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The Jin–Xin relaxation approximation of scalar conservation laws in several dimensions with large initial perturbation

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ABSTRACT

This paper is concerned with nonlinear stability of strong planar rarefaction waves for the Jin–Xin relaxation approximation of scalar conservation laws in several dimensions. For such a problem, local stability of weak or strong planar rarefaction waves have been obtained in Luo (1997) [20] and Zhao (2000) [43] respectively. For the global stability results, to the best of our knowledge, the only result available now is on the one-dimensional case, cf. Zhao (2000) [43], which is based on the maximum principle established in Natalini (1996) [30]. The main purpose of this paper is try to deduce some nonlinear stability results with large initial perturbation. Our analysis is based on the elementary energy method and the continuation argument.

Contents

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1. Introduction

Systems of hyperbolic conservation laws in several space dimensions take the form

$$\mathbf{u}_t+\sum_{j=1}^{N}\mathbf{F}_j(\mathbf{u})_{x_j}=0,\quad \mathbf{u}\in\mathbf{R}^n,\ (t,\mathbf{x})=(t,x_1,\dots,x_N)\in\mathbf{R}^+\times\mathbf{R}^N. \tag{1.1}$$

To approximate this system from numerical point of view, S. Jin and Z.-P. Xin [14] proposed the following relaxation system

$$\begin{cases}\mathbf{u}_t+\sum_{j=1}^{N}\mathbf{v}_{jx_j}=0,\quad \mathbf{u}\in\mathbf{R}^n,\ \mathbf{v}_j\in\mathbf{R}^n,\\ \mathbf{v}_{jt}+\mathbf{A}_j\mathbf{u}_{x_j}=-\dfrac{\mathbf{v}_j-\mathbf{F}_j(\mathbf{u})}{\epsilon},\quad j=1,2,\dots,N,\end{cases} \tag{1.2}$$

where $\mathbf{A}_j=a_j\mathbf{I}$ with $\mathbf{I}$ being the $n\times n$ identity matrix, $a_j>0$ $(j=1,2,\dots,N)$ and ϵ are some positive constants. A numerical scheme to solve (1.2) is also designed in [14] which yields satisfactory numerical solutions to hyperbolic conservation laws (1.1). The main features of this scheme are its generality and simplicity.

For scalar conservation laws in two dimensions

$$u_t+f(u)_x+g(u)_y=0,\quad u\in\mathbf{R}, \tag{1.3}$$

its Jin–Xin relaxation approximation becomes

$$\begin{cases}u_t+v_{1x}+v_{2y}=0,\\ v_{1t}+a_1u_x=-\dfrac{v_1-f(u)}{\epsilon},\quad u,v_1,v_2\in\mathbf{R},\\ v_{2t}+a_2u_y=-\dfrac{v_2-g(u)}{\epsilon}.\end{cases} \tag{1.4}$$

Here a_1 and a_2 are two positive constants satisfying the following sub-characteristic condition

$$\sup_{u\in\mathcal{M}}\left\{\frac{|f'(u)|^2}{a_1}+\frac{|g'(u)|^2}{a_2}\right\}<1, \tag{1.5}$$

where $\mathcal{M}\subset\mathbf{R}$ is the state space whose precise definition will be specified later.

This paper is concerned with time-asymptotic behavior of global solutions to the Cauchy problem (1.4) with prescribed initial data

$$\begin{cases}u(0,x,y)=u_0(x,y),\\ v_1(0,x,y)=v_{10}(x,y),\\ v_2(0,x,y)=v_{20}(x,y),\end{cases} \tag{1.6}$$

which satisfy

$$\begin{cases}\lim\limits_{x\to\pm\infty}\|u_0(x,y)-u^\pm\|_{L^\infty(\mathbf{R}_y)}=0,\\ \lim\limits_{x\to\pm\infty}\|v_{10}(x,y)-f(u^\pm)\|_{L^\infty(\mathbf{R}_y)}=0,\\ \lim\limits_{x\to\pm\infty}\|v_{20}(x,y)-g(u^\pm)\|_{L^\infty(\mathbf{R}_y)}=0.\end{cases} \tag{1.7}$$

Here u^- and u^+ are two constants satisfying $u^- < u^+$. Since our main concern is on the asymptotics of the global solutions $(u(t,x,y), v_1(t,x,y), v_2(t,x,y))$ to the Cauchy problem (1.4), (1.6), we can assume without loss of generality that $\epsilon = 1$ in the rest of this manuscript.

The flux functions $f(u)$ and $g(u)$ are assumed to be sufficiently smooth and the scalar conservation laws (1.3) is assumed to be genuinely nonlinear in the x-direction, i.e.

$$f''(u) > 0, \quad \forall u \in \mathcal{M}. \tag{1.8}$$

For such a flux function $f(u)$ and the two constant states u^- and u^+ given above, a planar rarefaction wave is the unique global entropy solution $r(t,x)$ of the following Riemann problem

$$\begin{cases} r_t + f(r)_x = 0, \\ r(0,x) = r_0^R(x) = \begin{cases} u^-, & x < 0, \\ u^+, & x > 0. \end{cases} \end{cases} \tag{1.9}$$

It is well known that $r(t,x)$ can be given explicitly by

$$r(t,x) = \begin{cases} u^-, & \frac{x}{t} < f'(u^-), \\ (f')^{-1}(\frac{x}{t}), & f'(u^-) \leqslant \frac{x}{t} \leqslant f'(u^+), \\ u^+, & \frac{x}{t} > f'(u^+). \end{cases} \tag{1.10}$$

Our main purpose in this paper is to discuss the global solvability of Cauchy problem (1.4), (1.6) and to use the functions $(r(t,x), f(r(t,x)), g(r(t,x)))$ to describe the large time behavior of such a global solution $(u(t,x,y), v_1(t,x,y), v_2(t,x,y))$. That is, we will discuss the nonlinear stability of the planar rarefaction wave $r(t,x)$ for the Jin–Xin relaxation approximation of scalar conservation laws in two dimensions.

Such a problem was originally considered by T.-P. Liu in [18] in the one-dimensional case, yet for the general 2×2 hyperbolic systems of conservation laws with relaxation and later by T. Luo in [20] and H.-J. Zhao [43] for the Cauchy problem (1.4), (1.6). Before stating these results precisely, we first outline the strategy to deal with this problem. Firstly, since the planar rarefaction wave $r(t,x)$ has singularity at $t = 0$, as in [25], we need to construct its smooth approximation $\phi(t,x)$ which is the unique global smooth solution of the Cauchy problem of the following generalized Burgers equation

$$\begin{cases} \phi_t + f(\phi)_x = 0, \\ \phi(0,x) = \phi_0(x). \end{cases} \tag{1.11}$$

Here $\phi_0(x)$ is a smooth, monotonic increasing function satisfying

$$\lim_{x \to \pm\infty} \phi_0(x) = u^{\pm}. \tag{1.12}$$

With the above smooth approximation in hand, the stability analysis is then divided into two steps: The first step is to deduce the corresponding one-dimensional stability result. That is, one needs to show that the unique global solution $(u(t,x), v_1(t,x))$ to the following related one-dimensional Cauchy problem

$$\begin{cases} u_t + v_{1x} = 0, \\ v_{1t} + a_1 u_x = -v_1 + f(u), \\ \big(u(0,x), v_1(0,x)\big) = \big(u_0(x), v_{10}(x)\big) \end{cases} \tag{1.13}$$

satisfies

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}}\bigl|\bigl(u(t,x)-\phi(t,x), v_1(t,x)-f\bigl(\phi(t,x)\bigr)\bigr)\bigr|=0 \tag{1.14}$$

if $(u_0(x), v_{10}(x))$ is a suitable perturbation of $(u_0(x), f(\phi_0(x)))$.

Now let $(\bar{u}(t,x), \bar{v}_1(t,x))$ denote the unique global solution to the Cauchy problem (1.13) with the specially chosen initial data $(u_0(x), v_{10}(x)) = (\phi_0(x), f(\phi_0(x)))$, then the second step is to show that one can use $(\bar{u}(t,x), \bar{v}_1(t,x), g(\bar{u}(t,x)))$ to describe the large time behavior of the global solution $(u(t,x,y), v_1(t,x,y), v_2(t,x,y))$ of the Cauchy problem (1.4), (1.6). More precisely, let

$$\begin{cases} U(t,x,y) = u(t,x,y) - \bar{u}(t,x), \\ V_1(t,x,y) = v_1(t,x,y) - \bar{v}_1(t,x), \\ V_2(t,x,y) = v_2(t,x,y) - g\bigl(\bar{u}(t,x)\bigr), \end{cases} \tag{1.15}$$

it is easy to check that $(U(t,x,y), V_1(t,x,y), V_2(t,x,y))$ solves

$$\begin{cases} U_t + V_{1x} + V_{2y} = 0, \\ V_{1t} + a_1 U_x = f(\bar{u}+U) - f(\bar{u}) - V_1, \\ V_{2t} + a_2 U_y = g(\bar{u}+U) - g(\bar{u}) - V_2 - g(\bar{u})_t \end{cases} \tag{1.16}$$

with initial data

$$\begin{cases} U(0,x,y) = U_0(x,y) = u_0(x,y) - \phi_0(x), \\ V_1(0,x,y) = V_{10}(x,y) = v_{10}(x,y) - f\bigl(\phi_0(x)\bigr), \\ V_2(0,x,y) = V_{20}(x,y) = v_{20}(x,y) - g\bigl(\phi_0(x)\bigr). \end{cases} \tag{1.17}$$

Then the problem is reduced to show that the Cauchy problem (1.16), (1.17) admits a unique global solution $(U(t,x,y), V_1(t,x,y), V_2(t,x,y))$ which tends to zero uniformly with respect to $(x,y)\in\mathbf{R}^2$ as $t\to\infty$. As is well known, the key point to deduce the above result is how to get the uniform energy type estimate on $(U(t,x,y), V_1(t,x,y), V_2(t,x,y))$ and the main difficulty lies in how to control the possible growth of the solution $(U(t,x,y), V_1(t,x,y), V_2(t,x,y))$ caused by the nonlinearity of Eq. (1.16).

Recall that according to whether the $H^2(\mathbf{R}^2)$-norm of the initial perturbation $(U_0(x,y), V_{10}(x,y), V_{20}(x,y))$ is small or not and $\delta = |u^+ - u^-|$, the strength of the planar rarefaction wave $r(t,x)$, is small or not, the corresponding nonlinear stability results are classified into local (or global) stability of weak (or strong) rarefaction waves respectively. Here we must emphasis that in the one-dimensional case, the terminology "global stability" can be defined independent of the way to construct the smooth approximation of the rarefaction wave $r(t,x)$, i.e. does not depend on the way to construct $\phi_0(x)$ satisfying (1.12), by considering whether the norm $\|(u_0(x)-u^-, v_{10}(x) - f(u^-))\|_{L^2(\mathbf{R}^-)} + \|(u_0(x)-u^+, v_{10}(x)-f(u^+))\|_{L^2(\mathbf{R}^+)} + \|(u_{0x}(x), v_{10x}(x))\|_{L^2(\mathbf{R})}$ is small or not. But for the two-dimensional case, since the initial data $(u_0(x,y), v_{10}(x,y), v_{20}(x,y))$ is a two-dimensional perturbation of the one-dimensional profile $(\phi_0(x), f(\phi_0(x)), g(\phi_0(x)))$, we do not know how to formulate a definition on the terminology "global stability" independent of the way to construct the smooth approximation of the planar rarefaction wave $r(t,x)$.

Following the strategy outlined above, local stability of the weak or strong planar rarefaction waves $r(t,x)$ defined in (1.10) have been studied in [20] and [43] respectively. For the corresponding result with large initial perturbation, to the best of our knowledge, no results are available now and the main purpose of our present manuscript is devoted to this problem.

Now we turn to state our main results. Before doing so, since, as pointed out before, the terminology "global stability" depends on the way to construct the smooth approximation of the planar rarefaction wave $r(t,x)$, we consider the general way to construct the smooth approximation $\phi(t,x)$ of the planar rarefaction wave $r(t,x)$, which is the unique global smooth solution $\phi(t,x)$ of the Cauchy problem (1.11) with general $\phi_0(x)$ given by

$$\phi_0(x)=\frac{u^++u^-}{2}+\frac{u^+-u^-}{2}m_0(\varepsilon x). \tag{1.18}$$

Here $m_0(x)$ is assumed to satisfy the following properties:

(H_1) $m_0(x)\in C^\infty(\mathbf{R})$, $m_0'(x)\geqslant 0$, and for each $p\in[1,\infty]$, $k\in\mathbf{Z}^+$, $\|\frac{\partial^k m_0(x)}{\partial x^k}\|_{L^p(\mathbf{R})}\leqslant C(p,k)<+\infty$;
(H_2) $m_0(x)$ has finite inflection points;
(H_3) $\lim_{x\to\pm\infty}m_0(x)=\pm 1$.

Remark 1.1. It is worth to pointing out that the ways to construct the smooth approximation of the planar rarefaction wave $r(t,x)$ employed in [15,25,26,28] satisfy the assumptions (H_1)–(H_3).

Next, define $(\bar{u}(t,x),\bar{v}_1(t,x))$, $(U(t,x,y),V_1(t,x,y),V_2(t,x,y))$, and $(U_0(x,y),V_{10}(x,y),V_{20}(x,y))$ as above, we will pay our attention to the Cauchy problem (1.16), (1.17) in the rest of this manuscript.

Similar to that of [20], we can get from (1.16) and (1.17) that

$$\begin{cases} V_1(t,x,y)=e^{-t}V_{10}(x,y)+\displaystyle\int_0^t e^{s-t}\big(f(\bar{u}+U)-f(\bar{u})-a_1U_x\big)(s,x,y)\,ds,\\ V_2(t,x,y)=e^{-t}V_{20}(x,y)+\displaystyle\int_0^t e^{s-t}\big(g(\bar{u}+U)-g(\bar{u})-a_2U_y-g(\bar{u})_t\big)(s,x,y)\,ds. \end{cases} \tag{1.19}$$

Here $U(t,x,y)$ satisfies

$$U_{tt}+U_t-a_1U_{xx}-a_2U_{yy}+\big[f(\bar{u}+U)-f(\bar{u})\big]_x+g(\bar{u}+U)_y=0,\quad (x,y)\in\mathbf{R}^2,\ t>0 \tag{1.20}$$

with initial data

$$\begin{cases} U(0,x,y)=U_0(x,y),\\ U_t(0,x,y)=V_0(x,y)\equiv -V_{10x}(x,y)-V_{20y}(x,y). \end{cases} \tag{1.21}$$

Set $\ell=\varepsilon\delta$, $\delta=|u^+-u^-|$, and

$$\begin{cases} N_1(0)=\big\|\big(U_0(x,y),\nabla U_0(x,y),V_0(x,y)\big)\big\|^2_{L^2(\mathbf{R}^2)},\\ N_2(0)=\big\|\big(\nabla U_{0x}(x,y),\nabla U_{0y}(x,y),\nabla V_0(x,y)\big)\big\|^2_{L^2(\mathbf{R}^2)}, \end{cases}$$

then for general smooth nonlinear flux functions $f(u)$ and $g(u)$, our first result can be stated as follows:

Theorem 1.1. *Assume that*

- $(U_0(x,y),V_{10}(x,y),V_{20}(x,y))\in H^2(\mathbf{R}^2)$;
- *there exist two positive constants D_1 and D_2, which are independent of the parameter ℓ, and two non-negative constants $\alpha\geqslant\beta\geqslant 0$ such that*

$$\begin{cases} N_1(0)\leqslant D_1\ell^\alpha,\\ N_2(0)\leqslant D_2\big(1+\ell^{-\beta}\big). \end{cases} \tag{1.22}$$

Then there exist positive constants $\ell_0 > 0$ and $M_1 > 0$ with M_1 depending only on $N_1(0)$ and $N_2(0)$ such that if $0 < \ell < \ell_0$ and the sub-characteristic condition

$$\sup_{u\in\mathcal{M}}\left\{\frac{|f'(u)|^2}{a_1}+\frac{|g'(u)|^2}{a_2}\right\}\leqslant k_0<1 \tag{1.23}$$

holds for some ℓ-independent constant k_0, $\mathcal{M}=[-B(N_0)-M_1, B(N_0)+M_1]$ with $B(N_0)$ being defined by

$$\begin{cases} N_0=\max\{|u^-|, |u^+|, \|f(\phi_0(x))\|_{L^\infty(\mathbf{R})}\},\\ F(N_0)=\sup\limits_{|u|\leqslant N_0}|f'(u)|,\\ B(N_0)=2N_0+F(2N_0), \end{cases} \tag{1.24}$$

the Cauchy problem (1.14), (1.15) *admits a unique global solution $(U(t,x,y), V_1(t,x,y), V_2(t,x,y))$ satisfying*

$$\lim_{t\to\infty}\sup_{(x,y)\in\mathbf{R}^2}\left|\left(U(t,x,y), V_1(t,x,y), V_2(t,x,y)\right)\right|=0. \tag{1.25}$$

Consequently

$$\begin{cases} \lim\limits_{t\to\infty}\sup\limits_{(x,y)\in\mathbf{R}^2}|u(t,x,y)-r(t,x)|=0,\\ \lim\limits_{t\to\infty}\sup\limits_{(x,y)\in\mathbf{R}^2}|v_1(t,x,y)-f(r(t,x))|=0,\\ \lim\limits_{t\to\infty}\sup\limits_{(x,y)\in\mathbf{R}^2}|v_2(t,x,y)-g(r(t,x))|=0. \end{cases} \tag{1.26}$$

Remark 1.2. In Theorem 1.1, since the parameter ℓ is assumed to be small, the assumption (1.22) does imply that $N_1(0)$, the lower order energy norm of the initial perturbation, is small while $N_2(0)$ can be sufficiently large. Note also that since $\alpha \geqslant \beta$, the $L^\infty(\mathbf{R}^2)$-norm of the initial perturbation can be large. It is worth pointing out that since $\ell=\delta\varepsilon$, by choosing ε sufficiently small, δ, the strength of the planar rarefaction wave, can indeed be large in our Theorem 1.1.

Remark 1.3. From the proof of Theorem 1.1, one can easily deduce that when $f''(u)$ and $g''(u)$ satisfy certain growth condition as $|u|\to\infty$, similar result also holds even if $\alpha<\beta$ and in such a case, $\lim_{\ell\to 0^+}\|U(t)\|_{L^\infty(\mathbf{R}^2)}=+\infty$.

Although Theorem 1.1 holds for any smooth nonlinear flux functions $f(u)$ and $g(u)$, it does ask the lower order energy norm of the initial perturbation $N_1(0)$ to be small. Thus a natural question is how about the case if the initial perturbation is large even for some special class of nonlinear flux functions $f(u)$ and $g(u)$? For result in this direction, we can show that if both $f''(u)$ and $g''(u)$ are uniformly bounded, then we can indeed obtain the energy type estimate (3.35) without any small smallness assumption on the parameter ℓ and such an estimate is independent of the parameter M appeared in the *a priori* assumption (3.4). This estimate together with the continuation argument yield the following result:

Theorem 1.2. *Assume that*

- $(U_0(x,y), V_{10}(x,y), V_{20}(x,y))\in H^2(\mathbf{R}^2)$;
- *$f''(u)$ and $g''(u)$ are uniformly bounded. That is, there exists a positive constant $D_2>0$ such that*

$$|f''(u)|+|g''(u)|\leqslant D_3,\quad \forall u\in\mathbf{R}. \tag{1.27}$$

Then there exists a positive constant $M_2 > 0$, which depends only on $N_1(0)$ and $N_2(0)$, such that if the sub-characteristic condition (1.5) *holds with $\mathcal{M} = [-B(N_0) - M_2, B(N_0) + M_2]$, the Cauchy problem* (1.14), (1.15) *has a unique global solution $(U(t,x,y), V_1(t,x,y), V_2(t,x,y))$ satisfying* (1.26).

In Theorem 1.2, we do not use the smallness of either the lower order energy norm of the initial perturbation $N_1(0)$ or the parameter ℓ to control the possible growth of the solution $U(t,x,y)$ caused by the nonlinearity of the equation under our consideration and the stability result holds for any $H^2(\mathbf{R}^2)$-initial perturbation $(U_0(x,y), V_{10}(x,y), V_{20}(x,y))$. Even so, it asks the second derivative of the flux functions $f(u)$ and $g(u)$ with respect to u to be uniformly bounded. Our last result in this paper is to show that such an assumption can be replaced by the assumption that the third derivative of the flux functions $f(u)$ and $g(u)$ with respect to u is uniformly bounded while similar result also holds.

Theorem 1.3. *Assume that*

- $(U_0(x,y), V_{10}(x,y), V_{20}(x,y)) \in H^2(\mathbf{R}^2)$;
- *$f'''(u)$ and $g'''(u)$ are uniformly bounded. That is, there exists a positive constant $D_4 > 0$ such that*

$$\left|f'''(u)\right| + \left|g'''(u)\right| \leqslant D_4, \quad \forall u \in \mathbf{R}. \tag{1.28}$$

Then there exists a positive constant $M_3 > 0$, which depends only on $N_1(0)$ and $N_2(0)$, such that the sub-characteristic condition (1.5) *holds with $\mathcal{M} = [-B(N_0) - M_3, B(N_0) + M_3]$, the Cauchy problem* (1.14), (1.15) *has a unique global solution $(U(t,x,y), V_1(t,x,y), V_2(t,x,y))$ satisfying* (1.26).

Remark 1.4. Several remarks concerning our main results are listed below:

(i) Our method in deducing Theorem 1.1 applies to higher space dimensional case and the results obtained in Theorems 1.2 and 1.3 also hold in the three-dimensional case.
(ii) As pointed by T. Luo in [20], the choice of the x-direction in this paper involves no loss of generality, because we can reduce the general situation to this case by suitable change of coordinates.
(iii) Our assumptions on the sub-characteristic conditions listed in Theorems 1.1–1.3 are essentially the same as those proposed by R. Natalini in [30]. The only difference lies in that our analysis is based on an elementary energy method, while Natalini's is based on the maximum principle.
(iv) In Theorem 1.2 and Theorem 1.3, we do not ask the strength of the planar rarefaction wave to be small and the results hold for any $H^2(\mathbf{R}^2)$-initial perturbation $(U_0(x,y), V_{10}(x,y), V_{20}(x,y))$. Thus we have shown the global stability of strong planar rarefaction waves for the Jin–Xin relaxation approximation of the two-dimensional scalar conservation laws, at least for the cases when either $(f''(u), g''(u))$ or $(f'''(u), g'''(u))$ is uniformly bounded.
(v) In Theorem 1.2 and Theorem 1.3, we ask that the nonlinear flux functions $f(u)$ and $g(u)$ satisfy certain growth conditions as $|u| \to \infty$. Then a natural question is whether similar result holds under less restrictions on the nonlinear flux functions $f(u)$ and $g(u)$. Such a problem is one of the topics of our current research.
(vi) Our choice of the smooth approximation of the planar rarefaction wave $r(t,x)$ is quite general. In fact from the proofs of our main results, one can easily deduce that our results also hold for any $\phi(t,x)$, the smooth approximation of the planar rarefaction wave $r(t,x)$, satisfying the estimates stated in Lemma 2.2 and Lemma 2.3.

Now we outline our main ideas to deduce the above results. Before doing so, we first point out the main difficulties we encountered:

(i) Since the system (1.4) is semilinear hyperbolic, if the sub-characteristic condition (1.5) holds globally, i.e. (1.5) holds true for $\mathcal{M} = \mathbf{R}$, then a Gronwall type argument can guarantee that the Cauchy problem (1.4), (1.6) is globally solvable for large initial data, cf. [22]. Even so, it seems that it is not an easy work to deduce the large time behavior (1.26), partially because the proof of (1.26) is based on some estimates on $(u(t,x,y), v_1(t,x,y), v_2(t,x,y))$ which are uniformly with respect to the time variable t.

(ii) For the case when $\mathcal{M}$ is just a subset of $\mathbf{R}$, the problem is more subtle. In such a case, even if the initial data lies in $\mathcal{M}$, the nonlinearity of the system (1.4) will certainly lead to the growth of the solution, which, if without suitably control, will eventually result in the escape of the solution out of $\mathcal{M}$. For the one-dimensional case, by exploiting the maximum principle, R. Natalini [30] succeeded in finding such an $\mathcal{M}$ which is closely related to the $L^\infty(\mathbf{R})$-norm of the initial data. We note, however, that since the analysis in [30] is based on the diagonalization of the system by the Riemann invariants which seems impossible for the higher dimensional case, such an argument seems not so easy, if not impossible, to be used to deal with the higher dimensional case.

Our analysis is based on the elementary energy method together with the continuation argument and, as mentioned before, the main difficulty lies in how to control the possible growth of the solutions caused by the nonlinearity of the system (1.4). To overcome such a difficulty, the argument used in [20] and [43] is to use the smallness of the initial perturbation and hence such an argument can only yield the local stability of planar rarefaction waves. To deduce the nonlinear stability result with large initial perturbation, our main tricks are the following:

(i) Our first trick is trying to use the smallness of the parameter $\ell = \varepsilon\delta$ to control the possible growth of the solution caused by the nonlinearity of the system (1.4). To this end, we first obtain certain estimates on $\phi(t,x)$, the smooth approximation of the planar rarefaction wave $r(t,x)$ and on $\frac{\partial^k \bar{u}(t,x)}{\partial x^k}$ for $k \geqslant 1$. Based on these estimates and some careful energy type estimates, we can show that $U(t,x,y)$ satisfies the estimates (3.5) and (3.20) provided that $U(t,x,y)$ satisfies the *a priori* assumption (3.4), the sub-characteristic condition (1.5) holds with $\mathcal{M} = [-B(N_0) - M, B(N_0) + M]$, and the assumption (3.19) is imposed. Although the right hand side of (3.20) does depend on the constant M, if we assume that the lower order initial perturbation $N_1(0)$ is suitably small, such a factor can be controlled suitably and this observation together with the continuation argument yield the result stated in Theorem 1.1.

(ii) In the above result, although the $H^2(\mathbf{R}^2)$-norm of the initial perturbation can be large, we do ask that $N_1(0)$, the lower order energy norm of the initial perturbation, is sufficiently small. To deduce the nonlinear stability of strong planar rarefaction wave $r(t,x)$ for any $H^2(\mathbf{R}^2)$-initial perturbation, we need to use the underlying structure of the system (1.4) fully. Our first observation in this direction is that if $f''(u)$ and $g''(u)$ are uniformly bounded, then the estimate (3.20) can indeed be improved such that its right hand side does not depend on M even without the assumption (3.19). This observation together with the continuation argument can lead to the global stability result stated in Theorem 1.2. Motivated by the result obtained in Theorem 1.2 and some careful energy type estimates, we can also show in Theorem 1.3 that if $f'''(u)$ and $g'''(u)$ are uniformly bounded, similar global stability result also holds.

Before concluding this section, we recall some former results concerning the nonlinear stability of rarefaction waves for dissipative hyperbolic conservation laws and on the hyperbolic conservation laws with relaxation as follows:

(i) The nonlinear stability of rarefaction waves of hyperbolic conservation laws with dissipative terms has been the subject of numerous studies, cf. [5,19,34,32,41] for compressible Navier–Stokes equations, [6,9,8,10,12,13,27,36,38–40,42] for hyperbolic conservation laws with artificial viscosity, [7] for the generalized KdV–Burgers equation, [20,43,24,44,33] for hyperbolic conservation laws with relaxation, and the references therein.

(ii) The relaxation mechanism arise in many physical situations, for example, gases not in thermodynamic equilibrium, kinetic theory, chromatography, river flow, traffic flows, and more general waves, cf. [37,1]. The general 2×2 hyperbolic systems of conservation laws with relaxation in the form

$$\begin{cases} u_t + f(u, v)_x = 0, \\ v_t + g(u, v)_x = \dfrac{h(u, v)}{\varepsilon}, \end{cases} \tag{1.29}$$

was first analyzed by T.-P. Liu in [18] to justify some nonlinear stability criteria for diffusion waves, expansion waves, and traveling waves. Since then, the stability of certain elementary waves was studied by H.-L. Liu, C. Woo and T. Yang [17], T. Luo [20], T. Luo and Z.-P. Xin [21], H.-J. Zhao [43], C. Mascia and R. Natalini [23], M. Mei and T. Yang [29], R.-H. Pan [35], C.-J. Zhu [45] and P. Zingano [46], etc. The problem on the convergence to the diffusion waves was given by I.-L. Chern [4]. Related results on the relaxation time limit can be found in G.-Q. Chen, C.D. Levermore, and T.-P. Liu [2], G.-Q. Chen and T.-P. Liu [3], C. Lattanzio and P. Marcati [16], R. Natalini [31], etc. For a more complete literature in this direction, we refer the interested reader to the monograph [11] by L. Hsiao and the survey paper [31] by R. Natalini.

The rest of this paper is organized as follows. In Section 2, we will deduce certain estimates on $\phi(t,x)$, the smooth approximation of the planar rarefaction wave $r(t,x)$, and on the quantity $(w(t,x), z(t,x)) = (\bar{u}(t,x) - \phi(t,x), \bar{v}_1(t,x) - f(\phi(t,x)))$ which will be used in the subsequent sections. The energy type estimates are performed in Section 3 and the proofs of our main results are given in Section 4.

Notations Throughout the rest of this paper, we use C, C_i, and D_i ($i \in \mathbf{Z}^+$) to denote a generic positive constant which are independent of t, x, y but may vary from line to line, and use $\|\cdot\|_s$ to denotes the norm in $H^s(\mathbf{R})$ or $H^s(\mathbf{R}^2)$ with $\|\cdot\| = \|\cdot\|_0$. Finally, for $z = (z_1, \ldots, z_N) \in \mathbf{R}^N$, $\nabla = (\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N})$, $2 \leqslant k \in \mathbf{Z}^+$, $|D^k u(z)| = \sum_{\alpha_1+\cdots+\alpha_N=k} |\frac{\partial^k u(z)}{\partial z_1^{\alpha_1}\cdots\partial z_N^{\alpha_N}}|$.

2. Preliminaries

This section is devoted to citing some fundamental inequalities and to deducing some basic estimates on the smooth approximation of the planar rarefaction waves and on the quantity $(w(t,x), z(t,x)) = (\bar{u}(t,x) - \phi(t,x), \bar{v}_1(t,x) - f(\phi(t,x)))$. To do so, we first cite the Gagliardo–Nirenberg inequality which will be used frequently later:

Lemma 2.1 *(Gagliardo–Nirenberg inequality). Assume that $u(z) \in L^q(\mathbf{R}^N)$, $D^m u(z) \in L^r(\mathbf{R}^N)$ with $1 \leqslant q, r \leqslant +\infty$. Then, for any integral $j \in [0, m]$, we have*

$$\left\| D^j u(z) \right\|_{L^p} \leqslant C \left\| D^m u(z) \right\|_{L^r}^{\alpha} \left\| u(z) \right\|_{L^q}^{1-\alpha}.$$

Here

$$\frac{1}{p} = \frac{j}{N} + \alpha\left(\frac{1}{r} - \frac{m}{N}\right) + (1-\alpha)\frac{1}{q}, \qquad \frac{j}{m} \leqslant \alpha \leqslant 1.$$

If we are concentrated on the two-dimensional case, we have from Lemma 2.1 that:

Corollary 2.1. *For the case $N = 2$, we have*

(i) $$\|f\|_{L^\infty(\mathbf{R}^2)} \leqslant D_0 \left\| D^2 f \right\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}} \|f\|_{L^2(\mathbf{R}^2)}^{\frac{1}{2}},$$

(ii) $$\|fg\|_{L^2(\mathbf{R}^2)}^2 \leqslant \|f\|_{L^2(\mathbf{R}^2)} \|\nabla f\|_{L^2(\mathbf{R}^2)} \|g\|_{L^2(\mathbf{R}^2)} \|\nabla g\|_{L^2(\mathbf{R}^2)},$$

(iii)
$$\begin{cases} \|f\|^3_{L^3(\mathbf{R}^2)} \leqslant \|f\|^2_{L^2(\mathbf{R}^2)}\|\nabla f\|_{L^2(\mathbf{R}^2)}, \\ \|fg^2\|_{L^1(\mathbf{R}^2)} \leqslant \|f\|_{L^2(\mathbf{R}^2)}\|g\|_{L^2(\mathbf{R}^2)}\|\nabla g\|_{L^2(\mathbf{R}^2)}, \\ \|fgh\|_{L^1(\mathbf{R}^2)} \leqslant \|f\|_{L^2(\mathbf{R}^2)}\|g\|_{L^2(\mathbf{R}^2)}\|\nabla g\|_{L^2(\mathbf{R}^2)}\|h\|_{L^2(\mathbf{R}^2)}\|\nabla h\|_{L^2(\mathbf{R}^2)}, \end{cases}$$

(iv)
$$\begin{cases} \int_{\mathbf{R}^2} |f||g|^3\,dx\,dy \leqslant \|f\|^{\frac12}_{L^2(\mathbf{R}^2)}\|\nabla f\|^{\frac12}_{L^2(\mathbf{R}^2)}\|g\|^{\frac32}_{L^2(\mathbf{R}^2)}\|\nabla g\|^{\frac32}_{L^2(\mathbf{R}^2)}, \\ \int_{\mathbf{R}^2} |f|^2|g||h|\,dx\,dy \leqslant \|f\|_{L^2(\mathbf{R}^2)}\|\nabla f\|_{L^2(\mathbf{R}^2)}\|g\|^{\frac12}_{L^2(\mathbf{R}^2)}\|\nabla g\|^{\frac12}_{L^2(\mathbf{R}^2)}\|h\|^{\frac12}_{L^2(\mathbf{R}^2)}\|\nabla h\|^{\frac12}_{L^2(\mathbf{R}^2)}. \end{cases}$$

Here D_0 is some positive constant depending only on the dimension of the space.

Proof. (i) is a direct consequence of the Gagliardo–Nirenberg inequality with $N = 2$, $j = 0$, $q = 2$, $p = \infty$, $\alpha = \frac{1}{2}$. As to (ii), since

$$\begin{cases} \|f(\cdot,y)\|^2_{L^\infty(\mathbf{R}_x)} \leqslant \|f(\cdot,y)\|_{L^2(\mathbf{R}_x)}\|f_x(\cdot,y)\|_{L^2(\mathbf{R}_x)}, \\ \|g(\cdot,y)\|^2_{L^2(\mathbf{R}_x)} \leqslant \|g\|_{L^2(\mathbf{R}^2)}\|g_y\|_{L^2(\mathbf{R}^2)}, \end{cases}$$

we can deduce from the Hölder inequality that

$$\begin{aligned} \|fg\|^2_{L^2(\mathbf{R}^2)} &\leqslant \int_{\mathbf{R}_x} \|f(\cdot,y)\|^2_{L^\infty(\mathbf{R}_x)}\|g(\cdot,y)\|^2_{L^2(\mathbf{R}_x)}\,dy \\ &\leqslant \|g\|_{L^2(\mathbf{R}^2)}\|g_y\|_{L^2(\mathbf{R}^2)} \int_{\mathbf{R}_x} \|f(\cdot,y)\|_{L^2(\mathbf{R}_x)}\|f_x(\cdot,y)\|_{L^2(\mathbf{R}_x)}\,dy \\ &\leqslant \|f\|_{L^2(\mathbf{R}^2)}\|\nabla f\|_{L^2(\mathbf{R}^2)}\|g\|_{L^2(\mathbf{R}^2)}\|\nabla g\|_{L^2(\mathbf{R}^2)}, \end{aligned}$$

which shows that (ii) holds.

Having obtained (ii), (iii) and (iv) are direct consequence of (ii) and

$$\begin{cases} \|f\|^3_{L^3(\mathbf{R}^2)} \leqslant \|f\|_{L^2(\mathbf{R}^2)}\|f\|^2_{L^4(\mathbf{R}^2)}, \\ \|fg^3\|_{L^1(\mathbf{R}^2)} \leqslant \|fg\|_{L^2(\mathbf{R}^2)}\|g\|^2_{L^4(\mathbf{R}^2)}, \\ \|fg^2\|_{L^1(\mathbf{R}^2)} \leqslant \|f\|_{L^2(\mathbf{R}^2)}\|g\|^2_{L^4(\mathbf{R}^2)}, \\ \|fgh\|_{L^1(\mathbf{R}^2)} \leqslant \|f\|_{L^2(\mathbf{R}^2)}\|gh\|_{L^2(\mathbf{R}^2)}, \\ \|f^2gh\|_{L^1(\mathbf{R}^2)} \leqslant \|f\|^2_{L^4(\mathbf{R}^2)}\|gh\|_{L^2(\mathbf{R}^2)}. \end{cases}$$

This completes the proof of Corollary 2.1. □

Now we turn to deduce certain properties of the global smooth solution $\phi(t,x)$ of the Cauchy problem (1.11), (1.18). Notice that for some specially chosen $\phi_0(x)$, the corresponding results have been obtained in [28], etc.

Lemma 2.2. *Under the assumptions (H_1)–(H_3) imposed on $m_0(x)$, the Cauchy problem* (1.11), (1.18) *admits a unique global smooth solution $\phi(t,x)$ which satisfies*:

(i) $u^- < \phi(t,x) < u^+$, $\frac{\partial\phi(t,x)}{\partial x} > 0$ *holds for all* $(t,x) \in [0,+\infty] \times \mathbf{R}$;

(ii) $\|\frac{\partial\phi(t,x)}{\partial x}\|_{L^p(\mathbf{R})} \leqslant C_p \min\{\delta\varepsilon^{1-\frac{1}{p}}, \delta^{\frac{1}{p}} t^{\frac{1}{p}-1}\}$. *Here* $\delta = |u^+ - u^-|$, $p \in [1,\infty]$, *and* C_p *is a constant only depends on* p;

(iii) $\|\frac{\partial^2\phi(t,x)}{\partial x^2}\|_{L^1(\mathbf{R})} \leqslant C\min\{\delta\varepsilon, t^{-1}\}$, *where* C *is some positive constant*;

(iv) $\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}|\phi(t,x) - r(t,x)| = 0$.

Proof. Let $x(t;x_0)$ be the characteristic line passing through $(0,x_0)$, we have from (1.11) that

$$\begin{cases} \dfrac{dx(t;x_0)}{dt} = f'\big(\phi\big(t, x(t;x_0)\big)\big), \\ x(t;x_0)|_{t=0} = x_0 \end{cases}$$

and

$$\begin{cases} \dfrac{d\phi(t,x(t;x_0))}{dt} = 0, \\ \phi\big(t,x(t;x_0)\big)\big|_{t=0} = \phi_0(x_0). \end{cases}$$

From which we can deduce that

$$\begin{cases} x(t;x_0) = x_0 + f'\big(\phi_0(x_0)\big)t, \\ \phi\big(t,x(t;x_0)\big) = \phi_0(x_0). \end{cases} \tag{2.1}$$

From (2.1), the assumption (H_1), and the implicit function theorem, one easily deduce that the Cauchy problem (1.11), (1.18) admits a unique global classical solution $\phi(t,x)$ satisfying (i).

Let $x_0(t,x)$ be function of t and x implicitly defined by (2.1), one easily deduce that

$$\begin{cases} \dfrac{\partial x_0(t,x)}{\partial x} = \dfrac{1}{1+f''(\phi_0(x_0))\phi_0'(x_0)t} > 0, \\ \dfrac{\partial^2 x_0(t,x)}{\partial x^2} = -\dfrac{(f''(\phi_0(x_0))\phi_0''(x_0) + f'''(\phi_0(x_0))|\phi_0'(x_0)|^2)t}{(1+f''(\phi_0(x_0))\phi_0'(x_0)t)^3}, \end{cases} \tag{2.2}$$

and consequently

$$\begin{cases} \dfrac{\partial\phi(t,x)}{\partial x} = \dfrac{\phi_0'(x_0)}{1+f''(\phi_0(x_0))\phi_0'(x_0)t}, \\ \dfrac{\partial^2\phi(t,x)}{\partial x^2} = \dfrac{\phi_0''(x_0)}{(1+f''(\phi_0(x_0))\phi_0'(x_0)t)^3} - \dfrac{f'''(\phi_0(x_0))(\phi_0'(x_0))^3}{(1+f''(\phi_0(x_0))\phi_0'(x_0)t)^3}. \end{cases} \tag{2.3}$$

For each $p \in [1,\infty)$, we have from $(2.3)_1$ and (1.18) that

$$\begin{aligned} \left\|\frac{\partial\phi(t,x)}{\partial x}\right\|^p_{L^p(\mathbf{R})} &= \int_{\mathbf{R}} \frac{|\phi_0'(x_0)|^p}{(1+f''(\phi_0(x_0))\phi_0'(x_0)t)^p}\,dx \\ &= \int_{\mathbf{R}} \frac{|\phi_0'(x_0)|^p}{(1+f''(\phi_0(x_0))\phi_0'(x_0)t)^{p-1}}\,dx_0 \\ &= \left(\frac{\varepsilon\delta}{2}\right)^p \int_{\mathbf{R}} \frac{|m_0'(\varepsilon y)|^p}{(1+\frac{1}{2}\varepsilon\delta f''(\phi_0(x_0))m_0'(\varepsilon y)t)^p}\,dy. \end{aligned} \tag{2.4}$$

With (2.4) in hand, it is routine to prove that (ii) holds for $p \in [1,\infty)$, the case for $p=\infty$ is less complicated and we omit the details for brevity.

Now we turn to prove (iii). For this purpose, we have from $(2.3)_2$ and (i) that

$$\begin{aligned}\left\|\frac{\partial^2\phi(t,x)}{\partial x^2}\right\|_{L^1(\mathbf{R})} &\leqslant \int\limits_{\mathbf{R}} \frac{|\phi_0''(x_0)|}{(1+f''(\phi_0(x_0))\phi_0'(x_0)t)^2}\,dx_0 + \int\limits_{\mathbf{R}} \frac{|f'''(\phi_0(x_0))||\phi_0'(x_0)|^3 t}{(1+f''(\phi_0(x_0))\phi_0'(x_0)t)^2}\,dx_0 \\ &\leqslant \underbrace{O(1)\varepsilon^2\delta\int\limits_{\mathbf{R}} \frac{|m_0''(\varepsilon y)|}{(1+\varepsilon\delta m_0'(\varepsilon y)t)^2}\,dy}_{A_1} + \underbrace{O(1)\varepsilon^3\delta^3\int\limits_{\mathbf{R}} \frac{|m_0'(\varepsilon y)|^3 t}{(1+\varepsilon\delta m_0'(\varepsilon y)t)^2}\,dy}_{A_2}. \qquad (2.5)\end{aligned}$$

Now we turn to estimate A_1 and A_2 term by term. First for A_2, we can deduce straightforward that

$$A_2 \leqslant O(1)\varepsilon^2\delta^3\int\limits_{\mathbf{R}} \frac{|m_0'(z)|^3 t}{(1+\varepsilon\delta m_0'(z)t)^2}\,dz \leqslant O(1)\min\{\varepsilon\delta^2, \delta t^{-1}\}. \qquad (2.6)$$

As to A_1, unlike the cases studied in [25,28], since for general smooth function $m_0(x)$, we do not know the relation between $m_0''(x)$ and $m_0'(x)$, the arguments used there cannot be used any longer. We note, however, that since the assumption (H_2) tells us that $m_0(x)$ has only finite inflection points, thus there exist $z_j \in \mathbf{R}$ $(j=0,1,\dots,m+1)$ with $z_0=-\infty$, $z_{m+1}=+\infty$ such that $m_0''(x)$ keeps sign on the interval $[z_j, z_{j+1}]$ for $j=0,1,\dots,m$. Thus we have

$$\begin{aligned}A_1 &= O(1)\varepsilon\delta\int\limits_{\mathbf{R}} \frac{|m_0''(z)|\,dz}{(1+\varepsilon\delta m_0'(z)t)^2} \\ &= O(1)\varepsilon\delta\sum_{j=0}^{m}\left|\int\limits_{z_j}^{z_{j+1}} \frac{m_0''(z)\,dz}{(1+\varepsilon\delta m_0'(z)t)^2}\right| \\ &= \frac{O(1)}{t}\sum_{j=0}^{m}\left|\frac{1}{1+\varepsilon\delta m_0'(z_j)t} - \frac{1}{1+\varepsilon\delta m_0'(z_{j+1})t}\right| \\ &\leqslant O(1)t^{-1}. \qquad (2.7)\end{aligned}$$

Here we have used the fact that $\lim_{x\pm\infty} m_0'(x)=0$.

Inserting (2.6) and (2.7) into (2.5) proves (iii).

To prove (iv), we first notice from

$$\begin{cases} \pm x_0(t, f'(u^\pm)t) = \pm\big(f'(u^\pm) - f'\big(\phi_0\big(x_0\big(t, f'\big(u^\pm\big)t\big)\big)\big)\big) \geqslant 0, \\ \pm\dfrac{dx_0(t, f'(u^\pm)t)}{dt} = \pm\dfrac{f'(u^\pm) - f'(\phi_0(x_0(t, f'(u^\pm)t)))}{1+f''(\phi_0(x_0(t, f'(u^\pm)t)))t} \geqslant 0 \end{cases}$$

that $\lim_{t\to\infty} x_0(t, f'(u^\pm)t)$ exist and moreover we can show that

$$\lim_{t\to\infty} x_0\big(t, f'\big(u^\pm\big)t\big) = \pm\infty. \qquad (2.8)$$

We only prove that $\lim_{t\to\infty} x_0(t, f'(u^-)t) = -\infty$ since the other case can be treated similarly. Assume on the other hand that such a claim does not hold true, we then can find an $\bar{x} \in (-\infty, 0]$ such that

$$\lim_{t\to\infty} x_0\big(t, f'\big(u^-\big)t\big) = \bar{x}.$$

Since $x_0(t, f'(u^-)t)$ is a decreasing function of t, we have

$$\begin{aligned} -\infty < \bar{x} &\leqslant x_0\big(t, f'\big(u^-\big)t\big) \\ &= \big[f'\big(u^-\big) - f'\big(\phi_0\big(x_0\big(t, f'\big(u^-\big)t\big)\big)\big)\big]t \\ &\leqslant \big[f'\big(u^-\big) - f'\big(\phi_0(\bar{x})\big)\big]t < 0. \end{aligned} \tag{2.9}$$

Here we have used the facts that $f'(\phi_0(x))$ is an increasing function of x and (i).

Letting $t \to \infty$ in (2.9), we can arrive at a contradiction, this proves that $\lim_{t\to\infty} x_0(t, f'(u^-)t) = -\infty$.

With (2.8) in hand, we now turn to prove (iv). To this end, we divide the $t - x$ plane into the following three regions: $\Omega_1 = \{(t,x)\colon\ x < f'(u^-)t,\ t \geqslant 0\}$, $\Omega_2 = \{(t,x)\colon\ f'(u^-)t \leqslant x \leqslant f'(u^+)t,\ t \geqslant 0\}$, and $\Omega_3 = \{(t,x)\colon\ x > f'(u^+)t,\ t \geqslant 0\}$. It is easy to see that

$$\phi(t,x) - r(t,x) = \begin{cases} \phi(t,x) - u^-, & (t,x) \in \Omega_1, \\ \phi(t,x) - (f')^{-1}(\frac{x}{t}), & (t,x) \in \Omega_2, \\ \phi(t,x) - u^+, & (t,x) \in \Omega_3. \end{cases} \tag{2.10}$$

Since $m_0(x)$ and $x_0(t,x)$ are increasing functions of x, we have from (1.18), (2.1), (2.2) that if $x < f'(u^-)t$, one has

$$\big|\phi(t,x) - u^-\big| = \phi(t,x) - u^- = \frac{\delta}{2}\big(1 + m_0\big(\varepsilon x_0(t,x)\big)\big) \leqslant \frac{\delta}{2}\big(1 + m_0\big(\varepsilon x_0\big(t, f'\big(u^-\big)t\big)\big)\big), \tag{2.11}$$

while for $(t,x) \in \Omega_3$, we can deduce that

$$\big|\phi(t,x) - u^+\big| = u^+ - \phi(t,x) = \frac{\delta}{2}\big(1 - m_0\big(\varepsilon x_0(t,x)\big)\big) \leqslant \frac{\delta}{2}\big(1 - m_0\big(\varepsilon x_0\big(t, f'\big(u^+\big)t\big)\big)\big). \tag{2.12}$$

As for the case when $(t,x) \in \Omega_2$, we only need to estimate $f'(\phi(t,x)) - f'(r(t,x))$. For this purpose, since $f'(\phi(t,x))$ solves

$$\begin{cases} w_t + w w_x = 0, \\ w(0,x) = w_0(x) \equiv f'\big(\phi_0(x)\big), \end{cases} \tag{2.13}$$

we can get that

$$\begin{cases} f'\big(\phi\big(t, x(t;x_0)\big)\big) = f'\big(\phi_0(x_0)\big), \\ x(t;x_0) = x_0 + f'\big(\phi_0(x_0)\big)t. \end{cases} \tag{2.14}$$

Thus for $f'(u^-)t \leqslant x \leqslant f'(u^+)t$, we have from (1.10) and (2.14) that

$$\big|f'\big(\phi(t,x)\big) - f'\big(r(t,x)\big)\big| = \left|\frac{x - x_0}{t} - \frac{x}{t}\right| = \frac{|x_0|}{t}. \tag{2.15}$$

Noticing that $x_0(t,x)$ is increasing with respect to x, we have for $(t,x)\in\Omega_2$ that

$$x_0\big(t, f'\big(u^-\big)t\big) \leqslant x(t,x) \leqslant x_0\big(t, f'\big(u^+\big)t\big). \tag{2.16}$$

(2.15) and (2.16) together with the fact

$$x_0\big(t, f'\big(u^\pm\big)t\big) = t\big[f'\big(u^\pm\big) - f'\big(\phi_0\big(t, f'\big(u^\pm\big)t\big)\big)\big] \tag{2.17}$$

implies that for $f'(u^-)t \leqslant x \leqslant f'(u^+)t$

$$\begin{aligned} &\big|f'\big(\phi(t,x)\big) - f'\big(r(t,x)\big)\big| \\ &\quad\leqslant \max\big\{\big|f'\big(u^-\big) - f'\big(\phi_0\big(t, f'\big(u^-\big)t\big)\big)\big|, \big|f'\big(u^+\big) - f'\big(\phi_0\big(t, f'\big(u^+\big)t\big)\big)\big|\big\}. \end{aligned} \tag{2.18}$$

Putting (2.11), (2.12), and (2.18) together, we can deduce (iv) immediately from (2.8) and the assumption (H_3). This completes the proof of Lemma 2.2. □

For the temporal decay of the higher order derivatives of $\phi(t,x)$ with respect to x, since the assumption (H_1) implies that

$$\left\|\frac{\partial^k \phi(t,x)}{\partial x^k}\right\|_{L^p} \leqslant O(1)\varepsilon^{k-1}\delta,$$

the above estimate, (iii) of Lemma 2.2 together with the Gagliardo–Nirenberg inequality implies that:

Lemma 2.3. *Under the assumptions listed in Lemma 2.2, for each $k \geqslant 2$ and $\theta > 0$ sufficiently small, there exists a positive constant $C_{p,k}$ such that*

$$\left\|\frac{\partial^k \phi(t,x)}{\partial x^k}\right\|_{L^p} \leqslant C_{p,k}(\varepsilon\delta)^\theta (1+t)^{-1+\theta}. \tag{2.19}$$

Let

$$\begin{cases} w(t,x) = \bar{u}(t,x) - \phi(t,x), \\ z(t,x) = \bar{v}_1(t,x) - f\big(\phi(t,x)\big), \end{cases} \tag{2.20}$$

we now turn to deduce some energy type estimates on $(w(t,x), z(t,x))$.

For this purpose, recall that $(\bar{u}(t,x), \bar{v}(t,x))$ satisfies

$$\begin{cases} \bar{u}_t + \bar{v}_{1x} = 0, \\ \bar{v}_{1t} + a_1\bar{u}_x = -v_1 + f(\bar{u}) \end{cases} \tag{2.21}$$

with the prescribed initial data

$$\big(\bar{u}(0,x), \bar{v}_1(0,x)\big) = \big(\bar{u}_0(x), \bar{v}_{10}(x)\big) = \big(\phi_0(x), f\big(\phi_0(x)\big)\big), \tag{2.22}$$

where $\phi_0(x)$ is given by (1.18).

It is easy to deduce from Theorem 3.1 of [30] that the Cauchy problem (2.21), (2.22) admits a unique global smooth solution $(\bar{u}(t,x), \bar{v}(t,x))$ satisfying

$$\left\|\bar{u}(t,x)\right\|_{L^\infty(\mathbf{R})} \leqslant B(N_0), \qquad \left\|\bar{v}_1(t,x)\right\|_{L^\infty(\mathbf{R})} \leqslant \sqrt{a_1}B(N_0) \tag{2.23}$$

with

$$\begin{cases} N_0 = \max\left\{\left|u^+\right|, \left|u^-\right|, \left\|f\left(\phi_0(x)\right)\right\|_{L^\infty(\mathbf{R})}\right\}, \\ F(N_0) = \sup\limits_{|u|\leqslant N_0} \left|f'(u)\right|, \\ B(N_0) = 2N_0 + F(2N_0) \end{cases} \tag{2.24}$$

if the following sub-characteristic condition

$$\max_{|u|\leqslant B(N_0)} \left|f'(u)\right| < \sqrt{a_1} \tag{2.25}$$

holds true.

Moreover it is easy to see that $(w(t,x), z(t,x))$ satisfies

$$\begin{cases} w_t + z_x = 0, \\ z_t + a_1 w_x + z = \left[f(w+\phi) - f(\phi)\right] - \left(a_1\phi_x + f(\phi)_t\right), \\ \left(w(0,x), z(0,x)\right) = \left(\bar{u}_0(x) - \phi_0(x), \bar{v}_{10}(x) - f\left(\phi_0(x)\right)\right) = (0,0). \end{cases} \tag{2.26}$$

Direct calculations yield

$$\begin{cases} w_{tt} + w_t - a_1 w_{xx} = \left[f(w+\phi) - f(\phi)\right]_x + a_1\phi_{xx} + f(\phi)_{tx}, \\ w(0,x) = \bar{u}_0(x) - \phi_0(x) = 0, \quad w_t(0,x) = -z_{0x}(x) = 0. \end{cases} \tag{2.27}$$

The following lemma is concerned with the basic energy estimate on $w(t,x)$:

Lemma 2.4 *(Basic estimates). Under the sub-characteristic condition* (2.25)*, we have*

$$\left\|\left(w(t), w_x(t), w_t(t)\right)\right\|^2 + \int_0^t \left\|\left(\sqrt{\phi_x}w(s), w_x(s), w_t(s)\right)\right\|^2 ds \leqslant C_1\ell^{\frac{1}{4}}. \tag{2.28}$$

Here C_1 is a positive constant independent of t, x, ε, and δ.

Proof. For some positive constant λ, multiplying $(2.27)_1$ by $\lambda w + w_t$ and integrating the resulting identity with respect to t and x over $[0,t]\times R$, we have

$$\begin{aligned} &\frac{\lambda}{2}\left\|w(t)\right\|^2 + \frac{1}{2}\left\|w_t(t)\right\|^2 + \frac{a_1}{2}\left\|w_x(t)\right\|^2 + \lambda\int_{\mathbf{R}} w w_t(t)\,dx \\ &\quad + \int_0^t \left(\lambda a_1\left\|w_x(s)\right\|^2 ds + (1-\lambda)\left\|w_t(s)\right\|^2\right) ds \\ &= \underbrace{\lambda\int_0^t\int_{\mathbf{R}} \left\{\left[\left(\int_\phi^{w+\phi} f(y)\,dy - f(\phi)w\right)_x - \left(f(w+\phi) - f(\phi) - f'(\phi)w\right)\phi_x\right]\right\}(s,x)\,dx\,ds}_{J_1} \end{aligned}$$

$$\underbrace{-\int_0^t\int_{\mathbf{R}}\bigl\{w_t\bigl[f'(w+\phi)-f'(\phi)\bigr]\phi_x+w_tw_xf'(w+\phi)\bigr\}(s,x)\,dx\,ds}_{J_2}$$

$$\underbrace{+\int_0^t\int_{\mathbf{R}}\bigl\{\bigl[a_1\phi_{xx}+f(\phi)_{tx}\bigr](\lambda w+w_t)\bigr\}(s,x)\,dx\,dy}_{J_3}. \tag{2.29}$$

Now we turn to deal with J_i $(i=1,2,3)$ term by term. First from Lemma 2.2, (2.23), (2.24), and the convexity of $f(u)$, we know that there exists a positive constant $\gamma>0$ such that

$$J_1\leqslant -\frac{\gamma}{2}\lambda\int_0^t\int_{\mathbf{R}}\phi_xw^2\,dx\,ds. \tag{2.30}$$

For J_2, the Cauchy–Schwarz inequality yields

$$|J_2|\leqslant\left(\frac{1}{2}\bar{k}_1+\eta\right)\int_0^t\bigl\|w_x(s)\bigr\|^2\,ds+\left(\frac{1}{2}+\eta\right)\int_0^t\bigl\|w_t(s)\bigr\|^2\,ds$$
$$+O(1)\int_0^t\bigl\|w(s)\bigr\|^2\bigl\|\phi_x(s)\bigr\|\bigl\|\phi_{xx}(s)\bigr\|\,ds. \tag{2.31}$$

Here and in the rest of this paper, η is a positive constant which can be chosen sufficiently small and

$$\bar{k}_1=\max_{|u|\leqslant B(N_0)}\bigl|f'(u)\bigr|^2. \tag{2.32}$$

At last for J_3, we have from the Cauchy–Schwarz inequality and (2.23), (2.24) that

$$|J_3|\leqslant O(1)\int_0^t\bigl\|w(s)\bigr\|_{L^\infty}\bigl(\bigl\|\phi_{xx}(s)\bigr\|_{L^1}+\bigl\|\phi_x(s)\bigr\|^2\bigr)\,ds$$
$$+\eta\int_0^t\bigl\|w_t^2(s)\bigr\|\,ds+O(1)\int_0^t\bigl(\bigl\|\phi_{xx}(s)\bigr\|^2+\bigl\|\phi_x(s)\bigr\|_{L^4}^4\bigr)\,ds. \tag{2.33}$$

Notice that

$$\bigl\|w(t)\bigr\|_{L^\infty}\leqslant\bigl\|w(t)\bigr\|^{\frac{1}{2}}\bigl\|w_x(t)\bigr\|^{\frac{1}{2}},$$

we have

$$\begin{aligned}
&\int_0^t \|w(s)\|_{L^\infty}\big(\|\phi_{xx}(s)\|_{L^1}+\|\phi_x(s)\|^2\big)\,ds\\
&\quad\leqslant \int_0^t \|w(s)\|^{\frac{1}{2}}\|w_x(s)\|^{\frac{1}{2}}\big(\|\phi_{xx}(s)\|_{L^1}+\|\phi_x(s)\|^2\big)\,ds\\
&\quad\leqslant \eta\int_0^t \|w_x(s)\|^2\,ds+O(1)\int_0^t \|w(s)\|^2\big(\|\phi_{xx}(s)\|_{L^1}^{\frac{4}{3}}+\|\phi_x(s)\|^{\frac{8}{3}}\big)\,ds\\
&\qquad+O(1)\int_0^t \big(\|\phi_{xx}(s)\|_{L^1}^{\frac{4}{3}}+\|\phi_x(s)\|^{\frac{8}{3}}\big)\,ds. \qquad (2.34)
\end{aligned}$$

Substituting (2.34) into (2.33), we can get that

$$\begin{aligned}
|J_3|\leqslant{}&\eta\int_0^t \big(\|w_x(s)\|^2+\|w_t(s)\|^2\big)\,ds\\
&+O(1)\int_0^t \|w(s)\|^2\big(\|\phi_{xx}(s)\|_{L^1}^{\frac{4}{3}}+\|\phi_x(s)\|^{\frac{8}{3}}\big)\,ds\\
&+O(1)\int_0^t \big(\|\phi_{xx}(s)\|^2+\|\phi_x(s)\|_{L^4}^4+\|\phi_{xx}(s)\|_{L^1}^{\frac{4}{3}}+\|\phi_x(s)\|^{\frac{8}{3}}\big)\,ds. \qquad (2.35)
\end{aligned}$$

Due to

$$\lambda\int_{\mathbf{R}} w(t,x)w_t(t,x)\,dx\leqslant\frac{\lambda}{4}\|w(t)\|^2+\lambda\|w_t(t)\|^2,$$

we can deduce by inserting (2.30), (2.31), and (2.35) into (2.29) that

$$\begin{aligned}
&\frac{\lambda}{4}\|w(t)\|^2+\left(\frac{1}{2}-\lambda\right)\|w_t(t)\|^2+\frac{a_1}{2}\|w_x(t)\|^2+\frac{\gamma\lambda}{2}\int_0^t\int_{\mathbf{R}}\phi_x(s,x)w^2(s,x)\,dx\,ds\\
&\quad+\int_0^t\left\{\left(\lambda a_1-\frac{k_1'}{2}-2\eta\right)\|w_x(s)\|^2+\left(\frac{1}{2}-\lambda-2\eta\right)\|w_t(s)\|^2\right\}ds\\
&\leqslant O(1)\int_0^t \|w(s)\|^2\big(\|\phi_x(s)\|\|\phi_{xx}(s)\|+\|\phi_{xx}(s)\|_{L^1}^{\frac{4}{3}}+\|\phi_x(s)\|^{\frac{8}{3}}\big)\,ds\\
&\quad+O(1)\int_0^t \big(\|\phi_{xx}(s)\|^2+\|\phi_x(s)\|_{L^4}^4+\|\phi_{xx}(s)\|_{L^1}^{\frac{4}{3}}+\|\phi_x(s)\|^{\frac{8}{3}}\big)\,ds. \qquad (2.36)
\end{aligned}$$

Now the sub-characteristic condition (2.25) implies that we can find $\lambda \in (0, \frac{1}{2})$ and $\eta > 0$ sufficiently small such that

$$\begin{cases} \lambda a_1 - \dfrac{\bar{k}_1}{2} - 2\eta > 0, \\ \dfrac{1}{2} - \lambda - 2\eta > 0, \end{cases} \tag{2.37}$$

and moreover, Lemma 2.2 and Lemma 2.3 tell us that there is a positive constant C, which is independent of t, x, ε and δ, such that

$$\int_0^t \left(\left\| \phi_x(s) \right\| \left\| \phi_{xx}(s) \right\| + \left\| \phi_{xx}(s) \right\|_{L^1}^{\frac{4}{3}} + \left\| \phi_x(s) \right\|^{\frac{8}{3}} \right) ds \leqslant C. \tag{2.38}$$

(2.36), (2.37), (2.38) together with the Gronwall inequality imply

$$\begin{aligned} &\left\| w(t) \right\|^2 + \left\| w_x(t) \right\|^2 + \left\| w_t(t) \right\|^2 + \int_0^t \left(\left\| \sqrt{\phi_x(s)}\, w(s) \right\|^2 + \left\| w_x(s) \right\|^2 + \left\| w_t(s) \right\|^2 \right) ds \\ &\quad \leqslant C \int_0^t \left(\left\| \phi_{xx}(s) \right\|^2 + \left\| \phi_x(s) \right\|_{L^4}^4 + \left\| \phi_{xx}(s) \right\|_{L^1}^{\frac{4}{3}} + \left\| \phi_x(s) \right\|^{\frac{8}{3}} \right) ds. \end{aligned} \tag{2.39}$$

Finally from Lemma 2.2 and Lemma 2.3, we know that

$$\begin{cases} \left\| \phi_x(t) \right\|_{L^4} \leqslant O(1) \ell^{\frac{3}{8}} (1+t)^{-\frac{3}{8}}, \\ \left\| \phi_x(t) \right\| \leqslant O(1) \ell^{\frac{1}{10}} (1+t)^{-\frac{2}{5}}, \\ \left\| \phi_{xx}(t) \right\| \leqslant O(1) \ell^{\frac{1}{8}} (1+t)^{-\frac{7}{8}}, \end{cases} \tag{2.40}$$

and (2.28) follows immediately from (2.39) and (2.40). This completes the proof of Lemma 2.4. □

Based on the basic energy type estimate (2.28) on $w(t,x)$, we can get the corresponding higher order energy type estimates on $w(t,x)$, since the proofs are simpler in some sense, we just state the results and omit the details for brevity.

Lemma 2.5 *(Higher order estimates). Under the assumptions listed in Lemma* 2.4*, we have the following higher order energy type estimates on* $w(t,x)$

$$\left\| \left(w_x(t), w_t(t) \right) \right\|_4^2 + \int_0^t \left\| \left(w_x(s), w_t(s) \right) \right\|_4^2 ds \leqslant C_2 \ell^{\frac{1}{4}}. \tag{2.41}$$

Consequently

$$\left\| \left(w_x(t), w_t(t) \right) \right\|_{W^{3,\infty}(\mathbf{R})}^2 + \int_0^t \left\| \left(w_x(s), w_t(s) \right) \right\|_{W^{3,\infty}(\mathbf{R})}^2 ds \leqslant C_3 \ell^{\frac{1}{4}}. \tag{2.42}$$

Here C_2 *and* C_3 *are positive constants independent of* t, x, ε, *and* δ.

Before concluding this section, we list some properties on $\bar{u}(t,x)$.

Lemma 2.6. *The global solution* $(\bar{u}(t,x), \bar{v}_1(t,x))$ *of the Cauchy problem* (2.21), (2.22) *satisfies*

$$\begin{cases} \bar{u}_x(t,x) \geqslant 0, \quad \left|\bar{u}_t(t,x)\right| \leqslant \sqrt{a_1}\bar{u}_x(t,x), \quad t \geqslant 0,\ x \in \mathbf{R}, \\ \left\| \dfrac{\partial^i \bar{u}(t)}{\partial x^i} \right\|_{L^\infty(\mathbf{R})} \leqslant O(1)\left(\left\| \dfrac{\partial^i w(t)}{\partial x^i} \right\| + \left\| \dfrac{\partial^{i+1} w(t)}{\partial x^{i+1}} \right\| + \ell^{\frac{1}{8}}(1+t)^{-\frac{5}{8}} \right), \quad i = 1,2,3,4. \end{cases} \tag{2.43}$$

The proof of $(2.43)_1$ can be found in [20] and by noticing $\bar{u}(t,x) = w(t,x) + \phi(t,x)$, $(2.43)_2$ is a direct consequence of Sobolev's inequality, Lemma 2.2, and Lemma 2.3.

3. Energy estimates

This section is devoted to deducing some energy type estimates on the solution $(U(t,x,y), V_1(t,x,y), V_2(t,x,y))$ of (1.13)–(1.17) based on the assumption that such a solution has been extended to the time interval $[0,T)$.

Recall that $U(t,x,y)$ solves

$$U_{tt} + U_t - a_1 U_{xx} - a_2 U_{yy} + \left[f(\bar{u}+U) - f(\bar{u})\right]_x + g(\bar{u}+U)_y = 0 \tag{3.1}$$

with the initial data

$$\begin{cases} U(0,x,y) = U_0(x,y) \in H^2(\mathbf{R}^2), \\ U_t(0,x,y) = V_0(x,y) \equiv -V_{10x}(x,y) - V_{20y}(x,y) \in H^1(\mathbf{R}^2). \end{cases} \tag{3.2}$$

If we consider the Cauchy problem (3.1), (3.2) in the following Banach space:

$$X(0,T) = \left\{ U(t,x,y) \;\middle|\; \begin{array}{c} U(t,x,y) \in C^0(0,T; H^2(\mathbf{R}^2)),\ U_t(t,x,y) \in L^2(0,T; H^1(\mathbf{R}^2)), \\ U_x(t,x,y), U_y(t,x,y) \in L^2(0,T; H^1(\mathbf{R}^2)) \end{array} \right\},$$

then we have the following result on the local solvability of the Cauchy problem (3.1), (3.2):

Lemma 3.1 *(Local existence). Assume that* $U_0(x,y) \in H^2(\mathbf{R}^2)$, $(V_{10x}(x,y), V_{20y}(x,y)) \in H^1(\mathbf{R}^2)$, *then there exists a sufficiently small* t_1, *which depends only on* $\|U_0\|_2 + \|V_0\|_1$, *such that the Cauchy problem* (3.1), (3.2) *admits a unique* $U(t,x,y) \in X(0,t_1)$ *and for each* $0 \leqslant t \leqslant t_1$

$$\begin{cases} \left\|\left(U(t), \nabla U(t), U_t(t)\right)\right\|^2 \leqslant 4N_1(0), \\ \left\|\nabla\left(U_t(t), U_x(t), U_y(t)\right)\right\|^2 \leqslant 4N_2(0). \end{cases} \tag{3.3}$$

Assume that the local solution $U(t,x,y)$ constructed above has been extended to the time step $t = T \geqslant t_1$, we now turn to deduce certain energy type estimates on $U(t,x,y)$ based on the following *a priori* assumption

$$\sup_{(t,x,y)\in[0,T]\times\mathbf{R}^2} \left|U(t,x,y)\right| \leqslant M, \tag{3.4}$$

where $M > 0$ is some positive constant.

To do so, for simplicity of notations, we set

$$Q_1(M)=\sup_{|u|\leqslant B(N_0)+M}\left|f''(u)\right|^2,\qquad Q_2(M)=\sup_{|u|\leqslant B(N_0)+M}\left|g''(u)\right|^2,$$

and our result on the basic energy estimates can be stated in the following lemma:

Lemma 3.2 *(Basic energy estimates). Suppose that $U(t,x,y)\in X(0,T)$ solves the Cauchy problem* (3.1), (3.2) *and satisfies the a priori assumption* (3.4)*. If we assume further that the sub-characteristic condition* (1.5) *holds with $\mathcal{M}=[-B(N_0)-M,B(N_0)+M]$, then there is a positive constant $C_4>0$, which is independent of T, x, ℓ, and M, such that for $t\in[0,T]$, we have*

$$\left\|U_t(t)\right\|^2+\left\|\nabla U(t)\right\|^2+\left\|U(t)\right\|^2+\int_0^t\left(\left\|U_t(s)\right\|^2+\left\|\nabla U(s)\right\|^2\right)ds\leqslant C_4N_1(0). \tag{3.5}$$

Proof. Set

$$\begin{cases} k_1=\sup\limits_{|u|\leqslant B(N_0)+M}\left\{\left[f'(u)\right]^2\right\},\\ k_2=\sup\limits_{|u|\leqslant B(N_0)+M}\left\{\left[g'(u)\right]^2\right\},\end{cases}$$

then sub-characteristic condition imposed in Lemma 3.2 implies that

$$k=\frac{k_1}{a_1}+\frac{k_2}{a_2}\in(0,1).$$

Consequently we can choose

$$\lambda=\frac{3-k}{2-k},$$

and $\eta>0$ sufficiently small such that

$$\begin{cases} a_1-\dfrac{\lambda}{2}a_1-2\eta>0,\quad a_2-\dfrac{\lambda}{2}a_2-2\eta>0,\\ \lambda-1-\dfrac{\lambda k_1}{2a_1}-\dfrac{\lambda k_2}{2a_2}-2\eta>0,\quad 1<\lambda<2,\\ \dfrac{\lambda a_1}{2}-\eta>0,\quad \dfrac{\lambda a_2}{2}-\eta>0,\quad \dfrac{\lambda}{2}-\eta>0.\end{cases} \tag{3.6}$$

With the constants λ and η chosen as above, multiplying (3.1) by λU_t+U and integrating the result with respect to t, x, y over $[0,T]\times\mathbf{R}^2$, we have by some integrations by parts that

$$\frac{1}{2}\left\|U(t)\right\|^2+\frac{\lambda}{2}\left\|U_t(t)\right\|^2+\frac{\lambda a_1}{2}\left\|U_x(t)\right\|^2+\frac{\lambda a_2}{2}\left\|U_y(t)\right\|^2$$
$$+\int_0^t\left((\lambda-1)\left\|U_t(s)\right\|^2+a_1\left\|U_x(s)\right\|^2+a_2\left\|U_y(s)\right\|^2\right)ds+\int_{\mathbf{R}^2}(UU_t)(t,x,y)\,dx\,dy$$

$$= \int_{\mathbf{R}^2} (UU_t)(0,x,y)\,dx\,dy + \frac{1}{2}\|U(0)\|^2 + \frac{\lambda}{2}\|U_t(0)\|^2 + \frac{\lambda a_1}{2}\|U_x(0)\|^2 + \frac{\lambda a_2}{2}\|U_y(0)\|^2$$
$$- \int_0^t \int_{\mathbf{R}^2} \left\{(U+\lambda U_t)\left[\left(f(\bar{u}+U)-f(\bar{u})\right)_x + g(\bar{u}+U)_y\right]\right\}(s,x,y)\,ds\,dx\,dy. \tag{3.7}$$

Due to

$$- \int_0^t \int_{\mathbf{R}^2} \left\{(U+\lambda U_t)\left[\left(f(\bar{u}+U)-f(\bar{u})\right)_x + g(\bar{u}+U)_y\right]\right\}(s,x,y)\,ds\,dx\,dy$$
$$= \underbrace{- \int_0^t \int_{\mathbf{R}^2} \left\{U\left[f(\bar{u}+U)-f(\bar{u})\right]_x\right\}(s,x,y)\,ds\,dx\,dy}_{I_1}$$
$$\underbrace{- \int_0^t \int_{\mathbf{R}^2} \left\{Ug(U+\bar{u})_y\right\}(s,x,y)\,ds\,dx\,dy}_{I_2}$$
$$\underbrace{- \lambda \int_0^t \int_{\mathbf{R}^2} \left\{U_t\left[f(\bar{u}+U)-f(\bar{u})\right]_x\right\}(s,x,y)\,ds\,dx\,dy}_{I_3}$$
$$\underbrace{- \lambda \int_0^t \int_{\mathbf{R}^2} \left\{U_t g(U+\bar{u})_y\right\}(s,x,y)\,ds\,dx\,dy,}_{I_4}$$

to prove Lemma 3.2, we only need to control I_j $(j = 1,2,3,4)$ suitably. For this purpose, we have from Lemma 2.6 that

$$I_1 = - \int_0^t \int_{\mathbf{R}^2} \left\{ \left[\int_{\bar{u}}^{\bar{u}+U} f(z)\,dz - f(\bar{u})U \right]_x + \left[f(\bar{u}+U) - f(\bar{u}) - f'(\bar{u})U\right]\bar{u}_x \right\}(s,x,y)\,ds\,dx\,dy$$
$$= - \int_0^t \int_{\mathbf{R}^2} \left(f(\bar{u}+U) - f(\bar{u}) - f'(\bar{u})U\right)(s,x,y)\bar{u}_x(s,x)\,ds\,dx\,dy$$
$$\leqslant 0 \tag{3.8}$$

and

$$\begin{aligned} I_2 &= \int_0^t \int_{\mathbf{R}^2} \big\{ U_y g(U+\bar u) \big\}(s,x,y)\,dx\,dy\,ds \\ &= \int_0^t \int_{\mathbf{R}^2} \left\{ \int_{\bar u}^{\bar u + U} g(z)\,dz \right\}_y (s,x,y)\,ds\,dx\,dy \\ &= 0. \end{aligned} \tag{3.9}$$

As to I_4, the Cauchy–Schwarz inequality gives

$$|I_4| \leqslant \frac{\lambda k_2}{2a_2} \int_0^t \big\| U_t(s) \big\|^2 ds + \frac{\lambda a_2}{2} \int_0^t \big\| U_y(s) \big\|^2 ds. \tag{3.10}$$

At last we deal with I_3. To this end, due to

$$\begin{aligned} I_3 = & \underbrace{-\lambda \int_0^t \int_{\mathbf{R}^2} \big\{ U_t \bar u_x \big[f'(\bar u + U) - f'(\bar u) \big] \big\}(s,x,y)\,ds\,dx\,dy}_{I_3^1} \\ & \underbrace{- \lambda \int_0^t \int_{\mathbf{R}^2} \big\{ U_t U_x f'(\bar u + U) \big\}(s,x,y)\,ds\,dx\,dy}_{I_3^2}, \end{aligned} \tag{3.11}$$

we now estimate I_3^i $(i=1,2)$ term by term. Firstly we have from the Cauchy–Schwarz inequality that

$$\big|I_3^2\big| \leqslant \frac{\lambda k_1}{2a_1} \int_0^t \big\| U_t(s) \big\|^2 ds + \frac{\lambda a_1}{2} \int_0^t \big\| U_x(s) \big\|^2 ds. \tag{3.12}$$

For I_3^1, notice from the sub-characteristic condition (1.5), (2.23), (2.24), and the fact the $f(u)$ is sufficiently smooth that

$$\begin{aligned} \big| f'(\bar u + U) - f'(\bar u) \big| &\leqslant \left\{ \max_{|U| \leqslant 1} \left| \frac{f'(\bar u + U) - f'(\bar u)}{U} \right| + \max_{|U| \geqslant 1} \left| \frac{f'(\bar u + U) - f'(\bar u)}{U} \right| \right\} |U| \\ &\leqslant \left\{ \max_{|U| \leqslant 1} \left| \frac{f'(\bar u + U) - f'(\bar u)}{U} \right| + \max_{|U| \geqslant 1} \big| f'(\bar u + U) - f'(\bar u) \big| \right\} |U| \\ &\leqslant \Big\{ \max_{|u| \leqslant 1 + B(N_0)} \big| f''(u) \big| + \max_{|U| \geqslant 1} \big| f'(\bar u + U) - f'(\bar u) \big| \Big\} |U| \\ &\leqslant O(1)|U|, \end{aligned} \tag{3.13}$$

we have

$$\begin{aligned}\left|I_3^1\right| &\leqslant O(1)\int_0^t\int_{\mathbf{R}^2}\left|\bar{u}_x(s,x)\right|^2 U^2(s,x,y)\,dx\,dy\,ds+\eta\int_0^t\left\|U_t(s)\right\|^2 ds\\ &\leqslant O(1)\int_0^t\left\|\bar{u}_x(s)\right\|_{L^\infty}^2\left\|U(s)\right\|^2 ds+\eta\int_0^t\left\|U_t(s)\right\|^2 ds. \end{aligned}\tag{3.14}$$

Putting (3.12) and (3.14) into (3.11), we get that

$$\begin{aligned}|I_3| &\leqslant O(1)\int_0^t\left\|\bar{u}_x(s)\right\|_{L^\infty}^2\left\|U(s)\right\|^2 ds\\ &\quad+\left(\frac{\lambda k_1}{2a_1}+\eta\right)\int_0^t\left\|U_t(s)\right\|^2 ds+\frac{\lambda a_1}{2}\int_0^t\left\|U_x(s)\right\|^2 ds. \end{aligned}\tag{3.15}$$

Substituting (3.8), (3.9), (3.10), and (3.15) into (3.7), we finally arrive at

$$\begin{aligned}&\frac{1}{2}\left\|U(t)\right\|^2+\frac{\lambda}{2}\left\|U_t(t)\right\|^2+\frac{\lambda a_1}{2}\left\|U_x(t)\right\|^2+\frac{\lambda a_2}{2}\left\|U_y(t)\right\|^2+\int_{\mathbf{R}^2}(UU_t)(t,x,y)\,dx\,dy\\ &\quad+\int_0^t\left\{\left(\lambda-1-\frac{\lambda k_1}{2a_1}-\frac{\lambda k_2}{2a_2}-\eta\right)\left\|U_t(s)\right\|^2+\frac{(2-\lambda)a_1}{2}\left\|U_x(s)\right\|^2+\frac{(2-\lambda)a_2}{2}\left\|U_y(s)\right\|^2\right\}ds\\ &\leqslant\int_{\mathbf{R}^2}UU_t(0,x,y)\,dx\,dy+\frac{1}{2}\left\|U(0)\right\|^2+\frac{\lambda}{2}\left\|U_t(0)\right\|^2+\frac{\lambda a_1}{2}\left\|U_x(0)\right\|^2+\frac{\lambda a_2}{2}\left\|U_y(0)\right\|^2\\ &\quad+O(1)\int_0^t\left\|\bar{u}_x(s)\right\|_{L^\infty}^2\left\|U(s)\right\|^2 ds. \end{aligned}\tag{3.16}$$

(3.6), (3.16) together with the Gronwall inequality implies

$$\begin{aligned}&\left\|\left(U(t),U_x(t),U_y(s),U_t(t)\right)\right\|^2+\int_0^t\left\|\left(U_t(s),U_x(s),U_y(s)\right)\right\|^2 ds\\ &\quad\leqslant O(1)N_1(0)\exp\left(O(1)\int_0^t\left\|\bar{u}_x(s)\right\|_{L^\infty}^2 ds\right). \end{aligned}\tag{3.17}$$

On the other hand, Lemma 2.2, Lemma 2.4, Lemma 2.5, and Lemma 2.6 tell us that

$$\begin{aligned}\int_0^t\left\|\bar{u}_x(s)\right\|_{L^\infty}^2 ds &\leqslant O(1)\int_0^t\left(\left\|w_x(s)\right\|^2+\left\|w_{xx}(s)\right\|^2+\ell^{\frac{1}{4}}(1+s)^{-\frac{5}{4}}\right)ds\\ &\leqslant O(1)\ell^{\frac{1}{4}}. \end{aligned}\tag{3.18}$$

Having obtained (3.17) and (3.18), (3.5) follows immediately. This completes the proof of Lemma 3.2. □

Now we turn to deduce the second-order energy type estimates on $U(t,x,y)$. For result in this direction, we have

Lemma 3.3. *Under the assumptions listed in Lemma* 3.1, *if we assume further that*

$$\ell \cdot \big(Q_1^8(M) + Q_2^8(M)\big) \leqslant 1, \tag{3.19}$$

we have

$$\begin{aligned} &\big\|(\nabla U_t, \nabla U_x, \nabla U_y)(t)\big\|^2 + \int_0^t \big\|(\nabla U_t, \nabla U_x, \nabla U_y)(s)\big\|^2 \, ds \\ &\quad \leqslant C_5\big(N_1(0) + N_2(0)\big) \exp\big(C_5\big(Q_1^2(M) + Q_1^2(M)\big)N_1(0)\big). \end{aligned} \tag{3.20}$$

Here C_5 is a positive constant independent of ℓ, t, x, y, and M.

Proof. With the constants λ and η chosen as in Lemma 3.2, we have by performing ∇ to (3.1), multiplying the resulting identity by $\nabla U + \lambda \nabla U_t$, and integrating the final result with respect to t, x, y over $[0,t] \times \mathbf{R}^2$ that

$$\begin{aligned} &\frac{1}{2}\big\|\nabla U(t)\big\|^2 + \frac{\lambda}{2}\big\|\nabla U_t(t)\big\|^2 + \frac{\lambda a_1}{2}\big\|\nabla U_x(t)\big\|^2 + \frac{\lambda a_2}{2}\big\|\nabla U_y(t)\big\|^2 \\ &\quad + \int_0^t \big((\lambda - 1)\big\|\nabla U_t(s)\big\|^2 + a_1\big\|\nabla U_x(s)\big\|^2 + a_2\big\|\nabla U_y(s)\big\|^2\big) \, ds \\ &\quad + \int_{\mathbf{R}^2} \nabla U(t,x,y) \cdot \nabla U_t(t,x,y) \, dx\,dy \\ &= \int_{\mathbf{R}^2} \nabla U(0,x,y) \cdot \nabla U_t(0,x,y) \, dx\,dy + \frac{1}{2}\big\|\nabla U(0)\big\|^2 + \frac{\lambda}{2}\big\|\nabla U_t(0)\big\|^2 \\ &\quad + \frac{\lambda a_1}{2}\big\|\nabla U_x(0)\big\|^2 + \frac{\lambda a_2}{2}\big\|\nabla U_y(0)\big\|^2 \\ &\quad + \underbrace{\int_0^t \int_{\mathbf{R}^2} \big\{\nabla U_x \cdot \nabla\big[f(\bar{u} + U) - f(\bar{u})\big]\big\}(s,x,y) \, ds\,dx\,dy}_{I_5} \\ &\quad + \underbrace{\int_0^t \int_{\mathbf{R}^2} \big\{\nabla U_y \cdot \nabla\big[g(U + \bar{u}) - g(\bar{u})\big]\big\}(s,x,y) \, ds\,dx\,dy}_{I_6} \end{aligned}$$

$$\underbrace{-\lambda\int_0^t\int_{\mathbf{R}^2}\big\{\nabla U_t\cdot\nabla\big[f(\bar u+U)-f(\bar u)\big]_x\big\}(s,x,y)\,ds\,dx\,dy}_{I_7}$$

$$\underbrace{-\lambda\int_0^t\int_{\mathbf{R}^2}\big\{\nabla U_t\cdot\nabla g(U+\bar u)_y\big\}(s,x,y)\,ds\,dx\,dy}_{I_8}. \tag{3.21}$$

Now we deal with I_j $(j=5,6,7,8)$ term by term. To this end, we first deduce from the sub-characteristic condition (1.5) with $\mathcal{M}=[-B(N_0)-M,B(N_0)+M]$, the estimates (3.5), (3.13), and the *a priori* assumption (3.4) that

$$\begin{aligned}|I_5|&\leqslant\eta\int_0^t\big\|\nabla U_x(s)\big\|^2\,ds+O(1)\int_0^t\big\|\nabla U(s)\big\|^2\,ds+O(1)\int_0^t\big\|\bar u_x(s)\big\|^2_{L^\infty}\big\|U(s)\big\|^2\,ds\\&\leqslant\eta\int_0^t\big\|\nabla U_x(s)\big\|^2\,ds+O(1)N_1(0)\\&\leqslant O(1)N_1(0)+\eta\int_0^t\big\|\nabla U_x(s)\big\|^2\,ds,\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}|I_6|&\leqslant\eta\int_0^t\big\|\nabla U_y(s)\big\|^2\,ds+O(1)\int_0^t\big\|\nabla U(s)\big\|^2\,ds+O(1)\int_0^t\big\|\bar u_x(s)\big\|^2_{L^\infty}\big\|U(s)\big\|^2\,ds\\&\leqslant\eta\int_0^t\big\|\nabla U_y(s)\big\|^2\,ds+O(1)N_1(0)\\&\leqslant O(1)N_1(0)+\eta\int_0^t\big\|\nabla U_y(s)\big\|^2\,ds.\end{aligned}\tag{3.23}$$

The estimation on I_7 is a little bitter complex. To control such a term, notice that

$$\begin{aligned}|I_7|\leqslant{}&\underbrace{\lambda\int_0^t\int_{\mathbf{R}^2}\big(\nabla U_t\cdot f'(\bar u+U)\nabla U_x\big)(s,x,y)\,dx\,dy\,ds}_{I_7^1}\\&\underbrace{+\lambda\int_0^t\int_{\mathbf{R}^2}\big(\nabla U_t\cdot\big[f'(\bar u+U)-f'(\bar u)\big]\nabla\bar u_x\big)(s,x,y)\,dx\,dy\,ds}_{I_7^2}\end{aligned}$$

$$+\underbrace{\lambda\int_0^t\int_{\mathbf{R}^2}\big(\nabla U_t\cdot f''(\bar u+U)\bar u_x\nabla U\big)(s,x,y)\,dx\,dy\,ds}_{I_7^3}$$

$$+\underbrace{\lambda\int_0^t\int_{\mathbf{R}^2}\big(\nabla U_t\cdot f''(\bar u+U)U_x\nabla U\big)(s,x,y)\,dx\,dy\,ds}_{I_7^4}$$

$$+\underbrace{\lambda\int_0^t\int_{\mathbf{R}^2}\big(\nabla U_t\cdot f''(\bar u+U)U_x\nabla\bar u\big)(s,x,y)\,dx\,dy\,ds}_{I_7^5}$$

$$+\underbrace{\lambda\int_0^t\int_{\mathbf{R}^2}\big(\nabla U_t\cdot\big[f''(\bar u+U)-f''(\bar u)\big]\nabla\bar u\bar u_x\big)(s,x,y)\,dx\,dy\,ds}_{I_7^6},\tag{3.24}$$

we have from the Cauchy–Schwarz inequality that

$$I_7^1\leqslant\frac{\lambda k_1}{2a_1}\int_0^t\big\|\nabla U_t(s)\big\|^2\,ds+\frac{\lambda a_1}{2}\int_0^t\big\|\nabla U_x(s)\big\|^2\,ds.\tag{3.25}$$

On the other hand, Lemma 2.6, (3.5), (3.13), (3.18) together with the assumption (3.19) imply that

$$\begin{aligned}\big|I_7^2\big|&\leqslant\frac{\eta}{10}\int_0^t\big\|\nabla U_t(s)\big\|^2\,ds+O(1)\int_0^t\big\|\bar u_{xx}(s)\big\|_{L^\infty}^2\big\|U(s)\big\|^2\,ds\\&\leqslant\frac{\eta}{10}\int_0^t\big\|\nabla U_t(s)\big\|^2\,ds+O(1)N_1(0),\end{aligned}\tag{3.26}$$

$$\begin{aligned}\big|I_7^3\big|+\big|I_7^5\big|&\leqslant\frac{\eta}{10}\int_0^t\big\|\nabla U_t(s)\big\|^2\,ds+O(1)Q_1^2(M)\int_0^t\big\|\bar u_x(s)\big\|_{L^\infty}^2\big\|\nabla U(s)\big\|^2\,ds\\&\leqslant\frac{\eta}{10}\int_0^t\big\|\nabla U_t(s)\big\|^2\,ds+O(1)N_1(0),\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}|I_7^6| &\leqslant \frac{\eta}{10}\int_0^t \|\nabla U_t(s)\|^2\,ds + O(1)Q_1^2(M)\int_0^t \|\bar{u}_x(s)\|_{L^\infty}^4 \|U(s)\|^2\,ds\\ &\leqslant \frac{\eta}{10}\int_0^t \|\nabla U_t(s)\|^2\,ds + O(1)N_1(0).\end{aligned} \tag{3.28}$$

Here in deducing (3.28), similar to that of (3.13), we have used the fact that

$$\begin{aligned}\left|f''(\bar{u}+U) - f''(\bar{u})\right| &\leqslant \max\Big\{\max_{|u|\leqslant 1+B(N_0)}\left|f'''(u)\right|, 2\max_{|u|\leqslant B(N_0)+M}\left|f''(u)\right|\Big\}|U|\\ &\leqslant O(1)\big(1+Q_1(M)\big)|U|.\end{aligned} \tag{3.29}$$

Now for I_7^4, noticing that (ii) of Corollary 2.1 with $f = U_x$, $g = \nabla U$ implies

$$\int_{\mathbf{R}^2} \big(|U_x|^2|\nabla U|^2\big)(s,x,y)\,dx\,dy \leqslant \|\nabla U(s)\|^2\big(\|\nabla U_x(s)\|^2 + \|\nabla U_y(s)\|^2\big), \tag{3.30}$$

we can get that

$$\begin{aligned}|I_7^4| &\leqslant \frac{\eta}{10}\int_0^t \|\nabla U_t(s)\|^2\,ds + O(1)Q_1^2(M)\int_0^t\int_{\mathbf{R}^2} \big(|U_x|^2|\nabla U|^2\big)(s,x,y)\,dx\,dy\\ &\leqslant \frac{\eta}{10}\int_0^t \|\nabla U_t(s)\|^2\,ds + O(1)Q_1^2(M)\int_0^t \|\nabla U(s)\|^2\big(\|\nabla U_x(s)\|^2 + \|\nabla U_y(s)\|^2\big)\,ds.\end{aligned} \tag{3.31}$$

Inserting (3.25)–(3.28), (3.31) into (3.24), we can conclude that

$$\begin{aligned}|I_7| &\leqslant O(1)N_1(0) + \left(\frac{\lambda k_1}{2a_1} + \frac{2}{5}\eta\right)\int_0^t \|\nabla U_t(s)\|^2\,ds + \frac{\lambda a_1}{2}\int_0^t \|\nabla U_x(s)\|^2\,ds\\ &\quad + O(1)Q_1^2(M)\int_0^t \|\nabla U(s)\|^2\big(\|\nabla U_x(s)\|^2 + \|\nabla U_y(s)\|^2\big)\,ds.\end{aligned} \tag{3.32}$$

At last, for I_8, similar to that of I_7, we have

$$\begin{aligned}|I_8| &\leqslant O(1)N_1(0) + \left(\frac{\lambda k_2}{2a_2} + \frac{2}{5}\eta\right)\int_0^t \|\nabla U_t(s)\|^2\,ds + \frac{\lambda a_2}{2}\int_0^t \|\nabla U_y(s)\|^2\,ds\\ &\quad + O(1)Q_2^2(M)\int_0^t \|\nabla U(s)\|^2\big(\|\nabla U_x(s)\|^2 + \|\nabla U_y(s)\|^2\big)\,ds.\end{aligned} \tag{3.33}$$

Putting (3.21)–(3.23), (3.32), (3.33) together, we finally arrive at

$$\frac{1}{2}\left\|\nabla U(t)\right\|^2+\frac{\lambda}{2}\left\|\nabla U_t(t)\right\|^2+\frac{\lambda a_1}{2}\left\|\nabla U_x(t)\right\|^2+\frac{\lambda a_2}{2}\left\|\nabla U_y(t)\right\|^2$$
$$+\int_0^t\left(\left(\lambda-1-\frac{\lambda k_1}{2a_1}-\frac{\lambda k_2}{2a_2}-\eta\right)\left\|\nabla U_t(s)\right\|^2+\left(\frac{(2-\lambda)a_1}{2}-\eta\right)\left\|\nabla U_x(s)\right\|^2\right.$$
$$\left.+\left(\frac{(2-\lambda)a_2}{2}-\eta\right)\left\|\nabla U_y(s)\right\|^2\right)ds+\int_{\mathbf{R}^2}\nabla U(t,x,y)\cdot\nabla U_t(t,x,y)\,dx\,dy$$
$$\leqslant O(1)\big(N_1(0)+N_2(0)\big)$$
$$+O(1)\big(Q_1^2(M)+Q_2^2(M)\big)\int_0^t\left\|\nabla U(s)\right\|^2\left\|\big(\nabla U_x(s),\nabla U_y(s)\big)\right\|^2ds. \tag{3.34}$$

Having obtained (3.34), (3.20) follows from (3.6) and the Gronwall inequality and this completes the proof of Lemma 3.3. □

It is worth to pointing out that, even under the assumption (3.19), the right hand side of the estimate (3.20) does depend on M, we note, however, that if both $f''(u)$ and $g''(u)$ are uniformly bounded, then the assumption (3.19) is unnecessary and we have from Lemma 3.2 and Lemma 3.3 that:

Corollary 3.1. *In addition to the assumption listed in Lemma* 3.2*, we assume further that both $f''(u)$ and $g''(u)$ are uniformly bounded, then we have*

$$\left\|U(t)\right\|_2^2+\left\|U_t(t)\right\|_1^2+\int_0^t\left(\left\|U_t(s)\right\|_1^2+\left\|\nabla U(s)\right\|_1^2\right)ds$$
$$\leqslant C_6\big(N_1(0)+N_2(0)\big)\exp\big(C_6N_1(0)\big). \tag{3.35}$$

Here $C_6>0$ is some positive constant independent of t, x, y, and M.

As pointed out before, the right hand side of the estimate (3.20) obtained in Lemma 3.3 does depend on M, what we are concerned with in the following is the following problem: Under which conditions on the nonlinear flux functions $f(u)$ and $g(u)$, can the second-order energy type estimates on $U(t,x,y)$ be bounded by some M-independent positive constant which depends only on the $H^2(\mathbf{R}^2)$-norm of the initial data? For result in this direction, we have

Lemma 3.4. *In addition to the assumptions listed in Lemma* 3.2*, we assume further that $f'''(u)$ and $g'''(u)$ are uniformly bounded, then we have*

$$\left\|\big(\nabla U_t(t),\nabla U_x(t),\nabla U_y(t)\big)\right\|^2+\int_0^t\left\|\big(\nabla U_t(s),\nabla U_x(s),\nabla U_y(s)\big)\right\|^2ds$$
$$\leqslant C_7\big(N_1(0)+N_2(0)\big)^5\exp\big(C_7N_1^4(0)\big). \tag{3.36}$$

Here C_7 is some positive constant independent of t, x, y, and M.

Proof. Compared with that of Lemma 3.3, the main new points in deducing Lemma 3.4 are the following:

- We do not use the smallness of the parameter ℓ to control the possible growth of the solutions caused by the nonlinearity of the equation under consideration, i.e. the assumption (3.19) is not imposed in Lemma 3.4.
- The right hand side of the estimate (3.36) is independent of M.

The main differences to deduce (3.36) are the way to control I_7 and I_8, especially the following terms

$$I_9 = I_7^3 + I_7^5 + I_7^6 - \lambda \int_0^t \int_{\mathbf{R}^2} \left\{ g''(\bar{u}+U) U_y \nabla U_t \cdot \nabla \bar{u} \right\}(s,x,y)\, dx\, dy\, ds \tag{3.37}$$

and

$$\begin{aligned} I_{10} &= -\lambda \int_0^t \int_{\mathbf{R}^2} \left\{ f''(\bar{u}+U) U_x \nabla U_t \cdot \nabla U + g''(\bar{u}+U) U_y \nabla U_t \cdot \nabla U \right\}(s,x,y)\, ds\, dx\, dy \\ &= \underbrace{-\lambda \int_0^t \int_{\mathbf{R}^2} \left\{ f''(\bar{u}+U) U_x^2 U_{xt} \right\}(s,x,y)\, dx\, dy\, ds}_{I_{10}^1} \\ &\quad \underbrace{-\lambda \int_0^t \int_{\mathbf{R}^2} \left\{ g''(\bar{u}+U) U_y^2 U_{yt} \right\}(s,x,y)\, dx\, dy\, ds}_{I_{10}^2} \\ &\quad \underbrace{-\lambda \int_0^t \int_{\mathbf{R}^2} \left\{ f''(\bar{u}+U) U_x U_y U_{yt} \right\}(s,x,y)\, dx\, dy\, ds}_{I_{10}^3} \\ &\quad \underbrace{-\lambda \int_0^t \int_{\mathbf{R}^2} \left\{ g''(\bar{u}+U) U_x U_y U_{xt} \right\}(s,x,y)\, dx\, dy\, ds}_{I_{10}^4}. \end{aligned} \tag{3.38}$$

In fact from Corollary 2.1, Lemma 3.2, the facts that $|f'''(u)| \leqslant O(1)$, $|f''(u)| \leqslant O(1)(1+|u|)$, we can deduce that

$$|I_9| \leqslant \frac{\eta}{5} \int_0^t \left\| \nabla U_t(s) \right\|^2 ds + O(1) \int_0^t \int_{\mathbf{R}^2} \left\{ \left(1+|U|^2\right) |\nabla U|^2 + |U|^2 |\bar{u}_x|^4 \right\}(s,x,y)\, dx\, dy\, ds$$

$$\leqslant O(1)N_1(0)+\frac{\eta}{5}\int_0^t \left\|\nabla U_t(s)\right\|^2 ds + O(1)\int_0^t \left\|U(s)\right\|\left\|\nabla U(s)\right\|^2\left\|\nabla\big(U_x(s),U_y(s)\big)\right\| ds$$

$$\leqslant O(1)\big(N_1(0)+N_1(0)\big)^3+\frac{\eta}{5}\int_0^t \left\|\nabla\big(U_t(s),U_x(s),U_y(s)\big)\right\|^2 ds. \tag{3.39}$$

Here it is worth pointing out that since our main purpose is trying to deal with the case with large initial perturbation, we have assumed that $N_1(0)\geqslant 1$ and $N_2(0)\geqslant 1$ in this lemma.

As to I_{10}^i $(i=1,2,3,4)$, due to

$$I_{10}^1=-\frac{\lambda}{3}\int_{\mathbf{R}^2}\left\{f''(\bar u+U)U_x^3\right\}(t,x,y)\,dx\,dy+\frac{\lambda}{3}\int_{\mathbf{R}^2}\left\{f''(\bar u+U)U_x^3\right\}(0,x,y)\,dx\,dy$$
$$+\frac{\lambda}{3}\int_0^t\int_{\mathbf{R}^2}\left\{f'''(\bar u+U)U_x^3(\bar u_t+U_t)\right\}(s,x,y)\,dx\,dy\,ds, \tag{3.40}$$

we have from Corollary 2.1, Lemma 3.2, the Cauchy–Schwarz inequality and the fact that $|f''(u)|\leqslant O(1)(1+|u|)$ that

$$\frac{\lambda}{3}\left|\int_{\mathbf{R}^2}\left\{f''(\bar u+U)U_x^3\right\}(t,x,y)\,dx\,dy\right|\leqslant O(1)\int_{\mathbf{R}^2}\left\{\big(1+|U|\big)|U_x|^3\right\}(t,x,y)\,dx\,dy$$
$$\leqslant O(1)\Big(\left\|U(t)\right\|^{\frac12}\left\|\nabla U(t)\right\|^2\left\|\nabla U_x(t)\right\|^{\frac32}+\left\|\nabla U(t)\right\|^2\left\|\nabla U_x(t)\right\|\Big)$$
$$\leqslant\frac{\eta}{10}\left\|\nabla U_x(t)\right\|^2+O(1)N_1(0)^5$$

and

$$\frac{\lambda}{3}\left|\int_{\mathbf{R}^2}\left\{f''(\bar u+U)U_x^3\right\}(0,x,y)\,dx\,dy\right|\leqslant O(1)N_1(0)^{\frac54}N_2(0)^{\frac34}.$$

As to the last term in the right hand side of (3.40), we have from Corollary 2.1, Lemma 3.2, and the fact that $f'''(u)$ is uniformly bounded that

$$\left|\frac{\lambda}{3}\int_0^t\int_{\mathbf{R}^2}\left\{f'''(\bar u+U)U_x^3(\bar u_t+U_t)\right\}(s,x,y)\,dx\,dy\,ds\right|$$
$$\leqslant O(1)\int_0^t\Big(\left\|U_x(s)\right\|^2\left\|\nabla_x U(s)\right\|+\left\|U_x(s)\right\|^{\frac32}\left\|\nabla_x U_x(s)\right\|^{\frac32}\left\|U_t(s)\right\|^{\frac12}\left\|\nabla U_t(s)\right\|^{\frac12}\Big)ds$$
$$\leqslant\frac{\eta}{10}\int_0^t\left\|\nabla U_t(s)\right\|^2 ds+O(1)N_1(0)^3\int_0^t\left\|\nabla U(s)\right\|^2\left\|\nabla U_x(s)\right\|^2 ds.$$

Inserting the above three estimates into to (3.40), we can deduce that

$$\begin{aligned}\left|I_{10}^1\right| &\leqslant O(1)\big(N_1(0)+N_2(0)\big)^5+\frac{\eta}{20}\left\|\nabla U_x(t)\right\|^2\\&\quad+\frac{\eta}{20}\int_0^t\left\|\nabla U_t(s)\right\|^2ds+O(1)N_1(0)^3\int_0^t\left\|\nabla U(s)\right\|^2\left\|\nabla U_x(s)\right\|^2ds.\end{aligned}\tag{3.41}$$

Similarly we have

$$\begin{aligned}\left|I_{10}^2\right| &\leqslant O(1)\big(N_1(0)+N_2(0)\big)^5+\frac{\eta}{20}\left\|\nabla U_y(t)\right\|^2\\&\quad+\frac{\eta}{20}\int_0^t\left\|\nabla U_t(s)\right\|^2ds+O(1)N_1(0)^3\int_0^t\left\|\nabla U(s)\right\|^2\left\|\nabla U_y(s)\right\|^2ds.\end{aligned}\tag{3.42}$$

Now we turn to deal with I_{10}^3, to this end, we have from

$$\begin{aligned}2U_{xy}U_tU_yf''(\bar u+U) &= \big[f'(\bar u+U)U_yU_{xy}\big]_t-\big[f'(\bar u+U)U_yU_{ty}\big]_x\\&\quad+\big[f'(\bar u+U)U_yU_tU_x\big]_y-f''(\bar u+U)\bar u_tU_yU_{ty}\\&\quad+f''(\bar u+U)\bar u_xU_yU_{ty}-f'''(\bar u+U)U_xU_y^2U_t\\&\quad-f''(\bar u+U)U_xU_tU_{yy},\end{aligned}\tag{3.43}$$

and

$$U_{yy}=\frac{1}{a_2}\big\{U_{tt}+U_t-a_1U_{xx}+\big(f(\bar u+U)-f(\bar u)\big)_x+g(\bar u+U)_y\big\}$$

that

$$\begin{aligned}2U_{xy}U_tU_yf''(\bar u+U) &= \big[f'(\bar u+U)U_yU_{xy}\big]_t-\big[f'(\bar u+U)U_yU_{ty}\big]_x\\&\quad+\big[f'(\bar u+U)U_yU_tU_x\big]_y-f''(\bar u+U)\bar u_tU_yU_{ty}\\&\quad+f''(\bar u+U)\bar u_xU_yU_{ty}-f'''(\bar u+U)U_xU_y^2U_t\\&\quad-\frac{f''(\bar u+U)U_xU_tU_{tt}}{a_2}+\frac{f''(\bar u+U)U_xU_tU_t}{a_2}\\&\quad+\frac{f''(\bar u+U)[f(\bar u+U)-f(\bar u)]_xU_xU_t}{a_2}\\&\quad+\frac{f''(\bar u+U)g(\bar u+U)_yU_xU_t}{a_2}\\&\quad-\frac{a_1f''(\bar u+U)U_xU_tU_{xx}}{a_2}.\end{aligned}\tag{3.44}$$

Consequently

$$I_{10}^3 = \underbrace{-\frac{\lambda}{2}\int_{\mathbf{R}^2}\left\{f'(\bar u+U)U_yU_{xy}\right\}(s,x,y)\,dx\,dy\Big|_{s=0}^{s=t}}_{J_{103}^1}$$

$$+\underbrace{\frac{\lambda}{2}\int_0^t\int_{\mathbf{R}^2}\left\{f''(\bar u+U)(\bar u_t-\bar u_x)U_yU_{ty}\right\}(s,x,y)\,dx\,dy\,ds}_{J_{103}^2}$$

$$+\underbrace{\frac{\lambda}{2}\int_0^t\int_{\mathbf{R}^2}\left\{f'''(\bar u+U)U_xU_y^2U_t\right\}(s,x,y)\,dx\,dy\,ds}_{J_{103}^3}$$

$$+\underbrace{\frac{\lambda}{2a_2}\int_0^t\int_{\mathbf{R}^2}\left\{f''(\bar u+U)U_xU_tU_{tt}\right\}(s,x,y)\,dx\,dy\,ds}_{J_{103}^4}$$

$$-\underbrace{\frac{\lambda}{2a_2}\int_0^t\int_{\mathbf{R}^2}\left\{f''(\bar u+U)U_xU_t^2\right\}(s,x,y)\,dx\,dy\,ds}_{J_{103}^5}$$

$$-\underbrace{\frac{\lambda}{2a_2}\int_0^t\int_{\mathbf{R}^2}\left\{f''(\bar u+U)\big(f(\bar u+U)-f(\bar u)\big)_xU_xU_t\right\}(s,x,y)\,dx\,dy\,ds}_{J_{103}^6}$$

$$-\underbrace{\frac{\lambda}{2a_2}\int_0^t\int_{\mathbf{R}^2}\left\{f''(\bar u+U)g(\bar u+U)_yU_xU_t\right\}(s,x,y)\,dx\,dy\,ds}_{J_{103}^7}$$

$$-\underbrace{\frac{\lambda a_1}{2a_2}\int_0^t\int_{\mathbf{R}^2}\left\{f''(\bar u+U)U_xU_tU_{xx}\right\}(s,x,y)\,dx\,dy\,ds}_{J_{103}^8} \tag{3.45}$$

and J_{103}^i $(i=1,2,\dots,8)$ can be estimated as in the following: First from the sub-characteristic condition (1.5) with $\mathcal{M}=[-B(N_0)-M, B(N_0)+M]$, Corollary 2.1, Lemma 3.2, Cauchy–Schwarz's inequality, the facts that $f'''(u)$ is uniformly bounded and $|f''(u)|\leqslant O(1)(1+|u|)$ that

$$\left|J_{103}^1\right|\leqslant O(1)\big(N_1(0)+N_2(0)\big)+\frac{\eta}{20}\left\|\nabla U_y(t)\right\|^2, \tag{3.46}$$

and

$$\sum_{i\neq 1,4,8}\left|J_{103}^{i}\right| \leqslant O(1)\int_0^t\int_{\mathbf{R}^2}\left\{\left(1+|U|\right)|\nabla U|\left(|\nabla U_t|+|U_t|\left(|U_t|+|U|+|\nabla U|\right)\right)\right\}(s,x,y)\,dx\,dy\,ds$$

$$+\,O(1)\int_0^t\int_{\mathbf{R}^2}\left|U_t(s,x,y)\right|\left|\nabla U(s,x,y)\right|^3\,dx\,dy\,ds$$

$$\leqslant O(1)N_1(0)+\frac{\eta}{20}\int_0^t\left\|\nabla U_t(s)\right\|^2\,ds$$

$$+\,O(1)\int_0^t\int_{\mathbf{R}^2}\left\{|\nabla U|\left(|U|^2\left(|\nabla U|+|U_t|\right)+\left(1+|U|\right)U_t^2\right)\right\}(s,x,y)\,dx\,dy\,ds$$

$$+\,O(1)\int_0^t\int_{\mathbf{R}^2}\left\{|\nabla U|^2|U_t|\left(|U|+|\nabla U|\right)\right\}(s,x,y)\,dx\,dy\,ds$$

$$\leqslant O(1)\left(N_1(0)+N_1(0)^3\right)+\frac{\eta}{20}\int_0^t\left\|\nabla\left(U_t(s),U_x(s),U_y(s)\right)\right\|^2\,ds$$

$$+\,O(1)N_1(0)^3\int_0^t\left\|\nabla U(s)\right\|^2\left\|\nabla\left(U_x(s),U_y(s)\right)\right\|^2\,ds. \tag{3.47}$$

Now for J_{103}^4, we have from Corollary 2.1, Lemma 2.2, the facts that $|f'''(u)|\leqslant O(1)$, $|f''(u)|\leqslant O(1)(1+|u|)$, and some integrations by parts that

$$J_{103}^4=\frac{\lambda}{4a_2}\int_{\mathbf{R}^2}\left\{f''(\bar u+U)U_xU_t^2\right\}(s,x,y)\,dx\,dy\Big|_{s=0}^{s=t}$$

$$+\,\frac{\lambda}{12a_2}\int_0^t\int_{\mathbf{R}^2}\left\{f'''(\bar u+U)U_t^3(\bar u_x+U_x)\right\}(s,x,y)\,dx\,dy\,ds$$

$$-\,\frac{\lambda}{4a_2}\int_0^t\int_{\mathbf{R}^2}\left\{f'''(\bar u+U)U_t^2U_x(\bar u_t+U_t)\right\}(s,x,y)\,dx\,dy\,ds$$

$$\leqslant O(1)\int_{\mathbf{R}^2}\left\{\left(1+|U|\right)|U_x|U_t^2\right\}(s,x,y)\,dx\,dy\Big|_{s=0}^{s=t}$$

$$+\,O(1)\int_0^t\int_{\mathbf{R}^2}\left\{\left(1+|U_x|\right)|U_t|^3+\left(1+|U_t|\right)|U_x|U_t^2\right\}(s,x,y)\,dx\,dy\,ds$$

$$\leqslant O(1)\big(N_1(0)+N_2(0)\big)^5+\frac{\eta}{20}\big\|\nabla(U_t,U_x)(t)\big\|^2+\frac{\eta}{20}\int_0^t\big\|\nabla U_t(s)\big\|^2\,ds$$
$$+\,O(1)N_1(0)^3\int_0^t\big\|\nabla U(s)\big\|^2\big\|\nabla U_x(s)\big\|^2\,ds. \tag{3.48}$$

At last for J^8_{103}, due to

$$J^8_{103}=\frac{\lambda a_1}{12a_2}\int_{\mathbf{R}^2}\big\{f''(\bar u+U)U_x^3\big\}(s,x,y)\,dx\,dy\bigg|_{s=0}^{s=t}$$
$$-\frac{\lambda a_1}{12a_2}\int_0^t\int_{\mathbf{R}^2}\big\{f'''(\bar u+U)(\bar u_t+U_t)U_x^3\big\}(s,x,y)\,dx\,dy\,ds$$
$$+\frac{\lambda a_1}{4a_2}\int_0^t\int_{\mathbf{R}^2}\big\{f'''(\bar u+U)(\bar u_x+U_x)U_x^2U_t\big\}(s,x,y)\,dx\,dy\,ds,$$

we have by repeating the argument to estimate J^4_{103} that

$$\big|J^8_{103}\big|\leqslant O(1)\big(N_1(0)+N_2(0)\big)^5+\frac{\eta}{20}\big\|\nabla(U_t,U_x)(t)\big\|^2+\frac{\eta}{20}\int_0^t\big\|\nabla U_t(s)\big\|^2\,ds$$
$$+\,O(1)N_1(0)^3\int_0^t\big\|\nabla U(s)\big\|^2\big\|\nabla U_x(s)\big\|^2\,ds. \tag{3.49}$$

Inserting (3.46)–(3.49) into (3.45), we finally arrive at

$$\big|I^3_{10}\big|\leqslant O(1)\big(N_1(0)+N_2(0)\big)^5+\frac{\eta}{10}\big\|\nabla(U_t,U_x,U_y)(t)\big\|^2+\frac{\eta}{10}\int_0^t\big\|\nabla(U_t,U_x,U_y)(s)\big\|^2\,ds$$
$$+\,O(1)N_1(0)^3\int_0^t\big\|\nabla U(s)\big\|^2\big\|\nabla(U_t,U_x,U_y)(s)\big\|^2\,ds. \tag{3.50}$$

Repeating the argument used above, we can also get that

$$\big|I^4_{10}\big|\leqslant O(1)\big(N_1(0)+N_2(0)\big)^5+\frac{\eta}{10}\big\|\nabla(U_t,U_x,U_y)(t)\big\|^2+\frac{\eta}{10}\int_0^t\big\|\nabla(U_t,U_x,U_y)(s)\big\|^2\,ds$$
$$+\,O(1)N_1(0)^3\int_0^t\big\|\nabla U(s)\big\|^2\big\|\nabla(U_t,U_x,U_y)(s)\big\|^2\,ds. \tag{3.51}$$

Putting (3.38), (3.41), (3.42), (3.50) and (3.51) together, we can deduce that

$$|I_{10}| \leqslant O(1)\big(N_1(0)+N_2(0)\big)^5 + \frac{\eta}{5}\big\|\nabla(U_t,U_x,U_y)(t)\big\|^2 + \frac{\eta}{5}\int_0^t \big\|\nabla(U_t,U_x,U_y)(s)\big\|^2\,ds$$
$$+ O(1)N_1(0)^3\int_0^t \big\|\nabla U(s)\big\|^2\big\|\nabla(U_t,U_x,U_y)(s)\big\|^2\,ds. \tag{3.52}$$

(3.25), (3.26), (3.39) together with (3.52) imply

$$|I_7| \leqslant O(1)\big(N_1(0)+N_2(0)\big)^5 + \frac{\eta}{2}\big\|\nabla(U_t,U_x,U_y)(t)\big\|^2$$
$$+ O(1)N_1(0)^3\int_0^t \big\|\nabla U(s)\big\|^2\big\|\nabla(U_t,U_x,U_y)(s)\big\|^2\,ds + \left(\frac{\lambda k_1}{2a_1}+\frac{\eta}{2}\right)\int_0^t \big\|\nabla U_t(s)\big\|^2\,ds$$
$$+ \left(\frac{\lambda a_1}{2}+\frac{\eta}{2}\right)\int_0^t \big\|\nabla U_x(s)\big\|^2\,ds + \frac{\eta}{2}\int_0^t \big\|\nabla U_y(s)\big\|^2\,ds, \tag{3.53}$$

while for I_8, we have by employing the argument used above that

$$|I_8| \leqslant O(1)\big(N_1(0)+N_2(0)\big)^5 + \frac{\eta}{2}\big\|\nabla(U_t,U_x,U_y)(t)\big\|^2$$
$$+ O(1)N_1(0)^3\int_0^t \big\|\nabla U(s)\big\|^2\big\|\nabla(U_t,U_x,U_y)(s)\big\|^2\,ds + \left(\frac{\lambda k_2}{2a_2}+\frac{\eta}{2}\right)\int_0^t \big\|\nabla U_t(s)\big\|^2\,ds$$
$$+ \left(\frac{\lambda a_2}{2}+\frac{\eta}{2}\right)\int_0^t \big\|\nabla U_y(s)\big\|^2\,ds + \frac{\eta}{2}\int_0^t \big\|\nabla U_x(s)\big\|^2\,ds. \tag{3.54}$$

Inserting (3.22), (3.24), (3.53), (3.54) into (3.21) yields

$$\frac{1}{2}\big\|\nabla U(t)\big\|^2 + \left(\frac{\lambda}{2}-\eta\right)\big\|\nabla U_t(t)\big\|^2 + \left(\frac{\lambda a_1}{2}-\eta\right)\big\|\nabla U_x(t)\big\|^2 + \left(\frac{\lambda a_2}{2}-\eta\right)\big\|\nabla U_y(t)\big\|^2$$
$$+ \int_0^t \left(\left(\lambda - 1 - \frac{\lambda k_1}{2a_1} - \frac{\lambda k_2}{2a_2} - \eta\right)\big\|\nabla U_t(s)\big\|^2 + \left(\frac{(2-\lambda)a_1}{2} - 2\eta\right)\big\|\nabla U_x(s)\big\|^2\right.$$
$$\left. + \left(\frac{(2-\lambda)a_2}{2} - 2\eta\right)\big\|\nabla U_y(s)\big\|^2\right)ds + \int_{\mathbf{R}^2} \nabla U(t,x,y)\cdot\nabla U_t(t,x,y)\,dx\,dy$$
$$\leqslant O(1)\big(N_1(0)+N_2(0)\big)^5 + O(1)N_1(0)^3\int_0^t \big\|\nabla U(s)\big\|^2\big\|\nabla\big(U_t(s),U_x(s),U_y(s)\big)\big\|^2\,ds. \tag{3.55}$$

(3.55), (3.5), (3.6) together with the Gronwall inequality imply

$$\begin{aligned}
&\big\|\nabla(U, U_t, U_x, U_y)(t)\big\|^2 + \int_0^t \big\|\nabla(U_t, U_x, U_y)(s)\big\|^2\, ds \\
&\quad \leqslant O(1)\big(N_1(0) + N_2(0)\big)^5 \exp\big(O(1)N_1(0)^4\big). \qquad (3.56)
\end{aligned}$$

This completes the proof of Lemma 3.4. □

4. The proofs of our main results

This section is devoted to proving our main results. To make the presentation easy to read, we divide this section into two subsections and the first one is concentrated on the proof of Theorem 1.1.

4.1. The proof of Theorem 1.1

Now we turn to prove Theorem 1.1 which is based on the continuation argument and the energy type estimates established in Lemma 3.2 and Lemma 3.3.

Under the conditions listed in Theorem 1.1, we have from the local existence result Lemma 3.1 that the Cauchy problem (3.1), (3.2) admits a unique smooth solution $U(t, x, y) \in X(0, t_1)$ on $\prod_{t_1} = \{(t, x, y) \mid 0 \leqslant t \leqslant t_1,\ (x, y) \in \mathbf{R}^2\}$, where t_1 depends only on $\|U_0\|_2$ and $\|V_0\|_1$ and $U(t, x, y)$ satisfies

$$\begin{cases} \big\|\big(U(t), \nabla U(t), U_t(t)\big)\big\|^2 \leqslant 4N_1(0), \\ \big\|\nabla\big(U_t(t), U_x(t), U_y(t)\big)\big\|^2 \leqslant 4N_2(0) \end{cases} \qquad (4.1)$$

for all $0 \leqslant t \leqslant t_1$.

The estimate (4.1) and the Gagliardo–Nirenberg inequality together with the assumption (1.22) imposed on the initial perturbation tell us that

$$\begin{aligned}
\big\|U(t)\big\|_{L^\infty} &\leqslant D_0\big\|U(t)\big\|^{\frac{1}{2}}\big\|D^2U(t)\big\|^{\frac{1}{2}} \\
&\leqslant 2D_0\sqrt[4]{N_1(0)N_2(0)} \\
&\leqslant 2D_0\sqrt[4]{D_1D_2\ell^\alpha\big(1 + \ell^{-\beta}\big)} \\
&\leqslant 2D_0\sqrt[4]{2D_1D_2} =: \widetilde{M}_1 \qquad (4.2)
\end{aligned}$$

holds true for all $0 \leqslant t \leqslant t_1$. Here D_0 is the Sobolev constant.

Since $\widetilde{M}_1 = 2D_0\sqrt[4]{2D_1D_2}$ is independent of ℓ, we can find a sufficiently small positive constant $\ell_1 \in (0, 1]$ such that for $0 < \ell \leqslant \ell_1$

$$\ell \cdot \big(Q_1(\widetilde{M}_1) + Q_2(\widetilde{M}_1)\big)^{16} \leqslant 1, \qquad \ell^\alpha\big(Q_1(\widetilde{M}_1) + Q_2(\widetilde{M}_1)\big)^2 \leqslant 1, \qquad D_1 \cdot \ell^\alpha \leqslant 1, \qquad (4.3)$$

thus if we assume that the sub-characteristic condition (1.5) holds with $\mathcal{M} = [-B(N_0) - \widetilde{M}_1, B(N_0) + \widetilde{M}_1]$, we know that the assumptions listed in Lemma 3.2 and Lemma 3.3 are satisfied with the constant M in the *a priori* assumption (3.4) being replaced by $\widetilde{M}_1$ and consequently we can deduce from Lemma 3.2, Lemma 3.3, and Lemma 3.5 that

$$\begin{cases} \big\|\big(U(t), U_t(t), \nabla U(t)\big)\big\|^2 \leqslant C_4 N_1(0) \leqslant C_4 D_1 \ell^\alpha, \\ \big\|\nabla\big(U_t(t), U_x(t), U_y(t)\big)\big\|^2 \leqslant C_5\big(N_1(0)+N_2(0)\big)\exp\big(C_5\big(Q_1^2(\widetilde{M}_1)+Q_2(\widetilde{M}_1)\big)N_1(0)\big) \\ \qquad \leqslant C_5\big(N_1(0)+N_2(0)\big)\exp\big(C_5 D_1\big(Q_1^2(\widetilde{M}_1)+Q_2^2(\widetilde{M}_1)\big)\ell^\alpha\big) \\ \qquad \leqslant C_5\big(N_1(0)+N_2(0)\big)\exp(C_5 D_1) \end{cases} \tag{4.4}$$

holds for all $0 \leqslant t \leqslant t_1$.

Now take $(U(t_1,x,y), U_t(t_1,x,y))$ as initial data, we have from Lemma 3.1 again that the local solution $U(t,x,y)$ constructed above can be extended to the time step $t = t_1 + t_2$ such that $U(t,x,y) \in X(0, t_1+t_2)$ satisfying

$$\begin{cases} \big\|\big(U(t), U_t(t), \nabla U(t)\big)\big\|^2 \leqslant 4\big\|\big(U(t_1), U_t(t_1), \nabla U(t_1)\big)\big\|^2 \leqslant 4C_4 N_1(0) \leqslant 4C_4 D_1 \ell^\alpha, \\ \big\|\nabla\big(U_t(t), U_x(t), U_y(t)\big)\big\|^2 \leqslant 4\big\|\nabla\big(U_t(t_1), U_x(t_1), U_y(t_1)\big)\big\|^2 \\ \qquad \leqslant 4C_5\big(N_1(0)+N_2(0)\big)\exp(C_5 D_1) \end{cases} \tag{4.5}$$

holds for $t_1 \leqslant t \leqslant t_1 + t_2$.

Notice that (4.5) together with (4.4) imply that (4.5) holds for $0 \leqslant t \leqslant t_1 + t_2$. Such an observation together with the Gagliardo–Nirenberg inequality tell us that

$$\begin{aligned} \big\|U(t)\big\|_{L^\infty} &\leqslant D_0 \big\|U(t)\big\|^{\frac{1}{2}} \big\|D^2 U(t)\big\|^{\frac{1}{2}} \\ &\leqslant D_0 \sqrt[4]{4C_4 N_1(0)} \sqrt[4]{4C_5\big(N_1(0)+N_2(0)\big)\exp(C_5 D_1)} \\ &\leqslant 2D_0 \sqrt[4]{C_4 C_5 D_1 \ell^\alpha \big(D_1 \ell^\alpha + D_2\big(1+\ell^{-\beta}\big)\big)\exp(C_5 D_1)} \\ &\leqslant 2D_0 \sqrt[4]{C_4 C_5 D_1 (1+2D_2)\exp(C_5 D_1)} =: \widetilde{M}_2 \end{aligned} \tag{4.6}$$

holds for $0 \leqslant t \leqslant t_1 + t_2$.

Since $\widetilde{M}_2 = 2D_0 \sqrt[4]{C_4 C_5 D_1 (1+2D_2)\exp(C_5 D_1)} > \widetilde{M}_1$ is independent of ℓ, we can find a constant $\ell_2 \in (0,1]$ which is chosen suitably small such that for $0 < \ell \leqslant \ell_2$

$$\ell \cdot \big(Q_1(\widetilde{M}_2) + Q_2(\widetilde{M}_2)\big)^8 \leqslant 1, \qquad \ell^\alpha \big(Q_1(\widetilde{M}_2) + Q_2(\widetilde{M}_2)\big)^2 \leqslant 1, \qquad D_1 \cdot \ell^\alpha \leqslant 1, \tag{4.7}$$

thus if we assume that the sub-characteristic condition (1.5) holds with $\mathcal{M} = [-B(N_0) - \widetilde{M}_2, B(N_0) + \widetilde{M}_2]$, we know that the assumptions listed in Lemma 3.2 and Lemma 3.3 are satisfied with the constant M in the *a priori* assumption (3.4) being replaced by $\widetilde{M}_2$ and consequently we can deduce from Lemma 3.2, Lemma 3.3, and Lemma 3.5 that

$$\begin{cases} \big\|\big(U(t), U_t(t), \nabla U(t)\big)\big\|^2 \leqslant C_4 N_1(0) \leqslant C_4 D_1 \ell^\alpha, \\ \big\|\nabla\big(U_t(t), U_x(t), U_y(t)\big)\big\|^2 \leqslant C_5\big(N_1(0)+N_2(0)\big)\exp\big(C_5\big(Q_1^2(\widetilde{M}_2)+Q_2(\widetilde{M}_2)\big)N_1(0)\big) \\ \qquad \leqslant C_5\big(N_1(0)+N_2(0)\big)\exp\big(C_5 D_1\big(Q_1^2(\widetilde{M}_2)+Q_2^2(\widetilde{M}_2)\big)\ell^\alpha\big) \\ \qquad \leqslant C_5\big(N_1(0)+N_2(0)\big)\exp(C_5 D_1) \end{cases} \tag{4.8}$$

holds for all $0 \leqslant t \leqslant t_1 + t_2$.

Now take $(U(t_1+t_2,x,y), U_t(t_1+t_2,x,y))$ as initial data, since we have by employing the local existence result Lemma 3.1 once more that the solution $U(t,x,y)$ constructed above can be extended into the time step $t = t_1 + t_2 + t_3$ such that $U(t,x,y) \in X(0, t_1+t_2)$ satisfying

$$\begin{cases} \left\|\left(U(t), U_t(t), \nabla U(t)\right)\right\|^2 \leqslant 4\left\|\left(U(t_1+t_2), U_t(t_1+t_2), \nabla U(t_1+t_2)\right)\right\|^2 \\ \qquad \leqslant 4C_4 N_1(0) \leqslant 4C_4 D_1 \ell^\alpha, \\ \left\|\nabla\left(U_t(t), U_x(t), U_y(t)\right)\right\|^2 \leqslant 4\left\|\nabla\left(U_t(t_1+t_2), U_x(t_1+t_2), U_y(t_1+t_2)\right)\right\|^2 \\ \qquad \leqslant 4C_5\left(N_1(0)+N_2(0)\right)\exp(C_5 D_1) \end{cases} \tag{4.9}$$

holds for $t_1+t_2 \leqslant t \leqslant t_1+t_2+t_3$.

Notice that $\widetilde{M}_2 > \widetilde{M}_1$, the above analysis shows that (4.9) holds for $0 \leqslant t \leqslant t_1+t_2+t_3$. These estimates together with the fact that the constants in the right hand side of $(4.9)_1$ and $(4.9)_2$ are independent of the time t tell us that (4.6) holds for all $0 \leqslant t \leqslant t_1+t_2+t_3$. Thus if we choose $0 < \ell \leqslant \min\{\ell_1, \ell_2\}$ and assume that the sub-characteristic condition (1.5) holds with $\mathcal{M} = [-B(N_0) - \widetilde{M}_2, B(N_0) + \widetilde{M}_2]$, we can deduce that the assumptions listed in Lemma 3.2 and Lemma 3.3 are satisfied with the constant M in the *a priori* assumption (3.4) being replaced by $\widetilde{M}_2$ and consequently we can deduce from Lemma 3.2, Lemma 3.3, and Lemma 3.5 that (4.8) holds for $0 \leqslant t \leqslant t_1+t_2+t_3$. If we take $(U(t_1+t_2+t_3, x, y), U_t(t_1+t_2+t_3, x, y))$ as initial data, we can then extend $U(t,x,y)$ to the time step $t = t_1+t_2+2t_3$. Repeating the above procedure, if we assume that

$$0 < \ell \leqslant \ell_0 := \min\{\ell_1, \ell_2\}$$

and the sub-characteristic condition (1.5) holds with $\mathcal{M} = [-B(N_0) - M_1, B(N_0) + M_1]$ and

$$M_1 := \widetilde{M}_2 = 2D_0\sqrt[4]{C_4 C_5 D_1 (1+2D_2)\exp(C_5 D_1)},$$

we can then extend $U(t,x,y)$ globally and as a by-product of the above procedure, we can also show that such a global solution $U(t,x,y)$ satisfies

$$\left\|U(t)\right\|_2^2 + \left\|U_t(t)\right\|_1^2 + \int_0^t \left(\left\|\nabla U(s)\right\|_1^2 + \left\|U_t(s)\right\|_1^2\right) ds \leqslant O(1)\left(\|U_0\|_2^2 + \|V_0\|_1^2\right). \tag{4.10}$$

Having obtained (4.10), the time-asymptotic behavior (1.25) can be proved by exploiting the argument used in [20] and [43]. This completes the proof of Theorem 1.1.

4.2. The proofs of Theorem 1.2 and Theorem 1.3

Compared with that of Theorem 1.1, the proofs of Theorem 1.2 and Theorem 1.3 are relatively easier. We will only prove Theorem 1.3 in details in the following, since the proof of Theorem 1.2 are completely the same, we thus omit the details for brevity.

For $(U_0(x,y), V_0(x,y)) \in H^2(\mathbf{R}^2)$, the local solvability result Lemma 3.1 tells us that there exists a sufficiently small positive constant t_1, which depends on $\|U_0\|_2$ and $\|V_0\|_1$, such that the Cauchy problem (3.1), (3.2) admits a unique solution $U(t,x,y) \in X(0,t_1)$ satisfying

$$\begin{cases} \left\|\left(U(t), \nabla U(t), U_t(t)\right)\right\|^2 \leqslant 4N_1(0), \\ \left\|\nabla\left(U_t(t), U_x(t), U_y(t)\right)\right\|^2 \leqslant 4N_2(0) \end{cases} \tag{4.11}$$

for all $0 \leqslant t \leqslant t_1$.

(4.11) together with the Gagliardo–Nirenberg inequality yield

$$\begin{aligned} \left\|U(t)\right\|_{L^\infty} &\leqslant D_0\left\|U(t)\right\|^{\frac{1}{2}}\left\|D^2 U(t)\right\|^{\frac{1}{2}} \\ &\leqslant 2D_0\sqrt[4]{N_1(0)N_2(0)} =: \widetilde{M}_3 \end{aligned} \tag{4.12}$$

holds true for all $0 \leqslant t \leqslant t_1$. Here D_0 is the Sobolev constant.

Thus if the sub-characteristic condition (1.5) holds with $\mathcal{M} = [-B(N_0) - \widetilde{M}_3, B(N_0) + \widetilde{M}_3]$, we know that the assumptions listed in Lemma 3.2, Lemma 3.4, and Lemma 3.5 are satisfied with the constant M in the *a priori* assumption (3.4) being replaced by $\widetilde{M}_3$ and consequently we can deduce from Lemma 3.2, Lemma 3.4, and Lemma 3.5 that

$$\begin{cases} \big\|\big(U(t), U_t(t), \nabla U(t)\big)\big\|^2 \leqslant C_4 N_1(0), \\ \big\|\nabla\big(U_t(t), U_x(t), U_y(t)\big)\big\|^2 \leqslant C_7\big(N_1(0) + N_2(0)\big)^5 \exp\big(C_7 N_1^4(0)\big) \end{cases} \tag{4.13}$$

holds for all $0 \leqslant t \leqslant t_1$.

Now take $(U(t_1, x, y), U_t(t_1, x, y))$ as initial data, we have by employing Lemma 3.1 again that the solution $U(t, x, y)$ constructed above can be extended into the time step $t = t_1 + t_2$ such that $U(t, x, y) \in X(0, t_1 + t_2)$ satisfying

$$\begin{cases} \big\|\big(U(t), U_t(t), \nabla U(t)\big)\big\|^2 \leqslant 4\big\|\big(U(t_1), U_t(t_1), \nabla U(t_1)\big)\big\|^2 \leqslant 4C_4 N_1(0), \\ \big\|\nabla\big(U_t(t), U_x(t), U_y(t)\big)\big\|^2 \leqslant 4\big\|\nabla\big(U_t(t_1), U_x(t_1), U_y(t_1)\big)\big\|^2 \\ \qquad \leqslant 4C_7\big(N_1(0) + N_2(0)\big)^5 \exp\big(C_7 N_1^4(0)\big) \end{cases} \tag{4.14}$$

holds for $t_1 \leqslant t \leqslant t_1 + t_2$.

(4.13) and (4.14) imply that (4.14) holds for $0 \leqslant t \leqslant t_1 + t_2$. This fact together with the Gagliardo–Nirenberg inequality yield

$$\begin{aligned} \big\|U(t)\big\|_{L^\infty} &\leqslant D_0\big\|U(t)\big\|^{\frac{1}{2}}\big\|D^2 U(t)\big\|^{\frac{1}{2}} \\ &\leqslant 2D_0\sqrt[4]{C_4 C_7 N_1(0)\big(N_1(0) + N_2(0)\big)^5} \exp\left(\frac{C_7 N_1^4(0)}{4}\right) =: \widetilde{M}_4 \geqslant \widetilde{M}_3 \end{aligned} \tag{4.15}$$

holds true for all $0 \leqslant t \leqslant t_1 + t_2$. Here to deduce $\widetilde{M}_4 \geqslant \widetilde{M}_3$, we have used the assumption that $N_i(0) \geqslant 1$ $(i = 1, 2)$ and $C_4 \geqslant 1$, $C_7 \geqslant 1$. Since we are concerned with the nonlinear stability result with large initial perturbation, such an assumption seems natural.

Now if the sub-characteristic condition (1.5) holds with $\mathcal{M} = [-B(N_0) - \widetilde{M}_4, B(N_0) + \widetilde{M}_4]$, we know that the assumptions listed in Lemma 3.2, Lemma 3.4, and Lemma 3.5 are satisfied with the constant M in the *a priori* assumption (3.4) being replaced by $\widetilde{M}_4$ and consequently we can deduce from Lemma 3.2, Lemma 3.4, and Lemma 3.5 that

$$\begin{cases} \big\|\big(U(t), U_t(t), \nabla U(t)\big)\big\|^2 \leqslant C_4 N_1(0), \\ \big\|\nabla\big(U_t(t), U_x(t), U_y(t)\big)\big\|^2 \leqslant C_7\big(N_1(0) + N_2(0)\big)^5 \exp\big(C_7 N_1^4(0)\big) \end{cases} \tag{4.16}$$

holds for all $0 \leqslant t \leqslant t_1 + t_2$.

Take $(U(t_1 + t_2, x, y), U_t(t_1 + t_2, x, y))$ as initial data, we have by employing Lemma 3.1 once more that the solution $U(t, x, y)$ constructed above can be extended into the time step $t = t_1 + t_2 + t_3$ such that $U(t, x, y) \in X(0, t_1 + t_2)$ satisfying

$$\begin{cases} \big\|\big(U(t), U_t(t), \nabla U(t)\big)\big\|^2 \leqslant 4\big\|\big(U(t_1 + t_2), U_t(t_1 + t_2), \nabla U(t_1 + t_2)\big)\big\|^2 \leqslant 4C_4 N_1(0), \\ \big\|\nabla\big(U_t(t), U_x(t), U_y(t)\big)\big\|^2 \leqslant 4\big\|\nabla\big(U_t(t_1 + t_2), U_x(t_1 + t_2), U_y(t_1 + t_2)\big)\big\|^2 \\ \qquad \leqslant 4C_7\big(N_1(0) + N_2(0)\big)^5 \exp\big(C_7 N_1^4(0)\big) \end{cases} \tag{4.17}$$

holds for $t_1 + t_2 \leqslant t \leqslant t_1 + t_2 + t_3$.

(4.16) together with (4.17) implies that (4.17) holds for all $0 \leqslant t \leqslant t_1 + t_2 + t_3$ and similar to that of (4.15), we can deduce that (4.15) holds for all $0 \leqslant t \leqslant t_1 + t_2 + t_3$. Thus if we assume that the sub-characteristic condition (1.5) holds with $\mathcal{M} = [-B(N_0) - \widetilde{M}_4, B(N_0) + \widetilde{M}_4]$, we know that the assumptions listed in Lemma 3.2, Lemma 3.4, and Lemma 3.5 are satisfied with the constant M in the *a priori* assumption (3.4) being replaced by $\widetilde{M}_4$ and consequently we can deduce from Lemma 3.2, Lemma 3.4, and Lemma 3.5 that (4.16) holds for all $0 \leqslant t \leqslant t_1 + t_2 + t_3$. And if we take $(U(t_1 + t_2 + t_3, x, y), U_t(t_1 + t_2 + t_3, x, y))$ as initial data, we have from Lemma 3.1 that $U(t, x, y)$ can be extended into the time step $t = t_1 + t_2 + 2t_3$. Repeating the above procedure, if we assume that the sub-characteristic condition (1.5) imposed in Theorem 1.3 holds with $M_3 = \widetilde{M}_4$, we can thus extend $U(t, x, y)$ step by step to a global one and as a by-product, we can show that (4.10) holds for all $t \geqslant 0$. This completes the proof of Theorem 1.3.

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