# On Applicability of Auxiliary System Approach to Detect Generalized Synchronization in Complex Network

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***Abstract*—Generalized synchronization is ubiquitous in nature. It is well known that the auxiliary system approach has been widely used to verify the presence of generalized synchronization. This approach was firstly proposed in a drive-response system, then extended to the bidirectionally coupled systems and complex networks. However, the well-known generalized auxiliary system method lacks a rigorous theoretical basis for its various applications. Two recent counterexamples indicate us the inapplicability of this method. Inspired by the counterexamples, we find that it is interesting to ask the following two fundamental questions: i) Why is the generalized auxiliary system approach unworkable in the networks with bidirectional couplings? ii) Are there any essential conditions for the applications of this approach? This technical note aims at establishing a rigorous theoretical basis for the applicability of auxiliary system approach. That is, the generalized auxiliary system approach is effective only if there does not exist any path from nodes to their driving neighbors (who drive these nodes) in a network. Several representative examples are also given to validate our theoretical results.**

***Index Terms*—Auxiliary system approach, complex networks, generalized synchronization.**

## I. Introduction

Synchronization is a specific kind of collective behaviors in nature. Historically, the research of synchronization can track back to the Huygens pendulum clocks in the 17th century. Among various kinds of synchronization in complex networks, the identical synchronization (IS) has been extensively investigated, in which the differences among their states converge to zero as time approaches infinity [1]–[10].

The generalized synchronization (GS) is another kind of typical synchronization, where there exists a functional relationship among interacting nodes [11]–[21], [25]. GS was first investigated between two unidirectionally coupled systems in 1995 by Rulkov *et al.*, who presented a mutual false nearest neighbors method to determine the presence of the functional relation between the states of systems [13]. Compared to the numerical method, later in 1996, Ref. [14] proposed a mathematical approach to detect the occurrence of GS. At the same time, Ref. [15] illustrated the theoretical basis behind the approach. This effective and simple mathematical approach, which is called auxiliary system method thereafter, is quite prevalent to detect the presence of functional relation when GS is discussed. In 2002, Ref. [16] extended the auxiliary system method to bidirectionally coupled systems and complex networks by introducing auxiliary node system for each original node. Later on, the technique has been extensively applied to determine the occurrence of GS in bidirectional complex networks [17]–[20].

Without rigorous certification, however, the extended auxiliary system method has been widely used. Very recently, Ref. [21] gives counterexamples to verify the inapplicability of this method in bidirectionally coupled systems and complex networks. These discoveries raise intriguing questions: Why is the auxiliary system approach unworkable in networks with bidirectional couplings? Are there any conditions for this technique in applications? Motivated by the questions, we explain the in-depth reason of this inapplicability in general complex networks from a theoretical angle. We conclude that the generalized auxiliary system approach is effective only when there is no path from nodes to their driving neighbors (who drive the nodes) in the network. In this technical note, we introduce a definition of GS of a complex network; present a condition under which the auxiliary system method can be applied to complex networks; give some remarks and two examples to illustrate our presented results.

The rest of this technical note is organized as follows. Some preliminaries and the network model are introduced in Section II. In Section III, the main results and some helpful remarks are presented. Generalized auxiliary system method is recalled in Section IV. Examples and further illustration are provided in Section V. Finally, concluding remarks are given in Section VI.

## II. Preliminaries

### A. Mathematical Preliminaries

To begin with, some necessary definitions are listed below.

*Definition 1 [22]:* Denote $V(G)$ and $\vec{E}(G)$ as the vertex set and the directed edge set of a directed graph $G$, respectively. Let $\xi$ and $\zeta$ be two not necessarily different vertices of $G$. By a $\xi\zeta$ *walk* we mean an alternating sequence of vertices and edges, say $\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_l, \eta_l, \xi_{l+1}$, such that $\xi_1 = \xi$, $\xi_{l+1} = \zeta$ and $\eta_i = \xi_i\,\xi_{i+1} \in \vec{E}(G)$, $1 \le i \le l$. A walk is called a *path* or $\xi\zeta$ *path* if all its vertices are distinct. The *distance between two vertices* $\xi$ and $\zeta$, denoted by $d(\xi, \zeta)$, is the minimum length of a $\xi\zeta$ path.

*Definition 2 [23]:* Let $f$ be a smooth vector field on a manifold $\mathscr{M}$. For each $p \in \mathscr{M}$, there exists an open interval—depending on $p$ and written $\mathcal{U}_p$—of $\mathbb{R}$ such that $0 \in \mathcal{U}_p$ and a smooth mapping

$$\varphi : \mathscr{W} \to \mathscr{M}$$

defined on the subset $\mathscr{W}$ of $\mathbb{R} \times \mathscr{M}$

$$\mathscr{W} = \{(t, p) \in \mathbb{R} \times \mathscr{M} : t \in \mathcal{U}_p\}$$

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with the following properties

1) $\varphi(0,p) = p$,
2) for each $p$ the mapping $\sigma_p : \mathcal{U}_p \to \mathscr{M}$ defined by
$$\sigma_p(t) = \varphi(t,p)$$
is an integral curve of $f$,
3) if $\mu : (t_1, t_2) \to \mathscr{M}$ is another integral curve of $f$ satisfying the condition $\mu(0) = p$, then $(t_1, t_2) \subset \mathcal{U}_p$ and the restriction of $\sigma_p$ to $(t_1, t_2)$ coincides with $\mu$,
4) $\varphi(s, \varphi(t,p)) = \varphi(s+t, p)$ whenever both sides are defined,
5) whenever $\varphi(t,p)$ is defined, there exists an open neighborhood $U$ of $p$ such that the mapping $\varphi^t : U \to \mathscr{M}$ defined by
$$\varphi^t(q) = \varphi(t,q)$$
is a diffeomorphism onto its image, and
$$(\varphi^{-1})^t = \varphi^{-t}.$$

The mapping $\varphi^t$ is called the *flow* of $f$.

Properties 1)–3) tell us that $\sigma_p$ is a unique integral curve of $f$ passing through $p$ at $t = 0$. Properties 4) and 5) say that the family of mappings $\{\varphi^t\}$ is group of local diffeomorphisms with parameter $t$, under the operation of composition. If the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ posses a unique solution $\mathbf{x}(t)$ for a given initial value $\mathbf{x}(t_0)$, then all possible motions are flows [24].

### B. The Network Model and Relevant Concepts

Generally, a complex network model is described by
$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) - \sum_{j \in \mathcal{N}_i} c_{ij}\, \mathbf{H}_i(\mathbf{x}_i - \mathbf{x}_j), \qquad 1 \le i \le N \tag{1}$$
where $\mathcal{N}_i = \{j \mid (j,i) \in \vec{E}(G)\}$ is the set of node $i$'s driving neighborhood, the state vector of the $i$-th node $\mathbf{x}_i \in \mathbb{R}^n$ is a smooth curve, $\mathbf{f}_i : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear vector field, individual node dynamics is $\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i)$, $\mathbf{H}_i \in \mathbb{R}^{n\times n}$ is the inner-coupling matrix of node $i$. The outer-coupling weight configuration matrix is $\mathbf{C} = (c_{ij}) \in \mathbb{R}^{N\times N}$, being $c_{ij} > 0$ the weight from node $j \in \mathcal{N}_i$ to node $i$ and $c_{ij} = 0$ otherwise.

To investigate GS of network (1), a definition of generalized synchronization of complex networks is introduced.

*Definition 3:* If there exists a node $i$ (that is not a 0 in-degree node) in network (1), a subset $B = B_1 \times B_2 \cdots \times B_N \subset \mathbb{R}^n \times \mathbb{R}^n \cdots \times \mathbb{R}^n$, a smooth mapping $\Phi$ and a manifold
$$\mathcal{M} = \{(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \mid \mathbf{x}_i = \Phi(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_N)\}, \mathcal{M} \subset B,$$
such that trajectory of network (1) with any initial states approaches $\mathcal{M}$ when $t$ tends to $+\infty$, network (1) is said to realize *generalized synchronization* (GS), $\mathcal{M}$ is called *GS manifold*.

Definition 3 reveals that $\mathcal{M}$ is an attracting invariant manifold. On the GS manifold $\mathcal{M}$, the existence of the smooth mapping $\Phi$ guarantees that one can predict $\mathbf{x}_i$ by using $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_N$. It is noted that the definition of GS of a network proposed in [25] is an implicit form, which is also a broadened definition. To investigate the theory of auxiliary system method, the explicit form is taken into account.

*Definition 4:* Let $\mathbf{x}^1(t)$ and $\mathbf{x}^2(t)$ be two solutions of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$. If there exists a region $B_{\mathbf{x}} \subseteq D_{\mathbf{x}}$, where $D_{\mathbf{x}}$ is the domain of $\mathbf{x}$, such that for any initial value $\mathbf{x}^1(t_0), \mathbf{x}^2(t_0) \in B_{\mathbf{x}}$ we have $\lim_{t\to+\infty} [\mathbf{x}^1(t) - \mathbf{x}^2(t)] = \mathbf{0}$, the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ is said to be *asymptotically stable*.

From Definition 4, the $i$-th node's state
$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) - \sum_{j \in \mathcal{N}_i} c_{ij}\, \mathbf{H}_i(\mathbf{x}_i - \mathbf{x}_j) \tag{2}$$
is asymptotically stable iff $\forall\, \mathbf{x}_i^1(t_0), \mathbf{x}_i^2(t_0) \in B_i$, one has
$$\lim_{t\to+\infty} \|\mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^1(t_0), \mathbf{x}_{i+1}(t_0), \ldots, \mathbf{x}_N(t_0)) - \mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^2(t_0), \mathbf{x}_{i+1}(t_0), \ldots, \mathbf{x}_N(t_0))\| = 0$$
where $\|\cdot\|$ is any norm of a vector.

## III. Main Results

Next, we will discuss why the auxiliary system approach is inapplicable to bidirectionally coupled networks. Although [21] gives counterexamples in terms of delay synchronization (an IS counterexample is exhibited in Section V), rigorous justification remains blank. The following theorem gives a theoretical answer to this question in the sense of Definition 3.

*Theorem 1:* Assume that there exists a node $i$ satisfying $d(i, i_k) = \infty\ (\forall i_k \in \mathcal{N}_i)$. Network (1) achieves GS iff system (2) is asymptotically stable.

*Proof "⇒":* If network (1) achieves GS, there exists a node $i$ and a smooth mapping $\Phi$ satisfying
$$\forall\, \mathbf{x}_i(t_0) \in B_i, \forall \epsilon > 0$$
$\exists T > 0$, s.t. $t \ge T$, and
$$\|\mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_N(t_0)) - \Phi(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_N)\| < \frac{\epsilon}{2}.$$

Then, for $\forall\, \mathbf{x}_i^1(t_0), \mathbf{x}_i^2(t_0) \in B_i$, one has
$$\begin{aligned}
&\|\mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^1(t_0), \mathbf{x}_{i+1}(t_0), \ldots, \mathbf{x}_N(t_0)) - \mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^2(t_0), \mathbf{x}_{i+1}(t_0), \ldots, \mathbf{x}_N(t_0))\| \\
&\le \|\mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^1(t_0), \mathbf{x}_{i+1}(t_0), \ldots, \mathbf{x}_N(t_0)) - \Phi(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_N)\| \\
&\quad + \|\Phi(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_N) - \mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^2(t_0), \mathbf{x}_{i+1}(t_0), \ldots, \mathbf{x}_N(t_0))\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Namely, $\forall\, \mathbf{x}_i^1(t_0), \mathbf{x}_i^2(t_0) \in B_i$, one obtains
$$\lim_{t\to+\infty} \|\mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^1(t_0), \mathbf{x}_{i+1}(t_0), \ldots, \mathbf{x}_N(t_0)) - \mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^2(t_0), \mathbf{x}_{i+1}(t_0), \ldots, \mathbf{x}_N(t_0))\| = 0.$$

"⇐": The properties of flow are applied in this part. Suppose that system (2) is asymptotically stable. Then $\forall\, \mathbf{x}_i^1(t_0), \mathbf{x}_i^2(t_0) \in B_i, \forall \epsilon > 0$,

$\exists T > 0$, s.t. $t \geq T$, and

$$\begin{aligned}&\| \mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^1(t_0), \mathbf{x}_{i+1}(t_0), \ldots,\\&\mathbf{x}_N(t_0)) - \mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^2(t_0),\\&\mathbf{x}_{i+1}(t_0), \ldots, \mathbf{x}_N(t_0)) \|\\&< \epsilon .\end{aligned}$$

Assume that the number of elements in $\mathcal{N}_i$ is $m$. Let $\Phi^t_{\mathcal{N}_i}$ be the flow of

$$\dot{\mathbf{x}}_{i_k} = \mathbf{f}_{i_k}(\mathbf{x}_{i_k}) - \sum_{j\in\mathcal{N}_{i_k}} c_{i_k,j}\, \mathbf{H}_{i_k}(\mathbf{x}_{i_k} - \mathbf{x}_j),\ 1 \leq k \leq m \tag{3}$$

where $t \in \mathbb{R}$, $i_k \in \mathcal{N}_i$, $\Phi^t_i$ the flow of (2), $\Phi^t = (\Phi^t_{\mathcal{N}_i}, \Phi^t_i)$ the flow of the inter-correlated systems (3) and (2). Then the trajectory starting at $(\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T))), \mathbf{x}_i(t_0))$ passes

$$\begin{aligned}&\big((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T)), \Phi^t_i(\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T),\\&\ldots, \mathbf{x}_{i_m}(T))), \mathbf{x}_i(t_0))\big) .\end{aligned}$$

Since system (2) is asymptotically stable, for the above-mentioned $\epsilon$ and $\forall\, \mathbf{x}_i^1(t_0), \mathbf{x}_i^2(t_0) \in B_i$,

one gets

$$\begin{aligned}&\| \Phi^t_i(\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T))), \mathbf{x}_i^1(t_0))\\&- \Phi^t_i(\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T))), \mathbf{x}_i^2(t_0)) \|\\&< \epsilon .\end{aligned}$$

For the arbitrariness of $\mathbf{x}_i^1(t_0)$ and $\mathbf{x}_i^2(t_0)$, the flow of (2) depends only on

$$\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T))) .$$

From $d(i, i_k) = \infty\, (\forall i_k \in \mathcal{N}_i)$, we claim that

$$\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T)))$$

is independent on the initial state $\mathbf{x}_i(t_0)$. Thus the flow of (2) is not tied to the initial state $\mathbf{x}_i(t_0)$. Then we can define the transform for GS as

$$\begin{aligned}&\Phi\ (\mathbf{x}_1(T), \ldots, \mathbf{x}_{i-1}(T), \mathbf{x}_{i+1}(T), \ldots, \mathbf{x}_N(T))\\&\triangleq \tilde{\Phi}\,(\mathbf{x}_1(T), \ldots, \mathbf{x}_{i-1}(T), \mathbf{x}_i(T), \mathbf{x}_{i+1}(T), \ldots, \mathbf{x}_N(T))\\&= \lim_{t\to+\infty} \Phi^t_i(\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T))), \mathbf{x}_i(t_0))\\&= \mathbf{x}_i(T) .\end{aligned}$$

Therefore, for the above-mentioned $\epsilon$ and $T$, when $t \geq T$

$$\|\mathbf{x}_i(t) - \Phi(\mathbf{x}_1(t), \ldots, \mathbf{x}_{i-1}(t), \mathbf{x}_{i+1}(t), \ldots, \mathbf{x}_N(t))\| < \epsilon.$$

In other words, network (1) reaches GS. ■

*Remark 1:* For one thing, GS of network (1) requires an attracting GS manifold $\mathcal{M}$ in which

$$\mathbf{x}_i = \Phi(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_N)$$

holds for any initial state

$$(\mathbf{x}_1(t_0), \mathbf{x}_2(t_0), \ldots, \mathbf{x}_N(t_0)) \in B .$$

That is to say, the trajectory

$$\Phi^t_i(\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T))), \mathbf{x}_i(t_0))$$

is independent on $\mathbf{x}_i(t_0)$. For another, system (2) is asymptotically stable iff the aforementioned trajectory

$$\Phi^t_i(\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T))), \mathbf{x}_i(t_0))$$

depends only on

$$\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T))) ,$$

which is not necessarily independent on $\mathbf{x}_i(t_0)$. Therefore, the GS of network (1) and the asymptotic stability of system (2) are not the same thing. The latter generally involves the possibility that the trajectory

$$\Phi^t_i(\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T))), \mathbf{x}_i(t_0))$$

depends on $\mathbf{x}_i(t_0)$, unless

$$\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T)))$$

is independent on $\mathbf{x}_i(t_0)$. The independence of

$$\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T)))$$

on $\mathbf{x}_i(t_0)$ is equivalent to

$$d(i, i_k) = \infty\, (\forall i_k \in \mathcal{N}_i) .$$

More details will be given in Remark 2. As a result, GS of network (1) implies asymptotic stability of system (2); asymptotic stability of system (2), together with $d(i, i_k) = \infty\, (\forall i_k \in \mathcal{N}_i)$, leads to GS of network (1).

*Remark 2:* Recall Remark 1, network (1) achieves GS only when $\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T)))$ is independent on the initial value of $\mathbf{x}_i$. In fact, $d(i, i_k) < \infty$ means a path existing from node $i$ to $i_k$, and then the solution of system (3) depends on system (2), accordingly $\Phi^{-t}_{\mathcal{N}_i}((\mathbf{x}_{i_1}(T), \ldots, \mathbf{x}_{i_m}(T)))$ depends on the initial value of $\mathbf{x}_i$. Therefore, asymptotic stability of system (2) does not bring about GS of the network.

*Remark 3:* For bidirectionally coupled network, $d(i, i_k) = 1 (\forall i_k \in \mathcal{N}_i)$ is clear. According to Remark 2, even if system (2) is asymptotically stable, GS of network (1) is not necessarily realized.

*Remark 4:* If there exists a node $i$ in network (1) such that

1) $d(i, i_k) = \infty\, (\forall\, i_k \in \mathcal{N}_i)$,
2) system (2) is asymptotically stable (which can be verified by the auxiliary system approach in Section IV),

then network (1) reaches GS in the sense of Definition 3.

To better understand Theorem 1, take directed acyclic networks that satisfy $d(i, i_k) = \infty\, (\forall i_k \in \mathcal{N}_i)$ for example. It is a kind of ubiquitous networks serving as a platform for data structure, systems engineering, etc. The graph corresponding to the directed acyclic network is the so-called DAG (directed acyclic graph). For brevity, we call this kind of networks as DAG networks. Without loss of generality, assume that the edges of a DAG network are directed from the nodes with smaller ordinal numbers to those with larger ones. Accordingly the outer-coupling weight configuration matrix $\mathbf{C}$ is a lower triangular matrix. Then the following theorem can be deduced from Theorem 1.

*Theorem 1':* A DAG network reaches GS, iff there exists a node $i$ such that

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) - \sum_{j\in\mathcal{N}_i} c_{ij}\, \mathbf{H}_i(\mathbf{x}_i - \mathbf{x}_j) \tag{4}$$

is asymptotically stable, i.e., $\forall\, \mathbf{x}_i^1(t_0), \mathbf{x}_i^2(t_0) \in B_i$

$$\begin{aligned}&\lim_{t\to\infty} \| \mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^1(t_0))\\&\qquad - \mathbf{x}_i(t, \mathbf{x}_1(t_0), \ldots, \mathbf{x}_{i-1}(t_0), \mathbf{x}_i^2(t_0)) \|\\&= 0 .\end{aligned}$$

*Remark 5:* In a DAG network, the unidirectivity of information transmission leads to the driving neighbors' independent evolutions on a node. Then $\Phi_{\mathcal{N}_i}^{-t}((\mathbf{x}_{i_1}(T),\ldots,\mathbf{x}_{i_m}(T)))$ is independent on the initial value of $\mathbf{x}_i$ for all $i \in N$. Thus asymptotic stability of system (2) results in GS of the network, and vice versa.

## IV. The Auxiliary System Approach

According to Theorem 1, if node $i$ in network (1) satisfies $d(i,i_k) = \infty\,(\forall i_k \in \mathcal{N}_i)$, network (1) achieves GS iff system (2) is asymptotically stable. Arbitrariness of initial value of system (2) allows us to construct an auxiliary system (5), which is identical to system (2) (but evolve with different initial values lying in the same basin of attraction if chaos is considered), to identify GS of the network. The auxiliary system for the $i$th node is described by [16]–[20]

$$\dot{\hat{\mathbf{x}}}_i = \mathbf{f}_i(\hat{\mathbf{x}}_i) - \sum_{j\in\mathcal{N}_i} c_{ij}\,\mathbf{H}_i(\hat{\mathbf{x}}_i - \mathbf{x}_j). \tag{5}$$

If $\hat{\mathbf{x}}_i$ identically synchronizes with $\mathbf{x}_i$, GS of network (1) is assumed to occur. As soon as all nodes show identical behavior with their auxiliary counterparts, network (1) is said to achieve GS.

For bidirectionally coupled network,

$$d(i,i_k) = 1\,(\forall\, i_k \in \mathcal{N}_i)$$

leads to the dependence of

$$\Phi_{\mathcal{N}_i}^{-t}((\mathbf{x}_{i_1}(T),\ldots,\mathbf{x}_{i_m}(T)))$$

on the initial condition of $\mathbf{x}_i$. Recalling Remark 1 again, even if $\hat{\mathbf{x}}_i$ and $\mathbf{x}_i$ demonstrates identical behavior, we claim the asymptotic stability of system (2) rather than GS of network (1) in the sense of Definition 3. This is consistent with the numerical results in [21].

## V. Examples

### A. Bidirectionally Coupled Network

*Example 1:* Assume that two nodes are bidirectionally coupled, where each node dynamics is an ordinary differential equation with one-order

$$\begin{cases} \dot{x}_1(t) = f(x_1(t)) - c[x_1(t) - x_2(t)] \\ \dot{x}_2(t) = f(x_2(t)) - c[x_2(t) - x_1(t)] \end{cases} \tag{6}$$

where $f(x) = x$. By applying auxiliary system method in the bidirectional network, we introduce the two auxiliary nodes:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_1(t) - c[\hat{x}_1(t) - x_2(t)] \\ \dot{\hat{x}}_2(t) = \hat{x}_2(t) - c[\hat{x}_2(t) - x_1(t)]. \end{cases}$$

Denoting the differences $d_1 \triangleq \hat{x}_1 - x_1$, $d_2 \triangleq \hat{x}_2 - x_2$, we have the error systems:

$$\dot{d}_1(t) = (1-c)d_1(t) \tag{7}$$

$$\dot{d}_2(t) = (1-c)d_2(t). \tag{8}$$

Lyapunov Exponents (LEs) [26], [27] of the above error systems are shown in Fig. 1(a). On the basis of the stability theory [26], [27], nonnegetive LE cannot bring the stability of the zero solutions of system. Thus if the coupling strength $c \le 1$, the zero solutions of systems (7) and (8) won't be stable, and accordingly each node won't synchronize with their auxiliary partner. According to the results in Ref. [16]–[20], GS is unachievable when $c \le 1$. However, when $c = 0.6, 0.8$ and 1, the two nodes in network (6) identically synchronize as shown in Fig. 1(b). The nodes in network (6) attain GS since IS is undoubtedly a special case of GS.

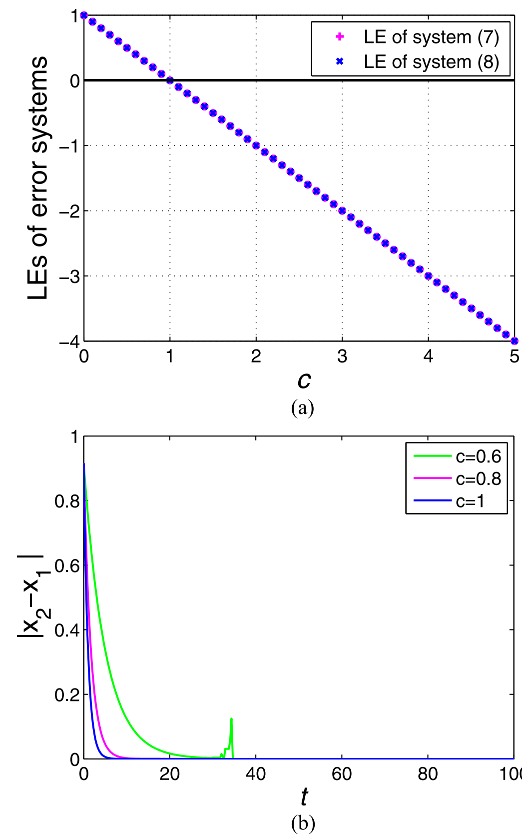


Fig. 1. (a) Lyapunov exponents of error systems (7) and (8). (b) Difference between node 1 and node 2 in network (6).

Theoretically, introducing $\tilde{d} \triangleq x_2 - x_1$, we have $\dot{\tilde{d}} = (1-2c)\,\tilde{d}$. Obviously, if $c > \frac{1}{2}$, $x_2$ and $x_1$ in network (6) identically synchronize to each other. Especially when $c = 1$, two special solutions of system (6)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

are linearly independent. Thus, the general solutions of the two-node system are

$$x_1 = \theta_1 e^t + \theta_2 e^{-t}$$

and

$$x_2 = \theta_1 e^t - \theta_2 e^{-t},$$

where $\theta_1$ and $\theta_2$ are determined by the initial values. Say, given $x_1(0) = 0$, $x_2(0) = 1$, the special solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^t - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^t + \frac{1}{2}e^{-t} \end{bmatrix}.$$

Clearly, one has

$$\lim_{t\to+\infty} [x_1(t) - x_2(t)] = 0.$$

Thus nodes in the bidirectionally coupled network (6) reach GS (actually IS) by rigorous justification. This conclusion, however, contradicts with the results obtained by the extended auxiliary system method. Recall the method again, the error systems are identical to $\dot{d}_i(t) = 0\,(i = 1, 2)$, then the differences between nodes and their auxiliary partners remain constants. Given distinct initial values of a node and its auxiliary counterpart, the difference remains and then GS fails.

## B. DAG Network

*Example 2:* For further illustration, a DAG network coupled with 100 nodes is taken as an example, in which the in-degrees obey a power-law distribution. Assume that the network model is depicted by

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) - c \sum_{j \in \mathcal{N}_i} a_{ij}\,(\mathbf{x}_i - \mathbf{x}_j), \quad 1 \le i \le N \tag{9}$$

where the adjacency matrix is $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{N \times N}$. If there is a link from node $j \in \mathcal{N}_i$ to node $i$ then $a_{ij} = 1$, otherwise $a_{ij} = 0$. We choose Chua circuit as the individual dynamics

$$\begin{cases} \dot{x} = -\alpha(x - y + g(x)) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y \end{cases}$$

where $g(x) = bx + 0.5(a - b)[|x + 1| - |x - 1|]$, the parameters are $(\alpha, \beta_i, a, b) = (9, 100/7 + 0.01\,i, -8/7, -5/7)\ (1 \le i \le 100)$.

Consider the auxiliary system of node $i$ as

$$\dot{\hat{\mathbf{x}}}_i = \mathbf{f}_i(\hat{\mathbf{x}}_i) - c \sum_{j \in \mathcal{N}_i} a_{ij}\,(\hat{\mathbf{x}}_i - \mathbf{x}_j). \quad 1 \le i \le N \tag{10}$$

Denoting $\mathbf{d}_i \triangleq \hat{\mathbf{x}}_i - \mathbf{x}_i (1 \le i \le N)$ as the differences, we plot the difference norms of the $i$th node vs. time $t$ in Fig. 2. Nodes in Fig. 2(a) are in their original sequence, and nodes in Fig. 2(b) in the descending order by in-degrees. Fig. 2(b) shows that, on the whole, higher in-degree nodes reach GS first and lower ones reach successively. It is seen from Fig. 2(a) that the original order of nodes has no serious influence on the speed of GS. In a word, in-degree outweighs the direction of paths in a DAG network when it comes to GS. The reason lies in the following fact. To investigate GS of network (9), subtracting it from (10), one has the error systems

$$\begin{aligned} \dot{\mathbf{d}}_i &= \mathbf{f}_i(\hat{\mathbf{x}}_i) - \mathbf{f}_i(\mathbf{x}_i) - c \sum_{j \in \mathcal{N}_i} a_{ij}\,\mathbf{d}_i \\ &= \mathbf{f}_i(\hat{\mathbf{x}}_i) - \mathbf{f}_i(\mathbf{x}_i) - c\,k_i^{in}\,\mathbf{d}_i,\ 1 \le i \le N \end{aligned} \tag{11}$$

where $k_i^{in}$ is the in-degree of node $i$. GS of network (9) is achieved iff the zero solution of system (11) is asymptotically stable. Choosing a Lyapunov candidate as $L_i = \frac{1}{2}\mathbf{d}_i^\top \mathbf{d}_i$, one has

$$\dot{L}_i = \mathbf{d}_i^\top\,(\mathbf{f}_i(\hat{\mathbf{x}}_i) - \mathbf{f}_i(\mathbf{x}_i)) - c\,k_i^{in}\,\mathbf{d}_i^\top \mathbf{d}_i\,. \tag{12}$$

Based on the Lyapunov's second method for stability [26], the zero solution of system (11) is asymptotically stable if $L$ belongs to the set $\{L \,|\, \dot{L}_i \le 0; \dot{L}_i = 0 \text{ iff } \|\mathbf{d}_i\| = 0\}$. Because higher in-degree has priority to stabilizing the the zero solution according to (12), the speed of GS in the DAG network grows in proportion to the node in-degree roughly [28], [29]. The reaching time is inversely proportional to the node in-degree "roughly" rather than "exactly" due to that besides $k_i^{in}$, individual dynamics $\mathbf{x}_i$ has influence on the speed of GS. While $\mathbf{x}_i$ is affected by the function $\mathbf{f}_i(\cdot)$, coupling strength $c$ and the entire network topology $\mathbf{A}$.

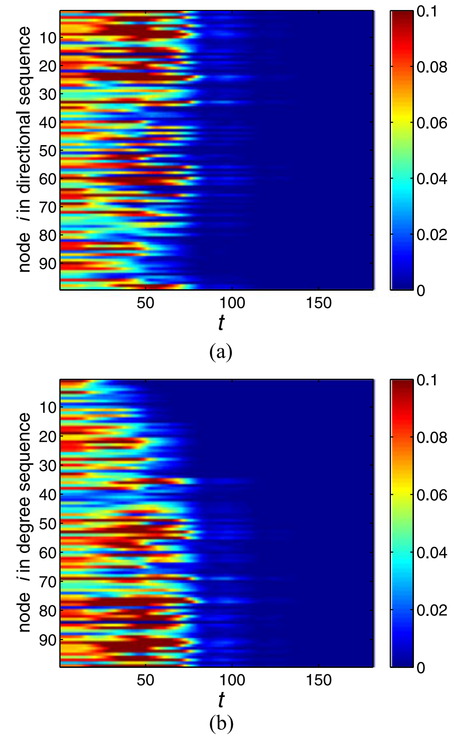


Fig. 2. Difference norms $\|\mathbf{d}_i\|_2 = \|\hat{\mathbf{x}}_i - \mathbf{x}_i\|_2$ of the $i$th node $(1 \le i \le 100)$ versus time $t$ in a DAG network coupled with 100 Chua circuits. (a) Nodes in the original sequence. (b) Nodes in their descending in-degree order.

## VI. Conclusion

In summary, we have investigated the criterion on which the auxiliary system method can be applied to complex networks [30], [31]. By rigorous theoretical certification, we have concluded that the method is effective to complex systems only when there is no path from nodes to their driving neighbors. According to our main results, the auxiliary system approach should not be applied to bidirectionally coupled systems. In addition, we have found that, roughly speaking, higher in-degree nodes reach GS first and lower ones reach successively in a DAG network. Actually it may be a universal phenomenon in general complex networks. Although our results are far from systematically uncovering the essence of GS in systems with interacting units, it should be fundamental and treated as a first step to exhaustively explore GS behavior of complex networks. Future work includes studying generalized synchronizability of complex networks, discussing testing methods for GS in bidirectionally coupled networks, and so on.

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