which is stated as follows.

Lemma 4.1. *If ϵ is suitably small, then it holds for each $0 \le t \le T$,*

$$\begin{aligned}&\big\|(\sqrt{\Phi}, \psi)(t)\big\|^2 + \int_0^t\int_0^\infty\bigg[\frac{\psi_\xi^2}{v} + \frac{|U_\xi\phi\psi_\xi|}{v} + |U_\xi|\big(p(v) - p(V) - p'(V)\phi\big)\bigg]d\xi\, d\tau \\ &\quad\lesssim \big\|(\sqrt{\Phi_0}, \psi_0)\big\|^2 + \epsilon^{\frac{2}{9}}M^{\frac{1}{2}},\end{aligned} \tag{4.1}$$

where $\Phi = \Phi(v, V)$ is defined by (3.2).

Proof. Multiply (2.26)$_1$ and (2.26)$_2$ by $p(V) - p(v)$ and ψ, respectively, and add these two equations to have

$$\begin{aligned}&\bigg[\Phi + \frac{1}{2}\psi^2\bigg]_t + \mu\frac{\psi_\xi^2}{v} - \mu\frac{U_\xi\phi\psi_\xi}{vV} + U_\xi\big(p(v) - p(V) - p'(V)\phi\big) \\ &\quad= \bigg[s_-\bigg(\Phi + \frac{1}{2}\psi^2\bigg) + \big(p(V) - p(v)\big)\psi + \mu\bigg(\frac{U_\xi + \psi_\xi}{V + \phi} - \frac{U_\xi}{V}\bigg)\psi\bigg]_\xi + \mu\psi\bigg(\frac{U_\xi}{V}\bigg)_\xi.\end{aligned} \tag{4.2}$$

We recall $U_\xi \geq 0$ and thus compute from (3.4) and (2.22) that if ϵ is suitably small, the discriminant D of

$$\mu\frac{\psi_\xi^2}{v} - \mu\frac{U_\xi\phi\psi_\xi}{vV} + U_\xi\big(p(v) - p(V) - p'(V)\phi\big)$$

satisfies

$$D = \frac{\mu|U_\xi|}{V^2 v f(v,V)} - 4 \leq \frac{C\epsilon}{V^{-\gamma+1}} - 4 < 0.$$

Then, we integrate (4.2) over $(0,t)\times(0,\infty)$ to find

$$\begin{aligned}
&\big\|(\sqrt{\Phi},\psi)(t)\big\|^2 + \int_0^t\int_0^\infty\left[\frac{\psi_\xi^2}{v} + \frac{|U_\xi\phi\psi_\xi|}{v} + |U_\xi|\big(p(v)-p(V)-p'(V)\phi\big)\right]\mathrm{d}\xi\,\mathrm{d}\tau \\
&\lesssim \big\|(\sqrt{\Phi_0},\psi_0)\big\|^2 + \int_0^t \|\psi(\tau)\|_{L^\infty}\left\|\left(\frac{U_\xi}{V}\right)_\xi(\tau)\right\|_{L^1}\mathrm{d}\tau.
\end{aligned} \tag{4.3}$$

To estimate the last term of (4.3), we get from (2.23) and $q \geq 10$ that

$$\left\|\left(\frac{U_\xi}{V}\right)_\xi(\tau)\right\|_{L^1} \leq \left\|\left(\frac{U_\xi}{V}\right)_\xi(\tau)\right\|_{L^1}^{\frac{1}{9}+\frac{8}{9}} \lesssim \epsilon^{\frac{1}{9}}(1+\tau)^{-\frac{4}{5}}.$$

Then we employ Sobolev's inequality and Young's inequality to deduce that for each $\nu > 0$,

$$\begin{aligned}
&\int_0^t \|\psi(\tau)\|_{L^\infty}\left\|\left(\frac{U_\xi}{V}\right)_\xi(\tau)\right\|_{L^1}\mathrm{d}\tau \\
&\leq M^{\frac{1}{4}}\int_0^t\left\|\frac{\psi_\xi}{\sqrt{v}}(\tau)\right\|^{\frac{1}{2}}\|\psi(\tau)\|^{\frac{1}{2}}\left\|\left(\frac{U_\xi}{V}\right)_\xi(\tau)\right\|_{L^1}\mathrm{d}\tau \\
&\leq \nu\int_0^t\left\|\frac{\psi_\xi}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau + C(\nu)M^{\frac{1}{3}}\int_0^t\|\psi(\tau)\|^{\frac{2}{3}}\left\|\left(\frac{U_\xi}{V}\right)_\xi(\tau)\right\|_{L^1}^{\frac{4}{3}}\mathrm{d}\tau \\
&\leq \nu\int_0^t\left\|\frac{\psi_\xi}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau + \int_0^t\|\psi(\tau)\|^2(1+\tau)^{-\frac{16}{15}}\mathrm{d}\tau + C(\nu)\epsilon^{\frac{2}{9}}M^{\frac{1}{2}}.
\end{aligned} \tag{4.4}$$

Plugging (4.4) into (4.3), we can complete this lemma by making use of Gronwall's inequality. □

We set $\tilde{v} := v/V$, and so Eq. $(2.26)_2$ is also written as

$$\left[\mu\frac{\tilde{v}_\xi}{\tilde{v}}-\psi\right]_t - s_-\left[\mu\frac{\tilde{v}_\xi}{\tilde{v}}-\psi\right]_\xi + \frac{\gamma\tilde{v}_\xi}{V^\gamma\tilde{v}^{\gamma+1}} = \frac{\gamma V_\xi}{V^{\gamma+1}}\left(1-\tilde{v}^{-\gamma}\right) - \mu\left(\frac{U_\xi}{V}\right)_\xi. \tag{4.5}$$

Multiplying (4.5) by $\tilde{v}_\xi/\tilde{v}$, we have a divergence form

$$\left[\frac{\mu}{2}\left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)^2-\psi\frac{\tilde{v}_\xi}{\tilde{v}}\right]_t + \frac{\gamma\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2} + \left[\psi\frac{\tilde{v}_t}{\tilde{v}}-\frac{\mu s_-}{2}\left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)^2\right]_\xi$$
$$= \frac{\psi_\xi^2}{v} - \frac{U_\xi\phi\psi_\xi}{vV} + \frac{\gamma V_\xi}{V^{\gamma+1}}\left(1-\tilde{v}^{-\gamma}\right)\frac{\tilde{v}_\xi}{\tilde{v}} - \mu\left(\frac{U_\xi}{V}\right)_\xi\frac{\tilde{v}_\xi}{\tilde{v}}. \tag{4.6}$$

We have from (3.35) and (2.22) that

$$\left(1-\tilde{v}^{-\gamma}\right)^2 = v^{-2\gamma}\left[\gamma\int_0^1\left(\theta v+(1-\theta)V\right)^{\gamma-1}\mathrm{d}\theta\cdot\phi\right]^2$$
$$\lesssim v^{-2\gamma}M^{2\gamma-2}\phi^2 \lesssim v^{-2\gamma}M^{3\gamma-1}\Phi, \tag{4.7}$$

and

$$\left\|V_\xi(\tau)\right\|_{L^\infty} \le \left\|V_\xi(\tau)\right\|_{L^\infty}^{\frac{1}{4}+\frac{3}{4}} \lesssim \epsilon^{\frac{1}{4}}(1+\tau)^{-\frac{3}{4}}. \tag{4.8}$$

These two inequalities imply that

$$\int_0^t\int_0^\infty \frac{\gamma V_\xi}{V^{\gamma+1}}\left(1-\tilde{v}^{-\gamma}\right)\frac{\tilde{v}_\xi}{\tilde{v}}\mathrm{d}\xi\,\mathrm{d}\tau$$
$$\le Cm^{-\gamma}M^{3\gamma-1}\int_0^t\left\|V_\xi(\tau)\right\|_{L^\infty}^2\left\|\sqrt{\Phi}(\tau)\right\|^2\mathrm{d}\tau + \int_0^t\int_0^\infty\frac{\gamma\tilde{v}_\xi^2}{4v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau$$
$$\le Cm^{-\gamma}M^{3\gamma-1}\epsilon^{\frac{1}{2}}\left[\left\|(\sqrt{\Phi_0},\psi_0)\right\|^2+\epsilon^{\frac{2}{9}}M^{\frac{1}{2}}\right] + \int_0^t\int_0^\infty\frac{\gamma\tilde{v}_\xi^2}{4v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau, \tag{4.9}$$

where we used (4.1) to get the last inequality. By a similar way as the above, we have from (2.23) that

$$\int_0^t\int_0^\infty\left|\mu\left(\frac{U_\xi}{V}\right)_\xi\frac{\tilde{v}_\xi}{\tilde{v}}\right|\mathrm{d}\xi\,\mathrm{d}\tau \le CM^\gamma\int_0^t\left\|\left(\frac{U_\xi}{V}\right)_\xi(\tau)\right\|^2\mathrm{d}\tau + \int_0^t\int_0^\infty\frac{\gamma\tilde{v}_\xi^2}{4v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau$$
$$\le CM^\gamma\epsilon + \int_0^t\int_0^\infty\frac{\gamma\tilde{v}_\xi^2}{4v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau. \tag{4.10}$$

Therefore, integrating (4.6) over $(0,t)\times(0,\infty)$ and using (4.9)–(4.10), we have the following lemma.

Lemma 4.2. *If ϵ is suitably small, it holds that*

$$\left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\right\|^2+\int_0^t\int_0^\infty\frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau\lesssim B+u_-^{-1}\int_0^t\psi_\xi^2(\tau,0)\mathrm{d}\tau, \tag{4.11}$$

where

$$B=\left\|\frac{\tilde{v}_{0\xi}}{\tilde{v}_0}\right\|^2+\left[1+m^{-\gamma}M^{3\gamma-1}\epsilon^{\frac{1}{2}}\right]\left[\left\|(\sqrt{\Phi_0},\psi_0)\right\|^2+\epsilon^{\frac{2}{9}}M^{\frac{1}{2}}\right]+M^\gamma\epsilon. \tag{4.12}$$

Multiplying $(2.26)_2$ by $-\psi_{\xi\xi}$, we get a divergence form

$$\left[\frac{1}{2}\psi_\xi^2\right]_t+\left[\frac{s_-}{2}\psi_\xi^2-\psi_t\psi_\xi\right]_\xi+\mu\frac{\psi_{\xi\xi}^2}{v}=\sum_{i=1}^3R_i-\mu\psi_{\xi\xi}\left(\frac{U_\xi}{V}\right)_\xi, \tag{4.13}$$

where R_i $(i=1,2,3)$ are defined in (3.20). Therefore, integrating (4.13) over $(0,t)\times(0,\infty)$ yields

$$\begin{aligned}M(t)&:=\left\|\psi_\xi(t)\right\|^2+u_-\int_0^t\psi_\xi^2(\tau,0)\mathrm{d}\tau+\int_0^t\left\|\frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau)\right\|^2\mathrm{d}\tau\\&\lesssim\|\psi_{0\xi}\|^2+\left(m^{1-\gamma}+M^{\gamma-1}\epsilon^2\right)\int_0^t\int_0^\infty\frac{\tilde{v}_\xi^2}{v^\gamma\tilde{v}^2}\mathrm{d}\xi\,\mathrm{d}\tau+M\epsilon\\&\quad+\int_0^t\int_0^\infty\left[J\phi^2+m^{-2}V_\xi^2\frac{\psi_\xi^2}{v}\right]\mathrm{d}\xi\,\mathrm{d}\tau+\int_0^t\left\|\frac{\phi_\xi}{v}\right\|^4\left\|\frac{\psi_\xi}{\sqrt{v}}\right\|^2\mathrm{d}\tau,\end{aligned} \tag{4.14}$$

where J is defined by (3.29). Employing (2.22), (2.23) and (3.35), we have, by Lemma 4.1,

$$\begin{aligned}\int_0^t\int_0^\infty\left[J\phi^2+m^2V_\xi^2\frac{\psi_\xi^2}{v}\right]\mathrm{d}\xi\,\mathrm{d}\tau&\lesssim\int_0^t\int_0^\infty\left[\left(\left(m^{-2\gamma-3}+M\right)V_\xi^2+m^{-1}U_{\xi\xi}^2\right)\phi^2+m^{-2}\epsilon^2\frac{\psi_\xi^2}{v}\right]\mathrm{d}\xi\,\mathrm{d}\tau\\&\lesssim M^{\gamma+1}\left(m^{-2\gamma-3}+M\right)\epsilon^{\frac{2}{3}}\left[\left\|(\sqrt{\Phi_0},\psi_0)\right\|^2+\epsilon^{\frac{2}{9}}M^{\frac{1}{2}}\right].\end{aligned} \tag{4.15}$$

For the last term on the right-hand side of (4.14), we find from (3.36) that

$$\int_0^t \left\| \frac{\phi_\xi}{v}(\tau) \right\|^4 \left\| \frac{\psi_\xi}{\sqrt{v}}(\tau) \right\|^2 \mathrm{d}\tau \lesssim \left[\left\| (\sqrt{\Phi_0}, \psi_0) \right\|^2 + \epsilon^{\frac{2}{9}} M^{\frac{1}{2}} \right] \Bigg[\left(u_-^{-1} \int_0^t \psi_\xi^2(\tau, 0) \mathrm{d}\tau \right)^2$$
$$+ \epsilon^4 m^{-4} M^{2\gamma+2} \left(\left\| (\sqrt{\Phi_0}, \psi_0) \right\|^2 + \epsilon^{\frac{2}{9}} M^{\frac{1}{2}} \right)^2 + B^2 \Bigg], \tag{4.16}$$

according to Lemma 4.1 and Lemma 4.2. Substitute (4.11), (4.15) and (4.16) into (4.14), and then apply Cauchy's inequality to discover

$$M(t) \lesssim A_1 + A_2 M(t)^2, \tag{4.17}$$

where

$$A_1 = \|\psi_{0\xi}\|^2 + \left(m^{1-\gamma} + M^{\gamma-1}\epsilon^2 \right) B + \frac{(m^{1-\gamma} + M^{\gamma-1}\epsilon^2)^2}{\|(\sqrt{\Phi_0}, \psi_0)\|^2 + \epsilon^{\frac{2}{9}} M^{\frac{1}{2}}}$$
$$+ M\epsilon + \left[\left\| (\sqrt{\Phi_0}, \psi_0) \right\|^2 + \epsilon^{\frac{2}{9}} M^{\frac{1}{2}} \right] \left[M^{\gamma+1} \left(m^{-2\gamma-3} + M \right) \epsilon^{\frac{2}{3}} \right.$$
$$\left. + \epsilon^4 m^{-4} M^{2\gamma+2} \left(\left\| (\sqrt{\Phi_0}, \psi_0) \right\|^2 + \epsilon^{\frac{2}{9}} M^{\frac{1}{2}} \right)^2 + B^2 \right] \tag{4.18}$$

and

$$A_2 = u_-^{-4} \left[\left\| (\sqrt{\Phi_0}, \psi_0) \right\|^2 + \epsilon^{\frac{2}{9}} M^{\frac{1}{2}} \right]. \tag{4.19}$$

Then employing Lemma 3.7, we conclude the following lemma.

Lemma 4.3. *There is a constant c_3 independent of T, m, M, δ and a, such that if ϵ is suitably small and*

$$A_1 A_2 < c_4, \tag{4.20}$$

where A_1 and A_2 are defined by (4.18) and (4.19), respectively, then it holds that for each $0 \le t \le T$,

$$\left\| \psi_\xi(t) \right\|^2 + u_- \int_0^t \psi_\xi^2(\tau, 0) \mathrm{d}\tau + \int_0^t \left\| \frac{\psi_{\xi\xi}}{\sqrt{v}}(\tau) \right\|^2 \mathrm{d}\tau \lesssim A_1 \tag{4.21}$$

and

$$\left\| \frac{\tilde{v}_\xi}{\tilde{v}}(t) \right\|^2 + \int_0^t \int_0^\infty \frac{\tilde{v}_\xi^2}{v^\gamma \tilde{v}^2} \mathrm{d}\xi \mathrm{d}\tau \lesssim B + u_-^{-2} A_1, \tag{4.22}$$

where B is defined by (4.12).

Proof of Theorem 3. Since $(\phi_0, \psi_0) \in H_0^1(\mathbb{R}_+)$, we find $t_0 > 0$ from Lemma 2.5 such that (2.26) has a unique solution $(\phi, \psi) \in X_{m_0,M_0}(0, t_0)$ with $1/m_0, M_0 \le C\epsilon^{-l}$. Then we deduce that if $(2.28)_1$ holds, then $\|\sqrt{\Phi_0}, \psi_0\|^2 + \epsilon^{2/9} M_0^{1/2}$ and B are bounded by $C_1(1 + \epsilon^{-2\alpha-(\gamma+1)l})$. Then we compute that if $(2.28)_1$ holds, then A_1 and A_2 are, respectively, bounded by $C_2(1 + \epsilon^{-6\alpha-3(\gamma+1)l})$ and $C_3(1 + \epsilon^{4l_0-2\alpha-(\gamma+1)l})$. Hence if $(2.28)_2$ holds, there exists $\epsilon_1 > 0$ such that (4.20) holds for each $\epsilon \le \epsilon_1$. Next we compute from $(2.28)_2$ that the right-hand side of (4.22) is bounded by $C_4(1 + \epsilon^{-2\alpha-(\gamma+1)l})$. We conclude from Lemma 4.1 and Lemma 4.3 that

$$\left|\Psi\left(\tilde{v}(t_0, \xi)\right)\right| \le \left\|\sqrt{\tilde{\Phi}}(t_0)\right\| \left\|\frac{\tilde{v}_\xi}{\tilde{v}}(t_0)\right\| \le C\epsilon^\theta, \tag{4.23}$$

where $\theta = -2\alpha - (\gamma + 1)l$. Hence

$$C^{-1}\epsilon^{2\theta/(1-\gamma)} \le v(t_0) \le C\epsilon^{2\theta}. \tag{4.24}$$

Since $(2.28)_3$ implies $2\theta/(\gamma - 1) \le -l$ and $2\theta \le -l$, we apply Lemma 2.5 again and recall (4.24) to find $t_1 > 0$ such that (2.6) has a unique solution $(\phi, \psi) \in X_{m_1,M_1}(0, t_0 + t_1)$, where $m_1 \ge C\epsilon^{2\theta/(1-\gamma)}$ and $M_1 \le C\epsilon^{2\theta}$. By elementary calculations, we conclude from $(2.28)_4$ that $\|\sqrt{\Phi_0}, \psi_0\|^2 + \epsilon^{2/9} M_0^{1/2}$ and B are bounded by $C_1(1 + \epsilon^{-2\alpha-(\gamma+1)l})$. Then we compute that if $(2.28)_4$ holds, then A_1 and A_2 are, respectively, bounded by $C_5(1 + \epsilon^{-6\alpha-3(\gamma+1)l})$ and $C_6(1 + \epsilon^{4l_0-2\alpha-(\gamma+1)l})$. Hence if $(2.28)_2$ holds, there exists $\epsilon_0 > 0$ such that (4.20) holds for each $\epsilon \le \epsilon_0$. Thus, we find from $(2.28)_2$ that the right-hand side of (4.22) is bounded by $C_4(1 + \epsilon^{-2\alpha-(\gamma+1)l})$. Then, combining Lemma 2.5 and the continuation process, we can prove (2.1) has the global solution in time $(\phi, \psi) \in X_{m_1,M_1}(0, \infty)$ satisfying (3.47) with the constant C depending only on ϵ. Thus, the asymptotic behavior of the solution (2.11) is concluded by employing Sobolev's inequality. □

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