EXISTENCE AND STABILITY OF NONISENTROPIC COMPRESSIBLE VORTEX SHEETS

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Abstract. We consider the short-time existence and nonlinear stability of vortex sheets for the nonisentropic compressible Euler equations in two spatial dimensions, based on the weakly linear stability result of Morando–Trebeschi [16]. The content of this paper summarizes the results collected in Morando–Trebeschi–Wang [18].

1. **Introduction.** We study compressible Euler equations in $\mathbb{R}^2$:

$$\begin{cases} (\partial_t + \boldsymbol{u}\cdot\nabla)p + \boldsymbol{\gamma} p\nabla\cdot\boldsymbol{u} = 0, \\ \rho(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = 0, \\ (\partial_t + \boldsymbol{u}\cdot\nabla)s = 0, \end{cases} \tag{1}$$

where pressure $p = p(t,x) \in \mathbb{R}$, velocity $\boldsymbol{u} = (v(t,x), u(t,x))^{\mathsf{T}} \in \mathbb{R}^2$, and entropy $s = s(t,x) \in \mathbb{R}$ are unknown functions of time t and position $x = (x_1, x_2)^{\mathsf{T}} \in \mathbb{R}^2$. We consider a polytropic gas, where the density ρ obeys the constitutive law $\rho = \rho(p,s) := Ap^{\frac{1}{\boldsymbol{\gamma}}}\mathrm{e}^{-\frac{s}{\boldsymbol{\gamma}}}$, with given $A > 0$ and $\boldsymbol{\gamma} > 1$ the adiabatic exponent of the gas.

According to Lax [12], a weak solution $(p, \boldsymbol{u}, s)$ of (1) that is smooth on either side of a smooth surface $\Gamma(t) := \{x_2 = \varphi(t, x_1)\}$ is said to be a *vortex sheet* (even called *contact discontinuity*) provided that it is a classical solution to (1) on each side of $\Gamma(t)$ and the following Rankine–Hugoniot conditions hold at each point of $\Gamma(t)$:

$$\partial_t\varphi = \boldsymbol{u}^+\cdot\nu = \boldsymbol{u}^-\cdot\nu, \qquad p^+ = p^-. \tag{2}$$

Here $\nu := (-\partial_{x_1}\varphi, 1)^{\mathsf{T}}$ is a spatial normal vector to $\Gamma(t)$ and $\boldsymbol{u}^\pm$, $p^\pm$, $s^\pm$ denote the restrictions of $\boldsymbol{u}$, p, s to both sides $\{\pm(x_2 - \varphi(t,x_1)) > 0\}$ of $\Gamma(t)$, respectively. These conditions yield that the normal velocity and pressure are continuous across

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$\Gamma(t)$. Hence the possible jumps displayed by a vortex sheet concern the tangential velocity and entropy. Remark also that the first two identities in (2) are the eikonal equations $\partial_t\varphi+\lambda_2(p^\pm, \mathbf{u}^\pm, s^\pm, \partial_{x_1}\varphi)=0$, where $\lambda_2(p,\boldsymbol{u},s,\xi):=\boldsymbol{u}\cdot(\xi,-1)^\mathsf{T}$ denotes the second characteristic field of system (1).

We are interested in the *structural stability* of vortex sheets to nonisentropic compressible Euler equations (1) with the initial data being a perturbation of *planar vortex sheets*:

$$(\bar{p}, \pm\bar{v}, 0, \bar{s}^\pm)^\mathsf{T} \quad \text{in} \quad \pm x_2 > 0, \tag{3}$$

where $\bar{p}>0$, $\bar{v}>0$, $\bar{s}^\pm$ are constants.

The interface $\Gamma(t)$ (namely, function φ) is a part of unknowns of nonlinear problem (1)–(2). The usual approach consists of straightening unknown interface $\Gamma(t)$ by a suitable change of coordinates in $\mathbb{R}^3$, in order to reformulate the free boundary problem in a fixed domain. Precisely, unknowns $(p,\boldsymbol{u},s)$ are replaced by functions

$$(p^\pm_\sharp, \boldsymbol{u}^\pm_\sharp, s^\pm_\sharp)(t,x_1,x_2) := (p^\pm, \boldsymbol{u}^\pm, s^\pm)(t,x_1,\Phi^\pm(t,x_1,x_2)),$$

where $\Phi^\pm$ are smooth functions satisfying

$$\Phi^\pm(t,x_1,0)=\varphi(t,x_1) \quad \text{and} \quad \pm\,\partial_{x_2}\Phi^\pm(t,x_1,x_2)\geq \kappa>0 \quad \text{if } x_2\geq 0. \tag{4}$$

Hereafter we drop the $\sharp$ index and set $U:=(p,v,u,s)^\mathsf{T}$ for convenience. Then the construction of vortex sheets for system (1) amounts to proving the existence of smooth solutions $(U^\pm,\Phi^\pm)$ to the following initial-boundary value problem:

$$\mathbb{L}(U^\pm,\Phi^\pm) := L(U^\pm,\Phi^\pm)U^\pm = 0 \qquad \text{if } x_2>0, \tag{5a}$$

$$\mathbb{B}(U^+,U^-,\varphi)=0 \qquad \text{if } x_2=0, \tag{5b}$$

$$(U^\pm,\varphi)|_{t=0}=(U^\pm_0,\varphi_0), \tag{5c}$$

where the differential operator $L(U,\Phi)$ takes the form:

$$L(U,\Phi) := I_4\partial_t + A_1(U)\partial_{x_1} + \widetilde{A}_2(U,\Phi)\partial_{x_2}, \tag{6}$$

symbol I_4 is the 4×4 identity matrix,

$$\widetilde{A}_2(U,\Phi) := \frac{1}{\partial_{x_2}\Phi}\left(A_2(U)-\partial_t\Phi I_4-\partial_{x_1}\Phi A_1(U)\right),$$

$$A_1(U) := \begin{pmatrix} v & \gamma p & 0 & 0\\ 1/\rho & v & 0 & 0\\ 0 & 0 & v & 0\\ 0 & 0 & 0 & v\end{pmatrix}, \quad A_2(U) := \begin{pmatrix} u & 0 & \gamma p & 0\\ 0 & u & 0 & 0\\ 1/\rho & 0 & u & 0\\ 0 & 0 & 0 & u\end{pmatrix},$$

and $\mathbb{B}$ denotes the boundary operator

$$\mathbb{B}(U^+,U^-,\varphi) := \begin{pmatrix} (v^+-v^-)|_{x_2=0}\partial_{x_1}\varphi-(u^+-u^-)|_{x_2=0}\\ \partial_t\varphi+v^+|_{x_2=0}\partial_{x_1}\varphi-u^+|_{x_2=0}\\ (p^+-p^-)|_{x_2=0}\end{pmatrix}.$$

Since equations (4)–(5) are not enough to determine functions $\Phi^\pm$, we require, as in Francheteau–Métivier [9], that functions $\Phi^\pm$ satisfy the following eikonal equations:

$$\partial_t\Phi^\pm+\lambda_2(p^\pm,\boldsymbol{u}^\pm,s^\pm,\partial_{x_1}\Phi^\pm)=0 \qquad \text{if } x_2\geq 0. \tag{7}$$

This choice of $\Phi^\pm$ has the advantage to considerably simplify the expression of equations (5a). More importantly, the rank of the boundary matrix for problem (5) keeps constant on the whole domain $\{x_2\geq 0\}$, which allows the application of the Kreiss symmetrizer technique to problem (5) in the spirit of Majda–Osher [13].

In the new variables, piecewise constant state (3) corresponds to the trivial solution of (4)–(5b) and (7)

$$\bar{U}^{\pm} = (\bar{p}, \pm\bar{v}, 0, \bar{s}^{\pm})^{\mathsf{T}}, \quad \bar{\Phi}^{\pm}(t, x_1, x_2) = \pm x_2, \tag{8}$$

with $\bar{p} > 0$ and $\bar{v} > 0$. Let us denote by $\bar{c}_{\pm} = c(\bar{p}, \bar{s}^{\pm})$ the sound speeds corresponding to the constant states $\bar{U}^{\pm}$, where $c(p,s) := \sqrt{p_\rho(\rho, s)} = \sqrt{\dfrac{\gamma e^{s/\gamma}}{A p^{\frac{1}{\gamma}-1}}}$ for the polytropic gas.

We aim to show the short-time existence of solutions to nonlinear problem (4)–(5) and (7) provided the initial data is sufficiently close to (8). Our main result is stated as follows.

Theorem 1.1. *Let $T > 0$ and $\mu \in \mathbb{N}$ with $\mu \geq 13$. Assume that background state (8) satisfies the stability conditions:*

$$2\bar{v} > (\bar{c}_+^{\frac{2}{3}} + \bar{c}_-^{\frac{2}{3}})^{\frac{3}{2}}, \quad 2\bar{v} \neq \sqrt{2}(\bar{c}_+ + \bar{c}_-). \tag{9}$$

Assume further that the initial data $U_0^{\pm}$ and φ_0 satisfy suitable compatibility conditions up to order μ[1]*, and $(U_0^{\pm} - \bar{U}^{\pm}, \varphi_0) \in H^{\mu+1/2}(\mathbb{R}^2_+) \times H^{\mu+1}(\mathbb{R})$ has compact support. Then there exists $\delta > 0$ such that, if $\|U_0^{\pm} - \bar{U}^{\pm}\|_{H^{\mu+1/2}(\mathbb{R}^2_+)} + \|\varphi_0\|_{H^{\mu+1}(\mathbb{R})} \leq \delta$, then there exists a solution $(U^{\pm}, \Phi^{\pm}, \varphi)$ of (4)–(5) and (7) on the time interval $[0, T]$ satisfying*

$$(U^{\pm} - \bar{U}^{\pm}, \Phi^{\pm} - \bar{\Phi}^{\pm}) \in H^{\mu-7}((0,T) \times \mathbb{R}^2_+), \qquad \varphi \in H^{\mu-6}((0,T) \times \mathbb{R}).$$

Compressible vortex sheets, along with shocks and rarefaction waves, are fundamental waves that play an important role in the study of general entropy solutions to multidimensional hyperbolic systems of conservation laws. It was observed long time ago in [14] (*cf.* Coulombel–Morando [5] for using only algebraic tools) that for two-dimensional nonisentropic Euler equations (1), piecewise constant vortex sheets (8) are violently unstable unless the following stability criterion is satisfied:

$$2\bar{v} \geq (\bar{c}_+^{\frac{2}{3}} + \bar{c}_-^{\frac{2}{3}})^{\frac{3}{2}}, \tag{10}$$

while they are linearly stable under this condition. In the seminal work of Coulombel and Secchi [7], building on their linear stability results in [6], the short-time existence and nonlinear stability of compressible vortex sheets are established for the two-dimensional *isentropic* case under condition (10) (as a strict inequality) by performing a modified Nash–Moser iteration scheme. These results were recently generalized by Chen–Secchi–Wang [3] to cover the relativistic case. Let us also quote the recent works by Huang–Wang–Yuan [11] and Ruan–Trakhinin [20] for similar results in the case of two-phase compressible flows.

As for three-dimensional gas dynamics, vortex sheets have been showed in Fejer–Miles [8] to be always violently unstable, which is analogous to the Kelvin–Helmholtz instability for incompressible fluids. In contrast, Chen–Wang [2] and Trakhinin [22] proved independently the nonlinear stability of compressible *current-vortex sheets* for three-dimensional compressible magnetohydrodynamics (MHD). This result indicates that non-paralleled magnetic fields stabilize the motion of three-dimensional compressible vortex sheets.

[1]For the precise definition of compatibility conditions of the initial data, see [18, Definition 4.1].

Extending the results in [6], the first two authors obtained in [16] the L^2–estimates for the linearized problems of (4)–(5) and (7) around background state (8) under condition (10) (as a strict inequality), and that around a small perturbation of (8) under (9). In the present paper we summarize the result obtained in [18] about structural nonlinear stability of two-dimensional nonisentropic vortex sheets, obtained by adopting the Nash–Moser iteration scheme developed in [10, 7] and already successfully applied to the plasma-vacuum interface problem [21], three-dimensional compressible steady flows [23] and MHD contact discontinuities [15].

It is worth noting that in the statement of Theorem 1.1, the inequality $2\bar{v} \neq \sqrt{2}(\bar{c}_+ + \bar{c}_-)$ is required in addition to stability condition (10) (with strict inequality). This is due to the fact that the linearized problem about piecewise constant basic state (8), with $\bar{v}$ taking the critical value above, satisfies an a priori estimate with additional loss of regularity from the data, which is related to the presence of a double root of the associated *Lopatinskiĭ determinant* (see [16, Theorem 3.1]). At the subsequent level of variable coefficient linearized problem about a perturbation of (8), the authors in [16] were not able to handle this further loss of regularity, thus the case of $\bar{v} = (\bar{c}_+ + \bar{c}_-)/\sqrt{2}$ is still open. Notice also that in the isentropic case (where $\bar{c}_+ = \bar{c}_- = \bar{c}$), value $(\bar{c}_+^{\frac{2}{3}} + \bar{c}_-^{\frac{2}{3}})^{\frac{3}{2}}$ coincides with $\sqrt{2}(\bar{c}_+ + \bar{c}_-)$ and condition (9) reduces to the supersonic condition $\bar{v} > \sqrt{2}\bar{c}$ studied in Coulombel–Secchi [7].

The plan of this paper is as follows. In Section 2, we introduce the effective linear problem and state the result of well-posedness, in usual Sobolev space H^s with s large enough, obtained for it. Section 3 is devoted to a short discussion of the modified Nash–Moser iteration scheme used to prove Theorem 1.1, based on the *a priori* tame estimates satisfied by the solution to the linearized problem.

2. Well-Posedness of the Effective Linear Problem.

A fundamental step to get the solvability of the nonlinear problem (4)–(5) and (7) is the study of the well-posedness of the corresponding linearized problem. We linearize (4)–(5) and (7) around a basic state $(U_{r,l}, \Phi_{r,l}) := (p_{r,l}, v_{r,l}, u_{r,l}, s_{r,l}, \Phi_{r,l})^\mathsf{T}$ given by a perturbation of the stationary solution (8). The index r (resp. l) denotes the state on the right (resp. on the left) of the interface (after change of variables). More precisely, the perturbation

$$(\dot{U}_{r,l}(t,x_1,x_2), \dot{\Phi}_{r,l}(t,x_1,x_2)) := (U_{r,l}(t,x_1,x_2), \Phi_{r,l}(t,x_1,x_2)) - (\overline{U}^{\pm}, \overline{\Phi}^{\pm})$$

is assumed to satisfy

$$\operatorname{supp}\big(\dot{U}_{r,l}, \dot{\Phi}_{r,l}\big) \subset \{-T \le t \le 2T,\ x_2 \ge 0,\ |x| \le R\}, \tag{11}$$

$$\dot{U}_{r,l} \in W^{2,\infty}(\Omega), \quad \dot{\Phi}_{r,l} \in W^{3,\infty}(\Omega), \quad \big\|\dot{U}_{r,l}\big\|_{W^{2,\infty}(\Omega)} + \big\|\dot{\Phi}_{r,l}\big\|_{W^{3,\infty}(\Omega)} \le K, \tag{12}$$

where T, R, and K are positive constants and Ω denotes the half-space $\{(t,x_1,x_2) \in \mathbb{R}^3 : x_2 > 0\}$. Moreover, we assume that $(\dot{U}_{r,l}, \dot{\Phi}_{r,l})$ satisfies constraints (4), (7), and Rankine–Hugoniot conditions (5b), that is,

$$\partial_t \Phi_{r,l} + v_{r,l}\partial_{x_1}\Phi_{r,l} - u_{r,l} = 0 \qquad \text{if } x_2 \ge 0, \tag{13a}$$

$$\pm\, \partial_{x_2}\Phi_{r,l} \ge \kappa_0 > 0 \qquad \text{if } x_2 \ge 0, \tag{13b}$$

$$\Phi_r = \Phi_l = \varphi \qquad \text{if } x_2 = 0, \tag{13c}$$

$$\mathbb{B}\big(U_r, U_l, \varphi\big) = 0 \qquad \text{if } x_2 = 0, \tag{13d}$$

for a suitable positive constant κ_0.

Let us consider solutions to (4)–(5) and (7) of the form $(U_{r,l}+\varepsilon V^{\pm}, \Phi_{r,l}+\varepsilon\Psi^{\pm})$, where $(V^{\pm}, \Psi^{\pm})$ represent some "small perturbations" of the basic state $(U_{r,l}, \Phi_{r,l})$. Up to second order errors and after the passage to the "good unknowns" of Alinhac (*cf.* [1])

$$\dot{V}^{+} := V^{+} - \frac{\Psi^{+}}{\partial_{x_2}\Phi_r}\partial_{x_2}U_r, \quad \dot{V}^{-} := V^{-} - \frac{\Psi^{-}}{\partial_{x_2}\Phi_l}\partial_{x_2}U_l \tag{14}$$

(made in order to get rid of first order terms in $\Psi^{\pm}$ originating from linearization), the *effective linearized problem* of (4)–(5) and (7) around the ground state $(U_{r,l}, \Phi_{r,l})$ reads as

$$\mathbb{L}'_e(U_{r,l}, \Phi_{r,l})\dot{V}^{\pm} := L(U_{r,l}, \Phi_{r,l})\dot{V}^{\pm} + \mathcal{C}(U_{r,l}, \Phi_{r,l})\dot{V}^{\pm} = f^{\pm} \quad \text{if } x_2 > 0, \tag{15a}$$

$$\mathbb{B}'_e(U_{r,l}, \Phi_{r,l})(\dot{V}, \psi) := \underline{b}\nabla_{t,x_1}\psi + \boldsymbol{b}_\sharp\psi + \underline{M}\dot{V}|_{x_2=0} = g \quad \text{if } x_2 = 0, \tag{15b}$$

$$\Psi^{+} = \Psi^{-} = \psi \quad \text{if } x_2 = 0\,. \tag{15c}$$

In view of the results obtained in [1, 9, 7], zero-th order terms in $\Psi^{\pm}$ are neglected in (15a) and considered as error terms at each Nash–Moser iteration step in the nonlinear analysis. Here we have set $\dot{V} := (\dot{V}^{+}, \dot{V}^{-})^{\mathsf{T}}$, $\nabla_{t,x_1}\psi = (\partial_t\psi, \partial_{x_1}\psi)^{\mathsf{T}}$. Moreover, differential operators $L(U_{r,l}, \Phi_{r,l})$ are defined in (6), while $\mathcal{C}(U_{r,l}, \Phi_{r,l})$ are suitable lower order operators, whose explicit form can be easily computed but is useless for the sequel of our discussion. Coefficients $\underline{b}$, $\boldsymbol{b}_\sharp$, and $\underline{M}$ are defined by

$$\underline{b}(t,x_1) := \begin{pmatrix} 0 & (v_r - v_l)|_{x_2=0} \\ 1 & v_r|_{x_2=0} \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{b}_\sharp(t,x_1) := \underline{M}(t,x_1)\begin{pmatrix} \dfrac{\partial_{x_2}U_r}{\partial_{x_2}\Phi_r} \\ \dfrac{\partial_{x_2}U_l}{\partial_{x_2}\Phi_l} \end{pmatrix}\Bigg|_{x_2=0},$$

$$\underline{M}(t,x_1) := \begin{pmatrix} 0 & \partial_{x_1}\varphi & -1 & 0 & 0 & -\partial_{x_1}\varphi & 1 & 0 \\ 0 & \partial_{x_1}\varphi & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

From (12), $\underline{b}, \underline{M} \in W^{2,\infty}(\mathbb{R}^2)$, $\boldsymbol{b}_\sharp \in W^{1,\infty}(\mathbb{R}^2)$, $\mathcal{C}(U_{r,l}, \Phi_{r,l}) \in W^{1,\infty}(\Omega)$, and the coefficients of the operators $L(U_{r,l}, \Phi_{r,l})$ are in $W^{2,\infty}(\Omega)$.

We observe that linearized boundary conditions (15b) depend on the traces of $\dot{V}^{\pm}$ only through the *noncharacteristic components* $\mathbb{P}(\varphi)\dot{V}^{\pm} := (\dot{V}_1^{\pm}, \dot{V}_3^{\pm} - \partial_{x_1}\varphi\,\dot{V}_2^{\pm})^{\mathsf{T}}$ of $\dot{V}^{\pm}$, as it is expected, because the boundary $\{x_2 = 0\}$ is characteristic for problem (15) in view of (13a).

On the effective linear problem (15), we are able to show the following well-posedness result in the usual Sobolev space H^s with order s large enough (see [18]).

Theorem 2.1. *Let $T > 0$ and $s \in [3, \tilde{\alpha}] \cap \mathbb{N}$ with any integer $\tilde{\alpha} \geq 3$. Assume that the stationary solution* (8) *satisfies* (9)*, and that perturbations $(\dot{U}_{r,l}, \dot{\Phi}_{r,l})$ belong to $H_\gamma^{s+3}(\Omega_T)$ for all $\gamma \geq 1$ and satisfy* (11)*–*(13)*, and*

$$\|(\dot{U}_{r,l}, \nabla\dot{\Phi}_{r,l})\|_{H^5_\gamma(\Omega_T)} + \|(\dot{U}_{r,l}, \partial_{x_2}\dot{U}_{r,l}, \nabla\dot{\Phi}_{r,l})|_{x_2=0}\|_{H^4_\gamma(\omega_T)} \leq K.$$

Assume further that $(f^{\pm}, g) \in H^{s+1}(\Omega_T) \times H^{s+1}(\omega_T)$ vanish in the past. Then there exists a positive constant K_0, which is independent of s and T, and there exist two constants $C > 0$ and $\gamma \geq 1$, which depend solely on K_0, such that, if $K \leq K_0$, then problem (15) *admits a unique solution $(\dot{V}^{\pm}, \psi) \in H^s(\Omega_T) \times H^{s+1}(\omega_T)$ that vanishes* *in the past and obeys the following tame estimate:*

$$\begin{aligned}
&\|\dot V\|_{H^s_\gamma(\Omega_T)} + \|\mathbb{P}(\varphi)\dot V|_{x_2=0}\|_{H^s_\gamma(\omega_T)} + \|\psi\|_{H^{s+1}_\gamma(\omega_T)} \\
&\quad \le C\big\{\|f\|_{H^{s+1}_\gamma(\Omega_T)} + \|g\|_{H^{s+1}_\gamma(\omega_T)} \\
&\qquad\qquad + \big(\|f\|_{H^4_\gamma(\Omega_T)} + \|g\|_{H^4_\gamma(\omega_T)}\big)\|(\dot U_{r,l},\dot\Phi_{r,l})\|_{H^{s+3}_\gamma(\Omega_T)}\big\}\,,
\end{aligned} \tag{16}$$

where $\dot V := (\dot V^+, \dot V^-)$, $\mathbb{P}(\varphi)\dot V := (\mathbb{P}(\varphi)\dot V^+, \mathbb{P}(\varphi)\dot V^-)$, $f := (f^+, f^-)$.

In the above statement, we have set $\Omega_T := (-\infty, T)\times\mathbb{R}^2_+$, $\omega_T := (-\infty, T)\times\mathbb{R} \simeq \partial\Omega_T$ for any real number T. Moreover, the functional spaces (and related norms) involved above are an "exponentially weighted" version of usual Sobolev spaces on Ω_T and ω_T, defined for all $k\in\mathbb{N}$ and $\gamma\ge 1$ as

$$H^k_\gamma(\Omega_T) := \left\{u\in\mathcal{D}'(\Omega_T)\,:\, \mathrm{e}^{-\gamma t}u\in H^k(\Omega_T)\right\},$$

provided with the natural norm $\|u\|_{H^k_\gamma(\Omega_T)} := \|e^{-\gamma t}u\|_{H^k(\Omega_T)}$ (and similarly for $H^k_\gamma(\omega_T)$). Since the most of functions we are dealing with have double $\pm$ states, in (16) we have used the shortcut notation $\|\dot V\|_{H^s_\gamma(\Omega_T)} := \sum_\pm \|\dot V^\pm\|_{H^s_\gamma(\Omega_T)}$ and similarly for the other terms in the estimate.

Let us shortly discuss the main steps of the proof of Theorem 2.1. The first two authors proved in [16, Theorem 4.1], by spectral analysis based on Kreiss symmetrizer techniques and paradifferential calculus, that problem (15) satisfies a basic L^2–*a priori* estimate with a loss of one tangential derivative. Then, in [18] we defined a dual problem for (15), to which we were able to associate the same kind of L^2–*a priori* estimate with a loss of one tangential derivative. Since system (15a) is symmetrizable hyperbolic and in view of the regularity of coefficients coming from (12), the well-posedness result in L^2 of [4] can be applied to the effective linear problem (15), giving the existence of a unique L^2–solution of (15). In order to get well-posedness in higher order Sobolev spaces, as it is required by Theorem 2.1, the essential point is deriving the *a priori tame* estimate (16) for all sufficiently smooth solutions to (15). We first obtain the estimate for tangential derivatives. Since the boundary matrix for our problem (15) is singular, there is no hope to estimate all the normal derivatives of $\dot V$ directly from equations (15a) by applying the standard approach for noncharacteristic boundary problems as in [19, 17]. However, for our problem (15), we can obtain the estimate of missing normal derivatives through the equations of the "linearized vorticity" and entropy, where the linearized vorticity has been introduced in [7]. Then, we estimate such normal derivatives by expressing them in terms of tangential derivatives and the linearized vorticity.

Let us notice, in the end, that, according to the loss of regularity from the data in the basic L^2–*a priori* estimate found in [16], the tame estimate (16) displays a loss of one derivative from data to the found solution. Moreover there is also a fixed loss of three derivatives from the coefficients of the system, namely the basic state $(\dot U_{r,l}, \dot\Phi_{r,l})$.

3. The Nolinear Problem: Nash–Moser Iteration Scheme.

In this section we turn to the resolution of the original nonlinear problem (4)–(5) and (7). Let us only sketch the idea of the proof of the main Theorem 1.1, referring to [18] for the details.

In order to reduce the original problem (4)–(5) and (7) into a nonlinear one with zero initial data, it is first convenient to seek the solution of (4)–(5) and (7) into the form

$$(U^{a\,\pm}, \Phi^{a\,\pm}, \varphi^a) + (V^\pm, \Psi^\pm, \psi)\,,$$

where $(U^{a\,\pm}, \Phi^{a\,\pm}, \varphi^a)$ (with $\Phi^{a\,\pm}|_{x_2=0} = \varphi^a$) is the so-called *approximate solution*, that is a solution of above problem in the sense of Taylor's series at time $t = 0$. Suitable necessary compatibility conditions of sufficiently large order have to be prescribed on the initial data $(U_0^\pm, \varphi_0)$ for the existence of such a sufficiently smooth approximate solution, see [18, Section 4].

Because of the loss of regularity from data and coefficients to the solution of the linearized problem, occurring in Theorem 2.1, we cannot hope to solve the nonlinear problem by resorting to an iteration scheme based on classical contraction principle. Instead, the Nash–Moser scheme turns out to be adapted to our situation, because it allows to handle the above loss of regularity.

As already announced in the end of Section 1, the solution $(V^\pm, \Psi^\pm, \psi)$ of the nonlinear problem with zero initial data is found as the limit of a sequence of solutions $(V_k^\pm, \Psi_k^\pm, \psi_k)$ coming from the resolution of "approximating" linearized problems, constructed by performing an iteration scheme based on a Nash–Moser type argument. At the $(k+1)-$th iteration of the scheme, the updated approximation $(V_{k+1}^\pm, \Psi_{k+1}^\pm, \psi_{k+1})$ is constructed from the approximation at previous step k as

$$V_{k+1}^\pm = V_k^\pm + \delta V_k^\pm, \quad \Psi_{k+1}^\pm = \Psi_k^\pm + \delta\Psi_k^\pm, \quad \psi_{k+1} = \psi_k + \delta\psi_k,$$

where the differences δV_k, $\delta\Psi_k$, and $\delta\psi_k$ are obtained from the resolution of the effective linear problem of kind (15)

$$\begin{cases} \mathbb{L}_e'(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2})\delta\dot{V}_k = f_k & \text{in } \Omega_T, \\ \mathbb{B}_e'(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2})(\delta\dot{V}_k, \delta\psi_k) = g_k & \text{on } \omega_T, \\ (\delta\dot{V}_k, \delta\psi_k) = 0 & \text{for } t < 0, \end{cases} \tag{17}$$

where, for simplicity, we have removed the $\pm$ superscripts,

$$\delta\dot{V}_k := \delta V_k - \frac{\partial_{x_2}(U^a + V_{k+1/2})}{\partial_{x_2}(\Phi^a + \Psi_{k+1/2})}\delta\Psi_k$$

is the "good unknown" (*cf.* (14)), and $(V_{k+1/2}, \Psi_{k+1/2})$ is a suitable "modification" of the approximating state at $k-$th step $(V_k^\pm, \Psi_k^\pm)$, costructed in such a way to compensate the loss of regularity from the coefficients and the data to the solution of the linearized problem and such that the basic state $(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2})$ involved in (17) satisfies all the assumptions needed in order to solve the linearized problem according to Theorem 2.1, that is constraints (11)–(13). The source terms (f_k, g_k) are defined through the accumulated errors at step k. In order to get convergence of the Nash–Moser scheme, so as to obtain $(V^\pm, \Psi^\pm, \psi)$ passing to the limit in the sequence $(V_k^\pm, \Psi_k^\pm, \psi_k)$, such accumulated errors have to converge to zero in the right functional space, which is proved in [18, Section 5.3].

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