Adaptive Exponential Synchronization of Complex Networks With Nondifferentiable Time-Varying Delay

Shuaibing Zhu, *Member, IEEE*, Jin Zhou, Quanxin Zhu, *Senior Member, IEEE*, Na Li, *Member, IEEE*, and Jun-An Lu

***Abstract*—In recent years, the adaptive exponential synchronization (AES) problem of delayed complex networks has been extensively studied. Existing results rely heavily on assuming the differentiability of the time-varying delay, which is not easy to verify in reality. Dealing with nondifferentiable delay in the field of AES is still a challenging problem. In this brief, the AES problem of complex networks with general time-varying delay is addressed, especially when the delay is nondifferentiable. A delay differential inequality is proposed to deal with the exponential stability of delayed nonlinear systems, which is more general than the widely used Halanay inequality. Next, the boundedness of the adaptive control gain is theoretically proved, which is neglected in much of the literature. Then, the AES criteria for networks with general delay are established for the first time by using the proposed inequality and the boundedness of the control gain. Finally, an example is given to demonstrate the effectiveness of the theoretical results.**

***Index Terms*—Adaptive control, complex network, delay, exponential synchronization.**

I. Introduction

In the real world, a great deal of large-scale systems can be modeled as complex networks, such as electrical power grids, neural networks, and biological networks. Since the proposal of several classical network models including the small-world network [1] and the scale-free network [2], network science has drawn increasing attention from researchers in various disciplines [3]–[11].

As a ubiquitous and important nonlinear phenomenon in nature, synchronization has become one of the most important topics for complex networks since the pioneering work of Pecora and Carroll [12], and lots of results [13]–[22] on the synchronization problem of networks have been established. In order to realize network synchronization or system stabilization, various control schemes have been proposed, such as the widely used linear feedback control [23]–[25] and adaptive feedback control [26]–[28]. Though the linear feedback control is easy to implement, the design of its control gain generally depends on network parameters. That is, it is difficult to determine a suitable control gain in the absence of the information on network parameters. In contrast, the adaptive control can well deal with networks with unknown parameters, because its control gain can be adapted automatically according to some updating laws until the synchronization is realized.

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Shuaibing Zhu and Quanxin Zhu are with the MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, China (e-mail: zhushuaibing@hunnu.edu.cn; zqx22@126.com).

Jin Zhou and Jun-An Lu are with the School of Mathematics and Statistics and the Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China (e-mail: jzhou@whu.edu.cn; jalu@whu.edu.cn).

Na Li is with the School of Mathematics and Statistics, Henan University, Kaifeng 475004, China (e-mail: nli@henu.edu.cn).

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The asymptotic synchronization problem of networks under adaptive control has been widely studied [28]–[30]. It is noteworthy that asymptotic synchronization criteria only guarantee the convergence of the synchronization error, but do not give the convergence speed, making it hard to predict the synchronization error. By comparison, exponential synchronization criteria are more practical as they provide the convergence speed. In addition, the time delay is usually inevitable on large-scale networks as the transmission speed of signals is finite. In recent years, considerable efforts have been devoted to the adaptive exponential synchronization (AES) problem of networks with coupling delay. In the literature, the updating laws of the adaptive control can be classified into the following two major categories (or in similar form):

$$\dot{\delta} = k\|\mu\|^q \tag{1}$$

$$\dot{\delta} = k\|\mu\|^q \exp(at) \tag{2}$$

where δ is the control gain, μ is the synchronization error, $k > 0$, $q \geq 1$, and $a > 0$. Most existing results use updating laws (1) with $q = 2$ to realize asymptotic synchronization, and some other results [31]–[34] also use (1) to realize exponential synchronization. In [31] and [32], exponential synchronization criteria are obtained directly based on the Gronwall inequality. In [33] and [34], delay differential inequalities are established in virtue of the Gronwall inequality, and then exponential synchronization criteria are derived from the delay differential inequalities proposed therein. It should be highlighted that the Gronwall inequality is used in the same wrong manner in all of [31]–[34], which will be analyzed later in Remark 6. To our best knowledge, it remains a challenging problem whether updating laws (1) can ensure exponential synchronization. Nevertheless, updating laws in the form of (2) with $q = 1$ or $q = 2$ has been successfully used in [35]–[37] to realize exponential synchronization. It is noticed that, [31]–[37] only considered constant delay or differentiable time-varying delay. Furthermore, the delay $\tau(t)$ is required to satisfy $\dot{\tau}(t) \leq \kappa < 1$ in most of the aforementioned results, where κ is a constant. As far as we know, the AES problem of networks with nondifferentiable time-varying delay has not been solved until now. The main difficulty lies in that, the delay is conventionally addressed via the integral term $\int_{t-\tau(t)}^{t} \mu^T(\theta)\mu(\theta)\mathrm{d}\theta$, while this term requires $\tau(t)$ be differentiable. In addition, it is literally impossible to know the exact analytical expression of $\tau(t)$, so it is difficult to verify the differentiability of $\tau(t)$ in general. Therefore, it is necessary and meaningful to investigate the AES problem of networks with nondifferentiable delay.

Motivated by the above discussions, this brief aims to investigate the AES problem of complex networks with general time-varying delay, especially when the delay is nondifferentiable. The main contributions are as follows.

1) A delay differential inequality is proposed to deal with the exponential stability of delayed nonlinear systems, which is more general than the widely used Halanay inequality.
2) The boundedness of the adaptive control gain is theoretically proved, which is neglected in [31]–[37]. In engineering, the

control gain cannot be infinitely large, so the theoretical analysis here is primary and ensures the practicality of the adaptive controllers.

3) The AES criteria for networks with general delay are derived from the proposed inequality and the boundedness of the control gain. It should be highlighted that nondifferentiable delay is addressed for the first time in the AES problem.

Notation: Denote the n-dimensional Euclidean space and the set of all the $(n \times m)$-dimensional real matrices as $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively. For a vector or a matrix, the superscript T represents the transpose and $\|\cdot\|$ denotes the two-norm. The symbol $\otimes$ represents the Kronecker product. The right-hand upper Dini derivative of a function φ is defined as $D^+\varphi(t) := \limsup_{h\to 0^+}[(\varphi(t+h)-\varphi(t))/h]$.

II. Preliminaries

A. Problem Formulation

Consider a controlled complex network with time-varying delay

$$\dot{x}_i(t) = f(x_i(t),t) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t-\tau(t)) + u_i(t) \quad (3)$$

where $1 \le i \le N$, $x_i \in \mathbb{R}^n$ is the state of the ith node, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuous function, $c > 0$ represents the coupling strength, $A = (a_{ij})_{N\times N}$ denotes the weighted outer coupling matrix, $\Gamma \in \mathbb{R}^{n\times n}$ denotes the inner coupling matrix, $\tau(t)$ is the time-varying delay, and $u_i(t)$ is the control input. If there is an edge pointing from node j to node i $(i \ne j)$, then $a_{ij} > 0$; otherwise, $a_{ij} = 0$. In addition, define $a_{ii} = -\sum_{j=1, j\ne i}^{N} a_{ij}$ for $1 \le i \le N$.

In the following, the main objective is to design appropriate adaptive feedback control to synchronize all the node states x_i to a desired state $\upsilon(t)$, which is a solution of the isolated node system:

$$\dot{\upsilon}(t) = f(\upsilon(t),t). \quad (4)$$

Defining the synchronization error as $\mu_i(t) = x_i(t)-\upsilon(t)$, one obtains from (3) to (4)

$$\dot{\mu}_i(t) = f(x_i(t),t) - f(\upsilon(t),t) + c\sum_{j=1}^{N} a_{ij}\Gamma\mu_j(t-\tau(t)) + u_i(t) \quad (5)$$

for $1 \le i \le N$. Define

$$\mu(t) = \left[\mu_1^T(t),\ldots,\mu_N^T(t)\right]^T.$$

Then, the synchronization of network (3) is achieved if one has $\mu(t) \to 0$ as $t \to +\infty$.

B. Assumptions and Lemmas

Here, some necessary assumptions and lemmas are to be introduced.

Assumption 1 (A1): The delay $\tau(t)$ of network (3) is assumed to be continuous and satisfy

$$0 \le \tau(t) \le \tau_{\max} \quad (6)$$

with $\tau_{\max}$ being a positive number.

Remark 1: In much of the literature, the delay $\tau(t)$ is usually required to be constant or differentiable. In assumption (A1), $\tau(t)$ is very general, since it can be nondifferentiable and may have uncountable nondifferentiable points.

Assumption 2 (A2): Assume that there exists a nonnegative number L such that

$$\|f(x,t) - f(y,t)\| \le L\|x-y\| \quad (7)$$

where $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Lemma 1: Let (A1) hold, and let α, β, a, τ_0 be positive constants satisfying $\alpha > \beta\exp(a\tau_0) + a$. If $\nu(t)$ is a nonnegative continuous function on $[t_0-\tau_0, t_0]$ satisfying

$$D^+\nu(t) \le -\alpha\nu(t) + \beta\tilde{\nu}(t) + \varepsilon\exp(-a(t-t_0)), \quad t \ge t_0$$

then

$$\nu(t) \le M\exp(-\gamma(t-t_0)) + \frac{\varepsilon\exp(-a(t-t_0))}{\rho}, \quad t \ge t_0 \quad (8)$$

where $\tilde{\nu}(t) = \max_{\theta\in[t-\tau_0,t]}\nu(\theta)$, $M = \tilde{\nu}(t_0)$, $\varepsilon \ge 0$, $\rho = \alpha - \beta\exp(a\tau_0) - a$, and γ is the unique positive solution of $\gamma = \alpha - \beta\exp(\gamma\tau_0)$.

Proof: Consider the following comparison system:

$$\begin{cases}\dot{\omega}(t) = -\alpha\omega(t) + \beta\tilde{\omega}(t) \\ \qquad + \varepsilon_1\exp(-a(t-t_0)), \quad t \ge t_0 \\ \omega(t) = \nu(t), \quad t \le t_0\end{cases} \quad (9)$$

where $\varepsilon_1 > \varepsilon$ and $\tilde{\omega}(t) = \max_{\theta\in[t-\tau_0,t]}\omega(\theta)$. We shall prove that

$$\omega(t) < \psi(t), \quad t \ge t_0 - \tau_0 \quad (10)$$

where $\psi(t) = M\exp(-\gamma(t-t_0)) + (\varepsilon_1/\rho)\exp(-a(t-t_0))$.

When $t \in [t_0-\tau_0, t_0]$, one obtains $\omega(t) \le M < \psi(t)$. If (10) does not hold, then

$$t_1 = \inf\{t > t_0 |\ \omega(t) \ge \psi(t)\}$$

is well defined. Since ω is continuous on $[t_0-\tau_0, +\infty)$, one has

$$\omega(t_1) = \psi(t_1), \quad \omega(t) < \psi(t),\ t \in [t_0-\tau_0, t_1). \quad (11)$$

From (9), it is derived that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\omega(t)\exp(\alpha t)] = \exp(\alpha t)[\beta\tilde{\omega}(t) + \varepsilon_1\exp(-a(t-t_0))]. \quad (12)$$

Integrating (12) from t_0 to t_1 gives

$$\omega(t_1)\exp(\alpha t_1) - \omega(t_0)\exp(\alpha t_0) = \int_{t_0}^{t_1}\exp(\alpha t)[\beta\tilde{\omega}(t) + \varepsilon_1\exp(-a(t-t_0))]\mathrm{d}t. \quad (13)$$

For $t \ge t_0$, there exists $\tilde{\tau} \in [-\tau_0, 0]$ such that $\tilde{\omega}(t) = \omega(t-\tilde{\tau})$. If $\tilde{\tau} = 0$, it follows from (11) that

$$\tilde{\omega}(t) = \omega(t) \le \psi(t) < \psi(t-\tau_0), \quad t \in [t_0, t_1].$$

Otherwise, one has

$$\tilde{\omega}(t) = \omega(t-\tilde{\tau}) < \psi(t-\tilde{\tau}) \le \psi(t-\tau_0), \quad t \in [t_0, t_1].$$

Defining $\tilde{t} = t - t_0$, it follows from $\tilde{\omega}(t) < \psi(t-\tau_0)$ that

$$\begin{aligned}&\beta\tilde{\omega}(t) + \varepsilon_1\exp(-a(t-t_0)) \\ &< M\beta\exp(\gamma\tau_0)\exp(-\gamma\tilde{t}) + \varepsilon_1\cdot\frac{\beta\exp(a\tau_0)+\rho}{\rho}\cdot\exp(-a\tilde{t}) \\ &= M(\alpha-\gamma)\exp(-\gamma\tilde{t}) + \frac{\varepsilon_1(\alpha-a)}{\rho}\exp(-a\tilde{t}).\end{aligned}$$

Then, it follows from (13) that

$$\begin{aligned}&\omega(t_1)\exp(\alpha t_1) - \omega(t_0)\exp(\alpha t_0) \\ &< M\exp(-\gamma(t_1-t_0))\exp(\alpha t_1) - M\exp(\alpha t_0) \\ &\quad + \frac{\varepsilon_1}{\rho}\exp(-a(t_1-t_0))\exp(\alpha t_1) - \frac{\varepsilon_1}{\rho}\exp(\alpha t_0) \\ &= \psi(t_1)\exp(\alpha t_1) - \left(M + \frac{\varepsilon_1}{\rho}\right)\exp(\alpha t_0).\end{aligned}$$

Considering $\omega(t_0) \le M$, it is derived that $\omega(t_1) < \psi(t_1)$, which contradicts (11). Therefore, inequality (10) holds. According to the comparison principle, one has

$$\nu(t) < \omega(t) < \psi(t).$$

Letting $\varepsilon_1 \to \varepsilon^+$ leads to (8). □

Remark 2: In the literature, the Halanay inequality [38] has been widely used to analyze the exponential synchronization of delayed networks and the exponential stability of delayed nonlinear systems. The simplest Lyapunov function $V(\mu(t)) = \mu^T(t)\mu(t)$ is often chosen to obtain an inequality in the form of the Halanay inequality. When the adaptive control is involved, however, the Lyapunov function would be more complicated to deal with the adaptive control, and the Halanay inequality is therefore difficult to use. For this reason, a more general differential inequality is established in Lemma 1. Particularly, Lemma 1 reduces to the Halanay inequality if taking $\varepsilon = 0$.

Remark 3: In Lemma 1, the positive constant a is required to satisfy $\alpha > \beta \exp(a\tau_0) + a$, which is equivalent to $a \in (0, \gamma)$. If $a \ge \gamma$, it is not clear whether (8) holds, but it can be easily deduced from Lemma 1 that ν exponentially converges to 0. Taking $\tilde{a} \in (0, \gamma)$, one has

$$\alpha > \beta \exp(\tilde{a}\tau_0) + \tilde{a}.$$

In addition, it follows from $\tilde{a} < a$ that

$$\begin{aligned}\dot{\nu}(t) &\le -\alpha\nu(t) + \beta\tilde{\nu}(t) + \varepsilon \exp(-a(t-t_0)) \\ &\le -\alpha\nu(t) + \beta\tilde{\nu}(t) + \varepsilon \exp(-\tilde{a}(t-t_0)), \quad t \ge t_0.\end{aligned}$$

Then, ν exponentially converges to 0 according to Lemma 1.

Lemma 2: Let $\Omega \subset \mathbb{R}$ be an interval in the form of (σ_1, σ_2) or $[\sigma_1, \sigma_2)$, where σ_1 may be $-\infty$ and σ_2 may be $+\infty$. Let $\xi : \Omega \to \mathbb{R}^n$ be a continuous function. If ξ is differentiable and there exists a nonnegative continuous function $\vartheta(t)$ satisfying

$$|\xi^T(t)\dot{\xi}(t)| \le \vartheta(t)\|\xi(t)\| \tag{14}$$

Then, we have the following:

1) The right-hand upper Dini derivative of $\|\xi\|$ exists, and

$$D^+\|\xi(t)\| \le \vartheta(t). \tag{15}$$

2) The derivative of $\|\xi\|^p$ exists, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^p = \begin{cases} p\|\xi(t)\|^{p-2}\xi^T(t)\dot{\xi}(t), & \text{if } \xi(t) \ne 0 \\ 0, & \text{if } \xi(t) = 0 \end{cases} \tag{16}$$

where $p > 1$.

Proof: Let $t_0 \in \Omega$. If $\xi(t_0) \ne 0$, one has

$$\begin{aligned}\left.\frac{\mathrm{d}\|\xi(t)\|^p}{\mathrm{d}t}\right|_{t=t_0} &= p\|\xi(t_0)\|^{p-1}\left.\frac{\mathrm{d}\|\xi(t)\|}{\mathrm{d}t}\right|_{t=t_0} \\ &= p\|\xi(t_0)\|^{p-1}\frac{\xi^T(t_0)\dot{\xi}(t_0)}{\sqrt{\xi^T(t_0)\xi(t_0)}} \\ &= p\|\xi(t_0)\|^{p-2}\xi^T(t_0)\dot{\xi}(t_0).\end{aligned} \tag{17}$$

Therefore, (16) is verified when $\xi(t_0) \ne 0$.

If $\xi(t_0) = 0$, then there exists $h_0 > 0$ such that $[t_0, t_0 + h_0] \subset \Omega$. Let $h \in (0, h_0]$ and define

$$t_1 = \sup\{t \in [t_0, t_0 + h] |\ \xi(t) = 0\}.$$

Since ξ is continuous, it can be derived that $\xi(t_1) = 0$.

If $t_1 = t_0 + h$, then

$$\frac{\|\xi(t_0+h)\|^p - \|\xi(t_0)\|^p}{h} = \frac{0-0}{h} = 0.$$

Otherwise, one has $t_1 \in [t_0, t_0 + h)$ and

$$\xi(t) \ne 0 \quad \forall t \in (t_1, t_0 + h].$$

Hence, $\|\xi(t)\|^p$ is differentiable on $(t_1, t_0 + h]$. It follows that

$$\begin{aligned}&\frac{\|\xi(t_0+h)\|^p - \|\xi(t_0)\|^p}{h} \\ &\quad= \frac{\|\xi(t_0+h)\|^p - \|\xi(t_1)\|^p}{h} \\ &\quad= \frac{t_0+h-t_1}{h}\left.\frac{\mathrm{d}\|\xi(t)\|^p}{\mathrm{d}t}\right|_{t=\varrho} \\ &\quad= p\|\xi(\varrho)\|^{p-2}\xi^T(\varrho)\dot{\xi}(\varrho)\frac{t_0+h-t_1}{h}\end{aligned}$$

where $\varrho \in (t_1, t_0+h)$. Considering (14) and $0 < t_0+h-t_1 \le h$, one obtains

$$\left|\frac{\|\xi(t_0+h)\|^p - \|\xi(t_0)\|^p}{h}\right| \le p\vartheta(\varrho)\|\xi(\varrho)\|^{p-1}. \tag{18}$$

Let $h \to 0^+$, one has $t_1 \to t_0^+$ and $\varrho \to t_0^+$, leading to

$$\|\xi(\varrho)\| \to \|\xi(t_0)\| = 0, \quad \vartheta(\varrho) \to \vartheta(t_0).$$

Then, it is deduced from (18) that

$$\frac{\|\xi(t_0+h)\|^p - \|\xi(t_0)\|^p}{h} \to 0, \quad \text{as } h \to 0^+.$$

That is, the right-hand derivative of $\|\xi(t)\|^p$ at t_0 is 0. If Ω is in the form of $[\sigma_1, \sigma_2)$ and $t_0 = \sigma_1$, then the derivative of $\|\xi(t)\|^p$ at t_0 is 0. Otherwise, t_0 is an inner point of Ω. Similarly, it can be proved that the left-hand derivative of $\|\xi(t)\|^p$ at t_0 is also 0, thus the derivative of $\|\xi(t)\|^p$ at t_0 is 0. Therefore, (16) is verified when $\xi(t_0) = 0$.

It can be verified that the proof from the beginning to (18) also holds for $p = 1$.

If $\xi(t_0) \ne 0$, one obtains from (17) that

$$\left.\frac{\mathrm{d}\|\xi(t)\|}{\mathrm{d}t}\right|_{t=t_0} = \|\xi(t_0)\|^{-1}\xi^T(t_0)\dot{\xi}(t_0) \le \vartheta(t_0).$$

Hence, $D^+\|\xi(t)\|$ exists and (15) is verified when $\xi(t_0) \ne 0$.

If $\xi(t_0) = 0$, one obtains from (18) that

$$-\vartheta(\varrho) \le \frac{\|\xi(t_0+h)\| - \|\xi(t_0)\|}{h} \le \vartheta(\varrho).$$

Let $h \to 0^+$, one has $\varrho \to t_0^+$. Hence, $D^+\|\xi(t)\|$ exists and (15) is verified when $\xi(t_0) = 0$. □

Remark 4: When analyzing the network synchronization, the established Lyapunov function may contain a term in the form of $\|\xi(t)\|^p$. Since it is not easy to decide in a rigorous way whether the derivative of $\|\xi(t)\|^p$ exists, a simple condition is given in Lemma 2 to ensure the existence of both $D^+\|\xi\|$ and $(\mathrm{d}/\mathrm{d}t)\|\xi(t)\|^p$ $(p > 1)$. From (16), it can be further obtained that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^p \le p\vartheta(t)\|\xi(t)\|^{p-1}. \tag{19}$$

Then, inequalities (15) and (19) together can be expressed in a slightly weaker but simpler form as

$$D^+\|\xi(t)\|^q \le q\vartheta(t)\|\xi(t)\|^{q-1}, \quad q \ge 1. \tag{20}$$

If $\dot{\xi}(t)$ is continuous, the condition (14) necessarily holds with $\vartheta(t) = \|\dot{\xi}(t)\|$. In this case, (20) becomes

$$D^+\|\xi(t)\|^q \le q\|\xi(t)\|^{q-1}\|\dot{\xi}(t)\|, \quad q \ge 1.$$

In addition, $D^+\|\xi(t)\|^q$ does not necessarily exist for $q < 1$. For example, when $\xi(t) = t$ and $q = (1/2)$, $D^+\|\xi(t)\|^q$ does not exist at $t = 0$.

III. Main Results

It is now ready to establish the exponential synchronization criteria for network (3) under adaptive control

$$u_i(t) = -\delta(t)\mu_i(t), \quad 1 \le i \le N \tag{21}$$

with updating laws

$$\dot{\delta}(t) = k\|\mu(t)\|^q \exp(at) \tag{22}$$

where $q \ge 1$, $\delta(0) \ge 0$, and k, a are positive parameters.

To realize exponential synchronization, the updating laws in [35]–[37] is designed in the form of (22) with $q = 1$ or $q = 2$, while the parameter q can take any value no less than 1.

A. Boundedness of the Control Gain

It is noteworthy that the control gain $\delta(t)$ keeps increasing if $\mu(t) \ne 0$ according to the updating laws (22). However, the control gain cannot be infinitely large in reality. Therefore, it is primary and fundamental to decide whether $\delta(t)$ is bounded.

Theorem 1: Let the assumptions (A1) and (A2) hold. If the control u_i $(1 \le i \le N)$ of network (3) is designed as (21) with updating laws (22), then the control gain $\delta(t)$ is necessarily bounded, or equivalently, $\delta(t)$ converges to some finite value.

Proof: Consider the Lyapunov function

$$V(t) = \mu^T(t)\mu(t) = \sum_{i=1}^{N} \mu_i^T(t)\mu_i(t).$$

Differentiating V yields

$$\begin{aligned}\dot{V}(t) &= 2\sum_{i=1}^{N}\mu_i^T(t)[f(x_i(t),t) - f(v(t),t)] \\ &\quad + 2c\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mu_i^T(t)\Gamma\mu_j(t-\tau(t)) \\ &\quad - 2\delta(t)\sum_{i=1}^{N}\mu_i^T(t)\mu_i(t).\end{aligned} \tag{23}$$

According to assumption (A2), one deduces

$$\begin{aligned}&\sum_{i=1}^{N}\mu_i^T(t)[f(x_i(t),t) - f(v(t),t)] \\ &\qquad \le \sum_{i=1}^{N}\|\mu_i^T(t)\| \times L\|\mu_i(t)\| = L\|\mu(t)\|^2.\end{aligned}$$

In addition, one has

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mu_i^T(t)\Gamma\mu_j(t-\tau(t)) \\ &\quad = \mu^T(t)(A\otimes\Gamma)\mu(t-\tau(t)) \le \|A\|\|\Gamma\|\|\mu(t)\|\|\mu(t-\tau(t))\|.\end{aligned}$$

Then, it follows that

$$\begin{aligned}\dot{V}(t) &\le 2(L-\delta(t))\|\mu(t)\|^2 + 2c\|A\|\|\Gamma\|\|\mu(t)\|\|\mu(t-\tau(t))\| \\ &\le -(2\delta(t) - 2L - c\|A\|\|\Gamma\|)\|\mu(t)\|^2 \\ &\quad + c\|A\|\|\Gamma\|\|\mu(t-\tau(t))\|^2.\end{aligned} \tag{24}$$

From (22), it is concluded that $\delta(t)$ does not decrease. If

$$\delta(t) < \delta_0 := \max\{L + c\|A\|\|\Gamma\|,\ \delta(0)\} + \rho \quad \forall t \ge 0$$

then $\delta(t)$ is bounded, where ρ is a parameter to be determined. Otherwise, there exists $t_0 > 0$ satisfying $\delta(t_0) = \delta_0$ and

$$\delta(t) \ge \delta_0 \ge L + c\|A\|\|\Gamma\| + \rho, \quad t \ge t_0.$$

Recalling (24), it is deduced that

$$\begin{aligned}\dot{V}(t) &\le -\alpha V(t) + \beta V(t-\tau(t)) \\ &\le -\alpha V(t) + \beta\tilde{V}(t), \quad t \ge t_0\end{aligned}$$

where $\tilde{V}(t) = \max_{\theta\in[t-\tau_{\max},t]} V(\theta)$, $\alpha = \beta + 2\rho$, and $\beta = c\|A\|\|\Gamma\|$. Taking $\varepsilon = 0$ in Lemma 1, one deduces by Lemma 1 that

$$V(t) \le \tilde{V}(t_0)\exp(-\gamma(t-t_0)), \quad t \ge t_0$$

where $\gamma > 0$ is the solution of

$$\gamma = \alpha - \beta\exp(\gamma\tau_{\max}).$$

Letting $\rho \to +\infty$, one has $\alpha \to +\infty$ and $\gamma \to +\infty$. Hence, one can select a large enough ρ such that $\gamma_1 := (q\gamma/2) > a$. Then, it follows from (22) that

$$\begin{aligned}\delta(t) &= \delta(t_0) + k\int_{t_0}^{t} V^{\frac{q}{2}}(\theta)\exp(a\theta)\mathrm{d}\theta \\ &\le \delta(t_0) + k\int_{t_0}^{t}\tilde{V}^{\frac{q}{2}}(t_0)\exp(-\gamma_1(\theta-t_0))\exp(a\theta)\mathrm{d}\theta \\ &= \delta_0 + \frac{k\tilde{V}^{\frac{q}{2}}(t_0)\exp(at_0)}{\gamma_1 - a}[1 - \exp(-(\gamma_1 - a)(t-t_0))] \\ &\le \delta_0 + \frac{k\tilde{V}^{\frac{q}{2}}(t_0)\exp(at_0)}{\gamma_1 - a}.\end{aligned}$$

Hence, $\delta(t)$ is bounded, which is equivalent that $\delta(t)$ converges to some finite value as $\delta(t)$ does not decrease. □

Remark 5: In Theorem 1, the boundedness of the adaptive control gain $\delta(t)$ is theoretically guaranteed. This is primary and fundamental, because controllers cannot provide infinitely large control gain. Despite the importance, theoretical analysis on the boundedness of $\delta(t)$ is neglected in existing studies [35]–[37]. In addition, it is more practical to design the control in a distributed way as

$$u_i(t) = -\delta_i(t)\mu_i(t), \quad 1 \le i \le N$$

with updating laws

$$\dot{\delta}_i(t) = k_i\|\mu_i(t)\|^q \exp(at)$$

where k_i, a are positive parameters and $q \ge 1$. However, it would be challenging to theoretically prove the boundedness of $\delta_i(t)$ for each i, which deserves further study.

B. Exponential Synchronization Criteria

Based on the boundedness of $\delta(t)$, it is next to derive the AES criteria for network (3) with general time-varying delay, especially when the delay is nondifferentiable.

Theorem 2: If the assumptions (A1) and (A2) hold, network (3) can realize global exponential synchronization under adaptive control (21) with updating laws (22). Precisely, the synchronization error is estimated as

$$\|\mu(t)\| \le M_0\exp\left(-\frac{at}{q}\right) \tag{25}$$

where M_0 is a large enough constant.

Proof: According to Theorem 1, $\delta(t)$ converges to a finite value, that is, $\delta(+\infty)$ is finite. Consider the following function:

$$V(t) = \|\mu(t)\|^q + \frac{q(\delta(t)-\delta^*)^2}{2k}\exp(-at)$$

where $\delta^* > \delta(+\infty)$ is a parameter to be determined. Since $\delta(t)$ does not decrease and $\delta(0) \ge 0$, it follows that $\delta(t) \ge 0$ and $\delta(t) \le \delta(+\infty)$. Recalling (23) and (24), one can easily obtain

$$\begin{aligned}&|\mu^T(t)\dot{\mu}(t)| \\ &\quad \le (L + \delta(t))\|\mu(t)\|^2 + c\|A\|\|\Gamma\|\|\mu(t)\|\|\mu(t-\tau(t))\|\end{aligned}$$

which can be rewritten as

$$|\mu^T(t)\dot{\mu}(t)| \leq \vartheta(t)\|\mu(t)\|$$

where $\vartheta(t) = (L + \delta(t))\|\mu(t)\| + c\|A\|\|\Gamma\|\|\mu(t - \tau(t))\|$ is continuous. Then, the Dini derivative $D^+\|\mu(t)\|^q$ exists according to Lemma 2, and

$$\begin{aligned} D^+V(t) =& D^+\|\mu(t)\|^q + q(\delta(t) - \delta^*)\|\mu(t)\|^q \\ &- a\frac{q(\delta(t) - \delta^*)^2}{2k}\exp(-at) \\ \leq& D^+\|\mu(t)\|^q + q(\delta(t) - \delta^*)\|\mu(t)\|^q. \end{aligned}$$

According to (20), one has

$$D^+\|\mu(t)\|^q \leq q\vartheta(t)\|\mu(t)\|^{q-1} \tag{26}$$

which further gives

$$D^+V(t) \leq -\alpha\|\mu(t)\|^q + \beta\|\mu(t)\|^{q-1}\|\mu(t - \tau(t))\|$$

where $\alpha = q(\delta^* - 2\delta(t) - L)$ and $\beta = qc\|A\|\|\Gamma\|$. Defining $\tilde{V}(t) = \max_{\theta\in[t-\tau_{\max},t]} V(\theta)$, one has

$$\|\mu(t)\|^{q-1}\|\mu(t - \tau(t))\| \leq \max_{\theta\in[t-\tau_{\max},t]} \|\mu(\theta)\|^q \leq \tilde{V}(t).$$

Moreover,

$$\begin{aligned} -\|\mu(t)\|^q =& -V(t) + \frac{q(\delta(t) - \delta^*)^2}{2k}\exp(-at) \\ \leq& -V(t) + \frac{q(\delta^*)^2}{2k}\exp(-at). \end{aligned}$$

Then, one obtains

$$D^+V(t) \leq -\alpha V(t) + \beta\tilde{V}(t) + \varepsilon\exp(-at)$$

where $\varepsilon = ((\alpha q(\delta^*)^2)/2k)$. Choosing

$$\delta^* = 2\delta(+\infty) + L + c\|A\|\|\Gamma\|\exp(a\tau_{\max}) + a + 1 \tag{27}$$

one has

$$\alpha \geq qc\|A\|\|\Gamma\|\exp(a\tau_{\max}) + qa + q > \beta\exp(a\tau_{\max}) + a. \tag{28}$$

In virtue of Lemma 1, one gets

$$V(t) \leq M\exp(-\gamma t) + \frac{\varepsilon\exp(-at)}{\rho}, \quad t \geq 0 \tag{29}$$

where $M = \tilde{V}(0) \leq \max_{\theta\in[-\tau_{\max},0]} \|\mu(\theta)\|^2 + ((q(\delta^*)^2)/2k)$, $\rho = \alpha - \beta\exp(a\tau_{\max}) - a$, and γ is the solution of

$$\gamma = \alpha - \beta\exp(\gamma\tau_{\max}). \tag{30}$$

Comparing (28) and (30), it can be derived that

$$\beta\exp(a\tau_{\max}) + a < \alpha = \beta\exp(\gamma\tau_{\max}) + \gamma$$

which further gives $a < \gamma$. Then, (29) gives

$$V(t) \leq \left(M + \frac{\varepsilon}{\rho}\right)\exp(-at)$$

which further yields

$$\|\mu(t)\| \leq M_0\exp\left(-\frac{at}{q}\right) \tag{31}$$

where $M_0 = (M + (\varepsilon/\rho))^{(1/q)}$. The proof is complete. □

Remark 6: The AES criteria for complex networks with nondifferentiable delay are established in Theorem 2. In [31]–[34], the AES problem is claimed to be solved by designing the updating laws in the form of $\dot{\delta} = k\|\mu\|^q$, which is simpler than updating laws (22). However, the results in [31]–[34] are based on using the Gronwall inequality [39] in a wrong manner. The Gronwall inequality is as follows: If $\nu(t)$ is a nonnegative and continuous function satisfying

$$\nu(t) \leq \alpha + \beta\int_0^t \nu(\theta)\mathrm{d}\theta, \quad t \geq 0 \tag{32}$$

then $\nu(t) \leq \alpha\exp(\beta t)$, where $\alpha \geq 0$ and $\beta \geq 0$. Note that β takes negative values in [31]–[34], which is incorrect. Consider the following simple example. Taking $\alpha = 1$ and $\beta = -1$, it can be verified that

$$\nu(t) = \frac{1}{3(1 + t^2)}$$

satisfies (32), but $\nu(t) \leq \exp(-t)$ does not hold. In [35]–[37], updating laws in the form of (22) with $q = 1$ or $q = 2$ is designed to realize the AES, but only constant delay and differentiable time-varying delay are considered therein. Furthermore, the delay is also required to satisfy $\dot{\tau}(t) \leq \kappa < 1$ in most of the aforementioned results. As far as we know, the AES problem of networks with nondifferentiable time-varying delay has not been solved ever before. The main difficulty lies in that the delay is conventionally addressed via the term $\int_{t-\tau(t)}^t \mu^T(\theta)\mu(\theta)\mathrm{d}\theta$, while the differentiability of this term requires $\tau(t)$ be differentiable. Quite different from the conventional technique, the AES criteria here are derived by analyzing the boundedness of the control gain $\delta(t)$ and establishing a new delay differential inequality.

Remark 7: In fact, the control gain $\delta(t)$ cannot be updated infinitely fast. It can be derived from (22) to (31) that

$$\dot{\delta}(t) \leq kM_0^q$$

implying that $\dot{\delta}(t)$ is bounded, which ensures that the updating laws (22) is practical.

Remark 8: In order to determine the control gain $\delta(t)$, it suffices to determine the initial value $\delta(0)$ and the parameters k, q, a of the updating laws (22). The control benefits the synchronization when $\delta(t) > 0$, and otherwise when $\delta(t) < 0$. Hence, it is required that $\delta(0) \geq 0$. In general, $\delta(0)$ can be set as any nonnegative value at will but should not be too large in general, since large $\delta(0)$ could lead to large $\delta(t)$, which may be unrealistic. Moreover, it is observed from Theorem 2 that the parameters k, q, a have no influence on whether the synchronization can be realized but affect the synchronization speed. Sometimes, the objective is to realize synchronization as fast as possible, while sometimes the objective is to realize synchronization at a minimal cost. Therefore, the optimal k, q, a depend on the situation, and they also depend on network parameters. Anyway, the synchronization can be necessarily realized according to Theorem 2 regardless of the choice of parameters.

Remark 9: From Theorem 2, the synchronization of network (3) can be realized for arbitrary network parameters and is independent of how the parameters of the updating laws are chosen. That is to say, the synchronization is robust to both the network parameters and the parameters of the updating laws.

IV. Numerical Simulations

The Lorenz system [40] is a well-known benchmark chaotic system, which is described by

$$\dot{v}(t) = f(v(t), t) = \begin{bmatrix} c_1(v_2(t) - v_1(t)) \\ c_3v_1(t) - v_2(t) - v_1(t)v_3(t) \\ -c_2v_3(t) + v_1(t)v_2(t) \end{bmatrix} \tag{33}$$

where $c_1 = 10, c_2 = 8/3$, and $c_3 = 28$. It can be verified that assumption (A2) holds since the Lorenz system is ultimately bounded [41].

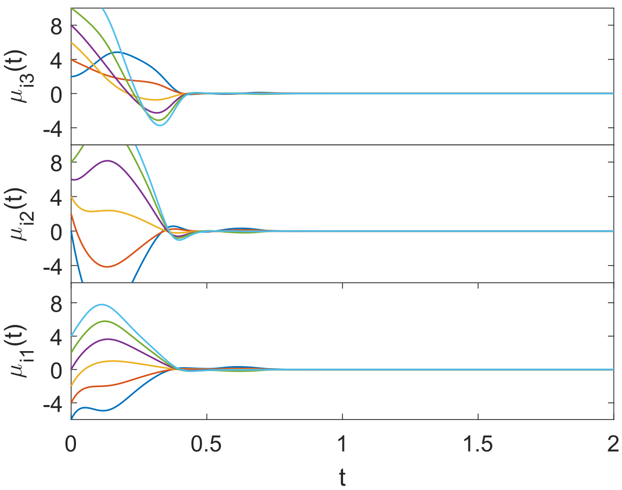

Fig. 1. Synchronization errors $\mu_{ij}(t)$ of network (34), where $1 \leq i \leq 4$ and $1 \leq j \leq 3$.

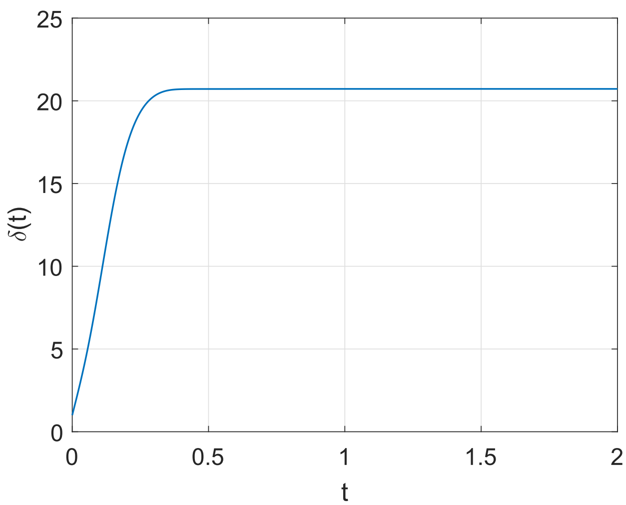

Fig. 2. Control gain $\delta(t)$ of network (34).

Consider a network consisting of six Lorenz systems

$$\dot{x}_i(t) = f(x_i(t), t) + c\sum_{j=1}^{6} a_{ij}\Gamma x_j(t-\tau(t)) - \delta(t)\mu_i(t) \quad (34)$$

where $1 \leq i \leq 6$, $c = 0.1$, and

$$\Gamma = \begin{bmatrix} 1 & 0.3 & -0.5 \\ 0 & 0.8 & -1 \\ 0.2 & -0.5 & 0.1 \end{bmatrix}$$

$$A = \begin{bmatrix} -6 & 1 & 0 & 3 & 0 & 2 \\ 1 & -5 & 0 & 2 & 2 & 0 \\ 2 & 0 & -7 & 3 & 0 & 2 \\ 0 & 1 & 2 & -5 & 1 & 1 \\ 1 & 2 & 3 & 0 & -7 & 1 \\ 0 & 0 & 1 & 2 & 1 & -4 \end{bmatrix}$$

$$\tau(t) = |2p+1-t|, \quad \text{if } t \in [2p, 2p+2)$$

with $p = 0, 1, 2, \ldots$ The updating laws is described by

$$\dot{\delta}(t) = k\|\mu(t)\|^q \exp(at) \quad (35)$$

where $k = 0.1$, $q = 2$, and $a = 0.5$.

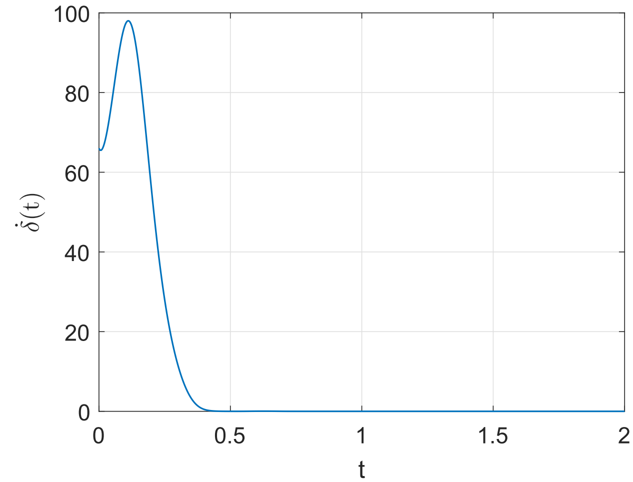

Fig. 3. Updating speed $\dot{\delta}(t)$ of the adaptive control gain of network (34).

Existing results [31]–[37] on AES cannot deal with network (34) as $\tau(t)$ is nondifferentiable, while ours can. Since assumption (A1) holds with $\tau_{\max} = 1$, network (3) realizes exponential synchronization according to Theorem 2.

Choose $\upsilon(0) = [0, -1, -5]^T$, $\delta(0) = 1$, and

$$x_i(t) = 2i + [-8, -3, -5]^T, \quad t \in [-1, 0]$$

where $1 \leq i \leq 6$. Fig. 1 demonstrates the components of the synchronization errors $\mu_i(t)$, verifying that the synchronization is realized. The control gain $\delta(t)$ and its updating speed $\dot{\delta}(t)$ are presented in Figs. 2 and 3. It can be seen that $\delta(t)$ and $\dot{\delta}(t)$ are both bounded. Moreover, the boundedness of $\dot{\delta}(t)$ together with (35) shows that the synchronization is exponential.

V. Conclusion

In this brief, the AES problem of complex networks with general time-varying delay has been studied, especially when the delay is nondifferentiable. A delay differential inequality extending the widely used Halanay inequality has been established to deal with the exponential stability of delayed nonlinear systems. The boundedness of the adaptive control gain has been proved so as to theoretically ensure that the adaptive control is practical in engineering. In virtue of the proposed inequality and the boundedness of the control gain, the AES criteria for networks with general delay have been established for the first time. In addition, it would be more practical if the adaptive control and the updating laws could be designed in a distributed way, but the theoretical analysis would be challenging and deserves further study.

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