Stability of Multidimensional Thermoelastic Contact Discontinuities

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Abstract

We study the system of nonisentropic thermoelasticity describing the motion of thermoelastic nonconductors of heat in two and three spatial dimensions, where the frame-indifferent constitutive relation generalizes that for compressible neo-Hookean materials. Thermoelastic contact discontinuities are characteristic discontinuities for which the velocity is continuous across the discontinuity interface. Mathematically, this renders a nonlinear multidimensional hyperbolic problem with a characteristic free boundary. We identify a stability condition on the piecewise constant background states and establish the linear stability of thermoelastic contact discontinuities in the sense that the variable coefficient linearized problem satisfies a priori tame estimates in the usual Sobolev spaces under small perturbations. Our tame estimates for the linearized problem do not break down when the strength of thermoelastic contact discontinuities tends to zero. The missing normal derivatives are recovered from the estimates of several quantities relating to physical involutions. In the estimate of tangential derivatives, there is a significant new difficulty, namely the presence of characteristic variables in the boundary conditions. To overcome this difficulty, we explore an intrinsic cancellation effect, which reduces the boundary terms to an instant integral. Then we can absorb the instant integral into the instant tangential energy by means of the interpolation argument and an explicit estimate for the traces on the hyperplane.

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Contents

1. Introduction

We study the equations of nonisentropic thermoelasticity in the Eulerian coordinates, governing the evolution of thermoelastic nonconductors of heat in two and three spatial dimensions. The constitutive relation under consideration generalizes that for *compressible neo-Hookean materials* (see CIARLET [13, p. 189]) and satisfies the necessary *frame indifference principle* (see DAFERMOS [18, §2.4]). This system can be reduced to a symmetrizable hyperbolic system on account of the divergence constraints.

Our main interest concerns the stability of *thermoelastic contact discontinuities* that are piecewise smooth, weak solutions with the discontinuity interface, across which the mass does not transfer and the velocity is continuous. The boundary matrix for the free boundary problem of thermoelastic contact discontinuities is always singular on the discontinuity interface. In other words, thermoelastic contact discontinuities are characteristic discontinuities to the system of thermoelasticity. As is well-known, characteristic discontinuities, along with shocks and rarefaction waves, are building blocks of general entropy solutions of multidimensional hyperbolic systems of conservation laws (see, e.g., CHEN–FELDMAN [4]). Therefore, it is important to analyze the stability of thermoelastic contact discontinuities when the initial thermodynamic process and interface are perturbed from the piecewise constant background state. Mathematically, this renders a nonlinear hyperbolic initial-boundary value problem with a characteristic free boundary.

Our work is motivated by the results on 3D compressible current-vortex sheets [6,28], 2D MHD contact discontinuities [22,23], and 2D compressible vortex sheets in elastodynamics [8,9]. For ideal compressible magnetohydrodynamics (MHD), there are two types of characteristic discontinuities: compressible current-vortex sheets and MHD contact discontinuities, corresponding respectively to $H \cdot N|_\Gamma = 0$ and $H \cdot N|_\Gamma \neq 0$, where H is denoted as the magnetic field, Γ as the discontinuity interface, and N as the spatial normal to Γ. CHEN–WANG [6,7] and TRAKHININ [28] established the nonlinear stability of 3D compressible current-vortex sheets independently, indicating the stabilization effect of non-paralleled magnetic fields to the motion of 3D compressible vortex sheets. The local existence of 2D MHD contact discontinuities was proved by MORANDO ET AL. [22,23] under the Rayleigh–Taylor

sign condition on the jump of the normal derivative of the pressure through a series of delicate energy estimates. Notice that the extension of the results in [22,23] to 3D MHD contact discontinuities is still a difficult open problem. For the system of thermoelasticity, CHEN ET AL. [8,9] recently obtained the linear stability of the 2D isentropic compressible vortex sheets associated with the boundary constraint $\boldsymbol{F} \cdot \boldsymbol{N}|_\Gamma = 0$ for the deformation gradient $\boldsymbol{F}$, by developing the methodology in COULOMBEL–SECCHI [15]. Comparing with the aforementioned two types of characteristic discontinuities in MHD, we naturally introduce and investigate the thermoelastic contact discontinuities that correspond to $\boldsymbol{F} \cdot \boldsymbol{N}|_\Gamma \neq 0$.

The goal of this paper is to explore the stabilizing mechanism in thermoelasticity such that the thermoelastic contact discontinuities are stable. More precisely, we identify a stability condition on the piecewise constant background states and establish the linear stability of thermoelastic contact discontinuities in the sense that the variable coefficient linearized problem satisfies appropriate a priori tame estimates under small perturbations. In particular, our tame estimates do not break down when the strength of thermoelastic contact discontinuities tends to zero. As far as we know, this is the first rigorous result on the stability of thermoelastic contact discontinuities in the mathematical theory of thermoelasticity.

In general, for hyperbolic problems with a characteristic boundary, there is a loss of control on the derivatives (precisely, on the normal derivatives of the characteristic variables) in a priori energy estimates. To overcome this difficulty, it is natural to introduce the Sobolev spaces with conormal regularity, where two tangential derivatives count as one normal derivative (see SECCHI [25] and the references therein). However, for our problem, we manage to work in the usual Sobolev spaces, since the missing normal derivatives of the characteristic variables can be recovered from the estimates of several quantities relating to the physical constraints.

In the estimate of tangential derivatives, there is a significant new difficulty, namely the presence of characteristic variables in the boundary conditions, which is completely different from the previous works such as [5–9,15,22,28]. New ideas are required to control the boundary integral term arising in the estimate of tangential derivatives owing to the complex nature of the boundary conditions. To address this issue, we utilize a combination of the boundary conditions and the restriction of the interior equations on the boundary to exploit an intrinsic cancellation effect. This cancellation enables us to reduce the boundary term into the sum of the error term $\mathcal{R}_2$ (*cf.* (7.19)) and the instant boundary integral term $\mathcal{R}_3$ (*cf.* (7.18)).

To establish the energy estimates uniform in the strength of the thermoelastic contact discontinuity for $\mathcal{R}_2$ and $\mathcal{R}_3$, we cannot use the boundary conditions for the spatial derivatives of the discontinuity function ψ, owing to the dependence of the coefficients on the strength (*cf.* (5.15d)). In order to overcome this difficulty, we develop an idea from TRAKHININ [29, Proposition 5.2] and explore new identities and estimates for the derivatives of ψ with the aid of the interpolation argument. We make the estimate of $\mathcal{R}_3$ differently for the cases whether it contains a time derivative. More precisely, we first consider the case with at least one time derivative. Thanks to the restriction of the interior equations on the boundary, the time derivative of the deformation gradient in $\mathcal{R}_3$ can be transformed into the tangential

space derivatives of the velocity (*cf.* (7.31)). As a result, the estimate of traces on the hyperplane (*cf.* Lemma 4.2) can be applied to control the primary term $\mathcal{R}_{31}$ (*cf.* (7.42)). Employing the identities and estimates for the normal derivative of the noncharacteristic variables, we can reduce the estimate of the instant tangential energy into that with one less time derivative and one more tangential spatial derivative (*cf.* (7.48)). Then we are led to deal with the case containing the space derivatives. For this case, we derive estimates (7.62) and (7.68) by means of the identities and estimates for linearized quantities (η, ζ) (*cf.* (6.29)–(6.32)) and Lemma 4.2. With these estimates in hand, we can finally obtain the desired estimate for all the tangential derivatives under the stability condition (3.23) on the background state. The methods and techniques developed here may be also helpful for other problems involving similar difficulties.

It is worth noting that our tame estimates are with a *fixed* loss of derivatives with respect to the source terms and coefficients. As such, the local existence and nonlinearly structural stability of thermoelastic contact discontinuities could be achieved with resorting to a suitable Nash–Moser iteration scheme as in [5,16].

Let us also mention some recent results on the classical solutions and weak–strong uniqueness for the system of polyconvex thermoelasticity. CHRISTOFOROU ET AL. [11] enlarge the equations of polyconvex thermoelasticity into a symmetrizable hyperbolic system, which yields the local existence of classical solutions of the Cauchy problem by applying the general theory in [18, Theorem 5.4.3]. The convergence in the zero-viscosity limit from thermoviscoelasticity to thermoelasticity is also provided in [11] by virtue of the relative entropy formulation developed in [10]. Moreover, CHRISTOFOROU ET AL. [11,12] establish the weak–strong uniqueness property in the classes of entropy weak and measure-valued solutions.

The rest of this paper is organized as follows: in Sect. 2, we introduce the system of thermoelasticity in the Eulerian coordinates, which can be symmetrizable hyperbolic, via the divergence constraints. Then we formulate the free boundary problem and the reduced problem in a fixed domain for thermoelastic contact discontinuities. It should be pointed out that no thermoelastic contact discontinuity is possible for the isentropic process (*cf.* Proposition 2.1). Section 3 is devoted to stating the main theorem of this paper, Theorem 3.1. Before that, based on an alternative form of the boundary operator, we deduce the variable coefficient linearized problem around the basic state (that is, a small perturbation of the stationary thermoelastic contact discontinuity satisfying suitable constraints). In Sect. 4, we collect some properties of the Sobolev functions and notations for later use, including the definitions of fractional Sobolev spaces and norms, the estimates of the traces on the hyperplane, and the Moser-type calculus inequalities. To show Theorem 3.1, in Sect. 5, we reduce the effective linear problem to a problem with homogeneous boundary conditions. Section 6 is dedicated to the proof of Proposition 6.1, *i.e.*, the estimate of normal derivatives. More precisely, we estimate the noncharacteristic variables W_{nc} and entropies $S^{\pm}$ in §6.1–§6.2, recover the missing L^2-norm of $\partial_1 D_{\text{tan}}^{\beta} W_1$ and $\partial_1 D_{\text{tan}}^{\beta} W_{jd+i+1}$ for $1 \leqslant i \leqslant d$ and $2 \leqslant j \leqslant d$ in §6.3–§6.4, and complete the proof of Proposition 6.1 by finite induction in §6.5. Let us remark that quantities ς, η, and ζ (*cf.* definitions (6.18), (6.29), and (6.30)) are introduced and estimated

to compensate the loss of the normal derivatives of characteristic variables W_1 and W_{jd+i+1}. In Sect. 7, we deduce the estimate of tangential derivatives, *i.e.*, Proposition 7.1. For this purpose, we start with the standard energy estimate to introduce the boundary term Q (*cf.* (7.5)) and the instant tangential energy $\mathcal{E}_{\text{tan}}^{\beta}(t)$ (*cf.* (7.8)). We present the intrinsic cancellation for Q in §7.2. Then the boundary integral term can be reduced to the sum of $\mathcal{R}_2$ (the error term, defined by (7.19)) and $\mathcal{R}_3$ (the instant boundary integral term, *cf.* (7.18)). After that, we deduce the estimate of $\mathcal{R}_2$ in §7.3 and the estimate of $\mathcal{R}_3$ in §7.4–§7.6. Proposition 7.1 is proved, respectively, for the two- and three-dimensional cases at the end of §7.5 and §7.6. With Propositions 6.1 and 7.1 in hand, we conclude the proof of the main theorem in Sect. 8. Propositions 2.1 and 2.2 are shown in "Appendices A and B", respectively.

2. Formulation of the Nonlinear Problems

In this section, we introduce the system of thermoelasticity in the Eulerian coordinates and formulate the nonlinear problems for thermoelastic contact discontinuities.

2.1. Equations of Motion

In the context of elastodynamics, a body is identified with an open subset $\mathcal{O}$ of the reference space $\mathbb{R}^d$ for $d = 2, 3$. A motion of the body over a time interval (t_1, t_2) is a Lipschitz mapping x of $(t_1, t_2) \times \mathcal{O}$ to $\mathbb{R}^d$ such that $x(t, \cdot)$ is a bi-Lipschitz homeomorphism of $\mathcal{O}$ for each t in (t_1, t_2). Every particle X of body $\mathcal{O}$ is deformed to the spatial position $x(t, X)$ at time t.

The *velocity* $\tilde{v} \in \mathbb{R}^d$ with i-th component $\tilde{v}_i$ and the *deformation gradient* $\widetilde{F} \in \mathbb{M}^{d \times d}$ with (i, j)-th entry $\widetilde{F}_{ij}$ are defined by

$$\tilde{v}_i(t, X) := \frac{\partial x_i}{\partial t}(t, X), \qquad \widetilde{F}_{ij}(t, X) := \frac{\partial x_i}{\partial X_j}(t, X),$$

respectively, where $\mathbb{M}^{m \times n}$ stands for the vector space of real $m \times n$ matrices. We assume that map $x(t, \cdot) : \mathcal{O} \to \mathbb{R}^d$ is orientation-preserving so that

$$\det \widetilde{F}(t, \cdot) > 0 \qquad \text{in } \mathcal{O}. \tag{2.1}$$

The compatibility between fields $\tilde{v}$ and $\widetilde{F}$ is expressed by

$$\frac{\partial \widetilde{F}_{ij}}{\partial t}(t, X) = \frac{\partial \tilde{v}_i}{\partial X_j}(t, X) \qquad \text{for } i, j = 1, \ldots, d. \tag{2.2}$$

We need to append the constraints

$$\frac{\partial \widetilde{F}_{ij}}{\partial X_k} = \frac{\partial \widetilde{F}_{ik}}{\partial X_j} \qquad \text{for } i, j, k = 1, \ldots, d, \tag{2.3}$$

in order to guarantee that $\widetilde{\boldsymbol{F}}$ is a gradient. We emphasize that constraints (2.3) are *involutions* to the system of thermoelasticity, meaning that constraints (2.3) are preserved by the evolution via relations (2.2), provided that they hold at the initial time (see DAFERMOS [17]).

We will work in the Eulerian coordinates (t, x). For convenience, let us denote by $v = (v_1, \ldots, v_d)^\mathsf{T}$ the velocity and by $\boldsymbol{F} = (F_{ij})$ the deformation gradient in the Eulerian coordinates so that

$$v_i(t, x) = \tilde{v}_i(t, X(t, x)) \qquad F_{ij}(t, x) = \widetilde{F}_{ij}(t, X(t, x)),$$

where $X(t, x)$ is the inverse map of $x(t, X)$ for each fixed t.

The system of thermoelasticity modeling the motion of thermoelastic nonconductors of heat consists of the kinematic relations

$$(\partial_t + v_\ell \partial_\ell) F_{ij} = \partial_\ell v_i F_{\ell j} \qquad \text{for } i, j = 1, \ldots, d, \tag{2.4}$$

and the following conservation laws of mass, linear momentum, and energy (see [18, §2.3]):

$$\begin{cases} \partial_t \rho + \partial_\ell(\rho v_\ell) = 0, \\ \partial_t(\rho v_i) + \partial_\ell(\rho v_\ell v_i) = \partial_\ell T_{i\ell} \qquad \text{for } i = 1, \ldots, d, \\ \partial_t(\rho \varepsilon + \frac{1}{2}\rho|v|^2) + \partial_\ell((\rho \varepsilon + \frac{1}{2}\rho|v|^2)v_\ell) = \partial_\ell(v_j T_{j\ell}), \end{cases} \tag{2.5}$$

where $\partial_t := \frac{\partial}{\partial t}$ and $\partial_\ell := \frac{\partial}{\partial x_\ell}$ represent the partial differentials, ρ is the (spatial) density related with reference density $\rho_{\text{ref}} > 0$ through

$$\rho = \rho_{\text{ref}} (\det \boldsymbol{F})^{-1}, \tag{2.6}$$

symbol T_{ij} denotes the (i, j)-th entry of the Cauchy stress tensor $\boldsymbol{T} \in \mathbb{M}^{d \times d}$, and ε is the (specific) internal energy. Equations (2.4) are directly from the compatibility relations (2.2). In the Eulerian coordinates, constraints (2.3) are reduced to

$$F_{\ell k} \partial_\ell F_{ij} = F_{\ell j} \partial_\ell F_{ik} \qquad \text{for } i, j, k = 1, \ldots, d, \tag{2.7}$$

which are the involutions of system (2.4)–(2.5); see LEI–LIU–ZHOU [20, Remark 2] for instance. Throughout this paper, we adopt the Einstein summation convention whereby a repeated index in a term implies the summation over all the values of that index.

For every given thermoelastic medium, the following constitutive relations hold (see COLEMAN–NOLL [14]):

$$\varepsilon = \varepsilon(\boldsymbol{F}, S), \quad \boldsymbol{T} = \boldsymbol{T}^\mathsf{T} = \rho \frac{\partial \varepsilon(\boldsymbol{F}, S)}{\partial \boldsymbol{F}} \boldsymbol{F}^\mathsf{T}, \quad \vartheta := \frac{\partial \varepsilon(\boldsymbol{F}, S)}{\partial S} > 0,$$

where S and ϑ represent the (specific) entropy and the (absolute) temperature, respectively. In this paper, we consider the internal energy functions of the form

$$\varepsilon(\boldsymbol{F}, S) = \sum_{i,j=1}^{d} \frac{a_j}{2} F_{ij}^2 + e(\rho, S), \tag{2.8}$$

where a_j, for $j = 1, \ldots, d$, are positive constants. In view of (2.6), the internal energy $\varepsilon(F, S)$ depends on the deformation gradient F only through $F^\mathsf{T} F$. Hence, relation (2.8) is *frame-indifferent*:

$$\varepsilon(F, S) = \varepsilon(QF, S)$$

for all $Q \in \mathbb{M}^{d \times d}$ with $Q Q^\mathsf{T} = I_d$ and $\det Q = 1$. Here and below, I_m denotes the identity matrix of order m. Moreover, the constitutive relation (2.8) generalizes that for the *compressible neo-Hookean materials* (see [13, p. 189]) to the nonisentropic thermoelasticity. A direct computation gives

$$T = \rho F \operatorname{diag}(a_1, \ldots, a_d) F^\mathsf{T} - p I_d, \tag{2.9}$$

with

$$p := \rho^2 \frac{\partial e(\rho, S)}{\partial \rho}, \qquad \vartheta = \frac{\partial e(\rho, S)}{\partial S} > 0, \tag{2.10}$$

where $p = p(\rho, S)$ is the pressure. The speed of sound $c = c(\rho, S)$ is assumed to satisfy

$$c(\rho, S) := \sqrt{p_\rho(\rho, S)} > 0 \qquad \text{for } \rho > 0. \tag{2.11}$$

If all of a_j are the same, then the material is *isotropic*; otherwise it is *anisotropic* (see [13, §3.4]). In the special case when all of a_j are equal to zero, system (2.5) is reduced to the compressible Euler equations in gas dynamics. Since this paper concerns the effect of elasticity to the evolution of materials, we set without loss of generality that $a_j = 1$ for all j. We refer to [13, Chapters 3–4] and [18, Chapter 2] for a thorough discussion of the constitutive relations.

For simplicity, the reference density ρ_{ref} is supposed to be unit, leading to

$$\operatorname{div}(\rho F_j) := \partial_\ell (\rho F_{\ell j}) = 0 \qquad \text{for } j = 1, \ldots, d, \tag{2.12}$$

where F_j stands for the j-th column of F; see, e.g., [20, Remark 1]. By virtue of the divergence constraints (2.12), we can reformulate (2.4) and (2.7) into the conservation laws:

$$\partial_t(\rho F_{ij}) + \partial_\ell (\rho F_{ij} v_\ell - v_i \rho F_{\ell j}) = 0 \qquad \text{for } i, j = 1, \ldots, d, \tag{2.13}$$

$$\partial_\ell (\rho F_{\ell k} F_{ij} - \rho F_{\ell j} F_{ik}) = 0 \qquad \text{for } i, j, k = 1, \ldots, d. \tag{2.14}$$

In light of (2.10)–(2.12), for smooth solutions, system (2.4)–(2.5) is equivalent to

$$(\partial_t + v_\ell \partial_\ell) p + \rho c^2 \partial_\ell v_\ell = 0, \tag{2.15a}$$

$$\rho(\partial_t + v_\ell \partial_\ell) v_i + \partial_i p - \rho F_{\ell k} \partial_\ell F_{ik} = 0 \qquad \text{for } i = 1, \ldots, d, \tag{2.15b}$$

$$\rho(\partial_t + v_\ell \partial_\ell) F_{ij} - \rho F_{\ell j} \partial_\ell v_i = 0 \qquad \text{for } i, j = 1, \ldots, d, \tag{2.15c}$$

$$(\partial_t + v_\ell \partial_\ell) S = 0. \tag{2.15d}$$

Let us take $U = (p, v, F_1, \ldots, F_d, S)^{\mathsf{T}}$ as the primary unknowns and define the following symmetric matrices:

$$A_0(U) := \mathrm{diag}\big(1/(\rho c^2),\ \rho \boldsymbol{I}_{d+d^2},\ 1\big), \tag{2.16}$$

$$A_i(U) := \begin{pmatrix} v_i/(\rho c^2) & \boldsymbol{e}_i^{\mathsf{T}} & 0 & \cdots & 0 & 0 \\ \boldsymbol{e}_i & \rho v_i \boldsymbol{I}_d & -\rho F_{i1} \boldsymbol{I}_d & \cdots & -\rho F_{id} \boldsymbol{I}_d & 0 \\ 0 & -\rho F_{i1} \boldsymbol{I}_d & & & & 0 \\ \vdots & \vdots & & \rho v_i \boldsymbol{I}_{d^2} & & \vdots \\ 0 & -\rho F_{id} \boldsymbol{I}_d & & & & 0 \\ 0 & 0 & 0 & \cdots & 0 & v_i \end{pmatrix} \tag{2.17}$$

for $i = 1, \ldots, d$, where we denote $\boldsymbol{e}_i := (\delta_{i,1}, \ldots, \delta_{i,d})^{\mathsf{T}}$ with $\delta_{i,j}$ being the Kronecker delta. Then system (2.15) reads as

$$A_0(U)\partial_t U + A_i(U)\partial_i U = 0, \tag{2.18}$$

which is symmetric hyperbolic, due to (2.6) and (2.11).

2.2. Thermoelastic Contact Discontinuities

Let U be smooth on each side of a smooth hypersurface $\Gamma(t) := \{x \in \mathbb{R}^d : x_1 = \varphi(t, x')\}$ for $x' := (x_2, \ldots, x_d)$:

$$U(t, x) = \begin{cases} U^+(t, x) & \text{in } \Omega^+(t) := \{x \in \mathbb{R}^d : x_1 > \varphi(t, x')\}, \\ U^-(t, x) & \text{in } \Omega^-(t) := \{x \in \mathbb{R}^d : x_1 < \varphi(t, x')\}, \end{cases} \tag{2.19}$$

where $U^\pm(t, x)$ are smooth functions in respective domains $\Omega^\pm(t)$. Then U is a weak solution of (2.5) and (2.12)–(2.14) if and only if it is a smooth solution of (2.7), (2.12), and (2.18) in domains $\Omega^\pm(t)$, and the following Rankine–Hugoniot jump conditions hold at every point of front $\Gamma(t)$:

$$[m_N] = 0, \tag{2.20a}$$
$$[m_N v] + [\rho F_{\ell N} F_\ell] = N[p], \tag{2.20b}$$
$$[m_N(\varepsilon + \tfrac{1}{2}|v|^2)] + [\rho F_{\ell N} F_\ell \cdot v] = [p v_N], \tag{2.20c}$$
$$[m_N F_{ij}] + [\rho F_{jN} v_i] = 0 \qquad \text{for } i, j = 1, \ldots, d, \tag{2.20d}$$
$$[\rho F_{jN}] = 0 \qquad \text{for } j = 1, \ldots, d, \tag{2.20e}$$
$$[\rho F_{kN} F_{ij} - \rho F_{jN} F_{ik}] = 0 \qquad \text{for } i, j, k = 1, \ldots, d, \tag{2.20f}$$

where $[g] := (g^+ - g^-)|_{\Gamma(t)}$ stands for the jump across $\Gamma(t)$, and

$$v_N^\pm := v^\pm \cdot N, \qquad F_{jN}^\pm := F_j^\pm \cdot N, \qquad m_N^\pm := \rho^\pm(\partial_t \varphi - v_N^\pm)$$

with $N := (1, -\partial_2 \varphi, \ldots, -\partial_d \varphi)^{\mathsf{T}}$, so that $m_N^\pm$ represent the mass transfer fluxes. Also see [18, §3.3] for the corresponding jump conditions written in the Lagrangian description.

We are interested in discontinuous weak solutions U for which the mass does not transfer across the discontinuity interface $\Gamma(t)$:

$$m_N^\pm = \rho^\pm(\partial_t\varphi - v_N^\pm) = 0 \qquad \text{on } \Gamma(t). \tag{2.21}$$

Then the matrix

$$\left(\partial_t\varphi A_0(U) - N_\ell A_\ell(U)\right)\big|_{\Gamma(t)}$$

$$= \begin{pmatrix} 0 & -N^\mathsf{T} & 0 & \cdots & 0 & 0 \\ -N & \boldsymbol{O}_d & \rho F_{1N}\boldsymbol{I}_d & \cdots & \rho F_{dN}\boldsymbol{I}_d & 0 \\ 0 & \rho F_{1N}\boldsymbol{I}_d & & & & 0 \\ \vdots & \vdots & & \boldsymbol{O}_{d^2} & & \vdots \\ 0 & \rho F_{dN}\boldsymbol{I}_d & & & & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\Bigg|_{\Gamma(t)}$$

has eigenvalues

$$\begin{aligned} &\pm\sqrt{|N|^2 + \rho^2 F_{\ell N} F_{\ell N}} && \text{with multiplicity } 1, \\ &\pm\rho\sqrt{F_{\ell N} F_{\ell N}} && \text{with multiplicity } d-1, \\ &0 && \text{with multiplicity } d^2 - d + 2, \end{aligned}$$

where $\boldsymbol{O}_m$ denotes the zero matrix of order m. As a result, the boundary matrix on $\Gamma(t)$:

$$A_{\text{bdy}} := \text{diag}\left(\partial_t\varphi A_0(U^+) - N_\ell A_\ell(U^+), -\partial_t\varphi A_0(U^-) + N_\ell A_\ell(U^-)\right)\big|_{\Gamma(t)}$$

is singular, which implies that the free boundary $\Gamma(t)$ is *characteristic*. In this sense, the weak solution U is a *characteristic discontinuity*.

We now reformulate the jump conditions (2.20) by means of assumption (2.1). More precisely, from (2.1), we derive

$$\begin{pmatrix} F_{1N}^\pm \\ \vdots \\ F_{dN}^\pm \end{pmatrix} = \begin{pmatrix} F_{11}^\pm \\ \vdots \\ F_{1d}^\pm \end{pmatrix} - \sum_{\ell=2}^{d} \partial_\ell\varphi \begin{pmatrix} F_{\ell 1}^\pm \\ \vdots \\ F_{\ell d}^\pm \end{pmatrix} \neq 0 \qquad \text{on } \Gamma(t). \tag{2.22}$$

Consequently, the boundary matrix A_{bdy} on $\Gamma(t)$ has $2d$ negative, $2d$ positive, and $2(d^2 - d + 2)$ zero eigenvalues. Since one more boundary condition is needed to determine the unknown interface function φ, the correct number of boundary conditions is $2d + 1$, according to the well-posedness theory for hyperbolic problems. Plugging involutions (2.20e) and condition (2.21) into (2.20d) leads to

$$F_{jN}^+[v] = 0 \qquad \text{on } \Gamma(t), \text{ for } j = 1, \ldots, d.$$

Then it follows from (2.22) that $[v] = 0$ on $\Gamma(t)$. We employ (2.20e) and (2.21) again to rewrite (2.20a)–(2.20d) as

$$\partial_t\varphi = v_N^+, \quad [v] = 0, \quad \rho^+ F_{\ell N}^+[F_\ell] = N[p] \qquad \text{on } \Gamma(t). \tag{2.23}$$

Definition 2.1. A thermoelastic contact discontinuity is a discontinuous weak solution of form (2.19) of system (2.5) and (2.12)–(2.14) with the boundary conditions (2.23).

We exclude (2.20e)–(2.20f) from (2.23) in order to prescribe the correct number of boundary conditions for the well-posedness of the thermoelastic contact discontinuity problem. On one hand, (2.20e)–(2.20f) are involutions inherited from the initial data. On the other hand, they prevent any thermoelastic contact discontinuity in the *isentropic* process. More generally, we have the following physically relevant result whose proof is postponed to "Appendix A".

Proposition 2.1. *If* $[S] = 0$ *on* $\Gamma(t)$, *then* $[U] = 0$ *on* $\Gamma(t)$ *so that no thermoelastic contact discontinuity exists.*

If condition (2.1) is ignored on interface $\Gamma(t)$, then there is another type of characteristic discontinuities for (2.5) and (2.12)–(2.14) with the constitutive relation (2.9), *i.e.*, the so-called *compressible vortex sheets* that are associated with the boundary constraints $(F_{1N}^\pm, \ldots, F_{dN}^\pm)|_{\Gamma(t)} = 0$. In this case, the jump conditions (2.20) are reduced to

$$\partial_t \varphi - v_N^+ = \partial_t \varphi - v_N^- = [p] = 0 \qquad \text{on } \Gamma(t).$$

Then the normal velocity and pressure are continuous across front $\Gamma(t)$, while the tangential components of the velocity can undergo a jump. See [8,9] for the two-dimensional isentropic case in this regard.

In this paper, we focus on the thermoelastic contact discontinuity problem corresponding to the boundary constraints

$$F_{1N}^\pm \neq 0, \quad F_{2N}^\pm = \cdots = F_{dN}^\pm = 0 \qquad \text{on } \Gamma(t). \tag{2.24}$$

Then the boundary conditions (2.23) on $\Gamma(t)$ become

$$\begin{cases} \partial_t \varphi - v_N^+ = 0, & [v] = 0, \\ \rho^+ F_{1N}^+ [F_{11}] = [p], & [F_{11} \partial_i \varphi + F_{i1}] = 0 \quad \text{for } i = 2, \ldots, d. \end{cases} \tag{2.25}$$

By virtue of (2.24), involutions (2.20f) are equivalent to

$$[F_j] = 0 \qquad \text{on } \Gamma(t), \text{ for } j = 2, \ldots, d. \tag{2.26}$$

Since φ describing the discontinuity front $\Gamma(t)$ is one of the unknowns, the thermoelastic contact discontinuity problem is a *free boundary problem*.

Taking into account the Galilean invariance of (2.7), (2.12), (2.18), and (2.20), we choose the following piecewise constant thermoelastic contact discontinuity as the background state:

$$\bar{\varphi} = 0, \qquad \overline{U}(x) = \begin{cases} \overline{U}^+ := (\bar{p}^+, 0, \overline{F}^+, \overline{S}^+) & \text{if } x_1 > 0, \\ \overline{U}^- := (\bar{p}^-, 0, \overline{F}^-, \overline{S}^-) & \text{if } x_1 < 0, \end{cases} \tag{2.27}$$

where $\overline{\boldsymbol{F}}^\pm = \mathrm{diag}\big(\overline{F}_{11}^\pm, \overline{F}_{22}, \ldots, \overline{F}_{dd}\big)$ and

$$\bar{p}^\pm = p(\bar{\rho}^\pm, \overline{S}^\pm), \quad \bar{\rho}^+ \overline{F}_{11}^+[\overline{F}_{11}] = [\bar{p}] \qquad \text{for } \bar{\rho}^\pm := (\det \overline{\boldsymbol{F}}^\pm)^{-1}, \quad (2.28)$$

in keeping with (2.6), (2.10), and (2.24)–(2.26). Without loss of generality, we assume that the principal stretches $\overline{F}_{11}^\pm$, $\overline{F}_{22}$, ..., $\overline{F}_{dd}$ are positive constants with $\overline{F}_{11}^+ > \overline{F}_{11}^-$. We point out that each of the background deformations is either a *dilation* or a *simple extension* when $\overline{F}_{22} = \cdots = \overline{F}_{dd}$ (see TRUESDELL--TOUPIN [30, §43–§44]).

2.3. Reduced Problem in a Fixed Domain

It is more convenient to convert the free boundary problem for thermoelastic contact discontinuities into a problem in a fixed domain. To this end, we replace unknowns $U^\pm$, being smooth in $\Omega^\pm(t)$, by

$$U_\sharp^\pm(t, x) := U(t, \Phi^\pm(t, x), x'), \tag{2.29}$$

where we take the lifting functions $\Phi^\pm$ as in MÉTIVIER [21, p. 70] to have the form

$$\Phi^\pm(t, x) := \pm x_1 + \chi(\pm x_1)\varphi(t, x'), \tag{2.30}$$

with $\chi \in C_0^\infty(\mathbb{R})$ satisfying that

$$\chi \equiv 1 \quad \text{on } [-1, 1], \qquad \|\chi'\|_{L^\infty(\mathbb{R})} < 1. \tag{2.31}$$

The cut-off function χ is introduced as in [21,23] to avoid the assumption in the main theorem that the initial perturbations have compact support. This change of variables is admissible on the time interval $[0, T]$ as long as $\|\varphi\|_{L^\infty([0,T] \times \mathbb{R}^{d-1})} \leqslant \frac{1}{2}$.

The existence of thermoelastic contact discontinuities amounts to constructing solutions $U_\sharp^\pm$, which are smooth in the fixed domain $\Omega := \{x \in \mathbb{R}^d : x_1 > 0\}$, of the following initial-boundary value problem:

$$\mathbb{L}(U^\pm, \Phi^\pm) := L(U^\pm, \Phi^\pm)U^\pm = 0 \qquad \text{if } x_1 > 0, \tag{2.32a}$$

$$\mathbb{B}(U^+, U^-, \varphi) = 0 \qquad \text{if } x_1 = 0, \tag{2.32b}$$

$$(U^+, U^-, \varphi) = (U_0^+, U_0^-, \varphi_0) \qquad \text{if } t = 0, \tag{2.32c}$$

where index "$\sharp$" has been dropped for notational simplicity. Thanks to transformation (2.29), operator $L(U, \Phi)$ is given by

$$L(U, \Phi) := A_0(U)\partial_t + \widetilde{A}_1(U, \Phi)\partial_1 + \sum_{i=2}^d A_i(U)\partial_i, \tag{2.33}$$

where $A_i(U)$, for $i = 0, \ldots, d$, are defined by (2.16)–(2.17), and

$$\widetilde{A}_1(U, \Phi) := \frac{1}{\partial_1 \Phi}\Big(A_1(U) - \partial_t \Phi A_0(U) - \sum_{i=2}^d \partial_i \Phi A_i(U)\Big).$$

According to (2.25), the boundary operator $\mathbb{B}$ reads as

$$\mathbb{B}(U^+, U^-, \varphi) := \begin{pmatrix} \partial_t \varphi - v_N^+ \\ [v] \\ [p] - \rho^+ F_{1N}^+ [F_{11}] \\ [F_{11} \partial_2 \varphi + F_{21}] \\ \vdots \\ [F_{11} \partial_d \varphi + F_{d1}] \end{pmatrix}. \tag{2.34}$$

The boundary matrix $\mathrm{diag}(-\widetilde{A}_1(U^+, \Phi^+), -\widetilde{A}_1(U^-, \Phi^-))$ for problem (2.32) has $2d$ negative eigenvalues ("incoming characteristics") on boundary $\partial\Omega := \{x \in \mathbb{R}^d : x_1 = 0\}$. As discussed before, the correct number of boundary conditions is $2d + 1$, which is just the case in (2.32b).

In accordance with (2.6)–(2.7), (2.20e)–(2.20f), and (2.24), we assume that the initial data (2.32c) satisfy that

$$\rho^\pm = (\det \boldsymbol{F}^\pm)^{-1} \qquad\qquad\qquad\qquad \text{if } x_1 \geqslant 0, \quad (2.35)$$

$$F_{\ell k}^\pm \partial_\ell^{\Phi^\pm} F_{ij}^\pm - F_{\ell j}^\pm \partial_\ell^{\Phi^\pm} F_{ik}^\pm = 0 \quad \text{for } i, j, k = 1, \ldots, d, \quad \text{if } x_1 > 0, \quad (2.36)$$

$$[\rho F_{jN}] = 0 \qquad\qquad\qquad \text{for } j = 1, \ldots, d, \qquad\quad \text{if } x_1 = 0, \quad (2.37)$$

$$[\rho F_{kN} F_{ij} - \rho F_{jN} F_{ik}] = 0 \quad \text{for } i, j, k = 1, \ldots, d, \quad \text{if } x_1 = 0, \quad (2.38)$$

$$F_{jN}^\pm = 0 \qquad\qquad\qquad\quad \text{for } j = 2, \ldots, d, \qquad\quad \text{if } x_1 = 0. \quad (2.39)$$

Here and below, to simplify the notation, we denote the partial differentials with respect to the lifting function Φ by

$$\partial_t^\Phi := \partial_t - \frac{\partial_t \Phi}{\partial_1 \Phi} \partial_1, \quad \partial_1^\Phi := \frac{1}{\partial_1 \Phi} \partial_1, \quad \partial_i^\Phi := \partial_i - \frac{\partial_i \Phi}{\partial_1 \Phi} \partial_1 \text{ for } i = 2, \ldots, d. \tag{2.40}$$

The following proposition manifests that identities (2.35)–(2.39) are involutions in the straightened coordinates. See "Appendix B" for the proof.

Proposition 2.2. *For each sufficiently smooth solution of problem* (2.32) *on the time interval* $[0, T]$*, if constraints* (2.35)–(2.39) *are satisfied at the initial time, then these constraints and*

$$\partial_\ell^{\Phi^\pm}(\rho^\pm F_{\ell j}^\pm) = 0 \quad \text{if } x_1 > 0, \text{ for } j = 1, \ldots, d, \tag{2.41}$$

hold for all $t \in [0, T]$*.*

Relations (2.41) are involutions in the straightened coordinates corresponding to the divergence constraints (2.12), from which we can pass from the Eulerian to the Lagrangian formulation of the thermoelastic contact discontinuity problem.

3. Linearized Problem and Main Theorem

In this section we introduce the basic state $(\mathring{U}^\pm, \mathring{\varphi})$ that is a small perturbation of the stationary thermoelastic contact discontinuity $(\overline{U}^\pm, \overline{\varphi})$ given in (2.27)–(2.28). Then we perform the linearization and state the main theorem of this paper.

3.1. Basic State

We denote $\Omega_T := (-\infty, T) \times \Omega$ and $\omega_T := (-\infty, T) \times \partial\Omega$ for any real number T. Let the basic state $(\mathring{U}^\pm, \mathring{\varphi})$ with $\mathring{U}^\pm := (\mathring{p}^\pm, \mathring{v}^\pm, \mathring{F}^\pm, \mathring{S}^\pm)^\mathsf{T}$ be sufficiently smooth. According to form (2.30), we introduce the notations

$$\bar{\Phi}^\pm := \pm x_1, \qquad \mathring{\Phi}^\pm := \pm x_1 + \mathring{\Psi}^\pm, \qquad \mathring{\Psi}^\pm := \chi(\pm x_1)\mathring{\varphi}(t, x'), \qquad (3.1)$$

$$\mathring{v}_N^\pm := \mathring{v}^\pm \cdot \mathring{N}^\pm, \qquad \mathring{F}_{jN}^\pm := \mathring{F}_j^\pm \cdot \mathring{N}^\pm, \qquad \mathring{N}^\pm := (1, -\partial_2\mathring{\Phi}^\pm, \ldots, -\partial_d\mathring{\Phi}^\pm)^\mathsf{T}, \qquad (3.2)$$

where $\chi \in C_0^\infty(\mathbb{R})$ satisfies (2.31), and $\mathring{F}_j^\pm$ are the jth columns of $\mathring{F}^\pm$.

Perturbations $\mathring{V}^\pm := \mathring{U}^\pm - \overline{U}^\pm$ and $\mathring{\varphi}$ are supposed to satisfy that

$$\|\mathring{V}^\pm\|_{H^6(\Omega_T)} + \|\mathring{\varphi}\|_{H^6(\omega_T)} \leqslant K \qquad (3.3)$$

for a sufficiently small positive constant $K \leqslant 1$, so that

$$\pm\partial_1\mathring{\Phi}^\pm \geqslant \frac{1}{2} \qquad \text{on } \overline{\Omega_T}, \qquad (3.4)$$

thanks to the Sobolev embedding $H^6(\Omega_T) \hookrightarrow W^{3,\infty}(\Omega_T)$. We assume further that the basic state $(\mathring{U}^\pm, \mathring{\varphi})$ satisfies constraints (2.35), (2.32b), and (2.37)–(2.39), *i.e.*,

$$\mathring{\rho}^\pm = (\det \mathring{F}^\pm)^{-1}, \quad \mathring{p}^\pm = p(\mathring{\rho}^\pm, \mathring{S}^\pm) \qquad \text{if } x_1 \geqslant 0, \qquad (3.5a)$$

$$\mathbb{B}(\mathring{U}^+, \mathring{U}^-, \mathring{\varphi}) = 0 \qquad \text{if } x_1 = 0, \qquad (3.5b)$$

$$[\mathring{\rho}\mathring{F}_{jN}] = 0 \qquad \text{for } j = 1, \ldots, d, \qquad \text{if } x_1 = 0, \qquad (3.5c)$$

$$[\mathring{\rho}\mathring{F}_{kN}\mathring{F}_{ij} - \mathring{\rho}\mathring{F}_{jN}\mathring{F}_{ik}] = 0 \qquad \text{for } i, j, k = 1, \ldots, d, \qquad \text{if } x_1 = 0, \qquad (3.5d)$$

$$\mathring{F}_{jN}^\pm = 0 \qquad \text{for } j = 2, \ldots, d, \qquad \text{if } x_1 = 0. \qquad (3.5e)$$

Under (3.5c) and (3.5e), relations (3.5d) are equivalent to

$$[\mathring{F}_j] = 0 \qquad \text{on } \partial\Omega, \quad \text{for } j = 2, \ldots, d. \qquad (3.6)$$

Moreover, we assume that the basic state satisfies

$$\left(\partial_t + \sum_{\ell=2}^d \mathring{v}_\ell^\pm \partial_\ell\right)\mathring{F}_j^\pm - \sum_{\ell=2}^d \mathring{F}_{\ell j}^\pm \partial_\ell\mathring{v}^\pm = 0 \qquad \text{on } \partial\Omega, \text{ for } j = 2, \ldots, d, \qquad (3.7)$$

which will play an important role in the estimate of the tangential derivatives, especially in the proof of Lemma 7.2. As a matter of fact, constraints (3.7) come from restricting the interior equations for $F_j^\pm$ on boundary $\partial\Omega$ and utilizing (3.5b)–(3.5e).

Before performing the linearization, we give an alternative form of the boundary operator $\mathbb{B}$ defined in (2.34), which will be essential for providing the cancellation

effect in the estimate of the tangential derivatives. More precisely, by virtue of (2.39), we observe that

$$\det \boldsymbol{F}^{\pm} = \varrho(\boldsymbol{F}^{\pm})^{-1} F_{1N}^{\pm} \qquad \text{on } \partial\Omega, \tag{3.8}$$

where $\varrho(\boldsymbol{F})$ is the scalar function defined by

$$\varrho(\boldsymbol{F}) := \begin{cases} F_{22}^{-1} & \text{if } d = 2, \\ (F_{22}F_{33} - F_{23}F_{32})^{-1} & \text{if } d = 3. \end{cases} \tag{3.9}$$

In particular, for the background state (2.27), we have

$$\varrho(\overline{\boldsymbol{F}}^{\pm}) = \begin{cases} \overline{F}_{22}^{-1} & \text{if } d = 2, \\ \overline{F}_{22}^{-1}\,\overline{F}_{33}^{-1} & \text{if } d = 3. \end{cases} \tag{3.10}$$

Combine (3.8) with (2.35) and use (2.37) to obtain

$$\rho^{\pm} F_{1N}^{\pm} = \varrho(\boldsymbol{F}^+) = \varrho(\boldsymbol{F}^-) \qquad \text{on } \partial\Omega, \tag{3.11}$$

which yields

$$\mathbb{B}(U^+, U^-, \varphi) = \begin{pmatrix} \partial_t \varphi - v_N^+ \\ [v] \\ [p] - \varrho(\boldsymbol{F}^+)[F_{11}] \\ [F_{11}\partial_2\varphi + F_{21}] \\ \vdots \\ [F_{11}\partial_d\varphi + F_{d1}] \end{pmatrix}. \tag{3.12}$$

Furthermore, from (3.5a), (3.5e), and (3.6), we infer that

$$\mathring{\rho}^{\pm} \mathring{F}_{1N}^{\pm} = \varrho(\mathring{\boldsymbol{F}}^+) = \varrho(\mathring{\boldsymbol{F}}^-) \qquad \text{on } \partial\Omega. \tag{3.13}$$

As a result, constraints (3.5)–(3.7) are equivalent to constraints (3.5a)–(3.5b) and (3.5e)–(3.7).

3.2. Linearization and Main Theorem

Let us now deduce the linearized problem based on identity (3.12). For this purpose, we consider families $(U_\epsilon^{\pm}, \Phi_\epsilon^{\pm}) = (\mathring{U}^{\pm} + \epsilon V^{\pm}, \mathring{\Phi}^{\pm} + \epsilon \Psi^{\pm})$, where

$$\Psi^{\pm}(t, x) := \chi(\pm x_1)\psi(t, x'). \tag{3.14}$$

The linearized operators are given by

$$\begin{cases} \mathbb{L}'(\mathring{U}^{\pm}, \mathring{\Phi}^{\pm})(V^{\pm}, \Psi^{\pm}) := \dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbb{L}(U_\epsilon^{\pm}, \Phi_\epsilon^{\pm})\Big|_{\epsilon=0}, \\[2mm] \mathbb{B}'(\mathring{U}^{\pm}, \mathring{\varphi})(V, \psi) := \dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\mathbb{B}(U_\epsilon^+, U_\epsilon^-, \varphi_\epsilon)\Big|_{\epsilon=0}, \end{cases}$$

where $V := (V^+, V^-)^\mathsf{T}$, and $\varphi_\epsilon := \mathring{\varphi} + \epsilon\psi$ denotes the common trace of $\Phi_\epsilon^\pm$ on boundary $\partial\Omega$. A standard calculation leads to

$$\mathbb{L}'(U, \Phi)(V, \Psi) = L(U, \Phi)V + \mathcal{C}(U, \Phi)V - \frac{1}{\partial_1\Phi}\big(L(U, \Phi)\Psi\big)\partial_1 U, \quad (3.15)$$

where $L(U, \Phi)$ is given in (2.33), and $\mathcal{C}(U, \Phi)$ is the zero-th order operator defined by

$$\mathcal{C}(U, \Phi)V := V_\ell\frac{\partial A_0(U)}{\partial U_\ell}\partial_t U + V_\ell\frac{\partial \widetilde{A}_1(U, \Phi)}{\partial U_\ell}\partial_1 U + \sum_{i=2}^{d} V_\ell\frac{\partial A_i(U)}{\partial U_\ell}\partial_i U. \tag{3.16}$$

Thanks to the alternative form (3.12), we compute

$$\mathbb{B}'\big(\mathring{U}^\pm, \mathring{\varphi}\big)(V, \psi) = \begin{pmatrix} \big(\partial_t + \sum_{i=2}^{d}\mathring{v}_i^+\partial_i\big)\psi - v^+\cdot\mathring{N} \\ [v] \\ [p] - \varrho(\mathring{F}^+)[F_{11}] - [\mathring{F}_{11}]\partial_{F_{ij}}\varrho(\mathring{F}^+)F_{ij}^+ \\ [F_{11}\partial_2\mathring{\varphi} + F_{21}] + [\mathring{F}_{11}]\partial_2\psi \\ \vdots \\ [F_{11}\partial_d\mathring{\varphi} + F_{d1}] + [\mathring{F}_{11}]\partial_d\psi \end{pmatrix} \tag{3.17}$$

with $\varrho(F)$ defined by (3.9).

As in ALINHAC [1], applying the "good unknowns":

$$\dot{V}^\pm := V^\pm - \frac{\Psi^\pm}{\partial_1\mathring{\Phi}^\pm}\partial_1\mathring{U}^\pm, \tag{3.18}$$

we calculate (*cf.* [21, Proposition 1.3.1])

$$\mathbb{L}'(\mathring{U}^\pm, \mathring{\Phi}^\pm)(V^\pm, \Psi^\pm)$$
$$= L(\mathring{U}^\pm, \mathring{\Phi}^\pm)\dot{V}^\pm + \mathcal{C}(\mathring{U}^\pm, \mathring{\Phi}^\pm)\dot{V}^\pm + \frac{\Psi^\pm}{\partial_1\mathring{\Phi}^\pm}\partial_1\big(L(\mathring{U}^\pm, \mathring{\Phi}^\pm)\mathring{U}^\pm\big). \tag{3.19}$$

In view of the nonlinear results obtained in [1,5,16], we neglect the zero-th order terms in $\Psi^\pm$ of (3.19) and consider the *effective linear problem*

$$\begin{aligned} \mathbb{L}'_{e,\pm}\dot{V}^\pm &= f^\pm & &\text{if } x_1 > 0, & &(3.20a) \\ \mathbb{B}'_e(\dot{V}, \psi) &= g & &\text{if } x_1 = 0, & &(3.20b) \\ (\dot{V}, \psi) &= 0 & &\text{if } t < 0, & &(3.20c) \end{aligned}$$

where we abbreviate $\dot{V} := (\dot{V}^+, \dot{V}^-)^\mathsf{T}$ and denote

$$\mathbb{L}'_{e,\pm}\dot{V}^\pm := L(\mathring{U}^\pm, \mathring{\Phi}^\pm)\dot{V}^\pm + \mathcal{C}(\mathring{U}^\pm, \mathring{\Phi}^\pm)\dot{V}^\pm, \tag{3.21}$$

$$\mathbb{B}'_e(\dot{V}, \psi) := \begin{pmatrix} (\partial_t + \sum_{i=2}^d \mathring{v}_i^+ \partial_i)\psi - \dot{v}^+ \cdot \mathring{N}^+ - \partial_1 \mathring{v}_N^+ \psi \\ [\dot{v}] + \psi(\partial_1 \mathring{v}^+ + \partial_1 \mathring{v}^-) \\ [\dot{p}] - \varrho(\mathring{F}^+)[\dot{F}_{11}] - [\mathring{F}_{11}]\partial_{F_{ij}}\varrho(\mathring{F}^+)\dot{F}_{ij}^+ + \mathring{b}_1 \psi \\ [\dot{F}_{11}\partial_2\mathring{\varphi} + \dot{F}_{21}] + [\mathring{F}_{11}]\partial_2\psi + \mathring{b}_2\psi \\ \vdots \\ [\dot{F}_{11}\partial_d\mathring{\varphi} + \dot{F}_{d1}] + [\mathring{F}_{11}]\partial_d\psi + \mathring{b}_d\psi \end{pmatrix}, \tag{3.22}$$

with operators L and $\mathcal{C}$ defined by (2.33) and (3.16). In (3.22), we denote

$$\mathring{b}_1 := \partial_1 \mathring{p}^+ + \partial_1 \mathring{p}^- - \varrho(\mathring{F}^+)(\partial_1 \mathring{F}_{11}^+ + \partial_1 \mathring{F}_{11}^-) - [\mathring{F}_{11}]\partial_{F_{ij}}\varrho(\mathring{F}^+)\partial_1 \mathring{F}_{ij}^+,$$

$$\mathring{b}_i := (\partial_1 \mathring{F}_{11}^+ + \partial_1 \mathring{F}_{11}^-)\partial_i\mathring{\varphi} + (\partial_1 \mathring{F}_{i1}^+ + \partial_1 \mathring{F}_{i1}^-) \quad \text{for } i = 2, \ldots, d.$$

The explicit form (3.22) results from the identity $\mathbb{B}'_e(\dot{V}, \psi) = \mathbb{B}'(\mathring{U}^\pm, \mathring{\varphi})(V, \psi)$. We write $\mathring{V} := (\mathring{V}^+, \mathring{V}^-)^\mathsf{T}$, $\mathring{\psi} := (\mathring{\psi}^+, \mathring{\psi}^-)^\mathsf{T}$, $\mathbb{L}'_e\dot{V} := (\mathbb{L}'_{e,+}\dot{V}^+, \mathbb{L}'_{e,-}\dot{V}^-)^\mathsf{T}$, $f := (f^+, f^-)^\mathsf{T}$, etc. to avoid overloaded expressions.

We now state the main result of this paper.

Theorem 3.1. *Let $T > 0$ and $s \in \mathbb{N}_+$ be fixed. Assume that the background state* (2.27)–(2.28) *satisfies the stability condition*

$$\frac{[\bar{F}_{11}]}{\bar{F}_{11}^+} < \begin{cases} \bar{F}_{22}^2(\bar{F}_{11}^+)^{-2} & \text{if } d = 2, \\ \bar{C}^{-1} & \text{if } d = 3, \end{cases} \tag{3.23}$$

with

$$\overline{C} := \left(1 + \frac{\overline{F}_{22}^2}{\overline{F}_{33}^2}\right)^{1/2}\left\{ \max(1, \frac{(\overline{F}_{11}^+)^2}{\overline{F}_{22}^2}) + \max(1, \frac{(\overline{F}_{11}^+)^2}{\overline{F}_{33}^2})\frac{\overline{F}_{33}}{\overline{F}_{22}}\right\},$$

and that perturbations $(\mathring{V}^\pm, \mathring{\varphi}) \in H^{s+2}(\Omega_T) \times H^{s+2}(\omega_T)$ satisfy constraints (3.3)–(3.7). *Then there exist positive constants K_0 and C_0, uniformly bounded even when $[\overline{F}_{11}]$ tends to zero, such that, for all $K \leqslant K_0$ and $(\dot{V}^\pm, \psi) \in H^{s+1}(\Omega_T) \times H^{s+3/2}(\omega_T)$ vanishing in the past,*

$$\|\dot{V}\|_{H^1(\Omega_T)} + \|\psi\|_{H^{3/2}(\omega_T)}$$
$$\leqslant C_0\Big\{\|\mathbb{L}'_e\dot{V}\|_{H^1(\Omega_T)} + \|\mathbb{B}'_e(\dot{V}, \psi)\|_{H^{3/2}(\omega_T)}\Big\} \quad \text{if } s = 1, \tag{3.24}$$

$$\|\dot{V}\|_{H^s(\Omega_T)} + \|\psi\|_{H^{s+1/2}(\omega_T)}$$
$$\leqslant C_0\Big\{\|\mathbb{L}'_e\dot{V}\|_{H^s(\Omega_T)} + \|\mathbb{B}'_e(\dot{V}, \psi)\|_{H^{s+1/2}(\omega_T)}$$
$$+ \big(\|\mathbb{L}'_e\dot{V}\|_{H^3(\Omega_T)} + \|\mathbb{B}'_e(\dot{V}, \psi)\|_{H^{7/2}(\omega_T)}\big)$$
$$\times \big(\|\mathring{V}^\pm\|_{H^{s+2}(\Omega_T)} + \|\mathring{\varphi}\|_{H^{s+2}(\omega_T)}\big)\Big\} \quad \text{if } s \geqslant 3. \tag{3.25}$$

Notice that the $H^2(\Omega_T) \times H^{5/2}(\omega_T)$–estimate of $(\dot{V}, \psi)$ follows from (3.25) with $s = 3$. We remark that the tame estimates (3.24)–(3.25) present no loss of regularity with respect to the interior source term $\mathbb{L}'_e \dot{V}$, while there is a loss of one derivative with respect to the boundary source term $\mathbb{B}'_e(\dot{V}, \psi)$. It should also be pointed out that estimate (3.25) is with a *fixed* loss of regularity with respect to the coefficients, which offers a way to establish the nonlinear stability of thermoelastic contact discontinuities by a suitable Nash–Moser iteration scheme. The dropped terms in (3.19) will be considered as error terms at each Nash–Moser iteration step. Moreover, Theorem 3.1 provides the tame estimates in the usual Sobolev spaces H^s for the solutions and source terms vanishing in the past, which corresponds to the nonlinear problem with zero initial data. The case with general initial data is postponed to the nonlinear analysis that involves the construction of so-called approximate solutions.

4. Sobolev Functions and Notations

In this section, we first state the definitions of some fractional Sobolev spaces and norms for self-containedness. Then we prove two important estimates for the traces of $H^1(\mathbb{R}^{n+1}_+)$–functions on the hyperplane $\{y \in \mathbb{R}^{n+1} : y_1 = 0\}$ with $\mathbb{R}^{n+1}_+ := \{y \in \mathbb{R}^{n+1} : y_1 > 0\}$. We also present the Moser-type calculus inequalities and the notations for later use.

4.1. Fractional Sobolev Spaces and Norms

We first give the definitions of the Sobolev spaces and norms for general domains; see also TARTAR [27] for more details.

Definition 4.1. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ with $n \in \mathbb{N}_+$. For every nonnegative integer m, the Sobolev space $H^m(\mathcal{O})$ is defined by

$$H^m(\mathcal{O}) := \{u \in L^2(\mathcal{O}) : \partial^\alpha u \in L^2(\mathcal{O}) \text{ for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leqslant m\},$$

equipped with the norm

$$\|u\|_{H^m(\mathcal{O})} := \Big(\sum_{|\alpha| \leqslant m} \int_{\mathcal{O}} |\partial^\alpha u(y)|^2 \mathrm{d}y \Big)^{\frac{1}{2}}, \tag{4.1}$$

where $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ denotes a multi-index,

$$|\alpha| := \alpha_1 + \cdots + \alpha_n, \qquad \partial^\alpha u(y) := \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \cdots \partial y_n^{\alpha_n}} u(y).$$

For each real number $s \geqslant 0$ that is not an integer, the fractional Sobolev space $H^s(\mathcal{O})$ and its norm $\| \cdot \|_{H^s(\mathcal{O})}$ can be defined by interpolation between $H^{\lfloor s \rfloor}(\mathcal{O})$ and $H^{\lfloor s \rfloor + 1}(\mathcal{O})$ (see [27, §22]),
    where $\lfloor s \rfloor$ denotes the greatest integer less than or equal to s.

Next we present an alternative definition of the Sobolev space $H^s(\mathbb{R}^n)$ via the Fourier transform.

Definition 4.2. For each real number $s \geqslant 0$, we define

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : |\xi|^s \mathcal{F}u(\xi) \in L^2(\mathbb{R}^n) \right\},$$

where $\mathcal{F}u$ denotes the Fourier transform of u; in particular,

$$\mathcal{F}u(\xi) := \int_{\mathbb{R}^n} u(y) e^{-2\pi i\, y \cdot \xi} \, dy \quad \text{for } u \in L^1(\mathbb{R}^n).$$

The negative-order Sobolev spaces $H^{-s}(\mathbb{R}^n)$ are defined by duality as

$$H^{-s}(\mathbb{R}^n) := (H^s(\mathbb{R}^n))' \quad \text{for all } s \geqslant 0.$$

Let us recall that

$$\mathcal{F}(\partial^\alpha u) = (2\pi i\, \xi)^\alpha \mathcal{F}u \qquad \text{for all } u \in L^2(\mathbb{R}^n), \tag{4.2}$$

$$\int_{\mathbb{R}^n} u \,\overline{w} \, dy = \int_{\mathbb{R}^n} \mathcal{F}u \,\overline{\mathcal{F}w} \, dy \qquad \text{for all } u, w \in L^2(\mathbb{R}^n), \tag{4.3}$$

where $\overline{w}$ denotes the complex conjugation of w. Using identities (4.2)–(4.3), we can show that Definition 4.1 is equivalent to Definition 4.2 for $s \in \mathbb{N}$ and $\mathcal{O} = \mathbb{R}^n$. Furthermore, we refer to [27] for the equivalence between Definition 4.1 and Definition 4.2 for fractional Sobolev spaces $H^s(\mathbb{R}^n)$.

4.2. Traces on the Hyperplane

The following lemma is to characterize the traces of $H^1(\mathbb{R}^{n+1}_+)$–functions on the hyperplane $\{y \in \mathbb{R}^{n+1} : y_1 = 0\}$.

Lemma 4.1. *Any function $u \in H^1(\mathbb{R}^{n+1}_+)$ has a trace w on the hyperplane $\{y \in \mathbb{R}^{n+1} : y_1 = 0\}$ such that w belongs to $H^{1/2}(\mathbb{R}^n)$ and satisfies*

$$\int_{\mathbb{R}^n} (1 + 4\pi^2 |\xi'|^2)^{\frac{1}{2}} \left| \mathcal{F}w(\xi') \right|^2 d\xi' \leqslant \|u\|^2_{H^1(\mathbb{R}^{n+1}_+)}. \tag{4.4}$$

Proof. We first extend $u \in H^1(\mathbb{R}^{n+1}_+)$ to be defined in $\mathbb{R}^{n+1}$ by setting

$$Eu(y_1,\, y') := \begin{cases} u(y_1,\, y') & \text{if } y_1 > 0, \\ u(-y_1,\, y') & \text{if } y_1 < 0, \end{cases}$$

for all $y' := (y_2, \dots, y_{n+1}) \in \mathbb{R}^n$. In view of [27, Lemma 12.5], we obtain that $Eu \in H^1(\mathbb{R}^{n+1})$. A direct computation yields that

$$\begin{cases} \|Eu\|_{L^2(\mathbb{R}^{n+1})} \leqslant \sqrt{2}\|u\|_{L^2(\mathbb{R}^{n+1}_+)}, \\ \|\partial_y Eu\|_{L^2(\mathbb{R}^{n+1})} \leqslant \sqrt{2}\|\partial_y u\|_{L^2(\mathbb{R}^{n+1}_+)}. \end{cases} \tag{4.5}$$

By virtue of (4.5), it suffices to prove that, for all rapidly decreasing C^∞–function $\tilde{u} \in \mathscr{S}(\mathbb{R}^{n+1})$,

$$\int_{\mathbb{R}^n} (1 + 4\pi^2|\xi'|^2)^{\frac{1}{2}} \left|\mathcal{F}w(\xi')\right|^2 \mathrm{d}\xi' \leqslant \frac{1}{2}\|\tilde{u}\|^2_{H^1(\mathbb{R}^{n+1})} \qquad (4.6)$$

with w defined by $w(y') := \tilde{u}(0, y')$ for $y' \in \mathbb{R}^n$. According to [27, Lemma 15.11], we have

$$\mathcal{F}w(\xi') = \int_{\mathbb{R}} \mathcal{F}\tilde{u}(\xi_1, \xi')\mathrm{d}\xi_1 \qquad \text{for } \xi' \in \mathbb{R}^n,$$

which, along with the Cauchy–Schwarz inequality, implies

$$|\mathcal{F}w(\xi')|^2 \leqslant \int_{\mathbb{R}} (1 + 4\pi^2|\xi|^2) \left|\mathcal{F}\tilde{u}(\xi_1, \xi')\right|^2 \mathrm{d}\xi_1 \int_{\mathbb{R}} \frac{\mathrm{d}\xi_1}{1 + 4\pi^2|\xi|^2}.$$

Performing the change of variable: $\xi_1 = t(1 + 4\pi^2|\xi'|^2)^{1/2}$, we obtain

$$\int_{\mathbb{R}} \frac{\mathrm{d}\xi_1}{1 + 4\pi^2|\xi|^2} = (1 + 4\pi^2|\xi'|^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{\mathrm{d}t}{1 + 4\pi^2t^2} = \frac{1}{2}(1 + 4\pi^2|\xi'|^2)^{-\frac{1}{2}}.$$

Combine the two estimates above to infer that

$$\int_{\mathbb{R}^n} (1 + 4\pi^2|\xi'|^2)^{\frac{1}{2}} \left|\mathcal{F}w(\xi')\right|^2 \mathrm{d}\xi' \leqslant \frac{1}{2} \int_{\mathbb{R}^{n+1}} (1 + 4\pi^2|\xi|^2) \left|\mathcal{F}\tilde{u}(\xi_1, \xi')\right|^2 \mathrm{d}\xi,$$

from which we can deduce (4.6) by means of (4.2)–(4.3). □

The next lemma will be crucial for reducing the boundary integrals to the volume ones in the estimate of tangential derivatives.

Lemma 4.2. *If $n \in \mathbb{N}_+$ and $u_1, u_2 \in H^1(\mathbb{R}^{n+1}_+)$, then*

$$\left| \int_{\mathbb{R}^n} u_1 \frac{\partial u_2}{\partial y_j}(0, y') \, \mathrm{d}y' \right|$$
$$\leqslant \|u_1\|_{H^1(\mathbb{R}^{n+1}_+)} \|u_2\|_{H^1(\mathbb{R}^{n+1}_+)} \quad \text{for } j = 2, \ldots, n+1. \qquad (4.7)$$

Proof. In light of (4.2)–(4.3), we have

$$\left| \int_{\mathbb{R}^n} u_1 \frac{\partial u_2}{\partial y_j}(0, y') \, \mathrm{d}y' \right| = \left| \int_{\mathbb{R}^n} \overline{\mathcal{F}u_1} \ 2\pi \mathrm{i}\xi_j \mathcal{F}u_2(0, \xi') \, \mathrm{d}\xi' \right|$$
$$\leqslant \left(\int_{\mathbb{R}^n} (1 + 4\pi^2|\xi'|^2)^{\frac{1}{2}} \left|\mathcal{F}u_1(0, \xi')\right|^2 \, \mathrm{d}\xi' \right)^{\frac{1}{2}}$$
$$\times \left(\int_{\mathbb{R}^n} (1 + 4\pi^2|\xi'|^2)^{-\frac{1}{2}} 4\pi^2\xi_j^2 \left|\mathcal{F}u_2(0, \xi')\right|^2 \, \mathrm{d}\xi' \right)^{\frac{1}{2}},$$

which, combined with (4.4), leads to (4.7). □

4.3. Moser-Type Calculus Inequalities

We present the following Moser-type calculus inequalities that will be repeatedly employed in the subsequent analysis.

Lemma 4.3. (Moser-type calculus inequalities) *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ with Lipschitz boundary for $n \in \mathbb{N}_+$. Assume that $b \in C^\infty(\mathbb{R})$ and $u, w \in L^\infty(\mathcal{O}) \cap H^m(\mathcal{O})$ for an integer $m > 0$.*

(a) *If $|\alpha| + |\beta| \leqslant m$ and $b(0) = 0$, then*

$$\|\partial^\alpha u \partial^\beta w\|_{L^2} + \|uw\|_{H^m} \leqslant C\|u\|_{H^m}\|w\|_{L^\infty} + C\|u\|_{L^\infty}\|w\|_{H^m}, \quad (4.8)$$

$$\|b(u)\|_{H^m} \leqslant C(M_0)\|u\|_{H^m}. \quad (4.9)$$

(b) *If $|\alpha + \beta + \gamma| \leqslant m$, then*

$$\|\partial^\alpha[\partial^\beta, b(u)]\partial^\gamma w\|_{L^2} \leqslant C(M_0)\big(\|w\|_{H^m} + \|u\|_{H^m}\|w\|_{L^\infty}\big). \quad (4.10)$$

Moreover, if $u \in W^{1,\infty}(\mathcal{O})$, then

$$\|\partial^\alpha[\partial^\beta, b(u)]\partial^\gamma w\|_{L^2} \leqslant C(M_1)\big(\|w\|_{H^{m-1}} + \|u\|_{H^m}\|w\|_{L^\infty}\big). \quad (4.11)$$

Here we write $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\mathcal{O})}$, $\|\cdot\|_{H^m} := \|\cdot\|_{H^m(\mathcal{O})}$, and $\|\cdot\|_{W^{1,\infty}} := \|\cdot\|_{W^{1,\infty}(\mathcal{O})}$ for notational simplicity, and M_0 and M_1 are positive constants such that $\|u\|_{L^\infty} \leqslant M_0$ and $\|u\|_{W^{1,\infty}} \leqslant M_1$. As usual,

$$[a, b]c := a(bc) - b(ac)$$

denotes the notation of commutator.

Proof. We refer to STEIN [26, §VI.3–§VI.4] for reducing the analysis of this lemma to the case when $\mathcal{O} = \mathbb{R}^n$. See ALINHAC–GÉRARD [2, pp. 84–89] for the detailed proof of assertion (a) when $\mathcal{O} = \mathbb{R}^n$. Here we give the proof of (4.10)–(4.11) by means of (4.8)–(4.9). It follows from (4.8) that

$$\big\|\partial^\alpha[\partial^\beta, u]\partial^\gamma w\big\|_{L^2} \leqslant C \sum_{\alpha' \leqslant \alpha} \sum_{0 < \beta' \leqslant \beta} \big\|\partial^{\alpha'}\partial^{\beta'} u \, \partial^{\alpha-\alpha'}\partial^{\beta-\beta'}\partial^\gamma w\big\|_{L^2}$$

$$\leqslant C\|u\|_{H^m}\|w\|_{L^\infty} + C\|u\|_{L^\infty}\|w\|_{H^m}, \quad (4.12)$$

$$\big\|\partial^\alpha[\partial^\beta, u]\partial^\gamma w\big\|_{L^2} \leqslant C \sum_{\alpha' \leqslant \alpha} \sum_{\substack{\beta'' \leqslant \beta' \leqslant \beta \\ |\beta''|=1}} \big\|\partial^{\alpha'}\partial^{\beta'-\beta''}\big(\partial^{\beta''}u\big)\partial^{\alpha-\alpha'}\partial^{\beta-\beta'}\partial^\gamma w\big\|_{L^2}$$

$$\leqslant C\|u\|_{H^m}\|w\|_{L^\infty} + C\|u\|_{W^{1,\infty}}\|w\|_{H^{m-1}}. \quad (4.13)$$

Combining (4.13) with (4.9) yields that

$$\|\partial^\alpha[\partial^\beta, b(u)]\partial^\gamma w\|_{L^2} = \|\partial^\alpha[\partial^\beta, b(u) - b(0)]\partial^\gamma w\|_{L^2}$$

$$\leqslant C\|b(u) - b(0)\|_{H^m}\|w\|_{L^\infty} + C\|b(u) - b(0)\|_{W^{1,\infty}}\|w\|_{H^{m-1}}$$

$$\leqslant C(M_1)(\|w\|_{H^{m-1}} + \|u\|_{H^m}\|w\|_{L^\infty}).$$

Inequality (4.10) can be proved similarly from (4.9) and (4.12). ☐

4.4. Notations

For convenience, we collect the following notations.

(i) We will use letter C to denote any universal positive constant. Symbol $C(\cdot)$ denotes any generic positive constant depending only on the quantities listed in the parenthesis. Notice that constants C and $C(\cdot)$ may vary at different occurrence. We denote $A \lesssim B$ (or $B \gtrsim A$) if $A \leqslant CB$ holds uniformly for some universal positive constant C. Symbol $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

(ii) Letter d always denotes the spatial dimension. Both the two and three dimensional cases ($d = 2, 3$) are considered. Symbol Ω stands for the half-space $\{x \in \mathbb{R}^d : x_1 > 0\}$. Boundary $\partial \Omega := \{x \in \mathbb{R}^d : x_1 = 0\}$ is identified to $\mathbb{R}^{d-1}$. We write $\Omega_t := (-\infty, t) \times \Omega$ and $\omega_t := (-\infty, t) \times \partial \Omega$.

(iii) Symbol D will be used to denote

$$\mathrm{D} := (\partial_t, \partial_1, \ldots, \partial_d),$$

where $\partial_t := \frac{\partial}{\partial t}$ and $\partial_\ell := \frac{\partial}{\partial x_\ell}$ are the partial differentials. For any multi-index $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}^{d+1}$, we define

$$\mathrm{D}^\alpha := \partial_t^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}, \qquad |\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_d.$$

For $m \in \mathbb{N}$, we denote $\mathrm{D}^m := \{\mathrm{D}^\alpha : |\alpha| = m\}$.

(iv) Denote $\mathrm{D}_x := (\partial_1, \ldots, \partial_d)$ as the gradient vector and $\mathrm{D}_{\mathrm{tan}} := (\partial_t, \partial_2, \ldots, \partial_d)$ as the tangential derivative. We write

$$\mathrm{D}_{\mathrm{tan}}^\beta := \partial_t^{\beta_0} \partial_2^{\beta_2} \cdots \partial_d^{\beta_d}, \qquad |\beta| := \beta_0 + \beta_2 + \cdots + \beta_d,$$

for any multi-index $\beta = (\beta_0, \beta_2, \ldots, \beta_d) \in \mathbb{N}^d$. Denote $\mathrm{D}_{x'} := (\partial_2, \ldots, \partial_d)$.

(v) For any nonnegative integer m, we introduce

$$\|u(t)\|_m := \left(\sum_{|\alpha| \leqslant m} \|\mathrm{D}^\alpha u(t)\|_{L^2(\Omega)}^2 \right)^{1/2}, \tag{4.14}$$

$$\|u(t)\|_{\mathrm{tan},\, m} := \left(\sum_{|\beta| \leqslant m} \|\mathrm{D}_{\mathrm{tan}}^\beta u(t)\|_{L^2(\Omega)}^2 \right)^{1/2}, \tag{4.15}$$

$$\mathring{C}_m := 1 + \|(\mathring{V}, \mathring{\Psi})\|_{H^m(\Omega_T)}^2, \tag{4.16}$$

so that our formulas will be much shortened in the calculations.

(vi) Recall the partial differentials with respect to functions $\mathring{\Phi}^\pm$ from the notations in (2.40) to obtain

$$\partial_t^{\mathring{\Phi}^\pm} + \mathring{v}_\ell^\pm \partial_\ell^{\mathring{\Phi}^\pm} = \partial_t + \mathring{w}_\ell^\pm \partial_\ell,$$

where

$$\mathring{w}_1^\pm := \frac{1}{\partial_1 \mathring{\Phi}^\pm} (\mathring{v}_N^\pm - \partial_t \mathring{\Phi}^\pm), \qquad \mathring{w}_i^\pm := \mathring{v}_i^\pm \quad \text{for } i = 2, \ldots, d. \tag{4.17}$$

In view of condition (3.5b), we have

$$\mathring{w}_1^\pm = 0 \qquad \text{on } \partial\Omega. \tag{4.18}$$

Let us define

$$\partial_0 := \partial_t + \sum_{i=2}^d \mathring{v}_i^+ \partial_i \qquad \text{on } \overline{\Omega_T}, \tag{4.19}$$

which coincides with $\partial_t + \mathring{w}_\ell^\pm \partial_\ell$ on boundary $\partial\Omega$ as a result of (3.5b) and (4.18).

(vii) For any nonnegative integer m, a generic and smooth matrix-valued function of $\{(\mathrm{D}^\alpha \mathring{V}, \mathrm{D}^\alpha \mathring{\Psi}) : |\alpha| \leqslant m\}$ is denoted by $\mathring{c}_m$, and by $\underline{\mathring{c}}_m$ if it vanishes at the origin. The exact forms of $\mathring{c}_m$ and $\underline{\mathring{c}}_m$ may be different at each occurrence. For instance, the equations for $p^\pm$ in (3.20a) can be written as

$$(\partial_t + \mathring{w}_\ell^\pm \partial_\ell)p^\pm + \mathring{\rho}^\pm \mathring{c}_\pm^2 \partial_\ell^{\mathring{\Phi}^\pm} v_\ell^\pm = \mathring{c}_0 f + \underline{\mathring{c}}_1 V,$$

since $\mathcal{C}(\mathring{U}, \mathring{\Phi})$ are C^∞–functions of $(\mathring{V}, \mathrm{D}\mathring{V}, \mathrm{D}\mathring{\Psi})$ vanishing at the origin.

5. Partial Homogenization and Reformulation

It is more convenient to reformulate problem (3.20) into the case with homogeneous boundary conditions. To this end, noting that $g = \mathbb{B}'_e(\mathring{V}, \psi) \in H^{s+1/2}(\omega_T)$ vanishes in the past, we employ the trace theorem to find a regular function $V_\natural = (V_\natural^+, V_\natural^-)^\mathsf{T} \in H^{s+1}(\Omega_T)$ vanishing in the past such that

$$\mathbb{B}'_e(V_\natural, 0)\big|_{\omega_T} = g, \quad \|V_\natural\|_{H^m(\Omega_T)} \lesssim \|g\|_{H^{m-1/2}(\omega_T)} \quad \text{for } m = 1, \dots, s+1. \tag{5.1}$$

Then the new unknowns $V_\flat^\pm := \mathring{V}^\pm - V_\natural^\pm$ solve problem (3.20) with zero boundary source term and new internal source terms $\tilde{f}^\pm$:

$$\mathbb{L}'_{e,\pm} V^\pm = \tilde{f}^\pm \qquad\qquad \text{if } x_1 > 0, \tag{5.2a}$$
$$\mathbb{B}'_e(V, \psi) = 0 \qquad\qquad \text{if } x_1 = 0, \tag{5.2b}$$
$$(V, \psi) = 0 \qquad\qquad \text{if } t < 0, \tag{5.2c}$$

where we have dropped index "$\flat$" for simplicity of notation, operators $\mathbb{L}'_{e,\pm}$ and $\mathbb{B}'_e$ are defined by (3.21)–(3.22), and

$$\tilde{f}^\pm := f^\pm - L(\mathring{U}^\pm, \mathring{\Phi}^\pm)V_\natural^\pm - \mathcal{C}(\mathring{U}^\pm, \mathring{\Phi}^\pm)V_\natural^\pm. \tag{5.3}$$

We introduce new unknowns $W^\pm$ in order to distinguish the noncharacteristic variables from the others for problem (5.2). More precisely, we define

$$\begin{cases} W_1^\pm := p^\pm, \quad W_2^\pm := v^\pm \cdot \mathring{N}^\pm, \quad W_{j+1}^\pm := v_j^\pm, \\ W_{d+2}^\pm := p^\pm - \mathring{\rho}^\pm \mathring{F}_{1N}^\pm F_{11}^\pm, \quad W_{d+j+1}^\pm := \partial_j \mathring{\Phi}^\pm F_{11}^\pm + F_{j1}^\pm, \\ W_{jd+i+1}^\pm := F_{ij}^\pm, \quad W_{d^2+d+2}^\pm := S^\pm \quad \text{for } i = 1, \dots, d, \ j = 2, \dots, d, \end{cases} \tag{5.4}$$

where $\mathring{N}^\pm$ and $\mathring{F}^\pm_{1N}$ are given in (3.2). Equivalently, we set

$$W^\pm := \mathring{J}^{-1}_\pm V^\pm, \qquad \mathring{J}_\pm := J(\mathring{U}^\pm, \mathring{\Phi}^\pm),$$

where $J(U, \Phi)$ is the C^∞–function of $(U, D\Phi)$ defined as

$$J(U, \Phi) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \partial_2\Phi & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\rho F_{1N}} & 0 & 0 & -\frac{1}{\rho F_{1N}} & 0 & 0 \\ -\frac{\partial_2\Phi}{\rho F_{1N}} & 0 & 0 & \frac{\partial_2\Phi}{\rho F_{1N}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_3 \end{pmatrix} \quad \text{if } d = 2,$$

and

$$J(U, \Phi) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \partial_2\Phi & \partial_3\Phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{\rho F_{1N}} & 0 & 0 & 0 & -\frac{1}{\rho F_{1N}} & 0 & 0 & 0 \\ -\frac{\partial_2\Phi}{\rho F_{1N}} & 0 & 0 & 0 & \frac{\partial_2\Phi}{\rho F_{1N}} & 1 & 0 & 0 \\ -\frac{\partial_3\Phi}{\rho F_{1N}} & 0 & 0 & 0 & \frac{\partial_3\Phi}{\rho F_{1N}} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_7 \end{pmatrix} \quad \text{if } d = 3.$$

In terms of the new unknowns $W^\pm$, we obtain the equivalent formulation of problem (5.2a) as

$$\mathring{\mathcal{A}}^\pm_0 \partial_t W^\pm + \sum_{j=1}^d \mathring{\mathcal{A}}^\pm_j \partial_j W^\pm + \mathring{\mathcal{A}}^\pm_4 W^\pm = \mathring{J}^\mathsf{T}_\pm \tilde{f}^\pm \qquad \text{in } \Omega_T, \tag{5.5}$$

where $\mathring{\mathcal{A}}^\pm_i := \mathcal{A}_i(\mathring{U}^\pm, \mathring{\Phi}^\pm)$, for $i = 0, \dots, d$, with

$$\begin{cases} \mathcal{A}_1(U, \Phi) := J(U, \Phi)^\mathsf{T} \tilde{A}_1(U, \Phi) J(U, \Phi), \\ \mathcal{A}_j(U, \Phi) := J(U, \Phi)^\mathsf{T} A_j(U) J(U, \Phi) & \text{for } j = 0, 2, \dots, d, \\ \mathcal{A}_4(U, \Phi) := J(U, \Phi)^\mathsf{T} (L(U, \Phi) J(U, \Phi) + \mathcal{C}(U, \Phi) J(U, \Phi)). \end{cases} \tag{5.6}$$

Note that the coefficient matrices $\mathring{\mathcal{A}}_j^\pm$, for $j = 0, \ldots, d$, are symmetric, and $\mathring{\mathcal{A}}_0^\pm$ are positive definite. In particular, a straightforward calculation gives

$$\mathcal{A}_0(\overline{U}^\pm, \overline{\Phi}^\pm) = \begin{pmatrix} \dfrac{1}{\bar{\rho}^\pm \bar{c}_\pm^2} + \dfrac{1}{\bar{\rho}^\pm (\overline{F}_{11}^\pm)^2} & 0 & -\dfrac{1}{\bar{\rho}^\pm (\overline{F}_{11}^\pm)^2} & 0 & 0 \\ 0 & \bar{\rho}^\pm I_d & 0 & 0 & 0 \\ -\dfrac{1}{\bar{\rho}^\pm (\overline{F}_{11}^\pm)^2} & 0 & \dfrac{1}{\bar{\rho}^\pm (\overline{F}_{11}^\pm)^2} & 0 & 0 \\ 0 & 0 & 0 & \bar{\rho}^\pm I_{d^2-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.7)$$

$$\mathcal{A}_2(\overline{U}^\pm, \overline{\Phi}^\pm) = \begin{pmatrix} 0 & e_2^\mathsf{T} & 0 & 0 & 0 \\ e_2 & \boldsymbol{O}_d & \boldsymbol{O}_d & -\bar{\rho}^\pm \overline{F}_{22} I_d & 0 \\ 0 & \boldsymbol{O}_d & & & \\ 0 & -\bar{\rho}^\pm \overline{F}_{22} I_d & & \boldsymbol{O}_{d^2+1} & \\ 0 & 0 & & & \end{pmatrix}, \quad (5.8)$$

and, for the three-dimensional case,

$$\mathcal{A}_3(\overline{U}^\pm, \overline{\Phi}^\pm) = \begin{pmatrix} 0 & e_3^\mathsf{T} & 0 & 0 & 0 & 0 \\ e_3 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & -\bar{\rho}^\pm \overline{F}_{33} I_3 & 0 \\ 0 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & 0 \\ 0 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & 0 \\ 0 & -\bar{\rho}^\pm \overline{F}_{33} I_3 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & \boldsymbol{O}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.9)$$

where $\bar{\rho}^\pm := (\det \overline{F}^\pm)^{-1}$ are the background densities, and $\bar{c}_\pm := p_\rho(\bar{\rho}^\pm, \overline{S}^\pm)^{1/2}$ are the background speeds of sound. The explicit expressions (5.7)–(5.9) will be used in the estimate of tangential derivatives.

We now compute the exact form of $\mathring{\mathcal{A}}_1^\pm$ on boundary $\partial\Omega$, which is necessary for deriving the energy estimate of tangential derivatives. We first infer from (3.5b) and (3.5e) that matrices $\widetilde{A}_1(\mathring{U}^\pm, \mathring{\Phi}^\pm)$ satisfy

$$\widetilde{A}_1(\mathring{U}^\pm, \mathring{\Phi}^\pm)\Big|_{x_1=0} = \pm \begin{pmatrix} 0 & (\mathring{N}^\pm)^\mathsf{T} & 0 & 0 \\ \mathring{N}^\pm & \boldsymbol{O}_d & -\mathring{\rho}^\pm \mathring{F}_{1N}^\pm I_d & 0 \\ 0 & -\mathring{\rho}^\pm \mathring{F}_{1N}^\pm I_d & \boldsymbol{O}_d & 0 \\ 0 & 0 & 0 & \boldsymbol{O}_{d^2-d+1} \end{pmatrix}\Bigg|_{x_1=0}. \quad (5.10)$$

In light of (5.10), we can decompose the boundary matrices $\mathring{\mathcal{A}}_1^\pm$ as

$$\mathring{\mathcal{A}}_1^\pm = \mathring{J}_\pm^\mathsf{T} \widetilde{A}_1(\mathring{U}^\pm, \mathring{\Phi}^\pm) \mathring{J}_\pm = \mathring{\mathcal{A}}_{1a}^\pm + \mathring{\mathcal{A}}_{1b}^\pm \quad \text{with } \mathring{\mathcal{A}}_{1b}^\pm\Big|_{x_1=0} = 0, \quad (5.11)$$

where

$$\mathring{\mathcal{A}}_{1a}^{\pm} := \pm \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{O}_d & \boldsymbol{A}(\mathring{U}^{\pm}, \mathring{\Phi}^{\pm}) & 0 \\ 0 & \boldsymbol{A}(\mathring{U}^{\pm}, \mathring{\Phi}^{\pm}) & \boldsymbol{O}_d & 0 \\ 0 & 0 & 0 & \boldsymbol{O}_{d^2-d+1} \end{pmatrix}, \qquad (5.12)$$

with

$$\boldsymbol{A}(U, \Phi) := \mathrm{diag}\,(1, -\rho F_{1N} \boldsymbol{I}_{d-1}). \qquad (5.13)$$

The explicit expression of $\mathring{\mathcal{A}}_{1b}^{\pm}$ is of no interest. According to the kernels of matrices $\mathring{\mathcal{A}}_{1a}^{\pm}$, we denote by

$$W_{\mathrm{nc}}^{\pm} := (W_2^{\pm}, \dots, W_{2d+1}^{\pm})^{\mathsf{T}} \qquad (5.14)$$

the noncharacteristic parts of unknowns $W^{\pm}$, and by

$$W_{\mathrm{c}}^{\pm} := (W_1^{\pm}, W_{2d+2}^{\pm}, \dots, W_{d^2+d+2}^{\pm})^{\mathsf{T}}$$

the characteristic parts of $W^{\pm}$.

We reformulate the boundary conditions (5.2b) for unknowns $W^{\pm}$ into

$$\partial_0 \psi = W_2^+ + \mathring{\underline{c}}_1 \psi \qquad\qquad\qquad\text{on } \omega_T, \quad (5.15a)$$

$$[W_{i+1}] = \mathring{\underline{c}}_1 \psi \qquad\qquad\text{for } i = 1, \dots, d, \text{ on } \omega_T, \quad (5.15b)$$

$$[W_{d+2}] = [\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{\boldsymbol{F}}^+) F_{ij}^+ + \mathring{\underline{c}}_1 \psi \qquad\qquad\text{on } \omega_T, \quad (5.15c)$$

$$[W_{d+j+1}] = -[\mathring{F}_{11}] \partial_j \psi + \mathring{\underline{c}}_1 \psi \qquad\text{for } j = 2, \dots, d, \text{ on } \omega_T, \quad (5.15d)$$

where $\varrho(\boldsymbol{F})$ and ∂_0 are defined by (3.9) and (4.19), respectively. Here we recall that symbol $\mathring{\underline{c}}_m$ denotes a generic and smooth matrix-valued function of $\{(D^\alpha \mathring{V}, D^\alpha \mathring{\psi}) : |\alpha| \leqslant m\}$ vanishing at the origin. It is worth mentioning that the boundary conditions (5.15) depend upon the traces of $W^{\pm}$ not only through the noncharacteristic variables $W_{\mathrm{nc}}^{\pm}$ but also through the characteristic variables F_{ij}^+ for $i, j = 2, \dots, d$, which is a different situation from the standard one (see, e.g., [3, §4.1]).

6. Estimate of the Normal Derivatives

This section is devoted to the proof of the following proposition:

Proposition 6.1. *If the assumptions in Theorem 3.1 are satisfied, then*

$$\||W(t)\||_s^2 \lesssim \||W(t)\||_{\mathrm{tan},s}^2 + \|(\tilde{f}, W)\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+2}\|(\tilde{f}, W)\|_{L^\infty(\Omega_t)}^2, \qquad (6.1)$$

where $\||\cdot\||_s$, $\||\cdot\||_{\mathrm{tan},s}$, and $\mathring{C}_{s+2}$ are defined by (4.14)–(4.16), respectively. In addition,

$$\||W(t)\||_1^2 \lesssim \||W(t)\||_{\mathrm{tan},1}^2 + \|(\tilde{f}, W)\|_{H^1(\Omega_t)}^2. \qquad (6.2)$$

In this section, we let $\beta = (\beta_0, \beta_2, \dots, \beta_d) \in \mathbb{N}^d$ be a multi-index with $|\beta| \leqslant s - 1$. The proof of this proposition is divided into the following five subsections.

6.1. Estimate of the Noncharacteristic Variables

In view of (5.5) and (5.11)–(5.12), we have

$$
\begin{pmatrix} 0 \\ \partial_1 W_{\mathrm{nc}}^{\pm} \\ 0 \end{pmatrix} = - \mathring{\boldsymbol{B}}^{\pm} \mathring{\mathcal{A}}_0^{\pm} \partial_t W^{\pm} - \sum_{j=2}^{d} \mathring{\boldsymbol{B}}^{\pm} \mathring{\mathcal{A}}_j^{\pm} \partial_j W^{\pm}
$$
$$
- \mathring{\boldsymbol{B}}^{\pm} \mathring{\mathcal{A}}_{1b}^{\pm} \partial_1 W^{\pm} - \mathring{\boldsymbol{B}}^{\pm} \mathring{\mathcal{A}}_4^{\pm} W^{\pm} + \mathring{\boldsymbol{B}}^{\pm} \mathring{J}_{\pm}^{\mathsf{T}} \tilde{f}^{\pm}, \tag{6.3}
$$

where $\mathring{\boldsymbol{B}}^{\pm} := \pm \boldsymbol{B}(\mathring{U}^{\pm}, \mathring{\Phi}^{\pm})$, and $\boldsymbol{B}(U, \Phi)$ is defined by

$$
\boldsymbol{B}(U, \Phi) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{O}_d & A(U, \Phi)^{-1} & 0 \\ 0 & A(U, \Phi)^{-1} & \boldsymbol{O}_d & 0 \\ 0 & 0 & 0 & \boldsymbol{O}_{d^2-d+1} \end{pmatrix}, \tag{6.4}
$$

with $A(U, \Phi)$ given in (5.13).

Noting that $\boldsymbol{B}(U, \Phi)$ and $\mathcal{A}_j(U, \Phi)$ are C^{∞}–functions of $(U, \mathrm{D}\Phi)$ for $j = 0, \ldots, d$, we apply operator $\mathrm{D}_{\mathrm{tan}}^{\beta} := \partial_t^{\beta_0} \partial_2^{\beta_2} \cdots \partial_d^{\beta_d}$ to identity (6.3) and deduce

$$
\|\partial_1 \mathrm{D}_{\mathrm{tan}}^{\beta} W_{\mathrm{nc}}\|_{L^2(\Omega)}^2 \lesssim \|\mathrm{D}_{\mathrm{tan}}^{\beta}(\mathring{c}_1 \mathrm{D}_{\mathrm{tan}} W)\|_{L^2(\Omega)}^2 + \|\mathrm{D}_{\mathrm{tan}}^{\beta}(\mathring{\boldsymbol{B}} \mathring{\mathcal{A}}_{1b} \partial_1 W)\|_{L^2(\Omega)}^2
$$
$$
+ \|\mathrm{D}_{\mathrm{tan}}^{\beta}(\mathring{\boldsymbol{B}} \mathring{\mathcal{A}}_4 W)\|_{L^2(\Omega)}^2 + \|\mathrm{D}_{\mathrm{tan}}^{\beta}(\mathring{\boldsymbol{B}} \mathring{J}^{\mathsf{T}} \tilde{f})\|_{L^2(\Omega)}^2. \tag{6.5}
$$

Here we recall that $\mathring{c}_m$ denotes a generic and smooth matrix-valued function of $\{(\mathrm{D}^{\alpha} \mathring{V}, \mathrm{D}^{\alpha} \mathring{\Psi}) : |\alpha| \leqslant m\}$.

We integrate by parts to obtain

$$
\|u(t)\|_{m-1}^2 \lesssim \sum_{|\alpha| \leqslant m-1} \int_{\Omega_t} |\mathrm{D}^{\alpha} u(\tau, x)||\partial_t \mathrm{D}^{\alpha} u(\tau, x)| \mathrm{d}x \mathrm{d}\tau \lesssim \|u\|_{H^m(\Omega_t)}^2. \tag{6.6}
$$

By virtue of (6.6) and the Moser-type calculus inequality (4.11), we have

$$
\|\mathrm{D}_{\mathrm{tan}}^{\beta}(\mathring{c}_1 \mathrm{D}_{\mathrm{tan}} W)\|_{L^2(\Omega)}^2 \lesssim \|\mathring{c}_1 \mathrm{D}_{\mathrm{tan}}^{\beta} \mathrm{D}_{\mathrm{tan}} W + [\mathrm{D}_{\mathrm{tan}}^{\beta}, \mathring{c}_1] \mathrm{D}_{\mathrm{tan}} W\|_{L^2(\Omega)}^2
$$
$$
\lesssim \|W\|_{\mathrm{tan},s}^2 + \|[\mathrm{D}_{\mathrm{tan}}^{\beta}, \mathring{c}_1] \mathrm{D}_{\mathrm{tan}} W\|_{H^1(\Omega_t)}^2
$$
$$
\lesssim \|W\|_{\mathrm{tan},s}^2 + \|W\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+2} \|W\|_{L^{\infty}(\Omega_t)}^2. \tag{6.7}
$$

Since $\boldsymbol{B}(U, \Phi)$ and $J(U, \Phi)$ are C^{∞}–functions of $(U, \mathrm{D}\Phi)$, and $\mathcal{A}_4(U, \Phi)$ is a C^{∞}–function of $(U, \mathrm{D}\Phi, \mathrm{D}U, \mathrm{D}^2\Phi)$, we use (6.6) and the Moser-type calculus inequality (4.10) to obtain that

$$
\|\mathrm{D}_{\mathrm{tan}}^{\beta}(\mathring{\boldsymbol{B}} \mathring{\mathcal{A}}_4 W)\|_{L^2(\Omega)}^2 + \|\mathrm{D}_{\mathrm{tan}}^{\beta}(\mathring{\boldsymbol{B}} \mathring{J}^{\mathsf{T}} \tilde{f})\|_{L^2(\Omega)}^2
$$
$$
\lesssim \|\mathring{c}_2 W\|_{H^s(\Omega_t)}^2 + \|\mathring{c}_1 \tilde{f}\|_{H^s(\Omega_t)}^2
$$
$$
\lesssim \|(\tilde{f}, W)\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+2} \|(\tilde{f}, W)\|_{L^{\infty}(\Omega_t)}^2. \tag{6.8}
$$

Notice from (3.1) and (3.3) that the $W^{2,\infty}(\Omega_T)$–norm of $(\mathring{V}, \mathring{\Psi})$ is bounded by CK for some positive constant C depending only on χ. In view of (5.11), we have

$$\|\partial_1(\mathring{\boldsymbol{B}}^{\pm}\mathring{\mathcal{A}}_{1b}^{\pm})\|_{L^\infty(\Omega_T)} \lesssim \|\mathring{c}_2\|_{L^\infty(\Omega_T)} \lesssim 1, \qquad \mathring{\boldsymbol{B}}^{\pm}\mathring{\mathcal{A}}_{1b}^{\pm}\big|_{x_1=0} = 0.$$

Then we integrate by parts to obtain

$$\left\|\left(\mathring{\boldsymbol{B}}^{\pm}\mathring{\mathcal{A}}_{1b}^{\pm}\right)(\cdot, x_1, \cdot)\right\|_{L^\infty([0,T]\times\mathbb{R}^{d-1})} \lesssim \sigma(x_1) \qquad \text{for } x_1 \geqslant 0, \qquad (6.9)$$

where σ is an increasing function of x_1 satisfying

$$\sigma = \sigma(x_1) \in C^\infty(\mathbb{R}), \qquad \sigma(x_1) = \begin{cases} x_1 & \text{for } 0 \leqslant x_1 \leqslant 1, \\ 2 & \text{for } x_1 \geqslant 4. \end{cases} \qquad (6.10)$$

Utilizing the estimate above along with (6.6) and (4.11), we infer

$$\begin{aligned}
\|\mathrm{D}_{\mathrm{tan}}^\beta &(\mathring{\boldsymbol{B}}\mathring{\mathcal{A}}_{1b}\partial_1 W)\|_{L^2(\Omega)}^2 \\
&\lesssim \left\|\mathring{\boldsymbol{B}}\mathring{\mathcal{A}}_{1b}\mathrm{D}_{\mathrm{tan}}^\beta\partial_1 W + [\mathrm{D}_{\mathrm{tan}}^\beta, \mathring{\boldsymbol{B}}\mathring{\mathcal{A}}_{1b}]\partial_1 W\right\|_{L^2(\Omega)}^2 \\
&\lesssim \left\|\sigma\mathrm{D}_{\mathrm{tan}}^\beta\partial_1 W\right\|_{L^2(\Omega)}^2 + \left\|[\mathrm{D}_{\mathrm{tan}}^\beta, \mathring{\boldsymbol{B}}\mathring{\mathcal{A}}_{1b}]\partial_1 W\right\|_{H^1(\Omega_t)}^2 \\
&\lesssim \left\|\sigma\partial_1\mathrm{D}_{\mathrm{tan}}^\beta W\right\|_{L^2(\Omega)}^2 + \|W\|_{H^s(\Omega_t)}^2 + \mathring{\mathrm{C}}_{s+2}\|W\|_{L^\infty(\Omega_t)}^2.
\end{aligned} \qquad (6.11)$$

Apply operator $\sigma\partial_1^k\mathrm{D}_{\mathrm{tan}}^{\beta'}$ with $k + |\beta'| \leqslant s$ to system (5.5) and employ the standard arguments of the energy method to deduce

$$\|\sigma\partial_1^k\mathrm{D}_{\mathrm{tan}}^{\beta'}W(t)\|_{L^2(\Omega)}^2 \lesssim \|(\tilde{f}, W)\|_{H^1(\Omega_t)}^2 \qquad \text{for } k + |\beta'| \leqslant 1, \qquad (6.12)$$

$$\begin{aligned}
\|\sigma\partial_1^k\mathrm{D}_{\mathrm{tan}}^{\beta'}&W(t)\|_{L^2(\Omega)}^2 \\
&\lesssim \|(\tilde{f}, W)\|_{H^s(\Omega_t)}^2 + \mathring{\mathrm{C}}_{s+2}\|(\tilde{f}, W)\|_{L^\infty(\Omega_t)}^2 \quad \text{for } k + |\beta'| \leqslant s. \quad (6.13)
\end{aligned}$$

Plugging (6.7)–(6.8), (6.11), and (6.13) into (6.5) implies

$$\begin{aligned}
\sum_{|\beta|\leqslant s-1} &\|\partial_1\mathrm{D}_{\mathrm{tan}}^\beta W_{\mathrm{nc}}(t)\|_{L^2(\Omega)}^2 \\
&\lesssim \|W(t)\|_{\mathrm{tan},s}^2 + \|(\tilde{f}, W)\|_{H^s(\Omega_t)}^2 + \mathring{\mathrm{C}}_{s+2}\|(\tilde{f}, W)\|_{L^\infty(\Omega_t)}^2. \qquad (6.14)
\end{aligned}$$

Moreover, from (6.5) with $\beta = 0$, (6.9), and (6.13), we have

$$\|\partial_1 W_{\mathrm{nc}}(t)\|_{L^2(\Omega)}^2 \lesssim \|W(t)\|_{\mathrm{tan},1}^2 + \|(\tilde{f}, W)\|_{H^1(\Omega_t)}^2. \qquad (6.15)$$

6.2. Estimate of the Characteristic Variables $S^\pm$

The next lemma gives the estimate of the characteristic variables $W^\pm_{d^2+d+2}$ that are entropies $S^\pm$.

Lemma 6.1. *If the assumptions in Theorem* 3.1 *are satisfied, then*

$$\|S^\pm(t)\|_s^2 \lesssim \|(\tilde{f}, W)\|^2_{H^s(\Omega_t)} + \mathring{C}_{s+2}\|(\tilde{f}, W)\|^2_{L^\infty(\Omega_t)}, \qquad (6.16)$$

$$\|S^\pm(t)\|_1 \lesssim \|(\tilde{f}, W)\|_{H^1(\Omega_t)}. \qquad (6.17)$$

Proof. Since matrices $\mathcal{C}(\mathring{U}^\pm, \mathring{\Phi}^\pm)$ are C^∞–functions of $(\mathring{V}, D\mathring{V}, D\mathring{\Psi})$ vanishing at the origin, we can write the equations for $S^\pm$ in (5.2a) as

$$(\partial_t + \mathring{w}^\pm_\ell \partial_\ell)S^\pm = \mathring{c}_0 \tilde{f} + \underline{\mathring{c}}_1 W \qquad \text{in } \Omega,$$

where $\mathring{w}^\pm_\ell$, $\ell = 1, \ldots, d$, are given in (4.17). Let $\alpha := (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}^{d+1}$ be any multi-index with $|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_d \leqslant s$. Apply operator $D^\alpha := \partial_t^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ to the equations above and multiply the resulting identities by $D^\alpha S^\pm$ respectively to find

$$\partial_t |D^\alpha S^\pm|^2 + \partial_\ell(\mathring{w}^\pm_\ell |D^\alpha S^\pm|^2) - \partial_\ell \mathring{w}^\pm_\ell |D^\alpha S^\pm|^2$$
$$= 2D^\alpha S^\pm (D^\alpha(\mathring{c}_0 \tilde{f}) + D^\alpha(\mathring{c}_1 W) - [D^\alpha, \mathring{w}^\pm_\ell]\partial_\ell S^\pm).$$

Note that the $W^{2,\infty}(\Omega_T)$–norm of $(\mathring{V}, \mathring{\Psi})$ is bounded by CK for some positive constant C depending only on χ. By virtue of (4.18), we can obtain (6.16)–(6.17) by integrating the last identities over Ω_t and applying the Moser-type calculus inequalities (4.10)–(4.11). ◻

6.3. Estimate of the Characteristic Variables $W^\pm_1$

To compensate the loss of the normal derivatives of the characteristic variables $W^\pm_1 = p^\pm$, inspired by involutions (2.41), we introduce linearized divergences $\varsigma^\pm$ by

$$\varsigma^\pm := \partial_i^{\mathring{\Phi}^\pm}(\mathring{c}_\pm^{-2} \mathring{F}^\pm_{i1} p^\pm + \mathring{\rho}^\pm F^\pm_{i1}), \qquad (6.18)$$

where $\partial_i^{\mathring{\Phi}^\pm}$, $i = 1, \ldots, d$, are defined by (2.40), and $\mathring{c}_\pm := p_\rho(\mathring{\rho}^\pm, \mathring{S}^\pm)^{1/2}$ are the basic speeds of sound. See TRAKHININ [29] for a slightly different definition of the linearized divergences.

Then we obtain the following estimate for $\varsigma^\pm$:

Lemma 6.2. *If the assumptions in Theorem* 3.1 *are satisfied, then*

$$\|\varsigma^\pm(t)\|_{s-1}^2 \lesssim \|(\tilde{f}, W)\|^2_{H^s(\Omega_t)} + \mathring{C}_{s+2}\|(\tilde{f}, W)\|^2_{L^\infty(\Omega_t)}, \qquad (6.19)$$

$$\|\varsigma^\pm(t)\|_{L^2(\Omega)} \lesssim \|(\tilde{f}, W)\|_{H^1(\Omega_t)}. \qquad (6.20)$$

Proof. The equations for $\boldsymbol{F}^{\pm}$ and $p^{\pm}$ in (5.2a) read

$$(\partial_t + \mathring{w}_\ell^{\pm}\partial_\ell)F_{ij}^{\pm} - \mathring{F}_{\ell j}^{\pm}\partial_\ell^{\mathring{\phi}^{\pm}}v_i^{\pm} = \mathring{c}_0\tilde{f} + \underline{\mathring{c}}_1 W, \tag{6.21}$$

$$(\partial_t + \mathring{w}_\ell^{\pm}\partial_\ell)p^{\pm} + \mathring{\rho}^{\pm}\mathring{c}_\pm^2\partial_\ell^{\mathring{\phi}^{\pm}}v_\ell^{\pm} = \mathring{c}_0\tilde{f} + \underline{\mathring{c}}_1 W. \tag{6.22}$$

In view of these last equations, we compute

$$(\partial_t + \mathring{w}_\ell^{\pm}\partial_\ell)\left(\mathring{c}_\pm^{-2}\mathring{F}_{i1}^{\pm}p^{\pm} + \mathring{\rho}^{\pm}F_{i1}^{\pm}\right)$$
$$= \mathring{\rho}^{\pm}\mathring{F}_{\ell1}^{\pm}\partial_\ell^{\mathring{\phi}^{\pm}}v_i^{\pm} - \mathring{\rho}^{\pm}\mathring{F}_{i1}^{\pm}\partial_\ell^{\mathring{\phi}^{\pm}}v_\ell^{\pm} + \mathring{c}_0\tilde{f} + \mathring{c}_1 W.$$

Performing operators $\partial_i^{\mathring{\phi}^{\pm}}$ to the identities above and using

$$\mathring{\rho}^{\pm}\mathring{F}_{\ell1}^{\pm}\partial_i^{\mathring{\phi}^{\pm}}\partial_\ell^{\mathring{\phi}^{\pm}}v_i^{\pm} - \mathring{\rho}^{\pm}\mathring{F}_{i1}^{\pm}\partial_i^{\mathring{\phi}^{\pm}}\partial_\ell^{\mathring{\phi}^{\pm}}v_\ell^{\pm}$$
$$= \mathring{\rho}^{\pm}\mathring{F}_{i1}^{\pm}[\partial_\ell^{\mathring{\phi}^{\pm}}, \partial_i^{\mathring{\phi}^{\pm}}]v_\ell^{\pm} = \mathring{c}_2 DV = \mathring{c}_2 D(\mathring{J}W) = \mathring{c}_2 DW + \mathring{c}_2 W,$$

we have

$$(\partial_t + \mathring{w}_\ell^{\pm}\partial_\ell)\varsigma^{\pm} = \mathring{c}_1 D\tilde{f} + \mathring{c}_1\tilde{f} + \mathring{c}_2 DW + \mathring{c}_2 W. \tag{6.23}$$

Apply operator D^{α} with $|\alpha| \leqslant s-1$ to equations (6.23), multiply the resulting identities by $D^{\alpha}\varsigma^{\pm}$ respectively, and take the integration over Ω_t to obtain

$$\|D^{\alpha}\varsigma^{\pm}(t)\|_{L^2(\Omega)}^2 \lesssim \left(1 + \|D\mathring{w}\|_{L^{\infty}(\Omega_t)}\right)\|D^{\alpha}\varsigma^{\pm}\|_{L^2(\Omega_t)}^2 + \|[D^{\alpha}, \mathring{w}_\ell^{\pm}]\partial_\ell\varsigma^{\pm}\|_{L^2(\Omega_t)}^2$$
$$+ \left\|D^{\alpha}\left(\mathring{c}_1 D\tilde{f} + \mathring{c}_1\tilde{f} + \mathring{c}_2 DW + \mathring{c}_2 W\right)\right\|_{L^2(\Omega_t)}^2. \tag{6.24}$$

Since

$$\varsigma^{\pm} = \mathring{c}_1 W + \mathring{c}_1 DW, \tag{6.25}$$

we have

$$\|D^{\alpha}\varsigma^{\pm}\|_{L^2(\Omega_t)} \leqslant \|D^{\alpha}(\mathring{c}_2 DW + \mathring{c}_2 W)\|_{L^2(\Omega_t)},$$
$$\|[D^{\alpha}, \mathring{w}_\ell^{\pm}]\partial_\ell\varsigma^{\pm}\|_{L^2(\Omega_t)} \lesssim \left\|\left([D^{\alpha}, \mathring{c}_2]W, [D^{\alpha}, \mathring{c}_2]DW, [D^{\alpha}, \mathring{c}_1]D^2 W\right)\right\|_{L^2(\Omega_t)}.$$

Estimate (6.20) follows by plugging these last inequalities into (6.24) with $\alpha = 0$. Apply the Moser-type calculus inequality (4.10) and use $|\alpha| \leqslant s-1$ to derive

$$\|(D^{\alpha}(\mathring{c}_1 D\tilde{f}), D^{\alpha}(\mathring{c}_1\tilde{f}))\|_{L^2(\Omega_t)}^2$$
$$\lesssim \|(\mathring{c}_1 D^{\alpha}D\tilde{f}, \mathring{c}_1 D^{\alpha}\tilde{f})\|_{L^2(\Omega_t)}^2 + \|([D^{\alpha}, \mathring{c}_1]D\tilde{f}, [D^{\alpha}, \mathring{c}_1]\tilde{f})\|_{L^2(\Omega_t)}^2$$
$$\lesssim \|\tilde{f}\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+1}\|\tilde{f}\|_{L^{\infty}(\Omega_t)}^2.$$

By virtue of (4.10)–(4.11), we obtain

$$\|([D^{\alpha}, \mathring{c}_2]W, [D^{\alpha}, \mathring{c}_2]DW)\|_{L^2(\Omega_t)}^2 + \|[D^{\alpha}, \mathring{c}_1]D^2 W\|_{L^2(\Omega_t)}^2$$
$$\lesssim \|W\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+2}\|W\|_{L^{\infty}(\Omega_t)}^2.$$

Inserting the estimates above into (6.24) yields (6.19). $\qquad\square$

Thanks to (6.19), we can obtain the estimate of the characteristic variables $W_1^\pm = p^\pm$. More precisely, according to (5.4) and (5.14), we have

$$\mathring{N}^\pm \cdot F_1^\pm = \frac{|\mathring{N}^\pm|^2}{\mathring{\rho}^\pm \mathring{F}_{1N}^\pm}(W_1^\pm - W_{d+2}^\pm) - \sum_{j=2}^{d} \partial_j \mathring{\Phi}^\pm W_{d+j+1}^\pm$$

$$= \frac{|\mathring{N}^\pm|^2}{\mathring{\rho}^\pm \mathring{F}_{1N}^\pm} W_1^\pm + \mathring{c}_1 W_{nc}.$$

Combining the last identity with (6.18) and recalling (2.40), we calculate

$$\partial_1 \mathring{\Phi}^\pm \varsigma^\pm = \partial_1 \left(\mathring{c}_\pm^{-2} \mathring{F}_{1N}^\pm p^\pm + \mathring{\rho}^\pm \mathring{N}^\pm \cdot F_1^\pm \right) + \sum_{i=2}^{d} \partial_i \left(\partial_1 \mathring{\Phi}^\pm \left(\mathring{c}_\pm^{-2} \mathring{F}_{i1}^\pm p^\pm + \mathring{\rho}^\pm F_{i1}^\pm \right) \right)$$

$$= \frac{\mathring{c}_\pm^{-2}|\mathring{F}_{1N}^\pm|^2 + |\mathring{N}^\pm|^2}{\mathring{F}_{1N}^\pm} \partial_1 W_1^\pm + \mathring{c}_1 \partial_1 W_{nc} + \mathring{c}_1 D_{\tan} W + \mathring{c}_2 W,$$

which implies

$$\partial_1 W_1^\pm = \mathring{c}_1 \varsigma^\pm + \mathring{c}_1 \partial_1 W_{nc} + \mathring{c}_1 D_{\tan} W + \mathring{c}_2 W. \tag{6.26}$$

In light of (6.26), we utilize (6.6), (6.14), (6.19), (6.25), and the Moser-type calculus inequalities (4.10)–(4.11) to obtain

$$\sum_{|\beta| \leqslant s-1} \|\partial_1 D_{\tan}^\beta W_1(t)\|_{L^2(\Omega)}^2$$

$$\lesssim \sum_{|\beta| \leqslant s-1} \|(D_{\tan}^\beta \varsigma, D_{\tan}^\beta \partial_1 W_{nc}, D_{\tan}^\beta D_{\tan} W)\|_{L^2(\Omega)}^2$$

$$+ \sum_{|\beta| \leqslant s-1} \left\| ([D_{\tan}^\beta, \mathring{c}_1]W, [D_{\tan}^\beta, \mathring{c}_1]DW, D_{\tan}^\beta(\mathring{c}_2 W)) \right\|_{H^1(\Omega_t)}^2$$

$$\lesssim \||W(t)|\|_{\tan,s}^2 + \|(\tilde{f}, W)\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+2}\|(\tilde{f}, W)\|_{L^\infty(\Omega_t)}^2. \tag{6.27}$$

Furthermore, we plug (6.15) and (6.20) into (6.26) to obtain

$$\|\partial_1 W_1(t)\|_{L^2(\Omega)}^2 \lesssim \||W(t)|\|_{\tan,1}^2 + \|(\tilde{f}, W)\|_{H^1(\Omega_t)}^2. \tag{6.28}$$

6.4. Estimate of the Remaining Characteristic Variables

To recover the normal derivatives of the characteristic variables $W_{jd+i+1}^\pm = F_{ij}^\pm$ for $i = 1, \ldots, d$ and $j = 2, \ldots, d$, motivated by constraints (2.36), we introduce quantities $\eta^\pm := (\eta_1^\pm, \ldots, \eta_d^\pm)$ with

$$\eta_i^\pm := \mathring{F}_{k1}^\pm \partial_k^{\mathring{\Phi}^\pm} F_{i2}^\pm - \mathring{F}_{k2}^\pm \partial_k^{\mathring{\Phi}^\pm} F_{i1}^\pm. \tag{6.29}$$

In addition, for $d = 3$, we introduce quantities $\zeta^\pm := (\zeta_1^\pm, \zeta_2^\pm, \zeta_3^\pm)$ with

$$\zeta_i^\pm := \mathring{F}_{k1}^\pm \partial_k^{\mathring{\Phi}^\pm} F_{i3}^\pm - \mathring{F}_{k3}^\pm \partial_k^{\mathring{\Phi}^\pm} F_{i1}^\pm. \tag{6.30}$$

We have the following estimates for $\eta^\pm$ and $\zeta^\pm$.

Lemma 6.3. *If the assumptions in Theorem* 3.1 *are satisfied, then*

$$\|(\eta^\pm, \zeta^\pm)\|_{s-1}^2 \lesssim \|(\tilde{f}, W)\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+2}\|(\tilde{f}, W)\|_{L^\infty(\Omega_t)}^2, \qquad (6.31)$$

$$\|(\eta^\pm, \zeta^\pm)\|_{L^2(\Omega)} \lesssim \|(\tilde{f}, W)\|_{H^1(\Omega_t)}. \qquad (6.32)$$

Proof. Thanks to (6.21), we deduce the equations for $\eta^\pm$ and $\zeta^\pm$ as follows:

$$(\partial_t + \mathring{w}_\ell^\pm \partial_\ell)\eta^\pm = \mathring{c}_1 D\tilde{f} + \mathring{c}_1 \tilde{f} + \mathring{c}_2 DW + \mathring{c}_2 W, \qquad (6.33)$$

$$(\partial_t + \mathring{w}_\ell^\pm \partial_\ell)\zeta^\pm = \mathring{c}_1 D\tilde{f} + \mathring{c}_1 \tilde{f} + \mathring{c}_2 DW + \mathring{c}_2 W, \qquad (6.34)$$

where we have used that

$$\mathring{F}_{ki}^\pm \mathring{F}_{\ell j}^\pm \partial_k^{\mathring{\phi}^\pm} \partial_\ell^{\mathring{\phi}^\pm} - \mathring{F}_{kj}^\pm \mathring{F}_{\ell i}^\pm \partial_k^{\mathring{\phi}^\pm} \partial_\ell^{\mathring{\phi}^\pm} = \mathring{F}_{\ell i}^\pm \mathring{F}_{kj}^\pm [\partial_\ell^{\mathring{\phi}^\pm}, \partial_k^{\mathring{\phi}^\pm}] = \mathring{c}_2 D.$$

Noting that $\eta^\pm = \mathring{c}_1 DV$ and $\zeta^\pm = \mathring{c}_1 DV$, we perform the same analysis as $\varsigma^\pm$ in Lemma 6.2 to deduce (6.31)–(6.32). This completes the proof. $\quad\square$

According to (2.40), we compute

$$\eta_i^\pm = \frac{1}{\partial_1 \mathring{\phi}^\pm}\left(\mathring{F}_{1N}^\pm \partial_1 F_{i2}^\pm - \mathring{F}_{2N}^\pm \partial_1 F_{i1}^\pm\right) + \sum_{\ell=2}^d \left(\mathring{F}_{\ell 1}^\pm \partial_\ell F_{i2}^\pm - \mathring{F}_{\ell 2}^\pm \partial_\ell F_{i1}^\pm\right), \quad (6.35)$$

$$\zeta_i^\pm = \frac{1}{\partial_1 \mathring{\phi}^\pm}\left(\mathring{F}_{1N}^\pm \partial_1 F_{i3}^\pm - \mathring{F}_{3N}^\pm \partial_1 F_{i1}^\pm\right) + \sum_{\ell=2}^d \left(\mathring{F}_{\ell 1}^\pm \partial_\ell F_{i3}^\pm - \mathring{F}_{\ell 3}^\pm \partial_\ell F_{i1}^\pm\right), \quad (6.36)$$

which, combined with (5.4) and (5.14), imply

$$\partial_1 F_{i2}^\pm = \mathring{c}_1 \eta_i^\pm + \mathring{c}_1 \partial_1 W_{\text{nc}} + \mathring{c}_1 \partial_1 W_1 + \mathring{c}_1 D_{\tan} W + \mathring{c}_2 W, \qquad (6.37)$$

$$\partial_1 F_{i3}^\pm = \mathring{c}_1 \zeta_i^\pm + \mathring{c}_1 \partial_1 W_{\text{nc}} + \mathring{c}_1 \partial_1 W_1 + \mathring{c}_1 D_{\tan} W + \mathring{c}_2 W. \qquad (6.38)$$

In view of (6.37)–(6.38), we utilize (6.14)–(6.15), (6.27)–(6.28), (6.31)–(6.32), the Moser-type calculus inequalities (4.10)–(4.11), and (6.6) to obtain

$$\|\partial_1 W_{jd+i+1}(t)\|_{L^2(\Omega)}^2 \lesssim \|W(t)\|_{\tan,1}^2 + \|(\tilde{f}, W)\|_{H^1(\Omega_t)}^2, \qquad (6.39)$$

and

$$\sum_{|\beta| \leqslant s-1} \|\partial_1 D_{\tan}^\beta W_{jd+i+1}(t)\|_{L^2(\Omega)}^2$$

$$\lesssim \|W(t)\|_{\tan,s}^2 + \|(\tilde{f}, W)\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+2}\|(\tilde{f}, W)\|_{L^\infty(\Omega_t)}^2, \qquad (6.40)$$

for $i = 1, \ldots, d$, and $j = 2, \ldots, d$.

6.5. Proof of Proposition 6.1

Estimate (6.2) follows by applying (6.6) and combining estimates (6.15), (6.17), (6.28), and (6.39). Thanks to (6.3), (6.26), and (6.37)–(6.38), we can combine estimates (6.13), (6.16), (6.19), (6.31), and (6.40) to prove by induction in $\ell = 1, \ldots, s$ that

$$\sum_{k=1}^{\ell} \sum_{|\beta| \leqslant s-k} \|\partial_1^k D_{\tan}^{\beta} W(t)\|_{L^2(\Omega)}^2$$
$$\lesssim \|W(t)\|_{\tan,s}^2 + \|(\tilde{f}, W)\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+2} \|(\tilde{f}, W)\|_{L^{\infty}(\Omega_t)}^2. \qquad (6.41)$$

Estimate (6.1) follows from (6.41) with $\ell = s$. Then the proof of Proposition 6.1 is complete.

7. Estimate of the Tangential Derivatives

In this section, we establish the estimate for the tangential derivatives of solutions of the linearized problem (5.2).

Proposition 7.1. *If the assumptions in Theorem 3.1 are satisfied, then*

$$\|W(t)\|_{\tan,s}^2 \lesssim \mathcal{M}_s(t) + \|\mathring{\underline{c}}_1\|_{H^3(\Omega_T)} \|W(t)\|_s^2 \qquad (7.1)$$

for any constant $\epsilon > 0$, where Ψ is given in (3.14) and

$$\mathcal{M}_s(t) := \begin{cases} \|(W, \Psi, \tilde{f})\|_{H^1(\Omega_t)}^2 & \text{if } s = 1, \\ \|(W, \Psi, \tilde{f})\|_{H^s(\Omega_t)}^2 + \mathring{C}_{s+2} \|(W, \Psi, \tilde{f})\|_{H^3(\Omega_t)}^2 & \text{if } s \geqslant 2. \end{cases} \qquad (7.2)$$

The rest of this section is concerned with the proof of Proposition 7.1.

7.1. Prelude

Applying operator $D_{\tan}^{\beta} := \partial_t^{\beta_0} \partial_2^{\beta_2} \cdots \partial_d^{\beta_d}$ with $|\beta| \leqslant s$ to system (5.5), we obtain

$$\mathring{\mathcal{A}}_0^{\pm} \partial_t D_{\tan}^{\beta} W^{\pm} + \mathring{\mathcal{A}}_j^{\pm} \partial_j D_{\tan}^{\beta} W^{\pm} = R^{\pm}, \qquad (7.3)$$

where

$$R^{\pm} := D_{\tan}^{\beta} (\mathring{J}_{\pm}^{\mathsf{T}} \tilde{f}^{\pm}) - D_{\tan}^{\beta} (\mathring{\mathcal{A}}_4^{\pm} W^{\pm}) - [D_{\tan}^{\beta}, \mathring{\mathcal{A}}_0^{\pm}] \partial_t W^{\pm} - [D_{\tan}^{\beta}, \mathring{\mathcal{A}}_j^{\pm}] \partial_j W^{\pm}.$$

Take the scalar product of (7.3) with $D_{\tan}^{\beta} W^{\pm}$ to obtain

$$\sum_{\pm} \int_{\Omega} \mathring{\mathcal{A}}_0^{\pm} D_{\tan}^{\beta} W^{\pm} \cdot D_{\tan}^{\beta} W^{\pm} = \mathcal{R}_1 + \int_{\omega_t} Q, \qquad (7.4)$$

where

$$\mathcal{R}_1 := \sum_{\pm} \int_{\Omega_t} D_{tan}^{\beta} W^{\pm} \cdot \left(2R^{\pm} + (\partial_t \mathring{A}_0^{\pm} + \partial_j \mathring{A}_j^{\pm}) D_{tan}^{\beta} W^{\pm} \right),$$

$$Q := \sum_{\pm} \mathring{A}_{1a}^{\pm} D_{tan}^{\beta} W^{\pm} \cdot D_{tan}^{\beta} W^{\pm} = \underbrace{2[D_{tan}^{\beta} W_2 D_{tan}^{\beta} W_{d+2}]}_{Q_1} + Q_2, \qquad (7.5)$$

with

$$Q_2 := \begin{cases} -2\mathring{\rho}^+ \mathring{F}_{1N}^+ \left[D_{tan}^{\beta} W_3 D_{tan}^{\beta} W_5 \right] & \text{if } d = 2, \\ -2\mathring{\rho}^+ \mathring{F}_{1N}^+ \left[D_{tan}^{\beta} W_3 D_{tan}^{\beta} W_6 + D_{tan}^{\beta} W_4 D_{tan}^{\beta} W_7 \right] & \text{if } d = 3. \end{cases} \qquad (7.6)$$

Here and hereafter, for simplicity, we omit the differential symbol of the variables of integration when no confusion arises.

A standard computation with an application of the Moser-type calculus inequalities (4.10)–(4.11) and the Sobolev embedding $H^3(\Omega_t) \hookrightarrow L^\infty(\Omega_t)$ yields

$$\mathcal{R}_1 \lesssim \mathcal{M}_s(t). \qquad (7.7)$$

We introduce the instant tangential energy $\mathcal{E}_{tan}^{\beta}(t)$ as

$$\mathcal{E}_{tan}^{\beta}(t) := \sum_{\pm} \int_{\Omega} \mathcal{A}_0(\overline{U}^{\pm}, \overline{\Phi}^{\pm}) D_{tan}^{\beta} W^{\pm} \cdot D_{tan}^{\beta} W^{\pm},$$

where $\mathcal{A}_0$ is given in (5.6). Thanks to (5.7), we have

$$\mathcal{E}_{tan}^{\beta}(t) = \sum_{\pm} \left\{ \frac{1}{\bar{\rho}^{\pm} \bar{c}_{\pm}^2} \|D_{tan}^{\beta} W_1^{\pm}\|_{L^2(\Omega)}^2 + \frac{1}{\bar{\rho}^{\pm} (\overline{F}_{11}^{\pm})^2} \|D_{tan}^{\beta}(W_1^{\pm} - W_{d+2}^{\pm})\|_{L^2(\Omega)}^2 \right. $$
$$\left. + \sum_{j=2, \, j\neq d+2}^{d^2+d+1} \bar{\rho}^{\pm} \|D_{tan}^{\beta} W_j^{\pm}\|_{L^2(\Omega)}^2 + \|D_{tan}^{\beta} S^{\pm}\|_{L^2(\Omega)}^2 \right\}, \qquad (7.8)$$

where $\bar{\rho}^{\pm} := (\det \overline{F}^{\pm})^{-1}$ and $\bar{c}_{\pm} := p_{\rho}(\bar{\rho}^{\pm}, \overline{S}^{\pm})^{1/2}$.

Since $\mathring{A}_0^{\pm} - \mathcal{A}_0(\overline{U}^{\pm}, \overline{\Phi}^{\pm})$ are smooth functions of $\{(D^\alpha \mathring{V}, D^\alpha \mathring{\Psi}) : |\alpha| \leqslant 1\}$ and vanish at the origin, we plug (7.7) into (7.4) to infer

$$\mathcal{E}_{tan}^{\beta}(t) \leqslant C\mathcal{M}_s(t) + C\|\mathring{c}_1\|_{L^\infty(\Omega_T)} \|\!|W(t)|\!\|_s^2 + \int_{\omega_t} Q, \qquad (7.9)$$

where $\mathcal{M}_s(t)$ and Q are defined by (7.2) and (7.5), respectively.

7.2. Cancellation

We are going to show a cancellation for the last term in (7.9). By virtue of the boundary conditions (5.15b)–(5.15c), we find

$$
\begin{aligned}
Q_1 &= 2D_{\tan}^{\beta}[W_{d+2}]D_{\tan}^{\beta}W_2^+ + 2D_{\tan}^{\beta}[W_2]D_{\tan}^{\beta}W_{d+2}^- \\
&= Q_{1a} + [D_{\tan}^{\beta}, \mathring{c}_0]WD_{\tan}^{\beta}W_2^+ + D_{\tan}^{\beta}(\mathring{c}_1\psi)D_{\tan}^{\beta}W \qquad \text{on } \partial\Omega \qquad (7.10)
\end{aligned}
$$

with

$$
Q_{1a} := 2[\mathring{F}_{11}]\partial_{F_{ij}}\varrho(\mathring{\boldsymbol{F}}^+)D_{\tan}^{\beta}F_{ij}^+D_{\tan}^{\beta}W_2^+.
$$

Similarly, it follows from (3.13), (5.15b), and (5.15d) that

$$
\begin{aligned}
Q_2 &= -2\varrho(\mathring{\boldsymbol{F}}^+)\sum_{j=2}^{d}\left(D_{\tan}^{\beta}[W_{d+j+1}]D_{\tan}^{\beta}W_{j+1}^+ + D_{\tan}^{\beta}[W_{j+1}]D_{\tan}^{\beta}W_{d+j+1}^-\right) \\
&= Q_{2a} + \mathring{c}_0[D_{\tan}^{\beta}, \mathring{c}_0]D_{\tan}\psi D_{\tan}^{\beta}W + \mathring{c}_0 D_{\tan}^{\beta}(\mathring{c}_1\psi)D_{\tan}^{\beta}W \quad \text{on } \partial\Omega, \quad (7.11)
\end{aligned}
$$

with

$$
Q_{2a} := 2\varrho(\mathring{\boldsymbol{F}}^+)[\mathring{F}_{11}]\sum_{j=2}^{d}D_{\tan}^{\beta}\partial_j\psi D_{\tan}^{\beta}W_{j+1}^+.
$$

We decompose Q_{2a} further as

$$
Q_{2a} = Q_{2b} + \sum_{j=2}^{d}\partial_j\left(2\varrho(\mathring{\boldsymbol{F}}^+)[\mathring{F}_{11}]D_{\tan}^{\beta}\psi D_{\tan}^{\beta}W_{j+1}^+\right) + \mathring{c}_1 D_{\tan}^{\beta}WD_{\tan}^{\beta}\psi, \quad (7.12)
$$

with

$$
Q_{2b} := -2\varrho(\mathring{\boldsymbol{F}}^+)[\mathring{F}_{11}]\sum_{j=2}^{d}D_{\tan}^{\beta}\psi D_{\tan}^{\beta}\partial_j W_{j+1}^+.
$$

In order to deduce the cancellation between terms Q_{1a} and Q_{2b}, we need the following lemma.

Lemma 7.1. *If $i = 1, \ldots, d$, and $j = 2, \ldots, d$, then*

$$
\partial_0 F_{ij}^{\pm} = \sum_{k=2}^{d}\mathring{F}_{kj}^{\pm}\partial_k v_i^{\pm} + \mathring{c}_0\tilde{f} + \underline{\mathring{c}}_1 W \qquad\qquad \text{on } \partial\Omega, \quad (7.13)
$$

$$
\sum_{j=2}^{d}\partial_j W_{j+1}^+ = -\varrho(\mathring{\boldsymbol{F}}^+)^{-1}\partial_{F_{ij}}\varrho(\mathring{\boldsymbol{F}}^+)\partial_0 F_{ij}^+ + \mathring{c}_0\tilde{f} + \underline{\mathring{c}}_1 W \qquad \text{on } \partial\Omega, \quad (7.14)
$$

where ∂_0 is defined by (4.19).

Proof. Considering the restriction of equations (6.21) on boundary $\partial\Omega$, we utilize (4.18) and (3.5e) to deduce identities (7.13). In the two-dimensional case ($d = 2$), relation (7.14) follows directly from (7.13). If $d = 3$, then we obtain from (7.13) that

$$\begin{pmatrix} \partial_2 v_2^\pm & \partial_3 v_2^\pm \\ \partial_2 v_3^\pm & \partial_3 v_3^\pm \end{pmatrix} \begin{pmatrix} \mathring{F}_{22}^\pm & \mathring{F}_{23}^\pm \\ \mathring{F}_{32}^\pm & \mathring{F}_{33}^\pm \end{pmatrix} = \begin{pmatrix} \partial_0 F_{22}^\pm & \partial_0 F_{23}^\pm \\ \partial_0 F_{32}^\pm & \partial_0 F_{33}^\pm \end{pmatrix} + \mathring{c}_0 \tilde{f} + \underline{\mathring{c}}_1 W \qquad \text{on } \partial\Omega.$$

Then we can deduce (7.14) by virtue of $W_3^+ = v_2^+$, $W_4^+ = v_3^+$, and

$$-\frac{\partial_{F_{ij}} \varrho(\mathring{F}^+) \partial_0 F_{ij}^+}{\varrho(\mathring{F}^+)^2} = \mathring{F}_{22}^+ \partial_0 F_{33}^+ - \mathring{F}_{23}^+ \partial_0 F_{32}^+ - \mathring{F}_{32}^+ \partial_0 F_{23}^+ + \mathring{F}_{33}^+ \partial_0 F_{22}^+.$$

This completes the proof. $\square$

Thanks to identity (7.14), we find

$$Q_{2b} = \underbrace{2\varrho(\mathring{F}^+)[\mathring{F}_{11}] D_{\tan}^\beta \psi D_{\tan}^\beta \left(\varrho(\mathring{F}^+)^{-1} \partial_{F_{ij}} \varrho(\mathring{F}^+) \partial_0 F_{ij}^+\right)}_{Q_{2c}}$$
$$- 2\varrho(\mathring{F}^+)[\mathring{F}_{11}] D_{\tan}^\beta \psi D_{\tan}^\beta (\mathring{c}_0 \tilde{f} + \mathring{c}_1 W) \qquad \text{on } \partial\Omega. \tag{7.15}$$

Term Q_{2c} can be decomposed further as

$$Q_{2c} = 2[\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{F}^+) D_{\tan}^\beta \psi \, \partial_0 D_{\tan}^\beta F_{ij}^+ + \mathring{c}_0 D_{\tan}^\beta \psi \, [D_{\tan}^\beta, \mathring{c}_0] D_{\tan} W$$

$$= \mathring{c}_0 D_{\tan}^\beta \psi [D_{\tan}^\beta, \mathring{c}_0] D_{\tan} W + \sum_{j=2}^d \partial_j \left\{ 2[\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{F}^+) \mathring{v}_j^+ D_{\tan}^\beta \psi D_{\tan}^\beta F_{ij}^+ \right\}$$

$$+ \partial_t \left\{ 2[\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{F}^+) D_{\tan}^\beta \psi D_{\tan}^\beta F_{ij}^+ \right\} + \mathring{c}_1 D_{\tan}^\beta \psi \, D_{\tan}^\beta W$$

$$\underbrace{- 2[\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{F}^+) D_{\tan}^\beta F_{ij}^+ D_{\tan}^\beta \partial_0 \psi}_{Q_{2d}} + \mathring{c}_0 D_{\tan}^\beta W [D_{\tan}^\beta, \mathring{c}_0] D_{\tan} \psi. \tag{7.16}$$

In view of condition (5.15a), we derive the following desired cancellation:

$$Q_{1a} + Q_{2d} = 2[\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{F}^+) D_{\tan}^\beta F_{ij}^+ D_{\tan}^\beta (\mathring{c}_1 \psi) \qquad \text{on } \partial\Omega. \tag{7.17}$$

Combine (7.10)–(7.12) and (7.15)–(7.17) to obtain

$$\int_{\omega_t} Q = \mathcal{R}_2 + \underbrace{2 \int_{\partial\Omega} [\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{F}^+) D_{\tan}^\beta \psi D_{\tan}^\beta F_{ij}^+}_{\mathcal{R}_3}, \tag{7.18}$$

where

$$\mathcal{R}_2 := \int_{\omega_t} [D_{\tan}^\beta, \mathring{c}_0] W D_{\tan}^\beta W + \int_{\omega_t} \mathring{c}_0 D_{\tan}^\beta (\mathring{c}_1 \psi) D_{\tan}^\beta W$$

$$+ \int_{\omega_t} \mathring{c}_0 [D_{\tan}^\beta, \mathring{c}_0] D_{\tan} \psi D_{\tan}^\beta W + \int_{\omega_t} \mathring{c}_0 D_{\tan}^\beta \psi [D_{\tan}^\beta, \mathring{c}_0] D_{\tan} W$$

$$+ \int_{\omega_t} \mathring{c}_1 D_{\tan}^\beta \psi D_{\tan}^\beta W + \int_{\omega_t} \mathring{c}_0 D_{\tan}^\beta \psi D_{\tan}^\beta (\mathring{c}_0 \tilde{f} + \mathring{c}_1 W). \tag{7.19}$$

7.3. Estimate of Term $\mathcal{R}_2$

In this subsection, we deduce the estimate of term $\mathcal{R}_2$ defined by (7.19).

By virtue of assumption (3.3) and the Sobolev embedding, there exists some positive constant K_1 depending on $[\overline{F}_{11}]$ such that, if $K \leqslant K_1$, then

$$[\mathring{F}_{11}] \geqslant \frac{[\overline{F}_{11}]}{2} > 0 \qquad \text{on } \partial\Omega.$$

It follows from the boundary condition (5.15d) that

$$\partial_j \psi = -\frac{1}{[\mathring{F}_{11}]}[W_{d+j+1}] + \mathring{\underline{c}}_1 \psi \qquad \text{on } \partial\Omega, \text{ for } j = 2, \ldots, d. \qquad (7.20)$$

If we utilize (7.20) to control terms $\mathcal{R}_2$ and $\mathcal{R}_3$, then the energy estimates break down when $[\overline{F}_{11}]$ tends to zero. Hence, identity (7.20) cannot be used in the subsequent analysis for the proof of Theorem 3.1. Then we need to exploit new identities and estimates for $\mathrm{D}_{x'}\psi$. For this purpose, we apply the interpolation argument to deduce the following lemma, which is motivated by [29, Proposition 5.2].

Lemma 7.2. *If the assumptions in Theorem 3.1 are satisfied, then*

$$R_j := F_j^+ \cdot \mathring{N} - \sum_{i=2}^{d} \mathring{F}_{ij}^+ \partial_i \psi \qquad \text{defined on } \omega_T, \text{ for } j = 2, \ldots, d, \qquad (7.21)$$

satisfies

$$\|\mathrm{D}_{\tan}^\gamma R_j(t)\|_{H^{s-|\gamma|-1/2}(\partial\Omega)}^2 \lesssim \mathcal{M}_s(t), \qquad (7.22)$$

for all $t \in [0, T]$ and $\gamma \in \mathbb{N}^d$ with $|\gamma| \leqslant s - 1$, where $\mathcal{M}_s(t)$ is defined by (7.2).

Proof. Thanks to (3.5e) and (7.13), we have

$$\partial_0 F_j^+ \cdot \mathring{N} = \sum_{k=2}^{d} \mathring{F}_{kj}^+ \partial_k(v^+ \cdot \mathring{N}) + \sum_{k,i=2}^{d} v_i^+ \mathring{F}_{kj}^+ \partial_i \partial_k \mathring{\varphi} + \mathring{c}_1 \tilde{f} + \mathring{\underline{c}}_1 W$$

$$= \sum_{i=2}^{d} \mathring{F}_{ij}^+ \partial_i(v^+ \cdot \mathring{N}) + \mathring{c}_1 \tilde{f} + \mathring{c}_1 W.$$

Since, for $k = 2, \ldots, d$,

$$\partial_0 \partial_k \mathring{\varphi} = \partial_k(\mathring{v}^+ \cdot \mathring{N}) + \mathring{v}_i^+ \partial_i \partial_k \mathring{\varphi} = \partial_k \mathring{v}^+ \cdot \mathring{N} \qquad \text{on } \partial\Omega, \qquad (7.23)$$

we have

$$\partial_0(F_j^+ \cdot \mathring{N}) = \sum_{i=2}^{d} \mathring{F}_{ij}^+ \partial_i(v^+ \cdot \mathring{N}) + \mathring{c}_1 \tilde{f} + \mathring{c}_1 W.$$

It follows from (3.7) and (5.15a) that

$$-\sum_{i=2}^{d} \partial_0(\mathring{F}_{ij}^+ \partial_i \psi) = -\sum_{i=2}^{d} \mathring{F}_{ij}^+ \left(\partial_i \partial_0 \psi - \sum_{\ell=2}^{d} \partial_i \mathring{v}_\ell^+ \partial_\ell \psi \right) - \sum_{i,\ell=2}^{d} \mathring{F}_{\ell j}^+ \partial_\ell \mathring{v}_i^+ \partial_i \psi$$

$$= -\sum_{i=2}^{d} \mathring{F}_{ij}^+ \left(\partial_i (v^+ \cdot \mathring{N}) + \mathring{c}_1 \partial_i \psi \right) + \mathring{c}_2 \psi.$$

Thanks to (7.23), we have

$$\partial_0 R_j + \mathring{c}_1 R_j = \mathring{c}_1 \tilde{f} + \mathring{c}_1 W + \mathring{c}_2 \psi \qquad \text{on } \partial\Omega.$$

Using the standard arguments of the energy method yields that

$$\|D_{\tan}^\gamma R_j(t)\|_{H^m(\partial\Omega)} \lesssim \|\mathring{c}_1 \tilde{f} + \mathring{c}_1 W + \mathring{c}_2 \psi\|_{H^{m+|\gamma|}(\omega_t)} \qquad \text{for } m \in \mathbb{N}.$$

Applying the interpolation property (see [27, Lemma 22.3]), the trace theorem, and the Moser-type calculus inequality, we have

$$\|D_{\tan}^\gamma R_j(t)\|_{H^{s-|\gamma|-1/2}(\partial\Omega)} \lesssim \|\mathring{c}_1 \tilde{f} + \mathring{c}_1 W + \mathring{c}_2 \psi\|_{H^{s-1/2}(\omega_t)}$$

$$\lesssim \|\mathring{c}_1 \tilde{f} + \mathring{c}_1 W + \mathring{c}_2 \Psi\|_{H^s(\Omega_t)} \lesssim \sqrt{\mathcal{M}_s(t)},$$

where we utilize $\|\mathring{c}_2\|_{W^{1,\infty}(\Omega_t)} \lesssim K$ and $\|(W, \Psi, \tilde{f})\|_{L^\infty(\Omega_t)} \lesssim \|(W, \Psi, \tilde{f})\|_{H^3(\Omega_t)}$ by the Sobolev embedding theorem. This completes the proof. $\square$

By virtue of (7.21), from (2.27) and (3.10), we obtain the following assertions:

– If $d = 2$, then

$$\partial_2 \psi = (\mathring{F}_{22}^+)^{-1}(F_{12}^+ - \partial_2 \mathring{\varphi} F_{22}^+ - R_2) = \varrho(\overline{F}^+) F_{12}^+ + \underline{\mathring{c}}_1 W + \mathring{c}_0 R_2. \quad (7.24)$$

– If $d = 3$, then

$$\begin{pmatrix} \partial_2 \psi \\ \partial_3 \psi \end{pmatrix} = \varrho(\mathring{F}^+) \begin{pmatrix} \mathring{F}_{33}^+ & -\mathring{F}_{32}^+ \\ -\mathring{F}_{23}^+ & \mathring{F}_{22}^+ \end{pmatrix} \begin{pmatrix} F_2^+ \cdot \mathring{N} - R_2 \\ F_3^+ \cdot \mathring{N} - R_3 \end{pmatrix},$$

which implies

$$\partial_2 \psi = \varrho(\overline{F}^+) \overline{F}_{33} F_{12}^+ + \underline{\mathring{c}}_1 W + \mathring{c}_0 R_2 + \mathring{c}_0 R_3, \qquad (7.25)$$

$$\partial_3 \psi = \varrho(\overline{F}^+) \overline{F}_{22} F_{13}^+ + \underline{\mathring{c}}_1 W + \mathring{c}_0 R_2 + \mathring{c}_0 R_3. \qquad (7.26)$$

Identities (7.24)–(7.26) and estimate (7.22) enable us to control term $\mathcal{R}_2$. More precisely, from (7.24)–(7.26) and (5.15a), we have

$$D_{\tan}\psi = \mathring{c}_1 W + \sum_{j=2}^{d} \mathring{c}_0 R_j \qquad \text{on } \partial\Omega, \qquad (7.27)$$

where coefficients $\mathring{c}_1$ and $\mathring{c}_0$ are independent of $[\overline{F}_{11}]$. Assume without loss of generality that $0 < \beta' \leqslant \beta$, $|\beta'| = 1$, and $|\beta| \leqslant s$. For the last term in $\mathcal{R}_2$, we employ (7.27) to obtain

$$
\int_{\omega_t} \mathring{c}_0 D_{\tan}^{\beta} \psi D_{\tan}^{\beta} (\mathring{c}_0 \tilde{f} + \mathring{c}_1 W)
$$

$$
\lesssim \left\| \mathring{c}_0 D_{\tan}^{\beta-\beta'} \Big(\mathring{c}_1 W + \sum_{j=2}^{d} \mathring{c}_0 R_j \Big) \right\|_{H^{1/2}(\omega_t)} \left\| D_{\tan}^{\beta} (\mathring{c}_0 \tilde{f} + \mathring{c}_1 W) \right\|_{H^{-1/2}(\omega_t)}
$$

$$
\lesssim \left\| \mathring{c}_1 W + \sum_{j=2}^{d} \mathring{c}_0 \widetilde{R}_j \right\|_{H^s(\Omega_t)} \left\| \mathring{c}_0 \tilde{f} + \mathring{c}_1 W \right\|_{H^s(\Omega_t)}, \tag{7.28}
$$

where $\widetilde{R}_j$ is the extension of R_j from ω_T to Ω_T satisfying

$$
\| \widetilde{R}_j \|_{H^m(\Omega_t)} \lesssim \| R_j \|_{H^{m-1/2}(\omega_t)} \qquad \text{for } m = 1, \dots, s. \tag{7.29}
$$

Applying the Moser-type calculus inequality to (7.28) and using estimates (7.22) and (7.29), we obtain

$$
\int_{\omega_t} \mathring{c}_0 D_{\tan}^{\beta} \psi D_{\tan}^{\beta} (\mathring{c}_0 \tilde{f} + \mathring{c}_1 W) \lesssim \mathcal{M}_s(t).
$$

As the other terms in (7.19) can be handled similarly, we omit the details and conclude

$$
\mathcal{R}_2 \lesssim \mathcal{M}_s(t). \tag{7.30}
$$

7.4. Estimate of Term $\mathcal{R}_3$ with the Time Derivative

This subsection is devoted to deriving the estimate of term $\mathcal{R}_3$ given in (7.18) for $\beta = (\beta_0, \beta_2, \dots, \beta_d)$ satisfying $\beta_0 \geqslant 1$ and $|\beta| \leqslant s$.

Recalling the definition of background state $(\overline{U}^{\pm}, \overline{\Phi}^{\pm})$ in (2.27) and using identity (7.13), we have

$$
\partial_t F_{ij}^+ = \overline{F}_{jj} \partial_j v_i^+ + \underline{\mathring{c}}_0 D_{x'} W + \underline{\mathring{c}}_1 W + \mathring{c}_0 \tilde{f} \quad \text{on } \partial\Omega, \text{ for } i, j = 2, \dots, d, \tag{7.31}
$$

where $D_{x'} := (\partial_2, \dots, \partial_d)$. In light of (7.31), we compute

$$
[\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{F}^+) D_{\tan}^{\beta} F_{ij}^+
$$

$$
= [\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{F}^+) D_{\tan}^{\beta-e_1} \Big(\overline{F}_{jj} \partial_j v_i^+ + \underline{\mathring{c}}_0 D_{x'} W + \underline{\mathring{c}}_1 W + \mathring{c}_0 \tilde{f} \Big)
$$

$$
= -\sum_{j=2}^{d} [\overline{F}_{11}] \varrho(\overline{F}^+) D_{\tan}^{\beta-e_1} \partial_j v_j^+ + \underline{\mathring{c}}_0 D_{\tan}^{\beta-e_1} (\mathring{c}_0 D_{x'} W)
$$

$$
+ \mathring{c}_0 D_{\tan}^{\beta-e_1} (\underline{\mathring{c}}_0 D_{x'} W + \underline{\mathring{c}}_1 W + \mathring{c}_0 \tilde{f}) \qquad \text{on } \partial\Omega. \tag{7.32}
$$

Noting from (5.15a) that

$$\partial_t \psi = W_2^+ + \mathring{\mathfrak{c}}_0 D_{x'} \psi + \mathring{\mathfrak{c}}_1 \psi \qquad \text{on } \partial\Omega, \tag{7.33}$$

we have

$$\mathcal{R}_3 = 2 \int_{\partial\Omega} [\mathring{F}_{11}] \partial_{F_{ij}} \varrho(\mathring{F}^+) D_{\text{tan}}^\beta F_{ij}^+ D_{\text{tan}}^\beta \psi = \sum_{i=1}^{5} \mathcal{R}_{3i}, \tag{7.34}$$

where

$$\mathcal{R}_{31} := -\sum_{j=2}^{d} 2[\overline{F}_{11}] \varrho(\overline{F}^+) \int_{\partial\Omega} D_{\text{tan}}^{\beta-e_1} W_2^+ D_{\text{tan}}^{\beta-e_1} \partial_j v_j^+,$$

$$\mathcal{R}_{32} := -\sum_{j=2}^{d} 2[\overline{F}_{11}] \varrho(\overline{F}^+) \int_{\partial\Omega} D_{\text{tan}}^{\beta-e_1} \left(\mathring{\mathfrak{c}}_0 D_{x'} \psi\right) D_{\text{tan}}^{\beta-e_1} \partial_j v_j^+,$$

$$\mathcal{R}_{33} := -\sum_{j=2}^{d} 2[\overline{F}_{11}] \varrho(\overline{F}^+) \int_{\partial\Omega} D_{\text{tan}}^{\beta-e_1} \left(\mathring{\mathfrak{c}}_1 \psi\right) D_{\text{tan}}^{\beta-e_1} \partial_j v_j^+,$$

$$\mathcal{R}_{34} := \int_{\partial\Omega} \left(\mathring{\mathfrak{c}}_0 D_{\text{tan}}^{\beta-e_1} \left(\mathring{\mathfrak{c}}_0 D_{x'} W\right) + \mathring{\mathfrak{c}}_0 D_{\text{tan}}^{\beta-e_1} D_{x'}(\mathring{\mathfrak{c}}_0 W)\right) D_{\text{tan}}^\beta \psi,$$

$$\mathcal{R}_{35} := \int_{\partial\Omega} \mathring{\mathfrak{c}}_0 D_{\text{tan}}^{\beta-e_1} \left(\mathring{\mathfrak{c}}_1 W + \mathring{\mathfrak{c}}_0 \tilde{f}\right) D_{\text{tan}}^\beta \psi.$$

Let us first estimate $\mathcal{R}_{32}$ as

$$|\mathcal{R}_{32}| \lesssim \left| \underbrace{\int_{\partial\Omega} \mathring{\mathfrak{c}}_0 D_{\text{tan}}^{\beta-e_1} D_{\text{tan}} \psi D_{\text{tan}}^{\beta-e_1} D_{x'} W}_{\mathcal{R}_{32}^a} + \underbrace{\int_{\partial\Omega} [D_{\text{tan}}^{\beta-e_1}, \mathring{\mathfrak{c}}_0] D_{x'} \psi D_{\text{tan}}^{\beta-e_1} D_{x'} W}_{\mathcal{R}_{32}^b} \right|.$$

In view of (7.27), we employ the classical product estimate

$$\|uv\|_{H^{1/2}(\mathbb{R}^{d-1})} \lesssim \|u\|_{H^{3/2}(\mathbb{R}^{d-1})} \|v\|_{H^{1/2}(\mathbb{R}^{d-1})}$$

to obtain

$$|\mathcal{R}_{32}^a| \lesssim \left\| \mathring{\mathfrak{c}}_0 D_{\text{tan}}^{\beta-e_1} \left(\mathring{\mathfrak{c}}_1 W + \sum_{j=2}^{d} \mathring{\mathfrak{c}}_0 R_j\right) \right\|_{H^{1/2}(\partial\Omega)} \left\| D_{\text{tan}}^{\beta-e_1} D_{x'} W \right\|_{H^{-1/2}(\partial\Omega)}$$

$$\lesssim \left\| \mathring{\mathfrak{c}}_0 \right\|_{H^3(\Omega_t)} \left\| D_{\text{tan}}^{\beta-e_1} \left(\mathring{\mathfrak{c}}_1 W + \sum_{j=2}^{d} \mathring{\mathfrak{c}}_0 R_j\right) \right\|_{H^{1/2}(\partial\Omega)} \left\| D_{\text{tan}}^{\beta-e_1} W \right\|_{H^{1/2}(\partial\Omega)}. \tag{7.35}$$

Utilize the trace theorem, (6.6), and the Moser-type calculus inequality (4.11) to obtain

$$\left\| D_{\text{tan}}^{\beta-e_1} \left(\mathring{\mathfrak{c}}_1 W\right) \right\|_{H^{1/2}(\partial\Omega)}^2 \lesssim \left\| \mathring{\mathfrak{c}}_1 D_{\text{tan}}^{\beta-e_1} W \right\|_{H^1(\Omega)}^2 + \left\| [D_{\text{tan}}^{\beta-e_1}, \mathring{\mathfrak{c}}_1] W \right\|_{H^2(\Omega_t)}^2$$

$$\lesssim \|W(t)\|_s^2 + \mathcal{M}_s(t). \tag{7.36}$$

It follows from (7.22), the trace theorem, (6.6), and (7.29) that

$$\left\| D_{tan}^{\beta-e_1}\left(\mathring{c}_0 R_j\right)\right\|_{H^{1/2}(\partial\Omega)}^2 \lesssim \left\|\mathring{c}_0 D_{tan}^{\beta-e_1} R_j\right\|_{H^{1/2}(\partial\Omega)}^2 + \left\|[D_{tan}^{\beta-e_1}, \mathring{c}_0]R_j\right\|_{H^{1/2}(\partial\Omega)}^2$$

$$\lesssim \mathcal{M}_s(t) + \left\|[D_{tan}^{\beta-e_1}, \mathring{c}_0]\widetilde{R}_j\right\|_{H^2(\Omega_t)}^2 \lesssim \mathcal{M}_s(t). \quad (7.37)$$

Plugging (7.36)–(7.37) into (7.35) yields

$$|\mathcal{R}_{32}^a| \lesssim \left\|\mathring{\underline{c}}_0\right\|_{H^3(\Omega_t)} \|\!|W(t)|\!\|_s^2 + \mathcal{M}_s(t). \quad (7.38)$$

For $\mathcal{R}_{32}^b$, we find

$$\mathcal{R}_{32}^b = -\int_{\partial\Omega} D_{x'}[D_{tan}^{\beta-e_1}, \mathring{\underline{c}}_0]D_{x'}\psi D_{tan}^{\beta-e_1} W$$

$$= -\int_{\omega_t} \partial_t \left\{ D_{x'}[D_{tan}^{\beta-e_1}, \mathring{\underline{c}}_0]D_{x'}\psi D_{tan}^{\beta-e_1} W \right\}. \quad (7.39)$$

Hence, it follows from (7.22), (7.27), and (7.29) that

$$|\mathcal{R}_{32}^b| \lesssim \left\|\partial_t D_{x'}[D_{tan}^{\beta-e_1}, \mathring{\underline{c}}_0]D_{x'}\psi\right\|_{H^{-1/2}(\omega_t)} \left\|D_{tan}^{\beta-e_1} W\right\|_{H^{1/2}(\omega_t)}$$

$$+ \left\|D_{x'}[D_{tan}^{\beta-e_1}, \mathring{\underline{c}}_0]D_{x'}\psi\right\|_{H^{1/2}(\omega_t)} \left\|\partial_t D_{tan}^{\beta-e_1} W\right\|_{H^{-1/2}(\omega_t)}$$

$$\lesssim \|W\|_{H^s(\Omega_t)} \left\|D[D_{tan}^{\beta-e_1}, \mathring{\underline{c}}_0]\Big(\mathring{c}_1 W + \sum_{j=2}^d \mathring{c}_0 \widetilde{R}_j\Big)\right\|_{H^1(\Omega_t)}$$

$$\lesssim \mathcal{M}_s(t). \quad (7.40)$$

We decompose $\mathcal{R}_{34}$ as

$$\int_{\partial\Omega} \mathring{\underline{c}}_0 D_{tan}^{\beta}\psi D_{tan}^{\beta-e_1} D_{x'} W$$

$$+ \int_{\partial\Omega} D_{tan}^{\beta}\psi \left(\mathring{c}_0[D_{tan}^{\beta-e_1}, \mathring{c}_0]D_{x'} W + \mathring{c}_0[D_{tan}^{\beta-e_1} D_{x'}, \mathring{c}_0]W\right).$$

The first term in this decomposition can be estimated in the same way as $\mathcal{R}_{32}^a$, and the second term in this decomposition along with terms $\mathcal{R}_{33}$ and $\mathcal{R}_{35}$ can be controlled as $\mathcal{R}_{32}^b$. In conclusion, we arrive at

$$\sum_{i=2}^5 |\mathcal{R}_{3i}| \lesssim \left\|\mathring{\underline{c}}_0\right\|_{H^3(\Omega_t)} \|\!|W(t)|\!\|_s^2 + \mathcal{M}_s(t). \quad (7.41)$$

Let us deduce the estimate of term $\mathcal{R}_{31}$. In view of (4.7), we infer

$$|\mathcal{R}_{31}| \leqslant 2[\overline{F}_{11}]\varrho(\overline{F}^+)\|D_{tan}^{\beta-e_1} W_2^+\|_{H^1(\Omega)} \sum_{j=2}^d \|D_{tan}^{\beta-e_1} v_j^+\|_{H^1(\Omega)}$$

$$\leqslant \begin{cases} [\overline{F}_{11}]\varrho(\overline{F}^+)\|D_{tan}^{\beta-e_1}(W_2^+, W_3^+)\|_{H^1(\Omega)}^2 & \text{if } d = 2, \\ \sqrt{2}[\overline{F}_{11}]\varrho(\overline{F}^+)\|D_{tan}^{\beta-e_1}(W_2^+, W_3^+, W_4^+)\|_{H^1(\Omega)}^2 & \text{if } d = 3. \end{cases} \quad (7.42)$$

We now make the estimate for the term on the right-hand side of (7.42). Since $|\beta| \leqslant s$, we apply inequality (6.6) to obtain

$$\sum_{j=2}^{d+1} \|D_{\tan}^{\beta-e_1} W_j^+\|_{L^2(\Omega)}^2 \lesssim \|W\|_{H^s(\Omega_t)}^2. \qquad (7.43)$$

According to definition (7.8) for the instant tangential energy $\mathcal{E}_{\tan}^{\beta}(t)$, we have

$$\sum_{\ell=2}^{d} \sum_{j=2}^{d+1} \|\partial_\ell D_{\tan}^{\beta-e_1} W_j^+\|_{L^2(\Omega)}^2 \lesssim \begin{cases} \mathcal{E}_{\tan}^{\beta-e_1+e_2}(t) & \text{if } d = 2, \\ \mathcal{E}_{\tan}^{\beta-e_1+e_2}(t) + \mathcal{E}_{\tan}^{\beta-e_1+e_3}(t) & \text{if } d = 3. \end{cases} \qquad (7.44)$$

As for the normal derivatives in (7.42), we utilize (6.3) to derive

$$\begin{pmatrix} 0 \\ \partial_1 W_{nc}^\pm \\ 0 \end{pmatrix} = \mp \boldsymbol{B}\mathcal{A}_0(\overline{U}^\pm, \overline{\Phi}^\pm)\partial_t W^\pm \mp \sum_{j=2}^{d} \boldsymbol{B}\mathcal{A}_j(\overline{U}^\pm, \overline{\Phi}^\pm)\partial_j W^\pm + \mathring{\underline{c}}_1 D_{\tan} W$$
$$- \mathring{\boldsymbol{B}}^\pm \mathring{\mathcal{A}}_{1b}^\pm \partial_1 W^\pm - \mathring{\boldsymbol{B}}^\pm \mathring{\mathcal{A}}_4^\pm W^\pm + \mathring{\boldsymbol{B}}^\pm \mathring{J}_\pm^\mathsf{T} \tilde{f}^\pm, \qquad (7.45)$$

where $\boldsymbol{B}(U, \Phi)$ is defined by (6.4). By virtue of identities (5.8)–(5.9), we can compute the following assertions:

– For $d = 2$, the second and third components of

$$\boldsymbol{B}\mathcal{A}_0(\overline{U}^+, \overline{\Phi}^+)\partial_t W^+ + \boldsymbol{B}\mathcal{A}_2(\overline{U}^+, \overline{\Phi}^+)\partial_2 W^+$$

are $\frac{1}{\bar{\rho}^+(\overline{F}_{11}^+)^2}\partial_t(W_4^+ - W_1^+)$ and $-\frac{1}{\overline{F}_{11}^+}\partial_t W_5^+$, respectively.
– For $d = 3$, the second, third, and fourth components of

$$\boldsymbol{B}\mathcal{A}_0(\overline{U}^+, \overline{\Phi}^+)\partial_t W^+ + \boldsymbol{B}\mathcal{A}_2(\overline{U}^+, \overline{\Phi}^+)\partial_2 W^+ + \boldsymbol{B}\mathcal{A}_3(\overline{U}^+, \overline{\Phi}^+)\partial_3 W^+$$

are $\frac{1}{\bar{\rho}^+(\overline{F}_{11}^+)^2}\partial_t(W_5^+ - W_1^+)$, $-\frac{1}{\overline{F}_{11}^+}\partial_t W_6^+$, and $-\frac{1}{\overline{F}_{11}^+}\partial_t W_7^+$, respectively.

Using (7.8), (7.45), and the assertions above, we conclude

$$\sum_{j=2}^{d+1} \|\partial_1 D_{\tan}^{\beta-e_1} W_j^+\|_{L^2(\Omega)}^2 \leqslant \|D_{\tan}^{\beta-e_1}\big(-\mathring{\boldsymbol{B}}\mathring{\mathcal{A}}_{1b}\partial_1 W + \mathring{\boldsymbol{B}}\mathring{\mathcal{A}}_4 W + \mathring{\boldsymbol{B}}\mathring{J}^\mathsf{T}\tilde{f}\big)\|_{L^2(\Omega)}^2$$
$$+ \|D_{\tan}^{\beta-e_1}(\mathring{\underline{c}}_1 D_{\tan} W)\|_{L^2(\Omega)}^2 + \frac{\mathcal{E}_{\tan}^\beta(t)}{\bar{\rho}^+(\overline{F}_{11}^+)^2}. \qquad (7.46)$$

Employ (6.6) and the Moser-type calculus inequality (4.10) to derive

$$\|D_{\tan}^{\beta-e_1}(\mathring{\underline{c}}_1 D_{\tan} W)(t)\|_{L^2(\Omega)}^2 \lesssim \|\mathring{\underline{c}}_1\|_{L^\infty(\Omega_T)}\|W(t)\|_s^2 + \mathcal{M}_s(t). \qquad (7.47)$$

Plug (6.8)–(6.9), (6.11)–(6.13), and (7.47) into (7.46), insert the resulting estimate and (7.43)–(7.44) into (7.42), and use (7.9), (7.18), (7.30), (7.34), and (7.41) to obtain

$$\mathcal{E}^{\beta}_{\tan}(t) \leqslant C\mathcal{M}_s(t) + C\|\mathring{\mathsf{c}}_1\|_{H^3(\Omega_T)}\|\!|W(t)|\!\|_s^2$$
$$+ \begin{cases} \dfrac{[\overline{F}_{11}]}{\overline{F}^+_{11}}\mathcal{E}^{\beta}_{\tan}(t) + C\mathcal{E}^{\beta-e_1+e_2}_{\tan}(t) & \text{if } d = 2, \\[3mm] \sqrt{2}\,\dfrac{[\overline{F}_{11}]}{\overline{F}^+_{11}}\mathcal{E}^{\beta}_{\tan}(t) + C\mathcal{E}^{\beta-e_1+e_2}_{\tan}(t) + C\mathcal{E}^{\beta-e_1+e_3}_{\tan}(t) & \text{if } d = 3. \end{cases}$$

Since $\overline{F}^+_{11} > \overline{F}^-_{11} > 0$, we always see that $[\overline{F}_{11}]/\overline{F}^+_{11} < 1$. Moreover, it follows from (3.23) that $[\overline{F}_{11}]/\overline{F}^+_{11} < \frac{1}{2}$ for dimension $d = 3$. Thus, we can obtain

$$\mathcal{E}^{\beta}_{\tan}(t) \lesssim \mathcal{M}_s(t) + \|\mathring{\mathsf{c}}_1\|_{H^3(\Omega_T)}\|\!|W(t)|\!\|_s^2$$
$$+ \begin{cases} \mathcal{E}^{\beta-e_1+e_2}_{\tan}(t) & \text{if } d = 2, \\[2mm] \mathcal{E}^{\beta-e_1+e_2}_{\tan}(t) + \mathcal{E}^{\beta-e_1+e_3}_{\tan}(t) & \text{if } d = 3, \end{cases} \tag{7.48}$$

for all $\beta = (\beta_0, \beta_2, \ldots, \beta_d) \in \mathbb{N}^d$ with $|\beta| \leqslant s$ and $\beta_0 \geqslant 1$. Inequality (7.48) reduces the estimate of each instant tangential energy to that with one less time derivative. Therefore, we are led to estimate $\mathcal{R}_3$ for the case containing at least one space derivative.

7.5. Estimate of Term $\mathcal{R}_3$ with the x_2-Derivative

In this subsection, we make the estimate of $\mathcal{R}_3$ defined in (7.18) for the case when $\beta_2 \geqslant 1$ and $|\beta| \leqslant s$.

Computing from (3.9) that

$$\partial_{F_{ij}}\varrho(\overline{F}^+)\mathrm{D}^{\beta}_{\tan}F^+_{ij}$$
$$= \begin{cases} -\varrho(\overline{F}^+)^2\mathrm{D}^{\beta}_{\tan}F^+_{22} & \text{if } d = 2, \\[2mm] -\varrho(\overline{F}^+)^2\left(\overline{F}_{33}\mathrm{D}^{\beta}_{\tan}F^+_{22} + \overline{F}_{22}\mathrm{D}^{\beta}_{\tan}F^+_{33}\right) & \text{if } d = 3, \end{cases} \tag{7.49}$$

and using (7.24)–(7.25), we deduce

$$\mathcal{R}_3 = \underbrace{2\int_{\partial\Omega}[\overline{F}_{11}]\partial_{F_{ij}}\varrho(\overline{F}^+)\mathrm{D}^{\beta}_{\tan}F^+_{ij}\mathrm{D}^{\beta}_{\tan}\psi}_{\widetilde{\mathcal{R}}_{31}+\widetilde{\mathcal{R}}_{32}} + \underbrace{\int_{\partial\Omega}\mathring{\mathsf{c}}_0\mathrm{D}^{\beta}_{\tan}F^+_{ij}\mathrm{D}^{\beta}_{\tan}\psi}_{\widetilde{\mathcal{R}}_{33}}, \tag{7.50}$$

where

$$\widetilde{\mathcal{R}}_{31} := \begin{cases} -2[\overline{F}_{11}]\varrho(\overline{F}^+)^3\displaystyle\int_{\partial\Omega}\mathrm{D}^{\beta-e_2}_{\tan}F^+_{12}\mathrm{D}^{\beta}_{\tan}F^+_{22} & \text{if } d = 2, \\[4mm] -2[\overline{F}_{11}]\varrho(\overline{F}^+)^2\displaystyle\int_{\partial\Omega}\mathrm{D}^{\beta-e_2}_{\tan}F^+_{12}\left(\mathrm{D}^{\beta}_{\tan}F^+_{33} + \dfrac{\overline{F}_{33}}{\overline{F}_{22}}\mathrm{D}^{\beta}_{\tan}F^+_{22}\right) & \text{if } d = 3, \end{cases}$$

$$\widetilde{\mathcal{R}}_{32} := \int_{\partial\Omega}\mathring{\mathsf{c}}_0\mathrm{D}^{\beta-e_2}_{\tan}\left(\mathring{\mathsf{c}}_1 W + \sum_{\ell=2}^{d}\mathring{\mathsf{c}}_0 R_\ell\right)\mathrm{D}^{\beta}_{\tan}F^+_{ij}.$$

Similar to the derivation of estimates (7.35)–(7.38), we can obtain

$$|\widetilde{\mathcal{R}}_{32}| + |\widetilde{\mathcal{R}}_{33}| \lesssim \left\| \mathring{\mathsf{c}}_1 \right\|_{H^3(\Omega_t)} \|W\|_s^2 + \mathcal{M}_s(t). \tag{7.51}$$

Utilizing inequality (4.7) leads to

$$|\widetilde{\mathcal{R}}_{31}| \leqslant 2[\overline{F}_{11}]\varrho(\overline{F}^+)^3 \|D_{\tan}^{\beta-e_2} F_{12}^+\|_{H^1(\Omega)} \|D_{\tan}^{\beta-e_2} F_{22}^+\|_{H^1(\Omega)}$$
$$\leqslant [\overline{F}_{11}]\varrho(\overline{F}^+)^3 \|D_{\tan}^{\beta-e_2}(F_{12}^+, F_{22}^+)\|_{H^1(\Omega)}^2 \qquad \text{if } d=2. \tag{7.52}$$

Moreover, for $d=3$, we have

$$|\widetilde{\mathcal{R}}_{31}| \leqslant 2[\overline{F}_{11}]\varrho(\overline{F}^+)^2 \|D_{\tan}^{\beta-e_2} F_{12}^+\|_{H^1(\Omega)}$$
$$\times \left(\|D_{\tan}^{\beta-e_2} F_{33}^+\|_{H^1(\Omega)} + \frac{\overline{F}_{33}}{\overline{F}_{22}} \|D_{\tan}^{\beta-e_2} F_{22}^+\|_{H^1(\Omega)} \right)$$
$$\leqslant [\overline{F}_{11}]\varrho(\overline{F}^+)^2 \left(1 + \frac{\overline{F}_{33}^2}{\overline{F}_{22}^2} \right)^{1/2} \|D_{\tan}^{\beta-e_2}(F_{12}^+, F_{22}^+, F_{33}^+)\|_{H^1(\Omega)}^2. \tag{7.53}$$

To estimate the terms on the right-hand side of (7.52)–(7.53), we compute from (6.35)–(6.36) that

$$\eta_i^\pm = \pm\overline{F}_{11}^\pm \partial_1 F_{i2}^\pm - \overline{F}_{22} \partial_2 F_{i1}^\pm + \mathring{\mathsf{c}}_1 D_x W + \mathring{\mathsf{c}}_2 W, \tag{7.54}$$
$$\zeta_i^\pm = \pm\overline{F}_{11}^\pm \partial_1 F_{i3}^\pm - \overline{F}_{33} \partial_3 F_{i1}^\pm + \mathring{\mathsf{c}}_1 D_x W + \mathring{\mathsf{c}}_2 W. \tag{7.55}$$

By virtue of identities (7.54)–(7.55), estimates (6.31)–(6.32), and

$$F_{11}^+ = \frac{1}{\bar{\rho}^+ \overline{F}_{11}^+}(W_1^+ - W_{d+2}^+) + \mathring{\mathsf{c}}_1 W, \tag{7.56}$$

we obtain the following two assertions:

– If $d=2$, then

$$\|D_{\tan}^{\beta-e_2} D_x(F_{12}^+, F_{22}^+)\|_{L^2(\Omega)}^2$$
$$\leqslant \|D_{\tan}^\beta(F_{12}^+, F_{22}^+)\|_{L^2(\Omega)}^2 + \frac{\overline{F}_{22}^2}{(\bar{\rho}^+)^2(\overline{F}_{11}^+)^4} \|D_{\tan}^\beta(W_1^+ - W_4^+)\|_{L^2(\Omega)}^2$$
$$+ \frac{\overline{F}_{22}^2}{(\overline{F}_{11}^+)^2} \|D_{\tan}^\beta F_{21}^+\|_{L^2(\Omega)}^2 + C\|\mathring{\mathsf{c}}_1\|_{L^\infty(\Omega_T)} \|W\|_s^2 + C\mathcal{M}_s(t),$$

which, combined with (7.52), leads to

$$|\widetilde{\mathcal{R}}_{31}| \leqslant \overline{C}_0 \mathcal{E}_{\tan}^\beta(t) + C\|\mathring{\mathsf{c}}_1\|_{L^\infty(\Omega_T)} \|W\|_s^2 + C\mathcal{M}_s(t), \tag{7.57}$$

where

$$\overline{C}_0 := \max(1, \frac{(\overline{F}_{11}^+)^2}{\overline{F}_{22}^2}) \frac{[\overline{F}_{11}]}{\overline{F}_{11}^+}. \tag{7.58}$$

– If $d = 3$, then

$$
\begin{aligned}
&\|D_{\tan}^{\beta - e_2} D_x (F_{12}^+, F_{22}^+, F_{33}^+)\|_{L^2(\Omega)}^2 \\
&\quad \leqslant \|D_{\tan}^{\beta}(F_{12}^+, F_{22}^+, F_{33}^+)\|_{L^2(\Omega)}^2 + \|D_{\tan}^{\beta - e_2 + e_3}(F_{12}^+, F_{22}^+, F_{33}^+)\|_{L^2(\Omega)}^2 \\
&\qquad + \frac{\overline{F}_{22}^2}{(\bar{\rho}^+)^2 (\overline{F}_{11}^+)^4} \|D_{\tan}^{\beta}(W_1^+ - W_5^+)\|_{L^2(\Omega)}^2 + \frac{\overline{F}_{33}^2}{(\overline{F}_{11}^+)^2} \|D_{\tan}^{\beta - e_2 + e_3} F_{31}^+\|_{L^2(\Omega)}^2 \\
&\qquad + \frac{\overline{F}_{22}^2}{(\overline{F}_{11}^+)^2} \|D_{\tan}^{\beta} F_{21}^+\|_{L^2(\Omega)}^2 + C \|\mathring{\underline{c}}_1\|_{L^\infty(\Omega_T)} \|\|W\|\|_s^2 + C\mathcal{M}_s(t),
\end{aligned}
$$

which, along with (7.53), yields

$$
\begin{aligned}
|\widetilde{\mathcal{R}}_{31}| &\leqslant \overline{C}_1 \mathcal{E}_{\tan}^{\beta}(t) + \overline{C}_2 \mathcal{E}_{\tan}^{\beta - e_2 + e_3}(t) \\
&\quad + C \|\mathring{\underline{c}}_1\|_{L^\infty(\Omega_T)} \|\|W\|\|_s^2 + C\mathcal{M}_s(t), \tag{7.59}
\end{aligned}
$$

where

$$
\overline{C}_1 := \left(1 + \frac{\overline{F}_{33}^2}{\overline{F}_{22}^2}\right)^{1/2} \max(1, \frac{\overline{F}_{22}^2}{(\overline{F}_{11}^+)^2}) \frac{\overline{F}_{11}^+[\overline{F}_{11}]}{\overline{F}_{22}\overline{F}_{33}}, \tag{7.60}
$$

$$
\overline{C}_2 := \left(1 + \frac{\overline{F}_{33}^2}{\overline{F}_{22}^2}\right)^{1/2} \max(1, \frac{\overline{F}_{33}^2}{(\overline{F}_{11}^+)^2}) \frac{\overline{F}_{11}^+[\overline{F}_{11}]}{\overline{F}_{22}\overline{F}_{33}}. \tag{7.61}
$$

Plugging estimates (7.51), (7.57), and (7.59) into (7.50), and using (7.9), (7.18), and (7.30), we deduce

$$
\begin{aligned}
\mathcal{E}_{\tan}^{\beta}(t) &\leqslant C \|\mathring{\underline{c}}_1\|_{H^3(\Omega_T)} \|\|W(t)\|\|_s^2 + C\mathcal{M}_s(t) \\
&\quad + \begin{cases} \overline{C}_0 \, \mathcal{E}_{\tan}^{\beta}(t) & \text{if } d = 2, \\ \overline{C}_1 \, \mathcal{E}_{\tan}^{\beta}(t) + \overline{C}_2 \, \mathcal{E}_{\tan}^{\beta - e_2 + e_3}(t) & \text{if } d = 3. \end{cases} \tag{7.62}
\end{aligned}
$$

For $d = 3$, it follows from (3.23) that $\overline{C}_1 < 1$, so that estimate (7.62) implies that

$$
\mathcal{E}_{\tan}^{\beta}(t) \leqslant C \|\mathring{\underline{c}}_1\|_{H^3(\Omega_T)} \|\|W(t)\|\|_s^2 + C\mathcal{M}_s(t) + \frac{\overline{C}_2}{1 - \overline{C}_1} \mathcal{E}_{\tan}^{\beta - e_2 + e_3}(t) \tag{7.63}
$$

for all $\beta \in \mathbb{N}^3$ with $|\beta| \leqslant s$ and $\beta_2 \geqslant 1$.

Proof of Proposition 7.1 for $d = 2$. In the two-dimensional case, if (3.23) holds, then $\overline{C}_0 < 1$. From (7.62), we have

$$
\mathcal{E}_{\tan}^{\beta}(t) \lesssim \|\mathring{\underline{c}}_1\|_{H^3(\Omega_T)} \|\|W(t)\|\|_s^2 + \mathcal{M}_s(t), \tag{7.64}
$$

for all $\beta \in \mathbb{N}^2$ with $|\beta| \leqslant s$ and $\beta_2 \geqslant 1$. Combining (7.64) and (7.48), we can conclude (7.64) for all $\beta \in \mathbb{N}^2$ with $|\beta| \leqslant s$. The proof for case $d = 2$ is complete.

$\square$

7.6. Estimate of Term $\mathcal{R}_3$ with the x_3-Derivative

For the three-dimensional case ($d = 3$), in order to prove (7.1), it suffices to obtain the estimate of $\mathcal{R}_3$ defined in (7.18) for $\beta_3 \geqslant 1$ and $|\beta| \leqslant s$. For this purpose, we utilize (7.26) and (7.49) to deduce

$$\mathcal{R}_3 = 2\underbrace{\int_{\partial\Omega} [\overline{F}_{11}]\partial_{F_{ij}}\varrho(\overline{F}^+)\mathrm{D}_{\mathrm{tan}}^\beta F_{ij}^+\mathrm{D}_{\mathrm{tan}}^\beta\psi}_{\widehat{\mathcal{R}}_{31}+\widehat{\mathcal{R}}_{32}} + \underbrace{\int_{\partial\Omega} \mathring{\mathfrak{c}}_0\mathrm{D}_{\mathrm{tan}}^\beta F_{ij}^+\mathrm{D}_{\mathrm{tan}}^\beta\psi}_{\widehat{\mathcal{R}}_{33}}, \qquad (7.65)$$

where

$$\widehat{\mathcal{R}}_{31} := -2[\overline{F}_{11}]\varrho(\overline{F}^+)^2 \int_{\partial\Omega} \mathrm{D}_{\mathrm{tan}}^{\beta-e_3} F_{13}^+\Big(\frac{\overline{F}_{22}}{\overline{F}_{33}}\mathrm{D}_{\mathrm{tan}}^\beta F_{33}^+ + \mathrm{D}_{\mathrm{tan}}^\beta F_{22}^+\Big),$$

$$\widehat{\mathcal{R}}_{32} := \int_{\partial\Omega} \mathring{\mathfrak{c}}_0\mathrm{D}_{\mathrm{tan}}^{\beta-e_3}\Big(\mathring{\mathfrak{c}}_1 W + \sum_{\ell=2}^d \mathring{\mathfrak{c}}_0 R_\ell\Big)\mathrm{D}_{\mathrm{tan}}^\beta F_{ij}^+.$$

Similar to the derivation of estimates (7.35)–(7.38), we can deduce

$$|\widehat{\mathcal{R}}_{32}| + |\widehat{\mathcal{R}}_{33}| \lesssim \big\|\mathring{\mathfrak{c}}_1\big\|_{H^3(\Omega_t)}\|\!|W(t)|\!\|_s^2 + \mathcal{M}_s(t). \qquad (7.66)$$

In view of inequality (4.7), we have

$$|\widehat{\mathcal{R}}_{31}| \leqslant 2[\overline{F}_{11}]\varrho(\overline{F}^+)^2\|\mathrm{D}_{\mathrm{tan}}^{\beta-e_3} F_{13}^+\|_{H^1(\Omega)}$$

$$\times \Big(\frac{\overline{F}_{22}}{\overline{F}_{33}}\|\mathrm{D}_{\mathrm{tan}}^{\beta-e_3} F_{33}^+\|_{H^1(\Omega)} + \|\mathrm{D}_{\mathrm{tan}}^{\beta-e_3} F_{22}^+\|_{H^1(\Omega)}\Big)$$

$$\leqslant [\overline{F}_{11}]\varrho(\overline{F}^+)^2\Big(1 + \frac{\overline{F}_{22}^2}{\overline{F}_{33}^2}\Big)^{1/2}\|\mathrm{D}_{\mathrm{tan}}^{\beta-e_3}(F_{13}^+, F_{22}^+, F_{33}^+)\|_{H^1(\Omega)}^2. \qquad (7.67)$$

Use identities (7.54)–(7.56) and estimates (6.31)–(6.32) to derive

$$\|\mathrm{D}_{\mathrm{tan}}^{\beta-e_3}\mathrm{D}_x(F_{13}^+, F_{22}^+, F_{33}^+)\|_{L^2(\Omega)}^2$$

$$\leqslant \|\mathrm{D}_{\mathrm{tan}}^\beta(F_{13}^+, F_{22}^+, F_{33}^+)\|_{L^2(\Omega)}^2 + \|\mathrm{D}_{\mathrm{tan}}^{\beta-e_3+e_2}(F_{13}^+, F_{22}^+, F_{33}^+)\|_{L^2(\Omega)}^2$$

$$+ \frac{\overline{F}_{33}^2}{(\bar{\rho}^+)^2(\overline{F}_{11}^+)^4}\|\mathrm{D}_{\mathrm{tan}}^\beta(W_1^+ - W_5^+)\|_{L^2(\Omega)}^2 + \frac{\overline{F}_{22}^2}{(\overline{F}_{11}^+)^2}\|\mathrm{D}_{\mathrm{tan}}^{\beta-e_3+e_2} F_{21}^+\|_{L^2(\Omega)}^2$$

$$+ \frac{\overline{F}_{33}^2}{(\overline{F}_{11}^+)^2}\|\mathrm{D}_{\mathrm{tan}}^\beta F_{31}^+\|_{L^2(\Omega)}^2 + C\|\mathring{\mathfrak{c}}_1\|_{L^\infty(\Omega_t)}\|\!|W|\!\|_s^2 + C\mathcal{M}_s(t),$$

which, along with (7.9), (7.18), (7.30), (7.65), and (7.67)–(7.68), yields

$$\mathcal{E}_{\mathrm{tan}}^\beta(t) \leqslant \overline{C}_3\mathcal{E}_{\mathrm{tan}}^\beta(t) + \overline{C}_4\mathcal{E}_{\mathrm{tan}}^{\beta-e_3+e_2}(t)$$

$$+ C\|\mathring{\mathfrak{c}}_1\|_{L^\infty(\Omega_T)}\|\!|W|\!\|_s^2 + C\mathcal{M}_s(t), \qquad (7.68)$$

where

$$\overline{C}_3 := \left(1 + \frac{\overline{F}_{22}^2}{\overline{F}_{33}^2}\right)^{1/2} \max(1, \frac{\overline{F}_{33}^2}{(\overline{F}_{11}^+)^2}) \frac{\overline{F}_{11}^+[\overline{F}_{11}]}{\overline{F}_{22}\overline{F}_{33}}, \tag{7.69}$$

$$\overline{C}_4 := \left(1 + \frac{\overline{F}_{22}^2}{\overline{F}_{33}^2}\right)^{1/2} \max(1, \frac{\overline{F}_{22}^2}{(\overline{F}_{11}^+)^2}) \frac{\overline{F}_{11}^+[\overline{F}_{11}]}{\overline{F}_{22}\overline{F}_{33}}. \tag{7.70}$$

Noting from (3.23) that $\overline{C}_3 < 1$, we have

$$\mathcal{E}_{\text{tan}}^\beta(t) \leqslant C\|\mathring{\underline{c}}_1\|_{H^3(\Omega_T)}\|W(t)\|_s^2 + C\mathcal{M}_s(t) + \frac{\overline{C}_4}{1 - \overline{C}_3}\mathcal{E}_{\text{tan}}^{\beta - e_3 + e_2}(t), \tag{7.71}$$

for all $\beta \in \mathbb{N}^3$ with $\beta_3 \geqslant 1$ and $|\beta| \leqslant s$.

Proof of Proposition 7.1 for $d = 3$. Combine (7.63) and (7.71) to infer

$$\mathcal{E}_{\text{tan}}^\beta(t) \leqslant C\|\mathring{\underline{c}}_1\|_{H^3(\Omega_T)}\|W(t)\|_s^2 + C\mathcal{M}_s(t) + \frac{\overline{C}_2\overline{C}_4}{(1 - \overline{C}_1)(1 - \overline{C}_3)}\mathcal{E}_{\text{tan}}^\beta(t),$$

which yields

$$\mathcal{E}_{\text{tan}}^\beta(t) \lesssim \|\mathring{\underline{c}}_1\|_{H^3(\Omega_T)}\|W(t)\|_s^2 + \mathcal{M}_s(t) \tag{7.72}$$

for all $\beta \in \mathbb{N}^3$ with $\beta_3 \geqslant 1$ and $|\beta| \leqslant s$, provided

$$\overline{C}_2\overline{C}_4 < (1 - \overline{C}_1)(1 - \overline{C}_3).$$

This last condition is equivalent to (3.23) because of $\overline{C}_1\overline{C}_3 = \overline{C}_2\overline{C}_4$. Combining (7.48), (7.63), and (7.72), we deduce (7.72) for all $\beta \in \mathbb{N}^3$ with $|\beta| \leqslant s$. Therefore, we complete the proof for $d = 3$. □

8. Proof of Theorem 3.1

This subsection is dedicated to the proof of the main theorem of this paper, Theorem 3.1.

Combine estimates (6.1)–(6.2) and (7.1) to obtain

$$\|W(t)\|_s^2 \lesssim \|\mathring{\underline{c}}_1\|_{H^3(\Omega_T)}\|W(t)\|_s^2 + \mathcal{M}_s(t),$$

where $\mathcal{M}_s(t)$ is defined by (7.2). Thanks to (3.3), we apply the Moser-type calculus inequality (4.9) and take $K > 0$ sufficiently small to obtain

$$\|W(t)\|_s^2 \lesssim \mathcal{M}_s(t). \tag{8.1}$$

It follows from definitions (3.14)–(4.14) that

$$\|\Psi(t)\|_s^2 = \sum_{k+|\beta|\leqslant s}\int_0^\infty |\partial_1^k\chi(\pm x_1)|^2\mathrm{d}x_1\int_{\mathbb{R}^{d-1}}|\mathrm{D}_{\text{tan}}^\beta\psi(t, x')|^2\mathrm{d}x',$$

which, along with (2.31), leads to

$$\|\Psi(t)\|_s^2 \sim \sum_{|\beta| \leqslant s} \|\mathrm{D}_{\tan}^{\beta} \psi(t)\|_{L^2(\partial \Omega)}^2. \tag{8.2}$$

Integrate (8.2) over $(-\infty, T)$ to obtain

$$\|\Psi\|_{H^s(\Omega_T)} \sim \|\psi\|_{H^s(\omega_T)}. \tag{8.3}$$

Similarly, we see from (3.1) that

$$\|\mathring{\Psi}\|_{H^s(\Omega_T)} \sim \|\mathring{\varphi}\|_{H^s(\omega_T)}. \tag{8.4}$$

In view of (6.6), (7.22), (7.27) and (8.1), we have that

$$\sum_{|\beta| \leqslant s} \|\mathrm{D}_{\tan}^{\beta} \psi(t)\|_{L^2(\partial \Omega)}^2 \lesssim \|\psi\|_{H^s(\omega_t)}^2 + \sum_{|\beta|=s-1} \left\| \mathrm{D}_{\tan}^{\beta} \Big(\mathring{c}_1 W + \sum_{j=2}^{d} \mathring{\underline{c}}_0 R_j \Big) \right\|_{L^2(\partial \Omega)}^2$$

$$\lesssim \|W(t)\|_s^2 + \mathcal{M}_s(t) \lesssim \mathcal{M}_s(t), \tag{8.5}$$

which, along with (8.2), yields

$$\|(W, \Psi)(t)\|_1^2 \lesssim \int_0^t \|(W, \Psi)(\tau)\|_1^2 \, \mathrm{d}\tau + \|\tilde{f}\|_{H^1(\Omega_t)}^2,$$

$$\|(W, \Psi)(t)\|_s^2 \lesssim \int_0^t \|(W, \Psi)(\tau)\|_s^2 \, \mathrm{d}\tau + \|\tilde{f}\|_{H^s(\Omega_t)}^2$$
$$+ \|(\mathring{V}, \mathring{\Psi})\|_{H^{s+2}(\Omega_T)}^2 \|(W, \Psi, \tilde{f})\|_{H^3(\Omega_t)}^2 \quad \text{for } s \geqslant 3.$$

Applying Grönwall's inequality to the estimates above implies that

$$\|(W, \Psi)(t)\|_1^2 \lesssim \|\tilde{f}\|_{H^1(\Omega_t)}^2, \tag{8.6}$$

$$\|(W, \Psi)(t)\|_s^2 \lesssim \|\tilde{f}\|_{H^s(\Omega_t)}^2 + \|(\mathring{V}, \mathring{\Psi})\|_{H^{s+2}(\Omega_T)}^2 \|(W, \Psi, \tilde{f})\|_{H^3(\Omega_t)}^2 \quad \text{for } s \geqslant 3. \tag{8.7}$$

Since W and ψ vanish in the past, we integrate (8.6)–(8.7) over $[0, T]$ to deduce

$$\|(W, \Psi)\|_{H^1(\Omega_T)}^2 \lesssim \|\tilde{f}\|_{H^1(\Omega_T)}^2, \tag{8.8}$$

$$\|(W, \Psi)\|_{H^s(\Omega_T)}^2 \lesssim \|\tilde{f}\|_{H^s(\Omega_T)}^2 + \|(\mathring{V}, \mathring{\Psi})\|_{H^{s+2}(\Omega_T)}^2 \|(W, \Psi, \tilde{f})\|_{H^3(\Omega_T)}^2 \quad \text{for } s \geqslant 3. \tag{8.9}$$

Utilizing (8.9) with $s = 3$ and (3.3), we take $K > 0$ sufficiently small to derive

$$\|(W, \Psi)\|_{H^3(\Omega_T)}^2 \lesssim \|\tilde{f}\|_{H^3(\Omega_T)}^2. \tag{8.10}$$

Insert (8.10) into (8.9) to find

$$\|(W, \Psi)\|_{H^s(\Omega_T)}^2 \leqslant C(K_0, T) \Big\{ \|\tilde{f}\|_{H^s(\Omega_T)}^2 + \|(\mathring{V}, \mathring{\Psi})\|_{H^{s+2}(\Omega_T)}^2 \|\tilde{f}\|_{H^3(\Omega_T)}^2 \Big\}. \tag{8.11}$$

Recalling $V^{\pm} = \mathring{J}_{\pm} W^{\pm}$ (*cf.* (5.4)), we employ the Moser-type calculus inequality (4.10), (6.6), and the Sobolev embedding theorem to obtain

$$\|V\|^2_{H^s(\Omega_T)} \lesssim \sum_{|\alpha| \leqslant s} \left(\|\mathring{J} D^{\alpha} W\|^2_{L^2(\Omega_T)} + \|[D^{\alpha}, \mathring{J}] W\|^2_{L^2(\Omega_T)} \right)$$
$$\lesssim \|W\|^2_{H^s(\Omega_T)} + \|(\mathring{V}, \mathring{\Psi})\|^2_{H^{s+1}(\Omega_T)} \|W\|^2_{H^3(\Omega_T)}. \qquad (8.12)$$

Combining (8.3) with (8.10)–(8.12) yields

$$\|V\|^2_{H^s(\Omega_T)} + \|\psi\|^2_{H^s(\omega_T)}$$
$$\leqslant C(K_0, T) \Big\{ \|\tilde{f}\|^2_{H^s(\Omega_T)} + \|(\mathring{V}, \mathring{\Psi})\|^2_{H^{s+2}(\Omega_T)} \|\tilde{f}\|^2_{H^3(\Omega_T)} \Big\}. \qquad (8.13)$$

Thanks to (7.22), (7.27), and (8.13), we can obtain

$$\|V\|^2_{H^s(\Omega_T)} + \|\psi\|^2_{H^{s+1/2}(\omega_T)}$$
$$\leqslant C(K_0, T) \Big\{ \|\tilde{f}\|^2_{H^s(\Omega_T)} + \|(\mathring{V}, \mathring{\Psi})\|^2_{H^{s+2}(\Omega_T)} \|\tilde{f}\|^2_{H^3(\Omega_T)} \Big\}. \qquad (8.14)$$

It follows from (5.3) that

$$\|\tilde{f}\|^2_{H^m(\Omega_T)} \lesssim \|f\|^2_{H^m(\Omega_T)} + \|\mathring{c}_1 D V_\natural\|^2_{H^m(\Omega_T)} + \|\mathring{c}_1 V_\natural\|^2_{H^m(\Omega_T)}.$$

By virtue of (5.1), we employ the Moser-type calculus inequality (4.10) and the Sobolev embedding theorem to obtain

$$\|\tilde{f}\|^2_{H^m(\Omega_T)} \lesssim \|f\|^2_{H^m(\Omega_T)} + \|g\|^2_{H^{m+1/2}(\omega_T)} + \|(\mathring{V}, \mathring{\Psi})\|^2_{H^{m+1}(\Omega_T)} \|g\|^2_{H^{7/2}(\omega_T)}.$$

Insert the estimate with $m = s$ and $m = 3$ above into (8.14) and use (8.4) to deduce the tame estimate (3.25). Moreover, we can easily derive (3.24) from (8.8). This completes the proof of Theorem 3.1.

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Conflicts of interest The authors declare that they have no conflict of interest.

Appendix A: Proof of Proposition 2.1

Assume that $[S] = 0$ on $\Gamma(t)$. Taking the scalar product of the last identity in (2.23) with N and utilizing (2.20e) yields that

$$|N|^2 \left(p(\rho^+, S^+) - p(\rho^-, S^+) \right)$$
$$= |N|^2 [p] = \rho^+ F^+_{\ell N} [F_{\ell N}] = [\rho F_{\ell N} F_{\ell N}] = \sum_{j=1}^{d} (\rho^+ F^+_{\ell N})^2 [\rho^{-1}].$$

Then we infer from (2.11) and (2.22) that

$$[\rho] = [p] = 0,$$

which, combined with (2.23), gives

$$F_{\ell N}^+[F_\ell] = 0. \tag{A.1}$$

Plug (2.20e) into (2.20f) to obtain

$$F_{kN}^+[F_{ij}] - F_{jN}^+[F_{ik}] = 0 \qquad \text{for } i, j, k = 1, \ldots, d. \tag{A.2}$$

For $d = 2$, from (A.1)–(A.2), we have

$$(F_{1N}^+)^2[F_{i2}] + (F_{2N}^+)^2[F_{i2}] = F_{2N}^+ \left(F_{1N}^+[F_{i1}] + F_{2N}^+[F_{i2}] \right) = 0,$$

which, along with (2.22), yields $[F_{i2}] = 0$ for $i = 1, 2$. Then we utilize (A.2) again to obtain $[\boldsymbol{F}] = 0$ on $\Gamma(t)$.
For $d = 3$, relations (A.2) are equivalent to

$$(F_{1N}^+, \ F_{2N}^+, \ F_{3N}^+)^\mathsf{T} \times ([F_{i1}], [F_{i2}], [F_{i3}])^\mathsf{T} = 0 \qquad \text{for } i = 1, 2, 3,$$

which implies

$$[F_{ij}] = \omega_i F_{jN}^+ \tag{A.3}$$

for some scalar functions ω_i and for all $i, j = 1, 2, 3$. We plug (A.3) into (A.1) and utilize (2.22) to deduce that $\omega_i \equiv 0$ for all $i = 1, \ldots, d$. Then it follows from (A.3) that $[\boldsymbol{F}] = 0$ on $\Gamma(t)$.
In view of the second condition in (2.23), we find that $[U] = 0$ on $\Gamma(t)$, *i.e.*, the solution U is continuous across front $\Gamma(t)$. Therefore, there is no thermoelastic contact discontinuity for the case $[S] = 0$. This completes the proof of Proposition 2.1.

Appendix B: Proof of Proposition 2.2

We omit indices $\pm$ in several places below to avoid overloaded expressions.
1: *Proof of* (2.35). In the original variables, we see from (2.15c) that

$$(\partial_t + v_\ell \partial_\ell) \det \boldsymbol{F} = \frac{\partial \det \boldsymbol{F}}{\partial F_{ij}} (\partial_t + v_\ell \partial_\ell) F_{ij} = \det \boldsymbol{F} (\boldsymbol{F}^{-1})_{ji} F_{\ell j} \partial_\ell v_i$$
$$= \det \boldsymbol{F} \delta_{\ell,i} \partial_\ell v_i = \det \boldsymbol{F} \partial_i v_i,$$

which, combined with the first equation in (2.5), yields

$$(\partial_t + v_\ell \partial_\ell)(\rho \det \boldsymbol{F}) = 0.$$

After transformation (2.29), we find

$$(\partial_t + w_\ell \partial_\ell)(\rho \det \boldsymbol{F}) = 0,$$

where

$$w_1 := \frac{1}{\partial_1 \Phi}\Big(v_1 - \partial_t \Phi - \sum_{j=2}^{d} v_j \partial_j \Phi\Big), \qquad w_i := v_i \quad \text{for } i = 2, \cdots, d.$$

Since $w_1|_{x_1=0} = 0$ resulting from (2.32b), we can obtain identity (2.35) by the standard energy method.

2: *Proof of* (2.36). A straightforward calculation shows that solutions of (2.18) satisfy (see, e.g., the proof of Qian–Zhang [24, Proposition 1])

$$(\partial_t + v_\ell \partial_\ell)(F_{\ell k}\partial_\ell F_{ij} - F_{\ell j}\partial_\ell F_{ik}) = \partial_m v_i (F_{\ell k}\partial_\ell F_{mj} - F_{\ell j}\partial_\ell F_{mk}).$$

After transformation (2.29), we have

$$(\partial_t + w_\ell \partial_\ell)M_{k,i,j} = \partial_m^\Phi v_i M_{k,m,j}$$

with $M_{k,i,j} := F_{\ell k}\partial_\ell^\Phi F_{ij} - F_{\ell j}\partial_\ell^\Phi F_{ik}$. Here we recall the differentials with respect to (2.29) from definition (2.40). Similar to the proof of Hu--Wang [19, Lemma A.2], we can use integration by parts and $w_1|_{x_1=0} = 0$ to obtain (2.36).

3: *Proof of* (2.37) *and* (2.39). In the original variables, system (2.15) gives

$$(\partial_t + v_\ell \partial_\ell)(\rho F_{ij}) + \rho F_{ij}\partial_\ell v_\ell - \rho F_{\ell j}\partial_\ell v_i = 0. \tag{B.1}$$

After transformation (2.29), equation (B.1) becomes

$$(\partial_t + w_\ell \partial_\ell)(\rho F_{ij}) + \rho F_{ij}\partial_\ell^\Phi v_\ell - \rho F_{\ell j}\partial_\ell^\Phi v_i = 0. \tag{B.2}$$

By virtue of (2.32b), we have

$$(\partial_t + w_\ell \partial_\ell)\partial_i \varphi = \partial_i v \cdot N \qquad \text{on } \partial\Omega, \text{ for } i = 2, \ldots, d.$$

Then it follows from the restriction of (B.2) on $\partial\Omega$ that

$$(\partial_t + w_\ell \partial_\ell)(\rho F_{jN}) + \rho F_{jN}\sum_{\ell=2}^{d}\partial_\ell v_\ell = 0 \qquad \text{on } \partial\Omega. \tag{B.3}$$

Since $w_1|_{x_1=0} = 0$ and $[v] = 0$, we can derive (2.37) and (2.39) by employing the method of characteristics.

4: *Proof of* (2.38). It follows from (B.3) that

$$(\partial_t + w_\ell \partial_\ell)(\rho F_{kN} F_{ij} - \rho F_{jN} F_{ik}) - \rho F_{kN}(\partial_t + w_\ell \partial_\ell)F_{ij}$$
$$+ \rho F_{jN}(\partial_t + w_\ell \partial_\ell)F_{ik} + \sum_{\ell=2}^{d}\partial_\ell v_\ell(\rho F_{kN} F_{ij} - \rho F_{jN} F_{ik}) = 0 \qquad \text{on } \partial\Omega.$$

Since

$$(\partial_t + w_\ell \partial_\ell)F_{ij} = F_{\ell j}\partial_\ell^\Phi v_i = \frac{\partial_1 v_i}{\partial_1 \Phi}F_{jN} + \sum_{\ell=2}^{d}F_{\ell j}\partial_\ell v_i,$$

we have

$$(\partial_t + w_\ell^+ \partial_\ell)[I_{k,i,j}] + \sum_{\ell=2}^{d} \partial_\ell v_i^+ [I_{j,\ell,k}] + \sum_{\ell=2}^{d} \partial_\ell v_\ell^+ [I_{k,i,j}] = 0 \qquad \text{on } \partial\Omega,$$

for $I_{k,i,j} := \rho F_{kN} F_{ij} - \rho F_{jN} F_{ik}$. Since (2.38) holds at the initial time, *i.e.*, $[I_{k,i,j}] = 0$ at $t = 0$ for $i, j, k = 1, \ldots, d$, we employ the standard argument of the energy method to derive that (2.38) is satisfied for all $t \in [0, T]$.

5: *Proof of* (2.41). It suffices to prove (2.12) in the original variables. We note that (2.6)–(2.7) hold in virtue of (2.35)–(2.36) so that

$$\begin{aligned}
\partial_\ell(\rho F_{\ell k}) &= \partial_\ell((\det \boldsymbol{F})^{-1} F_{\ell k}) \\
&= (\det \boldsymbol{F})^{-1}\partial_\ell F_{\ell k} - (\det \boldsymbol{F})^{-2} F_{\ell k} \frac{\partial \det \boldsymbol{F}}{\partial F_{ij}} \partial_\ell F_{ij} \\
&= (\det \boldsymbol{F})^{-1} \left(\partial_\ell F_{\ell k} - (\boldsymbol{F}^{-1})_{ji} F_{\ell k} \partial_\ell F_{ij} \right) \\
&= (\det \boldsymbol{F})^{-1} \left(\partial_\ell F_{\ell k} - (\boldsymbol{F}^{-1})_{ji} F_{\ell j} \partial_\ell F_{ik} \right) \\
&= (\det \boldsymbol{F})^{-1} \left(\partial_\ell F_{\ell k} - \delta_{\ell,i} \partial_\ell F_{ik} \right) = 0.
\end{aligned}$$

This completes the proof of Proposition 2.2.

References

1. ALINHAC, S.: Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Commun. Partial Differ. Eqs.* **14**, 173–230, 1989. https://doi.org/10.1080/03605308908820595

2. ALINHAC, S., GÉRARD, P.: *Pseudo-Differential Operators and the Nash-Moser Theorem*. Translated from the 1991 French original by Stephen S. Wilson. American Mathematical Society, Providence (2007). https://doi.org/10.1090/gsm/082

3. BENZONI-GAVAGE, S., SERRE, D.: Multidimensional Hyperbolic Partial Differential Equations: First-Order Systems and Applications. Oxford University Press, Oxford (2007). https://doi.org/10.1093/acprof:oso/9780199211234.001.0001

4. CHEN, G.-Q., FELDMAN, M.: *The Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures*. Princeton University Press, Princeton, NJ (2018). https://doi.org/10.2307/j.ctt1jktq4b

5. CHEN, G.-Q., SECCHI, P., WANG, T.: Nonlinear stability of relativistic vortex sheets in three-dimensional Minkowski spacetime. *Arch. Ration. Mech. Anal.* **232**, 591–695, 2019. https://doi.org/10.1007/s00205-018-1330-5

6. CHEN, G.-Q., WANG, Y.-G.: Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics. *Arch. Ration. Mech. Anal.* **187**, 369–408, 2008. https://doi.org/10.1007/s00205-007-0070-8

7. CHEN, G.-Q., WANG, Y.-G.: Characteristic discontinuities and free boundary problems for hyperbolic conservation laws. In: HOLDEN, H., KARLSEN, K.H. (eds.) *Nonlinear Partial Differential Equations*, pp. 53–81. Springer, Heidelberg 2012. https://doi.org/10.1007/978-3-642-25361-4

8. CHEN, R.M., HU, J., WANG, D.: Linear stability of compressible vortex sheets in two-dimensional elastodynamics. *Adv. Math.* **311**, 18–60, 2017. https://doi.org/10.1016/j.aim.2017.02.014

9. Chen, R.M., Hu, J., Wang, D.: Linear stability of compressible vortex sheets in 2D elastodynamics: variable coefficients. *Math. Ann.* **376**, 863–912, 2020. https://doi.org/10.1007/s00208-018-01798-w

10. Christoforou, C., Tzavaras, A.E.: Relative entropy for hyperbolic-parabolic systems and application to the constitutive theory of thermoviscoelasticity. *Arch. Ration. Mech. Anal.* **229**, 1–52, 2018. https://doi.org/10.1007/s00205-017-1212-2

11. Christoforou, C., Galanopoulou, M., Tzavaras, A.E.: A symmetrizable extension of polyconvex thermoelasticity and applications to zero-viscosity limits and weak-strong uniqueness. *Commun. Partial Differ. Eqs.* **43**, 1019–1050, 2018. https://doi.org/10.1080/03605302.2018.1456551

12. Christoforou, C., Galanopoulou, M., Tzavaras, A.E.: Measure-valued solutions for the equations of polyconvex adiabatic thermoelasticity. *Discrete Contin. Dyn. Syst.* **39**, 6175–6206, 2019. https://doi.org/10.3934/dcds.2019269

13. Ciarlet, P.G.: Mathematical Elasticity. Three-Dimensional Elasticity, Vol. I. North-Holland Publishing Co., Amsterdam (1988). https://www.sciencedirect.com/bookseries/studies-in-mathematics-and-its-applications/vol/20/suppl/C

14. Coleman, B.D., Noll, W.: The thermodynamics of elastic materials with heat conduction and viscosity. *Arch. Ration. Mech. Anal.* **13**, 167–178, 1963. https://doi.org/10.1007/BF01262690

15. Coulombel, J.-F., Secchi, P.: The stability of compressible vortex sheets in two space dimensions. *Indiana Univ. Math. J.* **53**, 941–1012, 2004. https://doi.org/10.1512/iumj.2004.53.2526

16. Coulombel, J.-F., Secchi, P.: Nonlinear compressible vortex sheets in two space dimensions. *Ann. Sci. Éc. Norm. Supér.* **4**(41), 85–139, 2008. https://doi.org/10.24033/asens.2064

17. Dafermos, C.M.: Quasilinear hyperbolic systems with involutions. *Arch. Ration. Mech. Anal.* **94**, 373–389, 1986. https://doi.org/10.1007/BF00280911

18. Dafermos, C.M.: *Hyperbolic Conservation Laws in Continuum Physics*, 4th edn. Springer, Berlin (2016). https://doi.org/10.1007/978-3-662-49451-6

19. Hu, X., Wang, D.: Global existence for the multi-dimensional compressible viscoelastic flows. *J. Differ. Eqs.* **250**, 1200–1231, 2011. https://doi.org/10.1016/j.jde.2010.10.017

20. Lei, Z., Liu, C., Zhou, Y.: Global solutions for incompressible viscoelastic fluids. *Arch. Ration. Mech. Anal.* **188**, 371–398, 2008. https://doi.org/10.1007/s00205-007-0089-x

21. Métivier, G.: Stability of multidimensional shocks. In: Freistühler, H., Szepessy, A. (eds.) *Advances in the Theory of Shock Waves*, pp. 25–103. Birkhäuser, Boston 2001. https://doi.org/10.1007/978-1-4612-0193-9

22. Morando, A., Trakhinin, Y., Trebeschi, P.: Well-posedness of the linearized problem for MHD contact discontinuities. *J. Differ. Eqs.* **258**, 2531–2571, 2015. https://doi.org/10.1016/j.jde.2014.12.018

23. Morando, A., Trakhinin, Y., Trebeschi, P.: Local existence of MHD contact discontinuities. *Arch. Ration. Mech. Anal.* **228**, 691–742, 2018. https://doi.org/10.1007/s00205-017-1203-3

24. Qian, J., Zhang, Z.: Global well-posedness for compressible viscoelastic fluids near equilibrium. *Arch. Ration. Mech. Anal.* **198**, 835–868, 2010. https://doi.org/10.1007/s00205-010-0351-5

25. Secchi, P.: Well-posedness of characteristic symmetric hyperbolic systems. *Arch. Ration. Mech. Anal.* **134**, 155–197, 1996. https://doi.org/10.1007/BF00379552

26. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970). https://www.jstor.org/stable/j.ctt1bpmb07

27. Tartar, L.: *An Introduction to Sobolev Spaces and Interpolation Spaces*. Springer, Berlin (2007). https://doi.org/10.1007/978-3-540-71483-5

28. Trakhinin, Y.: The existence of current-vortex sheets in ideal compressible magnetohydrodynamics. *Arch. Ration. Mech. Anal.* **191**, 245–310, 2009. https://doi.org/10.1007/s00205-008-0124-6

29. Trakhinin, Y.: Well-posedness of the free boundary problem in compressible elasto-dynamics. *J. Differ. Eqs*. **264**, 1661–1715, 2018. https://doi.org/10.1016/j.jde.2017.10.005

30. Truesdell, C., Toupin, R.: The classical field theories. With an appendix on tensor fields by J. L. Ericksen. In: Flügge, S. (ed.) Handbuch der Physik, Bd. III/1, pp. 226–793, appendix, pp. 794–858. Springer, Berlin (1960). https://doi.org/10.1007/978-3-642-45943-6

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