Cluster Synchronization of Two-Layer Networks via Aperiodically Intermittent Pinning Control

Weiying Li, Jin Zhou, Jinqiang Li, Tao Xie, and Jun-An Lu

***Abstract*—As is well known, complex networks in reality are usually interconnected. To describe the interconnected networks, multiplex networks were presented recently. In this brief, an aperiodically intermittent pinning control scheme is put forward to realize cluster exponential synchronization in a two-layer (leader-follower) network with multi-links and time-varying delays. Based on the principle of intermittent control and Lasalle invariant principle, some criteria for cluster synchronization are deduced by theoretical analysis. Numerical simulations are provided to verify the effectiveness of the proposed scheme. In comparison with the previous methods, our model better describes the real-world networks and our control effectively reduces the control cost.**

***Index Terms*—Complex network, multiplex network, multi-link, synchronization, control.**

I. Introduction

THERE exist numerous complex networks in the world, such as World Wide Web, social networks, neural networks, etc., [1]–[3]. In particular, attention has been focused on the control and synchronization of complex networks and fruitful results have been obtained [4]–[6].

In real-world, the existing single-layer network is insufficient to describe the characteristics. Multi-layer structure provides a more realistic framework for the study of complex networks. In particular, two-layer network as a special case of multi-layer network has been investigated by many researchers. For example, in [7], the authors investigated the consensus problem for layered multi-agent systems that consisted of a leader layer and a follower layer. Two-layer network is so common in life that the framework of a company can also be regarded as a two-layer network. The first layer is the head-quarters network which includes the technical department, the marketing department and the operation department as the clusters. The second layer is one of its branch network which also has the above departments as the corresponding clusters. Being nodes, persons in each layer communicate by multi-links, namely WeChat, phone and email. Due to the finite speed of information transmission and congestion, time-varying delays are also considered in the network model. To be efficient, the main-leader of each department is pinned to convey information intermittently.

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Intermittent control was first introduced by Żochowski [8] in 2000. In [9], the synchronization for a class of neural networks with discrete and distributed delays was studied via intermittent control. In [10], the authors investigated the exponential stabilization problem for a class of nonlinear systems by means of periodically intermittent control. Most of the previous researches focused on periodic intermittent control. While in practice, say, the generation of wind power is typically aperiodically intermittent. Thus aperiodically intermittent control has recently been proposed.

In [11], synchronization of complex dynamical networks with multi-links is considered. By using intermittent and impulsive control, the finite-time synchronization problem for multi-linked complex networks has been proposed in [12]. However, all of the aforementioned studies are not concentrated on the networks with both multi-layer structure and multi-links. Motivated by the above observations, this brief focuses on the cluster synchronization of two-layer networks with multi-links and time-varying delays via aperiodically intermittent control. In the presented network, the leader layer is in the absence of control. Based on the previous researches on multi-layer network [13]–[15], we have generalized the network model, in which the topologies of the sublayer networks are not restricted to be the same.

The rest of this brief is organized as follows. Section II introduces the network model. Some cluster synchronization criteria for multi-linked complex networks are obtained in Section III. Numerical simulations are presented in Section IV and some conclusions are finally drawn in Section V.

II. Problem Statement and Some Preliminaries

Consider a two-layer network with M nodes in the leader layer and N nodes in the follower layer. The leader layer contains r multi-linked leaders' clusters $C_1, C_2, \dots, C_r$ with $M_k - M_{k-1}$ nodes in each cluster C_k and the follower layer contains r matching multi-linked followers' clusters $D_1, D_2, \dots, D_r$ with $N_k - N_{k-1}$ nodes in each cluster D_k, where $k = 1, 2, \dots, r$. Set $M_0 = 0$, $M_r = M$, $N_0 = 0$, $N_r = N$. Obviously $\sum_{k=1}^{r}(M_k - M_{k-1}) = M$ and $\sum_{k=1}^{r}(N_k - N_{k-1}) = N$. There are m kinds of links among nodes in each cluster. The nodes subordinated to the same cluster pair have identical dynamics, while the ones subordinated to different unmatched cluster pair may behave non-identical. As an example, the network with $r = 2$ and $m = 2$ is depicted in Fig. 1.

The leader-follower network model is:

$$\dot{s}_i(t) = f_k(t, s_i(t), s_i(t-\tau_0)) + \sum_{j=1}^{M} a_{ij}^{(1)} \varphi_1(s_j(t))$$

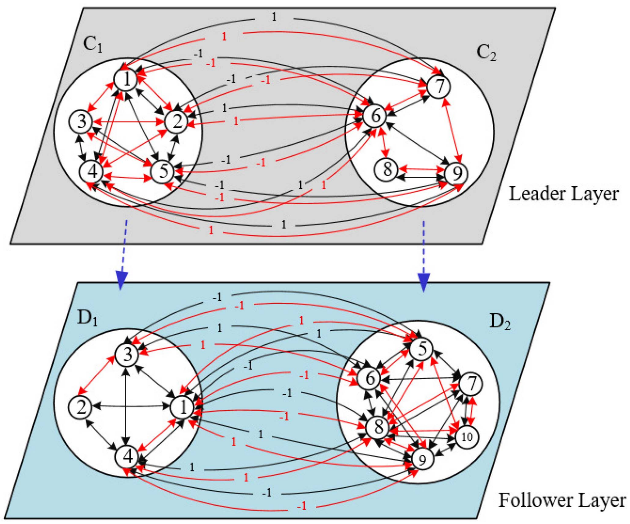

Fig. 1. The topology of the leader-follower network with cluster number $r = 2$ and multi-link number $m = 2$. The black line and red line represent two different kinds of links. The weights of the black lines and the red lines unmarked are defaulted as 10 and 0.1 respectively.

$$+\sum_{\iota=2}^{m}\sum_{j=1}^{M} a_{ij}^{(\iota)}\varphi_\iota(s_j(t-\tau_\iota))$$
$$i \in C_k,\ k = 1, 2, \ldots, r, \tag{1}$$

$$\dot{x}_i(t) = f_k(t, x_i(t), x_i(t-\tau_0)) + \sum_{j=1}^{N} b_{ij}^{(1)} g_1(x_j(t)) + \sum_{\iota=2}^{m}\sum_{j=1}^{N} b_{ij}^{(\iota)} g_\iota(x_j(t-\tau_\iota)) + u_i(x_i, s)$$
$$i \in D_k,\ k = 1, 2, \ldots, r, \tag{2}$$

where $s_i(t) = (s_{i1}(t), s_{i2}(t), \ldots, s_{in}(t))^T \in \mathbb{R}^n$ denotes the state vector of node i in the kth leaders' cluster and $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ denotes the state vector of node i in the kth followers' cluster. s is a function of $s_1, s_2, \ldots, s_M$. u_i is the controller to be designed. $f_k : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear vector function. $\tau_0(t)$ is the individual time-delay and $\tau_\iota(t)(2 \le \iota \le m)$ is the coupling time-delay. For $\iota = 1, 2, \ldots, m$, $\varphi_\iota(s_j(t)) = (\varphi_{\iota 1}(s_{j1}(t)), \varphi_{\iota 2}(s_{j2}(t)), \ldots, \varphi_{\iota n}(s_{jn}(t)))^T$ and $g_\iota(x_j(t)) = (g_{\iota 1}(x_{j1}(t)), g_{\iota 2}(x_{j2}(t)), \ldots, g_{\iota n}(x_{jn}(t)))^T$ are nonlinearly inner coupling functions. For simplicity, the outer couplings are considered as linear here, namely $a_{ij}^{(\iota)} = \sigma_\iota \hat{a}_{ij}^{(\iota)}, b_{ij}^{(\iota)} = \sigma_\iota \hat{b}_{ij}^{(\iota)}$. $\sigma_\iota > 0$ is the coupling strength. $A^{(\iota)} = (\hat{a}_{ij}^{(\iota)})_{M\times M}$ is the coupling matrix and if there is a link from node j to node i $(j \neq i)$, then $\hat{a}_{ij} \neq 0$; otherwise $\hat{a}_{ij} = 0$. $A^{(\iota)} = (\hat{a}_{ij}^{(\iota)})_{M\times M}$ is defined as the following form:

$$\begin{pmatrix} A_{11}^{(\iota)} & \cdots & A_{1r}^{(\iota)} \\ \vdots & \ddots & \vdots \\ A_{r1}^{(\iota)} & \cdots & A_{rr}^{(\iota)} \end{pmatrix}$$

where $A_{pq}^{(\iota)} \in R^{\mu\nu}$, μ and ν represent the number of nodes contained in the cluster C_p and C_q respectively. $\sum_{j\in C_k} a_{ij}^{(\iota)} = 0$ and $\sum_{j=1}^{M} a_{ij}^{(\iota)} = 0$. $B^{(\iota)} = (\hat{b}_{ij}^{(\iota)})_{N\times N}$ is the same definition as $A^{(\iota)} = (\hat{a}_{ij}^{(\iota)})_{M\times M}$ and both $A^{(\iota)}$ and $B^{(\iota)}$ are symmetric.

III. Main Results

A. The Synchronization of the Leader Layer

Define $\bar{s}^k(t) = \frac{1}{M_k - M_{k-1}}\sum_{i\in C_k} s_i(t)(k = 1, 2, \ldots, r)$ as the average state of leaders in the kth multi-linked leaders' cluster. Then we get

$$\dot{\bar{s}}^k(t) = \frac{1}{M_k - M_{k-1}}\sum_{i\in C_k} f_k(t, s_i(t), s_i(t-\tau_0)) + \frac{1}{M_k - M_{k-1}}\sum_{i\in C_k}\sum_{j=1}^{M} a_{ij}^{(1)}\varphi_1\big(s_j(t)\big) + \frac{1}{M_k - M_{k-1}}\sum_{i\in C_k}\sum_{\iota=2}^{m}\sum_{j=1}^{M} a_{ij}^{(\iota)}\varphi_\iota\big(s_j(t-\tau_\iota)\big).$$

Due to the symmetry of $A^{(\iota)}$, we have $\frac{1}{M_k - M_{k-1}}\sum_{i\in C_k}\sum_{j=1}^{M} a_{ij}^{(1)}\varphi_1(s_j(t)) = 0$ and $\frac{1}{M_k - M_{k-1}} \sum_{i\in C_k} \sum_{\iota=2}^{m} \sum_{j=1}^{M} a_{ij}^{(\iota)}\varphi_\iota\ (s_j(t-\tau_\iota)) = 0$. Thus

$$\dot{\bar{s}}^k(t) = \frac{1}{M_k - M_{k-1}}\sum_{i\in C_k} f_k(t, s_i(t), s_i(t-\tau_0)).$$

Denote $e_i^{sk}(t) = s_i^k(t) - \bar{s}^k(t)(k = 1, 2, \ldots, r)$, $\widetilde{\varphi}_1(e_j^{sk}(t)) = \varphi_1(s_j(t)) - \varphi_1(\bar{s}^k(t))$ and $\widetilde{\varphi}_\iota(e_j^{sk}(t-\tau_\iota)) = \varphi_\iota(s_j(t-\tau_\iota)) - \varphi_\iota(\bar{s}^k(t-\tau_\iota))$ for $2 \le \iota \le m$. We have

$$\dot{e}_i^{sk}(t) = \frac{1}{M_k - M_{k-1}}\sum_{j\in C_k}[f_k(t, s_i(t), s_i(t-\tau_0)) - f_k(t, s_j(t), s_j(t-\tau_0))] + \sum_{j=1}^{M} a_{ij}^{(1)}\widetilde{\varphi}_1\Big(e_j^{sk}(t)\Big) + \sum_{\iota=2}^{m}\sum_{j=1}^{M} a_{ij}^{(\iota)}\widetilde{\varphi}_\iota(e_j^{sk}(t-\tau_\iota)). \tag{3}$$

To achieve the synchronization of leaders in each cluster, the following definitions, assumptions and lemmas are proposed.

Definition 1: The leader layer is said to be cluster synchronized if for any initial conditions, $\|s_i(t) - s_j(t)\| \to 0$, for $t \geqslant 0$, $i, j \in C_k, i \neq j$, $k \in \{1, 2, \ldots, r\}$.

Assumption 1 (A1): Suppose that the time-varying delays $\tau_0(t), \tau_\iota(t)$ are differentiable and satisfy $\dot{\tau}_0(t) < \delta_0 < 1$, $\dot{\tau}_\iota(t) < \delta_\iota < 1$, $\tau = \max\{\tau_0(t), \tau_\iota(t)\}$ for any $t \in [0, +\infty)$, where δ_0 and δ_ι $(\iota = 2, \ldots, m)$ are positive constants.

Assumption 2 (A2): Suppose that the nonlinear vector function $f_k(t, x, \bar{x})$ satisfies $(x-y)^T(f_k(t, x, \bar{x}) - f_k(t, y, \bar{y})) \le \theta_k(x-y)^T(x-y) + \gamma_k(\bar{x}-\bar{y})^T(\bar{x}-\bar{y})$, for any $x, \bar{x}, y, \bar{y} \in \mathbb{R}^n$, where θ_k and γ_k are positive constants for $k = 1, \ldots, r$.

Assumption 3 (A3): There exist fixed positive constants $\alpha_\iota, \bar{\alpha}_\iota$ and $\beta_\iota, \bar{\beta}_\iota$ such that $|\varphi_\iota(x) - \varphi_\iota(y) - \alpha_\iota(x-y)| \le \beta_\iota|x-y|$ and $|g_\iota(x) - g_\iota(y) - \bar{\alpha}_\iota(x-y)| \le \bar{\beta}_\iota|x-y|$ for any $x, y \in \mathbb{R}^n$, $\iota = 1, 2, \ldots, m$.

Lemma 1 [16]: For any $x, y \in \mathbb{R}^n$, there exists a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ such that the inequality $2x^Ty \le x^TQx + y^TQ^{-1}y$ holds.

Based on the assumptions, the following theorem gives a criterion for reaching cluster synchronization of the leader layer.

Theorem 1: Assume that A1-A3 hold, the nodes in the multi-linked leaders' network (1) are said to be globally cluster synchronized, if there exist positive constants $\epsilon_\iota(\iota = 1, \ldots, m)$, such that the following conditions hold:

$$\begin{cases} \left(\theta + \frac{\gamma}{1-\delta_0} + \sum_{\iota=2}^{m} \frac{\sigma_\iota}{1-\delta_\iota}\right)U_I + \alpha_1\sigma_1 U_I A_I^{(1)} + \frac{\sigma_1\beta_1^2}{2\epsilon_1}P_I \\ +\frac{\sigma_1\epsilon_1}{2}U_I A_I^{(1)} {A_I^{(1)}}^T U_I + \sum_{\iota=2}^{m}\epsilon_\iota\sigma_\iota U_I A_I^{(\iota)}{A_I^{(\iota)}}^T U_I \leq 0 \\ \frac{{\alpha_\iota}^2+{\beta_\iota}^2}{2\epsilon_\iota}P_I - U_I \leq 0, \ \iota = 2, \ldots, m. \end{cases} \tag{4}$$

where $\theta = \max\{\theta_k | k = 1, 2, \ldots, r\}, \gamma = \max\{\gamma_k | k = 1, 2, \ldots, r\}$, $U = diag\{U_1, \ldots, U_r\}$, $U_k = E_k - \frac{1}{(M_k - M_{k-1})^2}\mathbf{1}\cdot\mathbf{1}^T$, $\mathbf{1} = (1, 1, \ldots, 1)^T$, $E_k = diag(\frac{1}{M_k - M_{k-1}})$, $P = diag\{P_1, \ldots, P_r\}$, $P_k = E_k^{-1}U_k$, $G_I = G \otimes I$.

Proof: Construct the following Lyapunov candidate for the error system (3)

$$V_1(t) = \frac{1}{2}(e^s(t))^T E_I e^s(t) + \frac{\gamma}{1-\delta_0}\int_{t-\tau_0}^{t}(e^s(\vartheta))^T E_I e^s(\vartheta)d\vartheta + \sum_{\iota=2}^{m}\frac{\sigma_\iota}{1-\delta_\iota}\int_{t-\tau_\iota}^{t}(e^s(\vartheta))^T E_I e^s(\vartheta)d\vartheta$$

where $e^s(t) = (e^{s1}(t)^T, e^{s2}(t)^T, \ldots, e^{sr}(t)^T)^T$ and $E = diag\{E_1, \ldots, E_r\}$, $E_k = diag(\frac{1}{M_k - M_{k-1}})$, $k = 1, 2, \ldots, r$.

$$\begin{aligned}\dot{V}_1(t) = &\sum_{k=1}^{r}\sum_{i\in C_k}\left(e_i^{sk}(t)\right)^T \frac{1}{(M_k - M_{k-1})^2}\sum_{j\in C_k}\Big[f_k(t, s_i(t), s_i(t-\tau_0)) \\ &-f_k\big(t, s_j(t), s_j(t-\tau_0)\big)\Big] \\ &+\sum_{k=1}^{r}\sum_{i\in C_k}\left(e_i^{sk}(t)\right)^T\frac{1}{M_k - M_{k-1}}\Big[\sum_{j=1}^{M}\sigma_1\hat{a}_{ij}^{(1)}\widetilde{\varphi}_1\left(e_j^{sk}(t)\right) \\ &+\sum_{\iota=2}^{m}\sum_{j=1}^{M}\sigma_\iota\hat{a}_{ij}^{(\iota)}\widetilde{\varphi}_\iota\left(e_j^{sk}(t-\tau_\iota)\right)\Big] \\ &+\frac{\gamma}{1-\delta_0}\sum_{k=1}^{r}\sum_{i\in C_k}\frac{1}{M_k - M_{k-1}}\Big[\left(e_i^{sk}(t)\right)^T e_i^{sk}(t) \\ &-(1-\dot{\tau}_0)\left(e_i^{sk}(t-\tau_0)\right)^T e_i^{sk}(t-\tau_0)\Big] \\ &+\sum_{\iota=2}^{m}\frac{\sigma_\iota}{1-\delta_\iota}\sum_{k=1}^{r}\sum_{i\in C_k}\frac{1}{M_k - M_{k-1}}\Big[\left(e_i^{sk}(t)\right)^T e_i^{sk}(t) \\ &-(1-\dot{\tau}_\iota)\left(e_i^{sk}(t-\tau_\iota)\right)^T e_i^{sk}(t-\tau_\iota)\Big)\Big].\end{aligned}$$

Denote $\theta = \max\{\theta_k | k = 1, 2, \ldots, r\}$, $\gamma = \max\{\gamma_k | k = 1, 2, \ldots, r\}$, $S(t) = (s_1(t)^T, \ldots, s_M(t)^T)^T$, $U = (u_{ij})_{M\times M} = diag\{U_1, \ldots, U_r\}$ and $U_k = E_k - \frac{1}{(M_k - M_{k-1})^2}\mathbf{1}\cdot\mathbf{1}^T$. Because of A2, we get

$$\begin{aligned}&\sum_{k=1}^{r}\sum_{i\in C_k}\left(e_i^{sk}(t)\right)^T\frac{1}{(M_k - M_{k-1})^2}\sum_{j\in C_k}\Big[f_k(t, s_i(t), s_i(t-\tau_0)) \\ &\quad - f_k\big(t, s_j(t), s_j(t-\tau_0)\big)\Big] \\ &= -\sum_{i>j}^{M}u_{ij}(s_i(t) - s_j(t))^T(f(s_i(t), s_i(t-\tau_0)) - f(s_j(t), s_j(t-\tau_0))) \\ &\leq \theta S(t)^T U_I S(t) + \gamma S(t-\tau_0)^T U_I S(t-\tau_0).\end{aligned}$$

Denote $P = diag\{P_1, \ldots, P_r\}$, $P_k = E_k^{-1}U_k$, $\Phi_\iota(S(t)) = (\varphi_\iota(s_1(t))^T, \varphi_\iota(s_2(t))^T, \ldots, \varphi_\iota(s_M(t))^T)^T$. Obviously, $A_I^{(\iota)}P = A_I^{(\iota)}$, $P^T P = P$. According to Lemma 1 and A3, we have

$$\begin{aligned}&\sum_{k=1}^{r}\sum_{i\in C_k}\left(e_i^{sk}(t)\right)^T\frac{1}{M_k - M_{k-1}}\Big[\sum_{j=1}^{M}\sigma_1\hat{a}_{ij}^{(1)}\widetilde{\varphi}_1\left(e_j^{sk}(t)\right) \\ &\quad+\sum_{\iota=2}^{m}\sum_{j=1}^{M}\sigma_\iota\hat{a}_{ij}^{(\iota)}\widetilde{\varphi}_\iota\left(e_j^{sk}(t-\tau_\iota)\right)\Big] \\ &= \sigma_1 S(t)^T U_I A_I^{(1)}\Phi_1(S(t)) + \sum_{\iota=2}^{m}\sigma_\iota S(t)^T U_I A_I^{(\iota)}\Phi_\iota(S(t-\tau_\iota)) \\ &= \sigma_1 S(t)^T U_I A_I^{(1)}(\Phi_1(S(t)) - \alpha_1 S(t)) + \alpha_1\sigma_1 S(t)^T U_I A_I^{(1)}S(t) \\ &\quad+\sum_{\iota=2}^{m}\sigma_\iota S(t)^T U_I A_I^{(\iota)}(\Phi_\iota(S(t-\tau_\iota)) - \alpha_\iota S(t-\tau_\iota)) \\ &\quad+\sum_{\iota=2}^{m}\alpha_\iota\sigma_\iota S(t)^T U_I A_I^{(\iota)}S(t-\tau_\iota) \\ &\leq S(t)^T(\frac{\epsilon_1\sigma_1}{2}U_I A_I^{(1)}{A_I^{(1)}}^T U_I + \frac{\sigma_1\beta_1^2}{2\epsilon_1}P_I + \sum_{\iota=2}^{m}\epsilon_\iota\sigma_\iota U_I A_I^{(\iota)}{A_I^{(\iota)}}^T U_I \\ &\quad+\alpha_1\sigma_1 U_I A_I^{(1)})S(t) + \sum_{\iota=2}^{m}\sigma_\iota S(t-\tau_\iota)^T\frac{{\alpha_\iota}^2 + {\beta_\iota}^2}{2\epsilon_\iota}P_I S(t-\tau_\iota).\end{aligned}$$

Thus from A1 and condition (4),

$$\begin{aligned}\dot{V}_1(t) \leq &S(t)^T\Big[(\theta + \frac{\gamma}{1-\delta_0} + \sum_{\iota=2}^{m}\frac{\sigma_\iota}{1-\delta_\iota})U_I + \alpha_1\sigma_1 U_I A_I^{(1)} + \frac{\sigma_1\beta_1^2}{2\epsilon_1}P_I \\ &+\frac{\epsilon_1\sigma_1}{2}U_I A_I^{(1)}{A_I^{(1)}}^T U_I + \sum_{\iota=2}^{m}\epsilon_\iota\sigma_\iota U_I A_I^{(\iota)}{A_I^{(\iota)}}^T U_I\Big]S(t) \\ &+\sum_{\iota=2}^{m}\sigma_\iota S(t-\tau_\iota)^T(\frac{{\alpha_\iota}^2 + {\beta_\iota}^2}{2\epsilon_\iota}P_I - U_I)S(t-\tau_\iota) \leq 0.\end{aligned}$$

By using Lasalle invariant principle, we obtain $\lim_{t\to\infty}\|s_i(t) - \bar{s}^k(t)\| = 0$, where $i \in C_k$, $k = 1, 2, \ldots, r$. Accordingly the leader layer network (1) realizes cluster synchronization. ■

B. The Synchronization of the Follower Layer

To synchronize the followers to its corresponding leaders, denote $\dot{\bar{s}}^k(t) = f_k(t, \bar{s}^k(t), \bar{s}^k(t-\tau_0))$. Without loss of generality, select the first node of each cluster to be controlled. The aperiodically intermittent controllers $u_i(x_i, s)$ in the follower layer network (2) are designed as follows.

$$u_i = \begin{cases} -\sum_{\iota=1}^{m}d_i^{(\iota)}\big(x_i(t) - \bar{s}^k(t)\big), & i = N_{k-1}+1, \ t \in (t_n, \chi_n]; \\ 0, & i = N_{k-1}+2, \ldots, N_k, \ t \in (t_n, \chi_n]; \\ 0, & i = N_{k-1}+1, \ldots, N_k, \ t \in (\chi_n, t_{n+1}], \end{cases} \tag{5}$$

$n \in N$, the control time sequence $\{t_n\}_{n=0}^{+\infty}$ satisfies $0 = t_0 < t_1 < \cdots < t_n < \cdots$, $\lim_{n\to\infty}t_n = +\infty$ and the control strength $d_i^{(\iota)} > 0$, $1 \leq \iota \leq m$, $i = N_{k-1}+1$, $k = 1, 2, \ldots, r$.

To achieve $x_1, \ldots, x_{N_1} \to \bar{s}^1$, $x_{N_1+1}, \ldots, x_{N_2} \to \bar{s}^2, \ldots$, $x_{N_{r-1}+1}, \ldots, x_{N_r} \to \bar{s}^r$, introduce the error signal of the kth cluster $e_i^k(t) = x_i(t) - \bar{s}^k(t)$. Denoting $\widetilde{g}_1(e_j^k(t)) = g_1(x_j(t)) - g_1(\bar{s}^k(t))$ and $\widetilde{g}_\iota(e_j^k(t-\tau_\iota)) = g_\iota(x_j(t-\tau_\iota)) - g_\iota(\bar{s}^k(t-\tau_\iota))(\iota = 2, \ldots, m)$, the error system is attained

$$\begin{aligned}\dot{e}_i^k(t) &= \dot{x}_i(t) - \dot{\bar{s}}^k(t)\\ &= f_k(t, x_i(t), x_i(t-\tau_0)) - f_k(t, \bar{s}^k(t), \bar{s}^k(t-\tau_0))\\ &\quad + \sum_{j=1}^N b_{ij}^{(1)} \widetilde{g}_1(e_j^k(t)) + \sum_{\iota=2}^m \sum_{j=1}^N b_{ij}^{(\iota)} \widetilde{g}_\iota(e_j^k(t-\tau_\iota))\\ &\quad + u_i(t)\ i \in D_k, \quad k = 1, 2, \ldots, r. \end{aligned} \tag{6}$$

Definition 2: Network (6) is said to be cluster exponentially synchronized if for some positive constants K, κ and any initial value $\pi \in \mathcal{C}([-\tau, 0]; \mathbb{R}^n)$, the following inequality is satisfied:

$$\lim_{t\to\infty} \sum_{k=1}^r \|e^k(t)\|^2 = \sum_{k=1}^r \lim_{t\to\infty} \sum_{i=N_{k-1}}^{N_k} \|x_i(t) - \bar{s}^k(t)\|^2 \le K e^{-\kappa t}.$$

Assumption 4 (A4) [17]: For the aperiodically intermittent control strategy, there exist two positive scalars $0 < \vartheta < \omega < +\infty$, such that for $n = 0, 1, 2, \ldots$

$$\begin{cases} \inf_n(\chi_n - t_n) = \vartheta \\ \sup_n(t_{n+1} - t_n) = \omega. \end{cases}$$

Lemma 2 [18]: Let $y(\cdot) : [t_0 - \tau, +\infty) \to [0, +\infty)$ be a continuous function such that $\dot{y}(t) < -ay(t) + b\bar{y}(t)$, where $a > b > 0$ and $\bar{y}(t) = \sup_{-\tau \le \kappa \le 0} y(t+\kappa)$, then $y(t) < \bar{y}(t_0)e^{-\eta(t-t_0)}$, where $\eta > 0$ is the unique solution of $\eta - a + be^{r\tau} = 0$.

Lemma 3 [18]: Let $y(\cdot) : [t_0 - \tau, +\infty) \to [0, +\infty)$ be a continuous function such that $\dot{y}(t) < ay(t) + b\bar{y}(t)$, then for $t \geqslant t_0$, $y(t) < \bar{y}(t_0)e^{(a+b)(t-t_0)}$, where $b > 0$, $a + b > 0$.

Lemma 4 [18]: Assume that A4 holds and the continuous and non-negative function $y(t)$ satisfies

$$\begin{cases} y(t) < \bar{y}(t_n)e^{-\gamma(t-t_n)}, & t_n \le t \le \chi_n \\ y(t) < \bar{y}(\chi_n)e^{\zeta(t-\chi_n)}, & \chi_n < t \le t_{n+1} \end{cases}, \quad n = 0, 1, 2..,$$

where t_n and χ_n are defined by the aperiodically intermittent controllers (5). Then $y(t) \le \bar{y}(0)e^{\rho}e^{-(\frac{\rho}{\omega})t}, t \geqslant 0$, being $\rho = \rho_1 - \rho_2 > 0$, $\rho_1 = \eta(\vartheta - \tau)$ and $\rho_2 = \zeta(\omega - \vartheta)$.

Introducing denotations $D^{(\iota)} = diag\{d_1^{(\iota)}, 0, \ldots, 0, d_{N_{k-1}+1}^{(\iota)}, 0, \ldots, 0, d_{N_{r-1}+1}^{(\iota)}, 0, \ldots, 0\}$, $\iota = 1, 2, \ldots, m$, we obtain a criterion for reaching cluster synchronization of the follower layer.

Theorem 2: Suppose that A1-A4 hold. If there exist positive constants Ξ and $\bar{\epsilon}_\iota(\iota = 1, \ldots, m)$ such that

$$\begin{cases} (2\theta + \Xi) \otimes I_{Nn} + 2\bar{\alpha}_1\sigma_1 B_I^{(1)} + \frac{\sigma_1\bar{\beta}_1^2}{\bar{\epsilon}_1}\bar{E}_I^{-1} + \bar{\epsilon}_1\sigma_1 B_I^{(1)} {B_I^{(1)}}^T \bar{U}_I \\ + \sum_{\iota=2}^m 2\bar{\epsilon}_\iota\sigma_\iota B_I^{(\iota)} {B_I^{(\iota)}}^T \bar{U}_I - 2\sum_{\iota=1}^m D_I^{(\iota)} < 0, \\ \rho = \eta(\vartheta - \tau) - (\phi + q)(\omega - \vartheta) > 0, \\ \Xi > q, \phi + q > 0, \end{cases} \tag{7}$$

where $q = 2\gamma + \sum_{\iota=2}^m \frac{\sigma_\iota(\bar{\alpha}_\iota^2 + \bar{\beta}_\iota^2)}{\bar{\epsilon}_\iota}\lambda_{max}(\bar{E}^{-1})$, $\phi = \lambda_{max}(2\theta I_{Nn} + 2\bar{\alpha}_1\sigma_1 B_I^{(1)} + \frac{\sigma_1\bar{\beta}_1^2}{\bar{\epsilon}_1}\bar{E}_I^{-1} + \bar{\epsilon}_1\sigma_1 B_I^{(1)} {B_I^{(1)}}^T \bar{U}_I + \sum_{\iota=2}^m 2\bar{\epsilon}_\iota\sigma_\iota B_I^{(\iota)} {B_I^{(\iota)}}^T \bar{U}_I)$, and $\eta > 0$ the unique solution of $\eta - \Xi + qe^{r\tau} = 0$, then the error system (6) is globally exponentially stabilized via aperiodically intermittent pinning controllers (5).

Proof: Construct the following Lyapunov candidate

$$V_2(t) = e(t)^T \bar{E}_I e(t) = \sum_{k=1}^r e^k(t)^T \bar{E}_k e^k(t) = X(t)^T \bar{U}_I X(t),$$

where $e(t) = (e^1(t)^T, e^2(t)^T, \ldots, e^r(t)^T)^T$, $t \in (t_n, \chi_n]$. $X(t) = (x_1(t)^T, \ldots, x_N(t)^T)^T$, $\bar{E}_k = diag(\frac{1}{N_k - N_{k-1}})$, $\bar{U}_k = \bar{E}_k - \frac{1}{(N_k - N_{k-1})^2}\mathbf{1}\cdot\mathbf{1}^T$. By (6), we have

$$\begin{aligned}\dot{V}_2(t) = 2\sum_{k=1}^r \sum_{i\in D_k} e_i^k(t)^T \frac{1}{N_k - N_{k-1}} \Big[& f_k(t, x_i(t), x_i(t-\tau_0)) \\ & - f_k(t, \bar{s}^k(t), \bar{s}^k(t-\tau_0)) + \sum_{j=1}^N \sigma_1 \hat{b}_{ij}^{(1)} \widetilde{g}_1(e_j^k(t)) \\ & + \sum_{\iota=2}^m \sum_{j=1}^N \sigma_\iota \hat{b}_{ij}^{(\iota)} \widetilde{g}_\iota(e_j^k(t-\tau_\iota)) - \sum_{\iota=1}^m d_i^{(\iota)} e_i^k(t) \Big]. \end{aligned}$$

Denote $\mathbf{g}_\iota(X(t)) = (g_\iota(x_1(t))^T, \ldots, g_\iota(x_N(t))^T)^T(\iota = 1, \ldots, m)$. From A2, A3 and Lemma 1, we have

$$\begin{aligned}\dot{V}_2(t) &\le 2\theta \sum_{k=1}^r e^k(t)^T \bar{E}_k e^k(t) + 2\gamma \sum_{k=1}^r e^k(t-\tau_0)^T \bar{E}_k e^k(t-\tau_0)\\ &\quad + 2\sigma_1 X(t)^T \bar{U}_I B_I^{(1)} \mathbf{g}_1(X(t)) - 2X(t)^T \bar{U}_I \sum_{\iota=1}^m D_I^{(\iota)} X(t)\\ &\quad + 2\sum_{\iota=2}^m \sigma_\iota X(t)^T \bar{U}_I B_I^{(\iota)} \mathbf{g}_\iota(X(t-\tau_\iota))\\ &\le X(t)^T \bar{U}_I [2\theta I_{Nn} + 2\bar{\alpha}_1\sigma_1 B_I^{(1)} + \frac{\sigma_1\bar{\beta}_1^2}{\bar{\epsilon}_1}\bar{E}^{-1} + \bar{\epsilon}_1\sigma_1 B_I^{(1)} {B_I^{(1)}}^T \bar{U}_I\\ &\quad + \sum_{\iota=2}^m 2\bar{\epsilon}_\iota\sigma_\iota B_I^{(\iota)} {B_I^{(\iota)}}^T \bar{U}_I - 2\sum_{\iota=1}^m D_I^{(\iota)}] X(t)\\ &\quad + 2\gamma X^T(t-\tau_0)\bar{U}_I X(t-\tau_0)\\ &\quad + X(t-\tau_\iota)^T \bar{U}_I \sum_{\iota=2}^m \frac{\sigma_\iota(\bar{\alpha}_\iota^2 + \bar{\beta}_\iota^2)}{\bar{\epsilon}_\iota} \bar{E}_I^{-1} X(t-\tau_\iota). \end{aligned}$$

According to conditions (7), we get

$$\begin{aligned}\dot{V}_2(t) &\le e(t)^T \bar{E}_I \Big[(2\theta + \Xi) \otimes I_{Nn} + 2\bar{\alpha}_1\sigma_1 B_I^{(1)} + \frac{\sigma_1\bar{\beta}_1^2}{\bar{\epsilon}_1}\bar{E}_I^{-1}\\ &\quad + \bar{\epsilon}_1\sigma_1 B_I^{(1)} {B_I^{(1)}}^T \bar{U}_I + \sum_{\iota=2}^m 2\bar{\epsilon}_\iota\sigma_\iota B_I^{(\iota)} {B_I^{(\iota)}}^T \bar{U}_I\\ &\quad - 2\sum_{\iota=1}^m D_I^{(\iota)} \Big] e(t) - \Xi V_2(t)\\ &\quad + \Big[2\gamma + \sum_{\iota=2}^m \frac{\sigma_\iota(\bar{\alpha}_\iota^2 + \bar{\beta}_\iota^2)}{\bar{\epsilon}_\iota} \lambda_{max}(\bar{E}_I^{-1})\Big] \Big(\sup_{t-\tau\le\chi\le t} V_2(\chi) \Big)\\ &\le -\Xi V_2(t) + q\Big(\sup_{t-\tau\le\chi\le t} V_2(\chi) \Big), \quad t_n \le t \le \chi_n. \end{aligned}$$

By Lemma 2, we have

$$V_2(t) \le \Big(\sup_{t_n-\tau\le\chi\le t_n} V_2(\chi) \Big) e^{-\eta(t-t_0)}, \quad t_n \le t \le \chi_n,$$

where $\eta > 0$ is the unique solution of $\eta - \Xi + qe^{r\tau} = 0$.

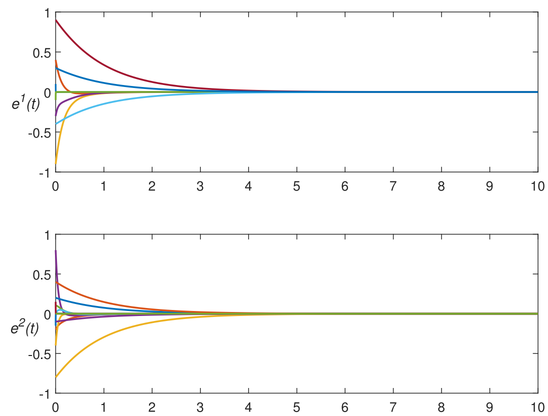

Fig. 2. Synchronization errors $e^1(t)$ and $e^2(t)$ in the first and second cluster pair versus time t. By applying the proposed scheme, the leaders and the followers of the two cluster pairs achieve cluster synchronization briefly.

When $t \in (\chi_n, t_{n+1}]$. Similar to the case of $t \in (t_n, \chi_n]$, we get

$$\begin{aligned}\dot{V}_2(t) &\le \lambda_{max}\Bigg(2\theta I_{Nn} + 2\bar{\alpha}_1\sigma_1 B_I^{(1)} + \frac{\sigma_1\bar{\beta}_1^2}{\bar{\epsilon}_1}\bar{E}_I^{-1} + \bar{\epsilon}_1\sigma_1 B_I^{(1)} B_I^{(1)^T}\bar{U}_I \\ &\quad + \sum_{\iota=2}^{m} 2\bar{\epsilon}_\iota\sigma_\iota B_I^{(\iota)} B_I^{(\iota)^T}\bar{U}_I\Bigg)V_2(t) + q\Bigg(\sup_{t-\tau\le\chi\le t} V_2(\chi)\Bigg) \\ &\le \phi V_2(t) + q\Bigg(\sup_{t-\tau\le\chi\le t} V_2(\chi)\Bigg), \quad \chi_n \le t \le t_{n+1}.\end{aligned}$$

According to Lemma 3, we have

$$V_2(t) \le \Bigg(\sup_{\chi_n-\tau\le\chi\le\chi_n} V_2(\chi)\Bigg)e^{(\phi+q)(t-t_0)}.$$

for $t \in (\chi_n, t_{n+1}]$. From Lemma 4, we get that

$$V_2(t) \le \Bigg(\sup_{-\tau\le\chi\le 0} V_2(\chi)\Bigg)e^{\rho} * e^{-(\frac{\rho}{\omega})t}, \quad t \geqslant 0,$$

where $\rho = r(\vartheta - \tau) - (\phi + q)(\omega - \vartheta) > 0$. Thus $\sum_{k=1}^{r} \|e^k(t)\|^2 \le Ke^{-\kappa t}$, where $K = e^{\rho}\sup_{-\tau\le\chi\le 0} V_2(\chi)$, $\kappa = \frac{\rho}{\omega} > 0$. By Definition 2, the network (2) is exponentially cluster-synchronized. ■

Remark: If the inner couplings are simplified as linear functions, the parameters are $\alpha_\iota = \bar{\alpha}_\iota = 1$ and $\beta_\iota = \bar{\beta}_\iota = 0$, $\iota = 1, 2, \ldots, m$. Theorem 1 and Theorem 2 are then simplified greatly.

IV. Numerical Simulation

In this section, an example is employed to show the effectiveness of the theorems. We consider a two-layer network with two-links, as shown in Fig. 1. The first layer consists of nine leaders which is divided into two clusters. The second layer consists of ten followers which is also divided into two clusters.

Consider the node dynamics as $f_k(x) = 0.1C_k x(t) + 0.1A_k \tanh(x(t)) + 0.1B_k \tanh(x(t - \tau_0(t)))(k = 1, 2)$, where

$$C_1 = C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0.2 & -0.1 \\ -0.5 & 0.45 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.1 & -0.1 \\ -0.4 & 0.4 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -0.4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.5 & -0.1 \\ -0.1 & -0.4 \end{pmatrix}$$

and the time-varying delays are selected as $\tau_0(t) = 0.01\frac{e^{2t}}{e^{2t}+2}$, $\tau_2(t) = 0.01\frac{e^t}{e^t+1}$. The inner coupling functions are $\varphi_1(x(t)) = g_1(x(t)) = (3x_1(t), 0))^T$ and $\varphi_2(x(t - \tau_2(t))) = g_2(x(t - \tau_2(t))) = (0.01x_1(t - \tau_2(t))), 0)^T$. Set $\theta = 0.085$, $\gamma = 0.25$, $(\alpha_1, \beta_1) = (\bar{\alpha}_1, \bar{\beta}_1) = (3, 1)$, $(\alpha_2, \beta_2) = (\bar{\alpha}_2, \bar{\beta}_2) = (0.01, 0)$, $\sigma_1 = 1$, $\sigma_2 = 0.12$, $\bar{\epsilon}_1 = \bar{\epsilon}_2 = \epsilon_1 = \epsilon_2 = 0.01$, $D^{(\iota)} = \mathrm{diag}\{500, 0, 0, 0, 500, 0, 0, 0, 0, 0\}(\iota = 1, 2)$ and $\Xi = 1$. From Fig. 2, it is seen that the cluster synchronization of the two-layer network is realized.

V. Conclusion

The exponential cluster synchronization of two-layer complex networks with multi-links and time-varying delays has been investigated in this brief. Based on Lasalle invariant principle and Lyapunov stability theory, novel cluster synchronization criteria of the two-layer multi-linked complex networks have been proposed. Numerical simulations have shown that the approach presented in this brief are of good performance. Compared to the similar work, the proposed model is more realistic and the method is less conservative. Future work includes the synchronous region and the control optimization of the proposed network model.

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