Estimating the Region of Attraction on Controlled Complex Networks With Time-Varying Delay

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***Abstract*—The importance of estimating the region of attraction (ROA) on complex networks has drawn attention very recently. However, challenging problems arise when applying the existing theory to networks with delay. In this article, we estimate the ROA on controlled complex networks with time-varying delay. A delay-independent ROA estimation method is derived at first. This method can deal with general delay and is convenient to use due to its independence of the delay. Accordingly, it may cause conservativeness. To further reduce the conservativeness, delay-dependent ROA estimation is established for networks with small delay by developing a new technique of dealing with the delay. Numerical simulations are provided to verify the theoretical results.**

***Index Terms*—Complex network, equilibrium point, region of attraction (ROA), synchronization, time-varying delay.**

I. Introduction

Since the pioneering work of Watts and Strogatz [1], complex networks have drawn much attention in the past two decades as they can well describe the real world. A body of work [2]–[9] on complex networks has been reported, including synchronization, statistical mechanics, controllability, and topology identification.

A complex network realizes synchronization if the state difference among the nodes vanishes as time approaches to infinity. In reality, many networks cannot realize synchronization by themselves, so external control is generally added to drive all the nodes to a desired state, which could be an equilibrium, a periodic orbit, or a chaotic orbit. When the desired state is an equilibrium of the node system, the synchronized state is an equilibrium of the entire network.

As a fundamental problem of network science, the synchronization of networks has been extensively investigated. Global synchronization criteria [10]–[14] rely heavily on the global Lipschitz condition (GLC), which requires that the nonlinear node system should show weak nonlinearity and behave like linear systems. It should be highlighted that many nonlinear systems, such as $\dot{x} = -x + x^2$, do not satisfy the GLC. In the literature, local synchronization criteria [15]–[17] without the GLC are relatively rare, and they neglect the importance of the initial conditions in determining whether a network can realize synchronization. Focusing on synchronization toward an equilibrium, it is meaningful to estimate the region of attraction (ROA), which is the set of the initial conditions that admit network synchronization towards the equilibrium [18].

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In 2019, Zhu *et al.* [18] investigated the ROA estimation problem for complex networks without delay. During the transmission of signals, the time delay is hardly avoidable due to the finite transmission speed. It is noteworthy that the existing results on ROA estimation cannot address general time-varying delay. In contrast to the popularity of the ROA estimation of linear systems with various constraints [19]–[21] and nonlinear systems without delay [22]–[24], the ROA estimation of nonlinear systems with delay [25]–[28] receives much less attention. In [25]–[27], the ROA estimation of nonlinear systems with constant delay is established by constructing complicated Lyapunov–Krasovskii functionals. The Lyapunov–Krasovskii approach is also used in [28] to deal with time-varying delay. However, the conditions therein are quite restrictive, since the delay $\tau(t)$ should satisfy $\tau(t) \in [h, 2h]$ and $\dot{\tau}(t) < 1$, where $h > 0$. In engineering, the analytical expression of the delay is hardly precisely known, not to mention the differentiability. To this end, it is of practical importance to deal with the ROA estimation problem of networks with general delay.

Motivated by the above discussions, this article investigates the ROA estimation problem for controlled complex networks with time-varying delay. The main contributions are as follows.

1) By adopting the Razumikhin technique, a differential inequality is established (in Lemma 2) to estimate the ROA of general nonlinear systems with time-varying delay.
2) For networks with general delay, delay-independent ROA estimation is derived in virtue of the differential inequality that we establish. For networks with small delay, delay-dependent ROA estimation is established by developing a new technique of dealing with the delay, which is less conservative than the presented delay-independent estimation.
3) To our best knowledge, it is the first time that nondifferentiable time-varying delay has been addressed for the ROA estimation of nonlinear systems. Compared to the existing results (such as [25]–[28]), our results are much easier to use for estimating the ROA.

The rest of this article is organized as follows. Section II presents some preliminaries. Section III presents the main results on estimating the ROA of controlled complex networks with time-varying delay. Section IV provides numerical simulations to verify the theoretical results. Section V gives the conclusion.

II. Preliminaries

A. Notations and Problem Formulation

First, some notations are introduced. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all the $(n \times m)$-dimensional real matrices, respectively; $\mathbb{R}_+$ is the set of nonnegative real numbers; the superscript T represents the transpose of a vector or a matrix; $\|\cdot\|$ denotes the two-norm of a vector or a matrix; $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a symmetric matrix; I_n is the identity matrix of dimension n. For a square matrix A, define $A^{\mathrm{s}} = (A + A^T)/2$. For a continuous function ψ, define $\|\psi\|_{t,h} = \max_{\theta\in[t-h,t]} \|\psi(\theta)\|$, where h is a positive constant.

Consider a controlled complex network consisting of N identical systems with time-varying delay

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} a_{ij} g(x_j(t-\tau(t))) + u_i(t),\ t \geq 0 \\ x_i(t) = \varphi_i(t), \qquad t \in [-\tau_{\max}, 0] \end{cases} \tag{1}$$

where $1 \leq i \leq N,\ x_i \in \mathbb{R}^n$ is the state vector of the ith node, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable nonlinear functions, $c > 0$ denotes the coupling strength, $A = (a_{ij})_{N\times N}$ denotes the weighted outer coupling matrix, $\tau(t)$ is the time-varying delay that satisfies $0 \leq \tau(t) \leq \tau_{\max}$, $\tau_{\max}$ is a positive constant, $u_i(t)$ is the control input to the ith node, and $\varphi_i : [-\tau_{\max}, 0] \to \mathbb{R}^n$ is a continuous function that denotes the initial condition of the ith node. If there is an edge from node i to node j $(j \neq i)$, then $a_{ij} > 0$; otherwise, $a_{ij} = 0$; the diagonal elements of A are defined by $a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij}$.

Remark 1: In network (1), the functions f and g are required to be continuously differentiable, implying that they are locally Lipschitz. Therefore, the existence and uniqueness of the solution of network (1) can be guaranteed (see [29, Th. 2.3] for more details).

Let s be an equilibrium point of the isolated node system $\dot{x}(t) = f(x(t))$. Since $f(s) + c\sum_{j=1}^{N} a_{ij} g(s) = f(s) = 0$, the synchronized state

$$x_1 = x_2 = \cdots = x_N = s$$

is an equilibrium point of the entire network (1) if $u_i = 0$. Let the controllers be $u_i(t) = -de_i(t)$, where $d > 0$ is the control gain, and $e_i(t) = x_i(t) - s$ denotes the synchronization error of the ith node. Denote $e(t) = [e_1^T(t), \ldots, e_N^T(t)]^T$ and

$$S = [s^T, \ldots, s^T]^T \in \mathbb{R}^{nN}. \tag{2}$$

Since $[x_1^T, \ldots, x_N^T]^T$ converges to S if and only if $x_1, \ldots, x_N$ synchronizes to s, the equilibrium S of network (1) is globally (locally) asymptotically stable if and only if network (1) realizes global (local) synchronization toward S.

The objective of this article is to estimate the ROA of the equilibrium S. The ROA includes all the initial conditions starting from which network (1) can realize synchronization toward S. The exact ROA of S is denoted as

$$R_A = \left\{ \varphi \,\middle|\, \lim_{t\to+\infty} e(t;\varphi) = 0 \right\}$$

where the function $\varphi = [\varphi_1^T, \ldots, \varphi_N^T]^T$ is the initial condition of network (1). In this article, we aim to find an open subset of R_A in the following form:

$$R_{\text{est}}(r) = \{\varphi \in R_A |\ \|\varphi - S\|_{0,\tau_{\max}} < r\} \tag{3}$$

which means that network (1) can realize synchronization if the bound of the initial synchronization error is less than r. The radius r of the ball-like set $R_{\text{est}}(r)$ is expected to be as large as possible.

Let $F_1 = Df(s)$ and $F_2 = Dg(s)$ be the Jacobians of f and g evaluated at s, respectively. Then

$$\begin{aligned} \dot{e}_i(t) &= f(x_i(t)) - f(s) - de_i(t) \\ &\quad + c\sum_{j=1}^{N} a_{ij}[g(x_j(t-\tau(t))) - g(s)] \\ &= (F_1 - dI_n)e_i(t) + c\sum_{j=1}^{N} a_{ij} F_2 e_j(t-\tau(t)) \\ &\quad + R_i^{(1)}(t) + c\sum_{j=1}^{N} a_{ij} R_j^{(2)}(t-\tau(t)),\ 1 \leq i \leq N \end{aligned} \tag{4}$$

where

$$\begin{aligned} lR_i^{(1)}(t) &= f(x_i(t)) - f(s) - F_1 e_i(t) \\ R_i^{(2)}(t) &= g(x_i(t)) - g(s) - F_2 e_i(t). \end{aligned} \tag{5}$$

It follows that $\|R_i^{(k)}(t)\| = o(\|e_i(t)\|)$. Define

$$\alpha_1(x) = \begin{cases} \frac{\|f(s+x) - f(s) - F_1 x\|}{\|x\|}, & x \in \mathbb{R}^n\backslash\{0\} \\ 0, & x = 0 \end{cases}$$

$$\alpha_2(x) = \begin{cases} \frac{\|g(s+x) - g(s) - F_2 x\|}{\|x\|}, & x \in \mathbb{R}^n\backslash\{0\} \\ 0, & x = 0. \end{cases}$$

Then, there exist nondecreasing and continuous functions $\phi_k : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\phi_k(0) = 0$ and

$$\alpha_k(x) \leq \phi_k(r), \text{ if } \|x\| \leq r \tag{6}$$

which further gives

$$\|R_i^{(k)}(t)\| \leq \phi_k(\|e_i(t)\|)\|e_i(t)\|,\ 1 \leq k \leq 2. \tag{7}$$

Remark 2: Since α_1 and α_2 are continuous at $x = 0$, we can simply take $\phi_k(r) = \max\{\alpha_k(x) |\ \|x\| \leq r\}$, which is nondecreasing and continuous. Sometimes, it is difficult to solve the exact ϕ_k. In this case, it is better to estimate a ϕ_k that is slightly bigger than the exact one.

B. Mathematical Preliminaries

Lemma 1 (see[30]): Let $x, y \in \mathbb{R}^m$ be constant vectors, where m is a positive integer. Then, $2x^T y \leq x^T P x + y^T P^{-1} y$, where $P \in \mathbb{R}^{m\times m}$ is any positive definite matrix.

Lemma 2: Suppose that $h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing continuous function and $h_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function. Let $v(t)$ be a nonnegative continuous function on $[t_0 - \tau_0, +\infty)$ such that

$$\dot{v}(t) \leq -h_1(v(t))v(t) + h_2(\|v\|_{t,\tau_0})\|v\|_{t,\tau_0},\ t \geq t_0$$

where τ_0 is a positive constant. If $h_1(0) > h_2(0)$ and $M := \|v\|_{t_0,\tau_0} < r_0$, then

$$v(t) \leq M\exp(-\sigma(t - t_0)),\ t \geq t_0 \tag{8}$$

where $r_0 = \sup\{r |\ h_1(r) > h_2(r)\}$ and σ is the unique positive solution of the equation $\sigma = h_1(M) - h_2(M)\exp(\sigma\tau_0)$.

Proof: According to the definition of r_0, one has $r_0 > 0$ and $h_1(M) > h_2(M)$. Define $\rho(\varsigma) = \varsigma + h_2(M)\exp(\varsigma\tau_0) - h_1(M)$. Since ρ is continuous and increasing, $\rho(0) = h_2(M) - h_1(M) < 0$, and $\rho(+\infty) = +\infty$, it can be concluded that σ is unique and positive. In addition, one can select a constant $\varepsilon > 0$ such that $\alpha > \beta$, where $\alpha = h_1(M + \varepsilon)$ and $\beta = h_2(M + \varepsilon)$. Define $\varepsilon_1 = \varepsilon(\alpha - \beta) > 0$.

Consider the following comparison system:

$$\begin{cases} \dot{w}(t) = \varepsilon_1 - h_1(w(t))w(t) \\ \qquad\qquad + h_2(\|w\|_{t,\tau_0})\|w\|_{t,\tau_0},\ t \geq t_0 \\ w(t) = v(t), \qquad\qquad t \leq t_0. \end{cases} \tag{9}$$

We shall prove that

$$w(t) < M\exp(-\delta(t - t_0)) + \varepsilon := h(t),\ t \geq t_0 - \tau_0 \tag{10}$$

where δ is the unique positive solution of the equation

$$\delta = \alpha - \beta \exp(\delta\tau_0). \tag{11}$$

When $t \in [t_0 - \tau_0, t_0]$, one has $w(t) \le M < h(t)$. If (10) is not true, then

$$t_1 = \inf\{t > t_0 |\ w(t) \ge h(t)\} > t_0$$

is finite. Then, one has $w(t_1) = h(t_1)$ and

$$w(t) < h(t),\ t \in [t_0 - \tau_0, t_1).$$

It follows that $h_1(w(t)) \ge h_1(h(t)) \ge h_1(M + \varepsilon) = \alpha$ for $t \in [t_0, t_1)$. In addition, when $t \in [t_0, t_1)$, there exists a $\hat{t} \in [t_0 - \tau_0, t_1)$ such that $\|w\|_{t,\tau_0} = w(\hat{t})$. If $\hat{t} \le t_0$, $w(\hat{t}) = v(\hat{t}) \le M$; otherwise, $w(\hat{t}) < h(\hat{t}) < M + \varepsilon$. It then follows that $h_2(\|w\|_{t,\tau_0}) \le h_2(M + \varepsilon) = \beta$ for $t \in [t_0, t_1)$. Then, one has

$$\dot{w}(t) \le \varepsilon_1 - \alpha w(t) + \beta \|w\|_{t,\tau_0},\ t \in [t_0, t_1)$$

which further yields

$$\frac{\mathrm{d}}{\mathrm{d}t}[w(t)\exp(\alpha t)] \le \exp(\alpha t)[\varepsilon_1 + \beta\|w\|_{t,\tau_0}],\ t \in [t_0, t_1). \tag{12}$$

Integrating (12) from t_0 to t_1 gives

$$\begin{aligned} &w(t_1)\exp(\alpha t_1) - w(t_0)\exp(\alpha t_0) \\ &\le \int_{t_0}^{t_1} \exp(\alpha t)[\beta\|w\|_{t,\tau_0} + \varepsilon_1]\mathrm{d}t. \end{aligned}$$

Since $w(t_0) \le M$ and $\|w\|_{t,\tau_0} \le h(t - \tau_0)$ for $t \in [t_0, t_1]$, it is deduced that

$$\begin{aligned} w(t_1) \le\ & M\left(1 - \frac{\beta\exp(\delta\tau_0)}{\alpha - \delta}\right)\exp(-\alpha(t_1 - t_0)) \\ &+ \frac{M\beta\exp(\delta\tau_0)}{\alpha - \delta}\exp(-\delta(t_1 - t_0)) \\ &+ \varepsilon[1 - \exp(-\alpha(t_1 - t_0))]. \end{aligned}$$

It then follows from (11) that $w(t_1) < h(t_1)$, which contradicts $w(t_1) = h(t_1)$. Therefore, (10) holds. According to the comparison principle, one has

$$v(t) < w(t) < h(t),\ t > t_0.$$

Letting $\varepsilon \to 0^+$ leads to $\alpha \to h_1(M)$, $\beta \to h_2(M)$, and $\delta \to \sigma$ according to (11) and the definition of σ. Therefore, inequality (8) is verified.

Remark 3: By adopting the Razumikhin technique, Lemma 2 extends the well-known Halanay inequality [31]. The Halanay inequality has constant coefficients for $v(t)$ and $\|v\|_{t,\tau_0}$, and can well address the global stability of delayed differential equations. Here, the coefficients are extended to the form of functions so as to deal with the local stability of delayed differential equations and estimate the ROA.

Lemma 3: If $v(t)$ is a nonnegative continuous function on $[t_0 - \tau_0, +\infty)$ such that

$$\dot{v}(t) \le q_1 v(t) + q_2\|v\|_{t,\tau_0},\ t \ge t_0$$

then,

$$v(t) \le M\exp[q(t - t_0)],\ t \ge t_0.$$

where $\tau_0 > 0$, $q = \max\{0, q_1 + q_2\}$, $q_1 \in \mathbb{R}$, $q_2 > 0$, and $M = \|v\|_{t_0,\tau_0}$.

Proof: When $q_1 + q_2 < 0$, one has $q_1 < 0$. Then, applying Lemma 2 gives $v(t) \le M \le M\exp[q(t - t_0)]$ for $t \ge t_0$.

Next, it is to discuss the case of $q_1 + q_2 \ge 0$. Let ϵ be any positive constant. If

$$v(t) < h(t) := (M + \epsilon)\exp[q(t - t_0)],\ t \ge t_0 \tag{13}$$

does not hold, then $t_1 = \inf\{t > t_0 |\ v(t) \ge h(t)\} > t_0$ is finite. It then follows that $v(t_1) = h(t_1)$ and $v(t) < h(t)$ for $t \in [t_0, t_1)$. Since $\frac{\mathrm{d}}{\mathrm{d}t}[v(t)\exp(-q_1 t)] \le q_2\exp(-q_1 t)\|v\|_{t,\tau_0}$, it is deduced that

$$\begin{aligned} &v(t_1)\exp(-q_1 t_1) - v(t_0)\exp(-q_1 t_0) \\ &\le q_2\int_{t_0}^{t_1}\exp(-q_1 t)\|v\|_{t,\tau_0}\mathrm{d}t \le q_2\int_{t_0}^{t_1}\exp(-q_1 t)h(t)\mathrm{d}t, \end{aligned}$$

which further gives

$$v(t_1) \le h(t_1) - [M + \epsilon - v(t_0)]\exp[q_1(t_1 - t_0)].$$

Since $v(t_0) \le M$, one has $v(t_1) < h(t_1)$, a contradiction. Hence, (13) holds. Letting $\epsilon \to 0^+$ completes the proof.

III. Main Results

In this section, two kinds of ROA estimation are established for networks with time-varying delay.

A. Delay-Independent ROA Estimation

We now establish the first kind of ROA estimation, which is named *delay-independent ROA estimation* as the radius of the estimated ROA is independent of the delay.

Theorem 1: Consider network (1). If the control gain d satisfies

$$d > d_0 := \lambda_{\max}(F_1^{\mathrm{s}}) + c\|A\|\|F_2\| \tag{14}$$

then network (1) realizes at least local synchronization toward the equilibrium S, and the ROA is estimated as $R_{\mathrm{est}}(r_0)$ with

$$r_0 = \sup\{r |\ \phi_1(r) + c\|A\|\phi_2(r) < d - d_0\} \tag{15}$$

where the symbols are defined in (2), (3), (4), and (6). Particularly, the synchronization is global if $r_0 = +\infty$.

Proof: Consider the following Lyapunov function candidate: $V(e(t)) = \sum_{i=1}^N e_i^T(t)e_i(t)$, which is also denoted as $V(t)$ for brevity. Differentiating V along the solution of (4) gives

$$\begin{aligned} \dot{V}(t) =\ & 2\sum_{i=1}^N e_i^T(t)(F_1 - dI_n)e_i(t) \\ &+ 2c\sum_{i=1}^N\sum_{j=1}^N a_{ij}e_i^T(t)F_2 e_j(t - \tau(t)) \\ &+ 2\sum_{i=1}^N e_i^T(t)R_i^{(1)}(t) \\ &+ 2c\sum_{i=1}^N\sum_{j=1}^N a_{ij}e_i^T(t)R_j^{(2)}(t - \tau(t)) \\ =\ & -2de^T(t)e(t) + 2e^T(t)[F_1^{\mathrm{s}} \otimes I_n]e(t) \\ &+ 2e^T(t)R^{(1)}(t) + 2ce^T(t)(A \otimes F_2)e(t - \tau(t)) \\ &+ 2ce^T(t)(A \otimes I_n)R^{(2)}(t - \tau(t)) \end{aligned} \tag{16}$$

where

$$R^{(k)}(t) = \left[{R_1^{(k)}}^T(t), \ldots, {R_N^{(k)}}^T(t)\right]^T,\ 1 \le k \le 2. \tag{17}$$

According to (7), one has $\|R_i^{(k)}(t)\| \leq \phi_k(\|e(t)\|)\|e_i(t)\|$, which further yields

$$\|R^{(k)}(t)\| \leq \phi_k(\|e(t)\|)\|e(t)\|,\ 1 \leq k \leq 2. \tag{18}$$

In virtue of Lemma 1, it is derived that

$$\begin{aligned} & 2ce^T(t)(A\otimes F_2)e(t-\tau(t)) \\ & \leq \frac{1}{\rho_1}c^2 e^T(t)(A\otimes F_2)(A\otimes F_2)^T e(t) + \rho_1\|e(t-\tau(t))\|^2 \\ & \leq \frac{1}{\rho_1}c^2\|A\|^2\|F_2\|^2\|e(t)\|^2 + \rho_1\hat{V}(t) \end{aligned} \tag{19}$$

where $\hat{V}(t) = \|V\|_{t,\tau_{\max}}$ and $\rho_1 > 0$ is a parameter to be determined. Similarly, one obtains

$$\begin{aligned} & 2ce^T(t)(A\otimes I_n)R^{(2)}(t-\tau(t)) \\ & \leq \frac{1}{\rho_2}c^2\|A\|^2\|e(t)\|^2 + \rho_2\|R^{(2)}(t-\tau(t))\|^2 \\ & \leq \left(\frac{1}{\rho_2}c^2\|A\|^2 + \rho_2\phi_k^2\left(\sqrt{\hat{V}(t)}\right)\right)\hat{V}(t) \end{aligned} \tag{20}$$

where $\rho_2 > 0$ is a parameter to be determined. According to the inequalities (16)–(20), one has

$$\begin{aligned} \dot{V}(t) \leq & -2d\|e(t)\|^2 + 2\lambda_{\max}(F_1^{\mathrm{s}})\|e(t)\|^2 \\ & + 2\phi_1(\|e(t)\|)\|e(t)\|^2 + \frac{1}{\rho_1}c^2\|A\|^2\|F_2\|^2\|e(t)\|^2 \\ & + \rho_1\hat{V}(t) + \left(\frac{1}{\rho_2}c^2\|A\|^2 + \rho_2\phi_k^2\left(\sqrt{\hat{V}(t)}\right)\right)\hat{V}(t) \\ = & -h_1(V(t))V(t) + h_2(\hat{V}(t))\hat{V}(t) \end{aligned} \tag{21}$$

where $h_1(x) = 2d - 2\lambda_{\max}(F_1^{\mathrm{s}}) - 2\phi_1(\sqrt{x}) - \frac{c^2\|A\|^2\|F_2\|^2}{\rho_1}$ and $h_2(x) = \rho_1 + \frac{1}{\rho_2}c^2\|A\|^2 + \rho_2\phi_2^2(\sqrt{x})$.

To prove $\lim_{t\to+\infty} V(t) = 0$, we wish $h_1(x) > h_2(x)$ on some interval. In order to maximize $h_1(x) - h_2(x)$, one should take $\rho_1 = c\|A\|\|F_2\|$ and $\rho_2 = \frac{c\|A\|}{\phi_2(\sqrt{x})}$. This leads to

$$\begin{aligned} h_1(x) &= 2d - 2\lambda_{\max}(F_1^{\mathrm{s}}) - 2\phi_1(\sqrt{x}) - c\|A\|\|F_2\| \\ h_2(x) &= c\|A\|\|F_2\| + 2c\|A\|\phi_2(\sqrt{x}). \end{aligned} \tag{22}$$

Then

$$h_1(x) - h_2(x) = 2(d-d_0) - 2[\phi_1(\sqrt{x}) + c\|A\|\phi_2(\sqrt{x})]. \tag{23}$$

According to (15), one has

$$\begin{aligned} r_0^2 &= \sup\{r^2 |\ \phi_1(r) + c\|A\|\phi_2(r) < d - d_0\} \\ &= \sup\{r |\ \phi_1(\sqrt{r}) + c\|A\|\phi_2(\sqrt{r}) < d - d_0\} \\ &= \sup\{r |\ h_1(r) > h_2(r)\}. \end{aligned}$$

Letting the initial condition $\varphi \in R_{\mathrm{est}}(r_0)$, one has

$$M := \max_{\theta\in[t_0-\tau_{\max},t_0]} V(\theta) < r_0^2.$$

Since $h_1(0) > h_2(0)$ and $M < r_0^2$, it follows from Lemma 2 that $V(t) \to 0$ as $t \to +\infty$, that is, $\lim_{t\to+\infty} e(t) = 0$. Hence, network (1) realizes at least local synchronization toward S, and $R_{\mathrm{est}}(r_0)$ is a subset of the ROA of S.

If $r_0 = +\infty$, one has $\lim_{t\to+\infty} e(t) = 0$ for any initial condition φ that is continuous on $[-\tau_{\max}, 0]$, so network (1) realizes global synchronization toward S.

Remark 4: The ROA estimation established in Theorem 1 is delay-independent, since r_0 is independent of the delay. Denote $\phi = \phi_1 + c\|A\|\phi_2$. Since ϕ is continuous and $\phi(0) = 0$, one has $r_0 > 0$ if $d > d_0$. That is, the estimated ROA is not empty if the control gain is large enough. In addition, if f and g are globally Lipschitz, it can be verified that ϕ_1 and ϕ_2 are bounded. Hence, $r_0 = +\infty$ if the control gain is large enough. That is, global synchronization can be realized.

Remark 5: In the literature, results on the ROA estimation of nonlinear systems with delay are quite rare. In [25]–[27], only the constant delay is considered. In [28], the time-varying delay is considered. However, the conditions therein are quite restrictive, since it is assumed that $\tau(t) \in [h, 2h]$ and $\dot{\tau}(t) < 1$, where $h > 0$. To the best of our knowledge, it is the first time that general delay has been addressed for the ROA estimation of nonlinear systems. Moreover, the results in [25]–[28] are established by constructing Lyapunov–Krasovskii functionals and need to solve a complicated Lyapunov differential equation. In contrast, the results here are much easier to use for estimating the ROA.

If the function g is linear, network (1) can be simplified to the following form:

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t-\tau(t)) + u_i(t),\ t\geq 0 \\ x_i(t) = \varphi_i(t), \qquad t\in[-\tau_{\max},0] \end{cases} \tag{24}$$

where $\Gamma \in \mathbb{R}^n$ is the inner coupling matrix. For network (24), the following corollary can be deduced.

Corollary 1: Consider network (24). If the control gain d satisfies

$$d > d_0 := \lambda_{\max}(F_1^{\mathrm{s}}) + c\|A\|\|\Gamma\|$$

then network (24) realizes at least local synchronization toward the equilibrium S, and the ROA is estimated as $R_{\mathrm{est}}(r_0)$ with

$$r_0 = \sup\{r |\ \phi_1(r) < d - d_0\}$$

where S and $R_{\mathrm{est}}(r_0)$ are defined in (2) and (3), respectively. Particularly, the synchronization is global if $r_0 = +\infty$.

Proof: The conclusion can be simply derived from Theorem 1 as $\phi_2 = 0$, where ϕ_2 is as defined in (6).

B. Delay-Dependent ROA Estimation

Though convenient to use, the previously established delay-independent ROA estimation may be conservative as it does not make full use of the delay. By developing a new technique of dealing with the delay, we establish another kind of ROA estimation, which is named *delay-dependent ROA estimation* as the radius of the estimated ROA depends on the delay. Compared to the delay-independent ROA estimation, the delay-dependent ROA estimation is less conservative when the delay is small.

For convenience, rewrite the error system (4) in the following compact form:

$$\begin{aligned} \dot{e}(t) = & I_N\otimes(F_1 - dI_n)e(t) + c(A\otimes F_2)e(t-\tau(t)) \\ & + R^{(1)}(t) + c(A\otimes I_n)R^{(2)}(t-\tau(t)). \end{aligned} \tag{25}$$

where $R^{(1)}$ and $R^{(2)}$ are defined in (17).

Theorem 2: Consider network (1), and suppose that the coupling delay satisfies

$$\varrho_0 := \tau_{\max}c\|A\|\|F_2\| < 1. \tag{26}$$

If the control gain d satisfies

$$d > d_0 := \frac{\lambda_{\max}(F_1^{\mathrm{s}}) + c\lambda_{\max}([A \otimes F_2]^{\mathrm{s}}) + \varrho_0\alpha_0}{1-\varrho_0} \tag{27}$$

then network (1) realizes at least local synchronization toward the equilibrium S, and the ROA is estimated as $R_{\mathrm{est}}(r_0)$ with $r_0 = \exp(-\varrho_0)\bar{r}$, where $\alpha_0 = \|F_1\| + c\|A\|\|F_2\|$

$$\bar{r} = \sup\left\{ r \;\middle|\; \phi_1(r) + c\|A\|\phi_2(r) < \frac{1-\varrho_0}{1+\varrho_0}(d-d_0) \right\}$$

and the other symbols are defined in (2), (3), (4), and (6). Particularly, the synchronization is global if $r_0 = +\infty$.

Proof: Consider the following Lyapunov function candidate: $V(e(t)) = \sum_{i=1}^N e_i^T(t)e_i(t)$, which is also denoted as $V(t)$ for brevity. According to (16), one has

$$\begin{aligned}\dot{V}(t) = & -2de^T(t)e(t) + 2e^T(t)[F_1^{\mathrm{s}} \otimes I_n]e(t) \\ & + 2e^T(t)R^{(1)}(t) + 2ce^T(t)(A \otimes F_2)e(t-\tau(t)) \\ & + 2ce^T(t)(A \otimes I_n)R^{(2)}(t-\tau(t)).\end{aligned} \tag{28}$$

It follows from (25) that

$$\begin{aligned}\|\dot{e}(t)\| \le & \|F_1 - dI_n\|\|e(t)\| + c\|A\|\|F_2\|\|e(t-\tau(t))\| \\ & + \phi_1(\|e(t)\|)\|e(t)\| \\ & + c\|A\|\phi_2(\|e(t-\tau(t))\|)\|e(t-\tau(t))\| \\ \le & \,\varphi\left(\|e\|_{t,\tau_{\max}}\right)\|e\|_{t,\tau_{\max}},\ t \ge 0\end{aligned} \tag{29}$$

where $\varphi(x) = d + \alpha_0 + \phi_1(x) + c\|A\|\phi_2(x)$. Since $\|e(t) - e(t-\tau(t))\| \le \int_{t-\tau(t)}^t \|\dot{e}(s)\|\mathrm{d}s$, one gets

$$\|e(t) - e(t-\tau(t))\| \le \tau_{\max}\|\dot{e}\|_{t,\tau_{\max}}. \tag{30}$$

Considering (29) and (30), one has

$$\begin{aligned}& ce^T(t)(A \otimes F_2)e(t-\tau(t)) \\ & = ce^T(t)(A \otimes F_2)e(t) + ce^T(t)(A \otimes F_2)[e(t-\tau(t)) - e(t)] \\ & \le e^T(t)[A \otimes F_2]^{\mathrm{s}}e(t) + \varrho_0\|e(t)\|\|\dot{e}\|_{t,\tau_{\max}} \\ & \le \lambda_{\max}([A \otimes F_2]^{\mathrm{s}})V(t) + \varrho_0\varphi\left(\sqrt{\tilde{V}(t)}\right)\tilde{V}(t)\end{aligned} \tag{31}$$

where $t \ge \tau_{\max}$ and $\tilde{V}(t) = \|V\|_{t,2\tau_{\max}}$. Recalling (18), one has

$$\begin{aligned}& ce^T(t)(A \otimes I_n)R^{(2)}(t-\tau(t)) \\ & \le c\|A\|\phi_2\left(\sqrt{\tilde{V}(t)}\right)\tilde{V}(t),\ t \ge \tau_{\max}.\end{aligned} \tag{32}$$

It then follows from (28), (31), and (32) that

$$\dot{V}(t) \le -g_1(V(t))V(t) + g_2(\tilde{V}(t))\tilde{V}(t),\ t \ge \tau_{\max}$$

where

$$\begin{aligned} g_1(x) &= 2d - 2\lambda_{\max}(F_1^{\mathrm{s}}) - 2c\lambda_{\max}([A \otimes F_2]^{\mathrm{s}}) - 2\phi_1(\sqrt{x}) \\ g_2(x) &= 2\varrho_0[d + \alpha_0 + \phi_1(\sqrt{x})] + 2c\|A\|(1+\varrho_0)\phi_2(\sqrt{x}).\end{aligned}$$

By simple calculation, one gets

$$\frac{g_1(x) - g_2(x)}{2(1+\varrho_0)} = \frac{1-\varrho_0}{1+\varrho_0}(d-d_0) - [\phi_1(\sqrt{x}) + c\|A\|\phi_2(\sqrt{x})].$$

Following the proof after (23), it is deduced that network (1) realizes at least local synchronization toward the equilibrium S, and the synchronization can be realized if

$$M_1 := \|e\|_{\tau_{\max},2\tau_{\max}} < \bar{r}. \tag{33}$$

Let the initial condition $\varphi \in R_{\mathrm{est}}(r_0)$. Next, we show that (33) holds. Denoting $M = \|e\|_{0,\tau_{\max}}$, one has

$$M < r_0 = \exp(-\varrho_0)\bar{r}.$$

Define $t_1 = \min\{\tau_{\max}, t_2\}$, where

$$t_2 = \inf\{t > 0|\ \|e(t)\| \ge \bar{r}\} > 0.$$

Note that t_2 could be $+\infty$. Then, one has

$$r_1 := \max_{\theta \in [-\tau_{\max}, t_1]} \|e(\theta)\| \le \bar{r}. \tag{34}$$

Denoting $\hat{V}(t) = \|V\|_{t,\tau_{\max}}$, it follows from (21) and (22) that

$$\dot{V}(t) \le -h_1(V(t))V(t) + h_2(\hat{V}(t))\hat{V}(t)$$

where $h_1(x) = 2d - 2\lambda_{\max}(F_1^{\mathrm{s}}) - 2\phi_1(\sqrt{x}) - c\|A\|\|F_2\|$ and $h_2(x) = c\|A\|\|F_2\| + 2c\|A\|\phi_2(\sqrt{x})$. When $t \in [0, t_1]$, one has $h_1(V(t)) \ge h_1(r_1^2) := \bar{h}_1$ and $h_2(\hat{V}(t)) \le h_2(r_1^2) := \bar{h}_2$. It then follows that

$$\dot{V}(t) \le -\bar{h}_1V(t) + \bar{h}_2\hat{V}(t),\ \forall t \in [0, t_1].$$

In virtue of Lemma 3, one gets

$$V(t) \le M^2\exp(\bar{h}t),\ \forall t \in [0, t_1] \tag{35}$$

where $\bar{h} = \max\{0, \bar{h}_2 - \bar{h}_1\}$. In view of (34) and the definition of $\bar{r}$, one gets

$$\begin{aligned}& (\bar{h}_2 - \bar{h}_1)/2 \\ & = \phi_1(r_1) + c\|A\|\phi_2(r_1) - d + \lambda_{\max}(F_1^{\mathrm{s}}) + c\|A\|\|F_2\| \\ & \le \frac{1-\varrho_0}{1+\varrho_0}(d-d_0) - d + \lambda_{\max}(F_1^{\mathrm{s}}) + c\|A\|\|F_2\| \\ & \le -d_0 + \lambda_{\max}(F_1^{\mathrm{s}}) + c\|A\|\|F_2\| \\ & \le -[\lambda_{\max}(F_1^{\mathrm{s}}) + c\lambda_{\max}([A \otimes F_2]^{\mathrm{s}})] \\ & \quad + \lambda_{\max}(F_1^{\mathrm{s}}) + c\|A\|\|F_2\|.\end{aligned} \tag{36}$$

Since 0 is an eigenvalue of A, 0 is also an eigenvalue of $A \otimes F_2$, indicating that there exists a vector $\mu \neq 0$ satisfying $(A \otimes F_2)\mu = 0$. It then follows that:

$$\mu^T[A \otimes F_2]^{\mathrm{s}}\mu = \mu^T(A \otimes F_2)\mu = 0$$

which further gives $\lambda_{\max}([A \otimes F_2]^{\mathrm{s}}) \ge 0$. Hence, $\bar{h} \le 2c\|A\|\|F_2\|$. Recalling (35), one has

$$\|e(t)\| \le M\exp(c\|A\|\|F_2\|t),\ \forall t \in [0, t_1].$$

If $t_1 < \tau_{\max}$, then

$$\|e(t)\| < M\exp(\varrho_0) < \bar{r},\ \forall t \in [0, t_1]$$

which contradicts (34). Hence, $t_1 = \tau_{\max}$. Then

$$M_1 = \|e\|_{\tau_{\max},2\tau_{\max}} \le M\exp(\varrho_0) < \bar{r}$$

that is, (33) holds. Therefore, $R_{\mathrm{est}}(r_0)$ is a subset of the ROA of the equilibrium S.

Remark 6: The delay-dependent ROA estimation is established in Theorem 2. Compared to the delay-independent ROA estimation in Theorem 1, the delay-dependent ROA estimation is less conservative when the delay is small.

For convenience, redefine the d_0, r_0 defined in Theorems 1 and 2 as $d_{\rm th1}, r_{\rm th1}$, and $d_{\rm th2} = d_{\rm th2}(\tau_{\max}), r_{\rm th2} = r_{\rm th2}(\tau_{\max})$, respectively. We now compare the two kinds of estimation.

1) It is obvious that $d_{\rm th1}, r_{\rm th1}$ do not depend on $\tau_{\max}$, while $d_{\rm th2}, r_{\rm th2}$ depend on $\tau_{\max}$. It is found that $d_{\rm th2}$ strictly increases in $\tau_{\max}$ and $r_{\rm th2}$ decreases (not strictly) in $\tau_{\max}$, implying that the results of Theorem 2 get better as $\tau_{\max}$ becomes smaller. In addition, $r_{\rm th2}$ strictly decreases in $\tau_{\max}$ if $r_{\rm th2}(0) < +\infty$.
2) The limits $\lim_{\tau_{\max}\to 0} d_{\rm th2}$ and $\lim_{\tau_{\max}\to 0} r_{\rm th2}$ exist, which are denoted as $\bar{d}_0$ and $\bar{r}_0$ for brevity. Since
$$\lambda_{\max}([A \otimes F_2]^{\rm s}) \le \|A \otimes F_2\| = \|A\|\|F_2\|. \tag{37}$$
Then, it is easy to verify that
$$\bar{d}_0 = \lambda_{\max}(F_1^{\rm s}) + c\lambda_{\max}([A\otimes F_2]^{\rm s}) \le d_{\rm th1}$$
$$\bar{r}_0 = \sup\left\{r \mid \phi_1(r) + c\|A\|\phi_2(r) < d - \bar{d}_0\right\} \ge r_{\rm th1}.$$
Therefore, Theorem 2 is generally better than Theorem 1 when $\tau_{\max}$ is small enough.
3) If
$$\lambda_{\max}([A \otimes F_2]^{\rm s}) < \|A\|\|F_2\| \tag{38}$$
then there exists a $\tau_2 \in (0, \tau_1)$ such that
$$d_{\rm th2} < d_{\rm th1}, \qquad \forall \tau_{\max} \in (0, \tau_1) \tag{39}$$
$$r_{\rm th2} \ge r_{\rm th1}, \qquad \forall \tau_{\max} \in (0, \tau_2) \tag{40}$$
where $\tau_1 = \frac{\|A\|\|F_2\| - \lambda_{\max}([A\otimes F_2]^{\rm s})}{(\alpha_0 + d_{\rm th1})\|A\|\|F_2\|}$. The inequality (39) is straightforward. It is easy to verify that $r_{\rm th2}(0) = \bar{r}_0 \ge r_{\rm th1}$ and
$$\begin{aligned} r_{\rm th2}(\tau_1) \le\ & \exp(-\rho_0) \\ & \times \sup\{r \mid \phi_1(r) + c\|A\|\phi_2(r) < d - d_{\rm th2}\} \\ =\ & \exp(-\rho_0) \\ & \times \sup\{r \mid \phi_1(r) + c\|A\|\phi_2(r) < d - d_{\rm th1}\} \\ =\ & r_{\rm th1}\exp(-\rho_0) \le r_{\rm th1} \end{aligned}$$
where $\rho_0 = \tau_1 c\|A\|\|F_2\|$ in this case. Then, there exists a $\tau_2 \in (0, \tau_1)$ satisfying (40). The above analysis indicates that, under (38)

1) the control gain threshold $d_{\rm th2}$ and the radius $r_{\rm th2}$ estimated by Theorem 2 are better than those estimated by Theorem 1 when $\tau_{\max} \in (0, \tau_2)$;
2) only $d_{\rm th2}$ is better when $\tau_{\max} \in (\tau_2, \tau_1)$; and
3) both $d_{\rm th2}$ and $r_{\rm th2}$ are worse when $\tau_{\max} > \tau_1$.

It is verified via simulations that inequality (38) holds in most cases. Furthermore, if $r_{\rm th1} < +\infty$, inequality (40) becomes
$$r_{\rm th2} > r_{\rm th1}, \qquad \forall \tau_{\max} \in (0, \tau_2)$$
i.e., the radius $r_{\rm th2}$ estimated by Theorem 2 is strictly better than that estimated by Theorem 1.

Corollary 2: Consider network (1). Suppose that the coupling matrix A is symmetric and the coupling delay satisfies
$$\varrho_0 := \tau_{\max} c\|A\|\|F_2\| < 1.$$
If $F_2^{\rm s}$ is semipositive definite and the control gain d satisfies
$$d > d_0 := \frac{\lambda_{\max}(F_1^{\rm s}) + \varrho_0(\|F_1\| + c\|A\|\|F_2\|)}{1 - \varrho_0}$$
then network (1) realizes at least local synchronization toward the equilibrium S, and the ROA is estimated as $R_{\rm est}(r_0)$ with $r_0 = \exp(-\varrho_0)\bar{r}$, where
$$\bar{r} = \sup\left\{ r \,\middle|\, \phi_1(r) + c\|A\|\phi_2(r) < \frac{1-\varrho_0}{1+\varrho_0}(d - d_0) \right\}$$
and the other symbols are defined in (2), (3), (4), and (6). Particularly, the synchronization is global if $r_0 = +\infty$.

Proof: Since A is symmetric, it follows that $\lambda_{\max}(A) = 0$ and $[A \otimes F_2]^{\rm s} = A \otimes F_2^{\rm s}$. In view that $F_2^{\rm s}$ is semi-positive definite, one has $\lambda_{\max}([A\otimes F_2]^{\rm s}) = 0$. Then, the conclusion can be simply derived from Theorem 2.

IV. Numerical Simulations

Consider a controlled network consisting of four nodes
$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + c\sum\limits_{j=1}^{4} a_{ij} g(x_j(t - \tau(t))) - de_i(t) \\ x_i(t) = \varphi_i(t), \qquad t \in [-\tau_{\max}, 0] \end{cases} \tag{41}$$
where $c = 1$, $\tau(t) = \tau_{\max}|\sin t|$, $\tau_{\max} > 0$ is a parameter to be discussed, and
$$f(z(t)) = [-z_1(t) + z_1(t)z_2(t), -z_2(t) + z_2^2(t)]^T$$
$$g(z(t)) = [z_2(t), -c_1\sin z_1(t) - c_2 z_2(t)]^T$$
$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -3 & 2 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$
with $z(t) = [z_1(t), z_2(t)]^T \in \mathbb{R}^2$ and $c_1 = c_2 = 0.1$. Let $s = [s_1, s_2]^T$ be an equilibrium of $\dot{x} = f(x)$. Solving $\begin{cases} -s_1 + s_1 s_2 = 0 \\ -s_2 + s_2^2 = 0 \end{cases}$ yields $s = [0, 0]^T$ or $s = [c_0, 1]$, where c_0 can be any real constant. It is verified that only $[0, 0]^T$ is a stable equilibrium.

To compare the delay-independent ROA estimation and the delay-dependent ROA estimation, both Theorems 1 and 2 will be used to estimate the ROA of the equilibrium $S = 0$ of network (41).

A. Delay-Independent ROA Estimation

Theorem 1 is first used to estimate the ROA of the equilibrium $S = 0$ of network (41). Set $\tau_{\max} = 100$.

By simple calculation, one gets $F_1 = -I_2$, $F_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.1 \end{bmatrix}$, $\alpha_1(z) = |z_2| \le \|z\|$, and $\alpha_2(z) = \frac{0.1|z_1 - \sin z_1|}{\|z\|}$. According to [18, Example 1], one has $\alpha_2(z) \le 0.1(1 - \frac{\sin\|z\|}{\|z\|})$. Then, take $\phi_1(r) = r$ and
$$\phi_2(r) = \begin{cases} 1 - \frac{\sin r}{r}, & r \in [0, \frac{\pi}{2}] \\ 1 - \frac{1}{r}, & r \in (\frac{\pi}{2}, +\infty). \end{cases}$$

According to (14), network (41) realizes at least local synchronization toward the equilibrium S if
$$d > d_0 = 2.9936.$$

Taking $d = 10$, it follows from Theorem 1 that $R_{\rm est}(r_0)$ with $r_0 = 4.0211$ is a subset of the ROA of $S = 0$.

Let the initial condition be
$$\varphi_i(t) = 0.3i + [0, -2]^T, \ t \in [-1, 0]$$
where $1 \le i \le 4$. Then, one has $\varphi \in R_{\rm est}(r_0)$, i.e., the initial condition belongs to the ROA. According to Theorem 1, the synchronization errors e_i converges to 0, as verified by Fig. 1(a). This validates the effectiveness of Theorem 1.

Reset $\varphi_1(t) = [0, 11]^T$ and $\varphi_2(t) = \varphi_3(t) = \varphi_4(t) = 0$ for $t \in [-1, 0]$. Then, one has $\varphi \notin R_{\rm est}(r_0)$. It is shown in Fig. 1(b) that

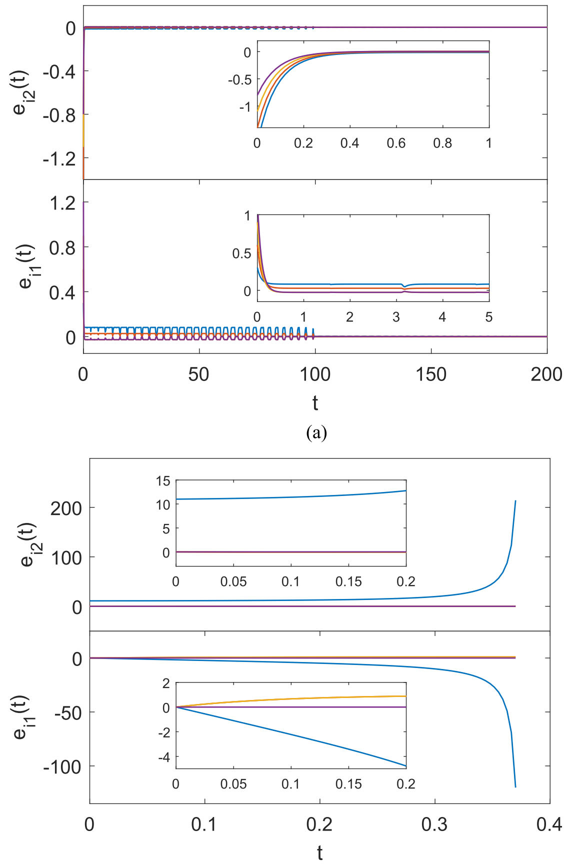

Fig. 1. Synchronization errors $e_{ij}(t)$ $(1 \leq i \leq 4, 1 \leq j \leq 2)$ of network (41): (a) the initial condition belongs to the estimated ROA, i.e., $\varphi \in R_{\text{est}}(r_0)$; (b) the initial condition does not belong to the estimated ROA, i.e., $\varphi \notin R_{\text{est}}(r_0)$. It is shown that the synchronization is realized when $\varphi \in R_{\text{est}}(r_0)$, but not realized when $\varphi \notin R_{\text{est}}(r_0)$.

some components of the synchronization error e diverge to infinity, which implies that the synchronization may not be realized when $\varphi \notin R_{\text{est}}(r_0)$. This example demonstrates the importance of the initial condition in determining whether the synchronization can be realized. In addition, this example also implies that the estimated r_0 cannot exceed 11.

B. Delay-Dependent ROA Estimation and Comparison

Theorem 2 is now used to estimate the ROA of the equilibrium $S = 0$ of network (41). Setting $\tau_{\max} = 0.01$, one has

$$\varrho_0 := \tau_{\max} c \|A\| \|F_2\| = 0.0399$$

which satisfies (26). Then, Theorem 2 is applicable as ϱ_0 also satisfies (27). By simple calculation, one obtains $d_0 = 1.2242$ and $r_0 = 4.7405$. The results here are better than those obtained by Theorem 1, demonstrating the advantage of Theorem 2 when the delay is small.

It is known that Theorem 1 gives $d_0 = 2.9936$ and $r_0 = 4.0211$ no matter how large $\tau_{\max}$ is, as shown by the dashed lines in Fig. 2. To show the influence of $\tau_{\max}$ on the d_0 and r_0 estimated by Theorem 2, we calculate d_0 and r_0 by Theorem 2 for $\tau_{\max} \in (0, \tau_0)$, where $\tau_0 = 0.1507$ is the supremum of $\tau_{\max}$ given by (26) and (27). The results are shown by the solid lines in Fig. 2. From this figure, we have several findings:

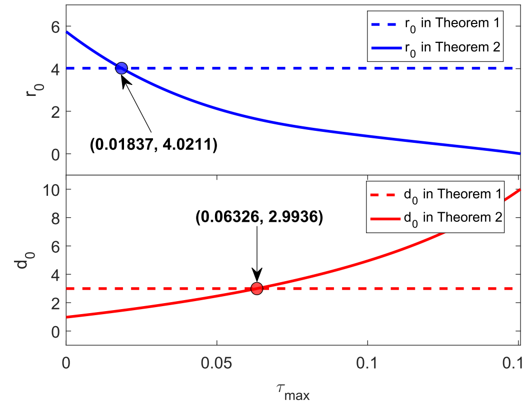

Fig. 2. Control threshold d_0 and radius r_0 estimated by Theorems 1 and 2 when $\tau_{\max} \in (0, \tau_0)$, where $\tau_0 = 0.1507$ is the supremum of $\tau_{\max}$ that Theorem 2 can deal with. It is shown that the ROA estimated by Theorem 2 is better than that estimated by Theorem 1 when $\tau_{\max}$ is less than the threshold 0.01837.

1) Theorem 2 has better r_0 and d_0 than Theorem 1 when $\tau_{\max}$ is smaller than the threshold $\tau_2 = 0.01837$.
2) Theorem 2 has better d_0 but worse r_0 when $\tau_{\max} \in (\tau_2, \tau_1)$ with $\tau_1 = 0.06326$.
3) Theorem 2 has worse r_0 and d_0 when $\tau_{\max} \in (\tau_1, \tau_0)$. In addition, Theorem 2 is not applicable when $\tau_{\max} \geq \tau_0$, while Theorem 1 is applicable.
4) The d_0 and r_0 estimated by Theorem 2 converge when $\tau_{\max}$ draws close to 0.

These numerical results, which are consistent with the theoretical analysis in Remark 6, show that one can determine which theorem has better performance according to the value of $\tau_{\max}$ once the thresholds τ_1, τ_2 are determined.

V. Conclusion

In this article, the ROA estimation problem for controlled complex networks with time-varying delay has been investigated. A differential inequality has been developed to estimate the ROA of nonlinear systems with delay. Based on this inequality, delay-independent ROA estimation of networks with general delay has been established, where the ROA is estimated in the form of a ball-like set. Then, delay-dependent ROA estimation has been established, which is less conservative than the delay-independent estimation for small delay. Unlike the popularity of network synchronization analysis, the ROA estimation of various network models is far from elaborately explored and thereby deserves further study. Specifically, the ROA estimation of networks with nonuniform time delays will be considered in our future work.

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