Z. Angew. Math. Phys. 66 (2015), 3299–3307

0044-2275/15/063299-9
published online September 4, 2015
DOI 10.1007/s00033-015-0579-x

Zeitschrift für angewandte Mathematik und Physik ZAMP

Vanishing shear viscosity in the magnetohydrodynamic equations with temperature-dependent heat conductivity

Tao Wang

Abstract. We establish an initial-boundary value problem for the compressible magnetohydrodynamic equations in one space dimension with large initial data when the heat conductivity is some positive power of the temperature. We prove that as the shear viscosity vanishes, global weak solutions convergence to a solution of the original equations with zero shear viscosity.

Mathematics Subject Classification. 76W05 · 35D30 · 35Q35.

Keywords. Magnetohydrodynamics · Temperature-dependent heat conductivity · Vanishing shear viscosity · Global weak solutions · Large data.

1. Introduction

Magnetohydrodynamics (MHD) is concerned with the study of the dynamics of electrically conducting fluids. The applications of magnetohydrodynamics cover a very wide range of physical objects, from cosmic plasmas to liquid metals. The plane magnetohydrodynamic flows are described by the following equations (see, e.g., [12,13]):

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + P)_x = (\lambda u_x)_x, \tag{1.2}$$

$$(\rho \boldsymbol{w})_t + (\rho u \boldsymbol{w} - b_1 \boldsymbol{b})_x = (\mu \boldsymbol{w}_x)_x, \tag{1.3}$$

$$\boldsymbol{b}_t + (u\boldsymbol{b} - b_1 \boldsymbol{w})_x = (\nu \boldsymbol{b}_x)_x, \tag{1.4}$$

$$\mathcal{E}_t + (u(\mathcal{E} + P) - b_1 \boldsymbol{b} \cdot \boldsymbol{w})_x = (\lambda u u_x + \mu \boldsymbol{w} \cdot \boldsymbol{w}_x + \nu \boldsymbol{b} \cdot \boldsymbol{b}_x + \kappa \theta_x)_x, \tag{1.5}$$

where $t > 0$ is the time variable, $x \in \Omega = (0,1)$ is the spatial variable, and the primary dependent variables are the density ρ, the longitudinal velocity $u \in \mathbb{R}$, the transverse velocity $\boldsymbol{w} \in \mathbb{R}^2$, the transverse magnetic field $\boldsymbol{b} \in \mathbb{R}^2$ and the temperature θ. The full pressure $P = p + \frac{1}{2}|\boldsymbol{b}|^2$ with $p = p(\rho,\theta)$ being the pressure of the fluid; the longitudinal magnetic field b_1 is a constant; the total energy $\mathcal{E} = \rho\left(e + \frac{1}{2}u^2 + \frac{1}{2}|\boldsymbol{w}|^2\right) + \frac{1}{2}|\boldsymbol{b}|^2$ with $e = e(\rho,\theta)$ being the specific internal energy. The bulk viscosity λ, the shear viscosity μ, the magnetic diffusivity ν and the heat conductivity κ may depend on ρ and θ generally.

In this article, we focus on the ideal, polytropic gas with the following constitutive relations:

$$p = R\rho\theta, \quad e = c_v\theta, \tag{1.6}$$

where R is the gas constant and c_v is the specific heat at constant volume. For boundary conditions, we take

$$(u, \boldsymbol{w}, \boldsymbol{b}, \theta_x)\,|_{\partial\Omega} = 0, \tag{1.7}$$

Birkhäuser

while the initial conditions are given by

$$(\rho, u, \boldsymbol{w}, \boldsymbol{b}, \theta)\,|_{t=0} = (\rho_0, u_0, \boldsymbol{w}_0, \boldsymbol{b}_0, \theta_0). \tag{1.8}$$

Here the initial data (1.8) are assumed to satisfy certain compatibility conditions as usual.

We assume that the coefficients λ, μ and ν are positive constants and the heat conductivity κ is given by

$$\kappa = \theta^\alpha \tag{1.9}$$

for some positive constant α. This choice of degenerate heat conductivity is motivated by the kinetic theory of gases, cf., for instance, [1,22].

The aim of this article is to show the convergence of weak solutions to the initial-boundary value problem (1.1)–(1.8) in the domain $Q_T := (0,T) \times \Omega$ with the initial data satisfying

$$\rho_0^{-1}, \rho_0 \in L^\infty(\Omega),\ u_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in L^2(\Omega),\ \theta_0 \in L^1(\Omega),\ \inf_\Omega \theta_0 > 0 \tag{1.10}$$

under the assumption (1.9) as the shear viscosity μ tends to zero.

We first recall some previous results related. For small initial data, the existence of global smooth solutions was proved in [10] and the large-time behavior of solutions was investigated in [14,15]. For large initial data, Chen and Wang [3] showed the existence, uniqueness and Lipschitz continuous dependence of global strong solutions to the initial-boundary value problem (1.1)–(1.8) with the coefficients λ, μ and ν being positive constants and the heat conductivity κ satisfying

$$C^{-1}(1+\theta^r) \le \kappa \equiv \kappa(\theta) \le C(1+\theta^r) \tag{1.11}$$

for some positive constants C and $r \ge 2$. Similar results are obtained in [2,20] when the pressure and internal energy have certain nonlinear dependence on temperature. Fan et al. [4] established the global existence and uniqueness of strong solutions with vacuum under the assumption (1.11) with $r > 0$, while Hu and Ju [8] proved the existence and uniqueness of global strong non-vacuum solutions with the heat conductivity κ satisfying (1.9) for some $\alpha > 0$. Very recently, Qin–Yang–Yao–Zhou in [16] study the thickness of the boundary layer for the planar MHD system with the same assumption on the heat conductivity. We also note that it is still an open problem to obtain the global strong or smooth solutions of the problem (1.1)–(1.8) with large initial data and constant heat conductivity, while it is a well-known result for the one-dimensional compressible Navier–Stokes equations (see [11]).

The vanishing shear viscosity limit of the weak solutions to the problem (1.1)–(1.8) was investigated by Fan et al. [5] under a restrictive assumption on the heat conductivity: $\kappa \equiv \kappa(\rho) \ge C/\rho$ or κ satisfies (1.11) with $r \ge 1$. When $(b_1, \boldsymbol{b}) \equiv 0$, the system (1.1)–(1.5) reduces to the compressible Navier–Stokes equations. In this case, Shelukhin [18] showed the zero shear viscosity limit of global strong solutions for the flow with constant heat conductivity between two parallel plates. See [19] for the extension to a free boundary problem of describing a joint motion of two compressible fluids with different viscosity. In the case of cylindric symmetry, the vanishing shear viscosity of global strong solutions was proved in [5,6,21] for isentropic flows and in [7,9,17] for non-isentropic flows.

We are now in a position to state our main result.

Theorem 1. *If the initial data and the heat conductivity κ satisfy (1.10) and (1.9) for some positive constant α, then the initial-boundary value problem (1.1)–(1.8) admits at least one weak solution $(\rho, u, \boldsymbol{w}, \boldsymbol{b}, \theta)$ satisfying for each $0 \le \beta \le 1$ and $0 < r < \frac{\alpha+3}{2}$,*

$$\begin{aligned}
&C^{-1} \le \rho \le C, \quad \theta \ge C^{-1},\\
&\int_0^T \int_\Omega \left[\frac{\beta\kappa(\theta)\theta_x^2}{\theta^{1+\beta}} + \frac{u_x^2 + \mu|\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2}{\theta^\beta}\right] \mathrm{d}x\mathrm{d}t \le C,\\
&\|\theta\|_{L^{2r}(Q_T)} + \|\theta_x\|_{L^r(Q_T)} \le C,
\end{aligned} \tag{1.12}$$

where the positive constant C is independent of μ. Moreover, there exist functions $\bar\rho, \bar u, \bar{\boldsymbol b}, \bar{\boldsymbol w}, \bar\theta$ solving (1.1)–(1.8) with $\mu = 0$ in the sense of distributions, such that, as the shear viscosity μ tends to zero,

$\rho \to \bar\rho$ *strongly in* $L^q(Q_T)$ *and weakly star in* $L^\infty(Q_T)$ *for each* $q \in [1, \infty)$,
$(u, \boldsymbol b) \to (\bar u, \bar{\boldsymbol b})$ *strongly in* $L^2(0, T; H_0^1(\Omega))$,
$\boldsymbol w \to \bar{\boldsymbol w}$ *strongly in* $L^2(Q_T)$,
$\theta \to \bar\theta$ *strongly in* $L^m(Q_T)$ *for each* $m \in [1, 3)$,
$\theta_x \to \bar\theta_x$ *weakly in* $L^n(Q_T)$ *for each* $n < 3/2$.

Remark 1.1. We note that the proofs in Section 2 for Theorem 1 can be applied to the case when the heat conductivity κ satisfies (1.11) for some $r > 0$. Hence, we generalize the earlier results of [5] where the authors considered the case when the heat conductivity κ satisfies (1.11) for some $r \geq 1$.

Since this article concerns the more general heat conductivity κ given in (1.9), the approach in [5] cannot be applied. To overcome this difficulty, we first obtain a uniform upper bound on the density from the standard energy estimates and then a uniform lower bound of the temperature by applying the comparison theorem to the equation for the temperature. With the upper bound for the density and the lower bound for the temperature in hand, we prove a higher-order integrability of the temperature as in (2.21) and the dissipative effect of the heat conductivity as in (2.17) and (2.18). Then, the proof of Theorem 1 can be completed in standard argument.

2. Proof of Theorem 1

This section is devoted to obtaining some a priori estimates of the solution $(\rho, u, \boldsymbol w, \boldsymbol b, \theta)$ to the problem (1.1)–(1.8) defined on Q_T. We will show that the a priori bounds are independent of the shear viscosity μ, which are essential to justify the vanishing shear viscosity limit.

To simplify the presentation, we denote C the various positive constant which is independent of μ. We will use $A \lesssim B$ (or $B \gtrsim A$) if $A \leq CB$ for some positive constant C. And $\|\cdot\|_{L^q}$ stands for the standard norm of the Lebesgue space $L^q(\Omega)$.

Firstly, we prove the basic a priori estimates of conservation of mass and energy along with the dissipation rate related to heat conductivity.

Lemma 2.1. *It holds for each $t \in [0, T]$ that*

$$\int_\Omega \rho(t, x)\mathrm{d}x = \int_\Omega \rho_0(x)\mathrm{d}x > 0, \tag{2.1}$$

$$\int_\Omega \mathcal{E}(t, x)\mathrm{d}x = \int_\Omega \mathcal{E}(0, x)\mathrm{d}x, \tag{2.2}$$

$$\int_0^t \int_\Omega \left[\frac{\kappa(\theta)\theta_x^2}{\theta^2} + \frac{u_x^2 + \mu|\boldsymbol w_x|^2 + |\boldsymbol b_x|^2}{\theta}\right]\mathrm{d}x\mathrm{d}t \lesssim 1. \tag{2.3}$$

Proof. Integrating (1.1) and (1.5) over $(0, t) \times \Omega$ yields the estimates (2.1) and (2.2).

We define $s := c_v \ln\theta - R\ln\rho$ and compute

$$(\rho s)_t + (\rho u s)_x - \left(\frac{\kappa\theta_x}{\theta}\right)_x = \frac{\kappa(\theta)\theta_x^2}{\theta^2} + \frac{\lambda u_x^2 + \mu|\boldsymbol w_x|^2 + \nu|\boldsymbol b_x|^2}{\theta}.$$

Integrating this last identity over $(0, t) \times \Omega$, we deduce

$$\int_\Omega [R\rho\ln\rho - c_v\rho\ln\theta]\,\mathrm{d}x + \int_0^t \int_\Omega \left[\frac{\kappa(\theta)\theta_x^2}{\theta^2} + \frac{\lambda u_x^2 + \mu|\boldsymbol w_x|^2 + \nu|\boldsymbol b_x|^2}{\theta}\right]\mathrm{d}x\mathrm{d}t \lesssim 1,$$

which combined with (2.1) and (2.2) gives

$$\int_{\Omega}\left[R\rho\phi\left(\frac{1}{\rho}\right)+c_v\rho\phi(\theta)\right]\mathrm{d}x+\int_0^t\int_{\Omega}\left[\frac{\kappa(\theta)\theta_x^2}{\theta^2}+\frac{\lambda u_x^2+\mu|\boldsymbol{w}_x|^2+\nu|\boldsymbol{b}_x|^2}{\theta}\right]\mathrm{d}x\mathrm{d}t\lesssim 1,$$

where $\phi(z):=z-\ln z-1\geq 0$ for $z>0$. The estimate (2.3) then follows. □

The following lemma gives us the uniform upper bound of the density ρ, which can be proved as in [5]. We omit its proof for brevity.

Lemma 2.2. *It holds for each* $(t,x)\in[0,T]\times\overline{\Omega}$ *that*

$$\rho(t,x)\lesssim 1. \tag{2.4}$$

The next lemma is devoted to obtaining the lower bound of the temperature θ.

Lemma 2.3. *For each* $(t,x)\in[0,T]\times\overline{\Omega}$*, it holds that*

$$\theta(t,x)\gtrsim 1. \tag{2.5}$$

Proof. It follows from (1.5) that the temperature θ satisfies

$$c_v\rho(\theta_t+u\theta_x)+pu_x=(\kappa\theta_x)_x+\lambda u_x^2+\mu|\boldsymbol{w}_x|^2+\nu|\boldsymbol{b}_x|^2, \tag{2.6}$$

which implies

$$\begin{aligned}\theta_t+u\theta_x-\frac{1}{c_v\rho}(\kappa\theta_x)_x&\geq\frac{\lambda u_x^2}{c_v\rho}-\frac{pu_x}{c_v\rho}\\&\geq\frac{\lambda}{c_v\rho}\left(u_x-\frac{p}{2\lambda}\right)^2-\frac{R^2\rho\theta^2}{4\lambda c_v}.\end{aligned}$$

By Lemma 2.2, we have

$$\theta_t+u\theta_x-\frac{1}{c_v\rho}(\kappa\theta_x)_x+C_1\theta^2\geq 0,$$

where C_1 is a positive constant independent of μ. If we set $\underline{\theta}:=\frac{\min_\Omega\theta_0}{C_2t+1}$ with $C_2=C_1\min_\Omega\theta_0$, and $\Theta:=\theta-\underline{\theta}$, then we derive

$$\begin{aligned}&\Theta_t+u\Theta_x-\frac{1}{c_v\rho}(\kappa\Theta_x)_x+C_1(\theta+\underline{\theta})\Theta\\&=\theta_t+C_2\frac{\min_\Omega\theta_0}{(C_2t+1)^2}+u\theta_x-\frac{1}{c_v\rho}(\kappa\theta_x)_x+C_1\theta^2-C_1\underline{\theta}^2\\&\geq C_2\frac{\min_\Omega\theta_0}{(C_2t+1)^2}-C_1\underline{\theta}^2\geq 0,\end{aligned}$$

and

$$\Theta_x|_{\partial\Omega}=0,\quad \Theta|_{t=0}\geq 0.$$

It follows from the comparison theorem and Lemma 2.2 that $\Theta\geq 0$ on $\overline{Q_T}$. This completes the proof of the lemma. □

Lemma 2.4. *Assume that* (1.9) *holds for some* $\alpha>0$ *and assume* $0\leq\beta\leq 1$. *Then*

$$\int_0^T\int_{\Omega}\left[\frac{\beta\kappa(\theta)\theta_x^2}{\theta^{1+\beta}}+\frac{u_x^2+\mu|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2}{\theta^\beta}\right]\mathrm{d}x\mathrm{d}t+\int_0^T\|\theta(t)\|_{L^\infty}^{1-\beta}\mathrm{d}t\lesssim 1. \tag{2.7}$$

Proof. By the estimate (2.3), it suffices to prove the case of $0 \le \beta < 1$.

Let $0 \le \beta < 1$ and multiply (2.6) by $\theta^{-\beta}$ to find

$$\left[\frac{\kappa\theta_x}{\theta^{1+\beta}}\right]_x + \frac{\beta\kappa(\theta)\theta_x^2}{\theta^{1+\beta}} + \frac{\lambda u_x^2 + \mu|\boldsymbol{w}_x|^2 + \nu|\boldsymbol{b}_x|^2}{\theta^\beta} = \left[\frac{c_v\rho\theta^{1-\beta}}{1-\beta}\right]_t + \left[\frac{c_v\rho u\theta^{1-\beta}}{1-\beta}\right]_x + R\rho\theta^{1-\beta}u_x.$$

Integrating this last identity over $[0,T]\times\Omega$, we have

$$\int_0^{\mathrm{T}}\int_\Omega \left[\frac{\beta\kappa(\theta)\theta_x^2}{\theta^{1+\beta}} + \frac{u_x^2 + \mu|\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2}{\theta^\beta}\right]\mathrm{d}x\mathrm{d}t \lesssim 1 + \int_\Omega \rho\theta^{1-\beta}\mathrm{d}x + \left|\int_0^{\mathrm{T}}\int_\Omega \rho\theta^{1-\beta}u_x\mathrm{d}x\mathrm{d}t\right|. \tag{2.8}$$

By Young's inequality and the estimates (2.1)–(2.2), we have

$$\int_\Omega \rho\theta^{1-\beta}\mathrm{d}x \lesssim \int_\Omega \rho(1+\theta)\mathrm{d}x \lesssim 1. \tag{2.9}$$

Apply Cauchy's inequality and utilize (2.2) and (2.4) to deduce

$$\begin{aligned}\left|\int_0^{\mathrm{T}}\int_\Omega \rho\theta^{1-\beta}u_x\mathrm{d}x\mathrm{d}t\right| &\le \epsilon\int_0^{\mathrm{T}}\int_\Omega \theta^{-\beta}|u_x|^2\mathrm{d}x\mathrm{d}t + C(\epsilon)\int_0^{\mathrm{T}}\int_\Omega \rho^2\theta^{2-\beta}\mathrm{d}x\mathrm{d}t \\ &\le \epsilon\int_0^{\mathrm{T}}\int_\Omega \theta^{-\beta}|u_x|^2\mathrm{d}x\mathrm{d}t + C(\epsilon)\int_0^{\mathrm{T}} \|\theta(t)\|_{L^\infty}^{1-\beta}\mathrm{d}t.\end{aligned} \tag{2.10}$$

Plugging (2.9) and (2.10) into (2.8), and taking ϵ suitable small, we have for $0 \le \beta < 1$ that

$$\int_0^{\mathrm{T}}\int_\Omega \left[\frac{\beta\kappa(\theta)\theta_x^2}{\theta^{1+\beta}} + \frac{u_x^2 + \mu|\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2}{\theta^\beta}\right]\mathrm{d}x\mathrm{d}t \lesssim 1 + \int_0^{\mathrm{T}} \|\theta(t)\|_{L^\infty}^{1-\beta}\mathrm{d}t. \tag{2.11}$$

Since

$$\theta^{1-\beta}(t,x) - \frac{\int_\Omega \rho\theta^{1-\beta}\mathrm{d}x}{\int_\Omega \rho\mathrm{d}x} \lesssim \left|\int_\Omega \theta^{-\beta}\theta_x\mathrm{d}x\right|, \tag{2.12}$$

we infer for $0 < \beta < 1$ that

$$\begin{aligned}\|\theta(t)\|_{L^\infty}^{1-\beta} &\lesssim 1 + \epsilon\int_\Omega \frac{\beta\kappa(\theta)\theta_x^2}{\theta^{1+\beta}}\mathrm{d}x + C(\epsilon)\int_\Omega \theta^{1-\beta-\alpha}\mathrm{d}x \\ &\lesssim 1 + \epsilon\int_\Omega \frac{\beta\kappa(\theta)\theta_x^2}{\theta^{1+\beta}}\mathrm{d}x + C(\epsilon)\left\|\theta^{1-\beta-\alpha}(t)\right\|_{L^\infty}.\end{aligned}$$

Applying Young's inequality and Lemma 2.3, we have

$$\|\theta(t)\|_{L^\infty}^{1-\beta} \lesssim C(\epsilon) + \epsilon\int_\Omega \frac{\beta\kappa(\theta)\theta_x^2}{\theta^{1+\beta}}\mathrm{d}x. \tag{2.13}$$

If we plug (2.13) into (2.11) and take ϵ suitable small, then we obtain (2.7) for $0 < \beta < 1$.

For $\beta = 0$, we choose $0 < \gamma < \min\{1, \alpha\}$ and deduce from (2.12) that

$$\|\theta(t)\|_{L^\infty} \lesssim 1 + \int_\Omega \frac{\gamma\kappa(\theta)\theta_x^2}{\theta^{1+\gamma}}\mathrm{d}x + \left\|\theta^{1+\gamma-\alpha}(t)\right\|_{L^\infty},$$

which implies from Young's inequality and Lemma 2.3 that

$$\|\theta(t)\|_{L^\infty} \lesssim 1 + \int_\Omega \frac{\gamma\kappa(\theta)\theta_x^2}{\theta^{1+\gamma}}\mathrm{d}x.$$

We have proved (2.7) for each $0 < \beta < 1$ and hence

$$\int_0^T \|\theta(t)\|_{L^\infty}\mathrm{d}t \lesssim 1. \tag{2.14}$$

Plugging (2.14) into (2.11) yields (2.7) with $\beta = 0$. This completes the proof of the lemma. □

The following lemma concerns the uniform lower bound of the density.

Lemma 2.5. *Under the assumptions of Theorem 1, we have for each* $(t, x) \in [0, T] \times \Omega$ *that*

$$\rho(t, x) \gtrsim 1. \tag{2.15}$$

Proof. First it follows from (1.7), Lemma 2.1 and Hölder's inequality that

$$\begin{aligned}|\boldsymbol{b}|^2 &= 2\int_0^x \boldsymbol{b}\cdot\boldsymbol{b}_x \\ &\le 2\left\|\sqrt{\theta}\right\|_{L^\infty}\|\boldsymbol{b}\|_{L^2}\left\|\frac{\boldsymbol{b}_x}{\sqrt{\theta}}\right\|_{L^\infty} \\ &\lesssim \left\|\sqrt{\theta}\right\|_{L^\infty}\left\|\frac{\boldsymbol{b}_x}{\sqrt{\theta}}\right\|_{L^\infty}.\end{aligned}$$

Using (2.3) and (2.7), we have

$$\int_0^T \|\boldsymbol{b}(t)\|_{L^\infty}^2\,\mathrm{d}t \lesssim \int_0^T\int_\Omega \frac{|\boldsymbol{b}_x|^2}{\theta}\mathrm{d}x\mathrm{d}t + \int_0^T \|\theta(t)\|_{L^\infty}\mathrm{d}t \lesssim 1. \tag{2.16}$$

We set $G = \exp(-\psi/\lambda)$ with ψ defined by

$$\psi(t, x) := \int_0^t \left[\lambda u_x - \rho u^2 - p - \frac{1}{2}|\boldsymbol{b}|^2\right]\mathrm{d}s + \int_0^x \rho_0 u_0 \mathrm{d}y.$$

It is easily shown that G is positive and bounded by using Lemma 2.2 in [5]. The elementary calculation gives

$$\begin{aligned}D_t\left(\frac{G}{\rho}\right) :=& (\partial_t + u\partial_x)\left(\frac{G}{\rho}\right) \\ =& \frac{G}{\rho}(\partial_t + u\partial_x)\left(-\frac{\psi}{\lambda}\right) - \frac{G}{\rho^2}(\partial_t + u\partial_x)\rho \\ =& \frac{G}{\lambda\rho}(-\psi_t - u\psi_x - \lambda u_x) \\ =& \frac{G}{\lambda\rho}\left(p + \frac{1}{2}|\boldsymbol{b}|^2\right),\end{aligned}$$

which implies

$$D_t\left(\frac{G}{\rho}\right) \le \frac{G}{\rho}\frac{|\boldsymbol{b}|^2}{2\lambda} + C\|\theta\|_{L^\infty},$$

or equivalently,

$$\left\|\frac{1}{\rho}\right\|_{L^\infty} \lesssim 1 + \int_0^t \|\theta(s)\|_{L^\infty} ds + \int_0^t \left\|\frac{1}{\rho}\right\|_{L^\infty} \|\boldsymbol{b}\|_{L^\infty}^2 \mathrm{d}s.$$

Applying Gronwall's inequality, (2.7) and (2.16), we have

$$\left\|\frac{1}{\rho}\right\|_{L^\infty} \lesssim 1.$$

The proof of the lemma is then completed. □

Lemma 2.6. *Under the assumptions of Theorem 1, we have*

$$\|\theta\|_{L^{2r}(Q_T)} \lesssim 1, \quad \forall\ 0 < r < \frac{\alpha+3}{2}, \tag{2.17}$$

$$\|\theta_x\|_{L^{r}(Q_T)} \lesssim 1, \quad \forall\ 0 < r < \frac{\alpha+3}{2}, \tag{2.18}$$

$$\|\kappa(\theta)\theta_x\|_{L^{r}(Q_T)} \lesssim 1, \quad \forall\ 1 \le r < \frac{\alpha+3}{\alpha+2}. \tag{2.19}$$

Proof. 1. We have from (2.2) and (2.15) that

$$\int_\Omega \theta(t,x)\mathrm{d}x \lesssim 1, \tag{2.20}$$

which enables us to find $a(t) \in \Omega$ such that $\theta(t, a(t)) \lesssim 1$. If $0 < \beta \le 1$, then we have

$$\begin{aligned} \theta^{\frac{2+\alpha-\beta}{2}}(t,x) &\le \theta^{\frac{2+\alpha-\beta}{2}}(t,a(t)) + \int_\Omega \theta^{\frac{\alpha-\beta}{2}}|\theta_x|\mathrm{d}x \\ &\lesssim 1 + \int_\Omega \theta^{\frac{\alpha-\beta}{2}}|\theta_x|\mathrm{d}x, \end{aligned}$$

which combined with (2.7) and (2.20) implies

$$\begin{aligned} \int_0^{\mathrm{T}} \|\theta(t)\|_{L^\infty}^{2+\alpha-\beta}\mathrm{d}t &\lesssim 1 + \int_0^{\mathrm{T}}\int_\Omega \frac{\beta\theta^\alpha\theta_x^2}{\theta^{1+\beta}}\mathrm{d}x \int_\Omega \theta\mathrm{d}x\mathrm{d}t \\ &\lesssim 1 + \int_0^{\mathrm{T}}\int_\Omega \frac{\beta\theta^\alpha\theta_x^2}{\theta^{1+\beta}}\mathrm{d}x\mathrm{d}t \lesssim 1. \end{aligned} \tag{2.21}$$

The combination of (2.20) and (2.21) implies

$$\int_0^{\mathrm{T}}\int_\Omega \theta^{3+\alpha-\beta}\mathrm{d}x\mathrm{d}t \lesssim 1 \tag{2.22}$$

for all $0 < \beta \le 1$. Then (2.17) follows.

2. If $0 < r < \frac{\alpha+3}{2}$, then there exists $\beta \in (0,1)$ such that $r \le \frac{3+\alpha-\beta}{2}$. Applying Young's inequality, we have from (2.7), (2.22) and (2.5) that

$$\int_0^T \int_\Omega \theta_x^r \mathrm{d}x\mathrm{d}t \le \int_0^T \int_\Omega \frac{\theta^\alpha \theta_x^2}{\theta^{1+\beta}} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \theta^{\frac{r(1+\beta-\alpha)}{2-r}} \mathrm{d}x\mathrm{d}t$$
$$\lesssim 1 + \int_0^T \int_\Omega \theta^{3+\alpha-\beta} \mathrm{d}x\mathrm{d}t \lesssim 1.$$

We obtain (2.18).

3. If $1 \le r < \frac{\alpha+3}{\alpha+2}$, then there exist $r_1, r_2 \ge 1$ such that

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}, \quad \alpha r_1 < 3+\alpha, \quad r_2 < \frac{3+\alpha}{2}.$$

Applying Hölder's inequality, we have from (2.17)–(2.18) that

$$\|\kappa(\theta)\theta_x\|_{L^r(Q_T)} \le \|\kappa(\theta)\|_{L^{r_1}(Q_T)} \|\theta_x\|_{L^{r_2}(Q_T)} \lesssim 1.$$

This completes the proof of the lemma. □

With Lemmas 2.1–2.6 in hand, we can apply the Poincaré inequality, the Sobolev imbedding theorem, and (2.16), (2.21) to the equations (1.2)–(1.5) in order to get the a priori estimates on the time derivatives of the solution $(\rho, u, \boldsymbol{w}, \boldsymbol{b}, \theta)$, which are stated in the following lemma.

Lemma 2.7. *Under the assumptions of Theorem 1, we have*

$$\|\rho_t\|_{L^\infty(0,T;H^{-1}(\Omega))} + \|((\rho u)_t, (\rho \boldsymbol{w})_t, \boldsymbol{b}_t)\|_{L^2(0,T;H^{-1}(\Omega))} \lesssim 1, \tag{2.23}$$

$$\|(\rho\theta)_t\|_{L^1(0,T;W^{-1,r}(\Omega))} \lesssim 1, \quad \forall\ 1 \le r < \frac{3+\alpha}{2+\alpha}. \tag{2.24}$$

In order to prove the existence of global weak solutions to (1.1)–(1.8) with the heat conductivity κ satisfying (1.9) for some $\beta > 0$, we first mollify the initial data, such that the global strong solution exists by [8], and then take the limit to obtain the global weak solution. With the above a priori estimates on $(\rho, u, \boldsymbol{w}, \boldsymbol{b}, \theta)$ given in Lemmas 2.1–2.7, following the argument in [5], we can justify the passage of the vanishing viscosity. This completes the proof of Theorem 1.

Acknowledgments

This work was supported by "The Fundamental Research Funds for the Central Universities."

References

1. Chapman, S., Cowling, T.G.: The Mathematical Theory of Nonuniform Gases, 3rd ed. Cambridge Mathematical Library, Cambridge University Press, Cambridge (1990); An account of the kinetic theory of viscosity, thermal conduction and diffusion in gases, In co-operation with D. Burnett, with a foreword by Carlo Cercignani
2. Chen, G.-Q., Wang, D.: Global solutions of nonlinear magnetohydrodynamics with large initial data. J. Differ. Equ. **182**, 344–376 (2002)
3. Chen, G.-Q., Wang, D.: Existence and continuous dependence of large solutions for the magnetohydrodynamic equations. Z. Angew. Math. Phys. **54**, 608–632 (2003)
4. Fan, J., Huang, S., Li, F.: Global Strong Solutions to the Planar Compressible Magnetohydrodynamic Equations with Large Initial Data and Vaccum. arXiv:1206.3624 (preprint)
5. Fan, J., Jiang, S., Nakamura, G.: Vanishing shear viscosity limit in the magnetohydrodynamic equations. Commun. Math. Phys. **270**, 691–708 (2007)

6. Frid, H., Shelukhin, V.: Boundary layers for the Navier–Stokes equations of compressible fluids. Commun. Math. Phys. **208**, 309–330 (1999)
7. Frid, H., Shelukhin, V.: Vanishing shear viscosity in the equations of compressible fluids for the flows with the cylinder symmetry. SIAM J. Math. Anal. **31**, 1144–1156 (electronic) (2000)
8. Hu, Y., Ju, Q.: Global large solutions of magnetohydrodynamics with temperature-dependent heat conductivity. Z. Angew. Math. Phys. **66**, 865–889 (2015)
9. Jiang, S., Zhang, J.: Boundary layers for the Navier–Stokes equations of compressible heat-conducting flows with cylindrical symmetry. SIAM J. Math. Anal. **41**, 237–268 (2009)
10. Kawashima, S., Okada, M.: Smooth global solutions for the one-dimensional equations in magnetohydrodynamics. Proc. Jpn. Acad. Ser. A Math. Sci. **58**, 384–387 (1982)
11. Kazhikhov, A.V., Shelukhin, V.V.: Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. Prikl. Math. Mech. **41**, 282–291 (1977)
12. Landau, L.D., Lifshitz, E.M.: Course of Theoretical Physics, vol. 8. Pergamon International Library of Science, Technology, Engineering and Social Studies, Pergamon Press, Oxford (1984). Electrodynamics of continuous media, Translated from the second Russian edition by J. B. Sykes, J. S. Bell and M. J. Kearsley, Second Russian edition revised by Lifshits and L. P. Pitaevskiĭ
13. Li, T., Qin, T.: Physics and Partial Differential Equations, vol. 1. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Higher Education Press, Beijing (2012). Translated from the Chinese original by Yachun Li
14. Liu, T.-P., Zeng, Y.: Large time behavior of solutions for general quasilinear hyperbolic–parabolic systems of conservation laws. Mem. Am. Math. Soc. **125**, viii+120 (1997)
15. Liu, T.-P., Zeng, Y.: Shock waves in conservation laws with physical viscosity. Mem. Am. Math. Soc. **234**, viii+168 (2015)
16. Qin, X., Yang, T., Yao, Z., Zhou, W.: A study on the boundary layer for the planar magnetohydrodynamics system. Acta Math. Sci. Ser. B Engl. Ed. **35**, 787–806 (2015)
17. Qin, X., Yang, T., Yao, Z., Zhou, W.: Vanishing shear viscosity and boundary layer for the Navier–Stokes equations with cylindrical symmetry. Arch. Ration. Mech. Anal. **216**, 1049–1086 (2015)
18. Shelukhin, V.: The limit of zero shear viscosity for compressible fluids. Arch. Ration. Mech. Anal. **143**, 357–374 (1998)
19. Shelukhin, V.: Vanishing shear viscosity in a free-boundary problem for the equations of compressible fluids. J. Differ. Equ. **167**, 73–86 (2000)
20. Wang, D.: Large solutions to the initial-boundary value problem for planar magnetohydrodynamics. SIAM J. Appl. Math. **63**, 1424–1441 (2003); (electronic)
21. Yao, L., Zhang, T., Zhu, C.: Boundary layers for compressible Navier–Stokes equations with density-dependent viscosity and cylindrical symmetry. Ann. Inst. H. Poincaré Anal. Non Linéaire **28**, 677–709 (2011)
22. Zel'dovich, Y.B., Raizer, Y.P.: Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, vol. II. Academic Press, New York (1967)

Tao Wang
School of Mathematics and Statistics
Wuhan University
Wuhan 430072
People's Republic of China
e-mail: tao.wang@whu.edu.cn

(Received: January 5, 2015; revised: August 20, 2015)