Bounded Synchronization of Heterogeneous Complex Dynamical Networks: A Unified Approach

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***Abstract*—This article addresses the bounded synchronization of heterogeneous complex dynamical networks. Different from most existing criteria which deal with only global bounded synchronization, a general theorem is established for analyzing both the local and global bounded synchronization of a class of heterogeneous networks in a unified approach. By developing a joint-diagonalization-like technique, several easy-to-use bounded synchronization criteria with low-dimensional linear matrix inequalities are derived based on the general theorem. Particularly, an estimate of the synchronization error and the admissible initial values are given. Examples are provided to verify the theoretical results.**

***Index Terms*—Bounded synchronization, complex network, heterogeneous, linear matrix inequality, synchronization.**

I. Introduction

The past two decades have witnessed the rapid development of complex networks due to their wide applications in science and engineering [1]–[10]. Synchronization, as an emerging phenomenon of a population of dynamically interacting units [4], is one of the most important topics in the field of complex networks. A great deal of research has focused on the synchronization of complex dynamical networks. Various synchronization criteria have been established for networks without control [11]–[15] or with control [16]–[20].

A great majority of the existing studies focused on the complete synchronization of networks of identical nodes. In fact, almost all networks in engineering have different nodes [21], which are called *heterogeneous networks*. Complete synchronization generally cannot be achieved in a heterogeneous network due to node difference. However, if the node difference is slight, the synchronization error may be ultimately bounded as the time evolves to infinity. This phenomenon is called *bounded synchronization* (BS). It is helpful and indeed necessary to estimate a compact set where the synchronization error ultimately dwells.

In recent years, the BS of heterogeneous networks has drawn increasing attention. In 2012, Zhao *et al.* [21] addressed the issue of the global BS of general networks with nonidentical nodes. In light of this work, some new criteria for the global BS of heterogeneous networks and heterogeneous multiagent systems were proposed in [22]–[27]. Some other BS criteria for networks with parameter mismatches were proposed in [28] and [29]. It is noted that these criteria are only applicable to global BS and little attention has been paid to local BS. The main difficulty of studying local BS lies in: 1) there is no general theorem that deals with local BS as far as we know, while there is such a theorem [21] for global BS; and 2) for local BS, the admissible initial values generally should be in a bounded set D, and the synchronization error finally converges to another bounded set. We have to deal with the relationship between the two bounded sets, while this is not necessary for global BS as D is the whole space in this case. Recently, Wang *et al.* [30] separately investigated the local and global BS of a heterogeneous complex switched network, but the local BS is only briefly analyzed. The dimension of the linear matrix inequality (LMI) established in [30, Th. 1] is $O(N)$, where N means the number of nodes. Generally speaking, the algorithm based on the standard LMI has a polynomial-time complexity [31], which increases drastically with the increase of N. Therefore, the BS criteria established in [30] cannot be directly applied to large-scale networks. To this end, it is of practical importance to establish both the local and global BS criteria based on low-dimensional LMIs for heterogeneous networks.

Motivated by the above discussions, this article aims to address both the local and global BS of heterogeneous networks in a unified approach. The main contributions are as follows.

1) A general theorem is established for analyzing both the local and global BS of heterogeneous networks. Not only the ultimate synchronization error but the admissible initial values are estimated in the form of spheres.
2) Several easy-to-use BS criteria with low-dimensional LMIs are derived based on the general theorem. Since a network is usually of high dimension, a joint-diagonalization-like technique is developed to realize dimensionality reduction, especially for directed networks. In the existing studies on BS, the dimensionality reduction is only realized for undirected networks.

The rest of this article is organized as follows. Section II introduces the network model and some mathematical preliminaries. Section III presents our main results on the BS criteria. Section IV provides two examples to verify the theoretical results. Section V concludes the article.

II. Model and Preliminaries

A. Notations and Network Model

First, some notations are introduced. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) denote the n-dimensional Euclidean space and the set of all the $(n\times m)$-dimensional real (complex) matrices, respectively; $\mathbb{R}_+$ is the set of nonnegative real numbers; the superscript T (H) represents the transpose (conjugate transpose) of a vector or a matrix; $\|\cdot\|$ denotes the 2-norm of a vector or a matrix; $|\cdot|$ denotes the modulus of a complex number; $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$, respectively, denote the maximal and minimal eigenvalues of a symmetric matrix; $\otimes$ represents the Kronecker

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product; I_N is the identity matrix of dimension N; $\mathrm{vec}_N\{x_i\}$ denotes $(x_1^T, x_2^T, \ldots, x_N^T)^T$ with x_i being a vector or a scalar; and B_r denotes an open ball centered at 0 of radius r and appropriate dimension. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a class-$\mathcal{K}_\infty$ function if it is continuous and strictly increasing with $\alpha(0) = 0$ and $\lim_{s\to+\infty} \alpha(s) = +\infty$.

Consider a heterogeneous network consisting of N systems of dimension n

$$\dot{x}_i(t) = f_i(t, x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t) \tag{1}$$

where $1 \le i \le N$, $x_i \in \mathbb{R}^n$ is the state vector of the ith node, f_i: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, and $c > 0$, $A = (a_{ij})_{N\times N}$, $\Gamma \in \mathbb{R}^{n\times n}$ denote the coupling strength, the weighted outer coupling matrix, and the inner coupling matrix, respectively. If there is an edge from node i to node j ($j \ne i$), then $a_{ij} > 0$; otherwise, $a_{ij} = 0$; the diagonal elements of A are defined by $a_{ii} = -\sum_{j=1, j\ne i}^{N} a_{ij}$.

In this article, network (1) is assumed to be strongly connected, or equivalently, A is assumed to be irreducible. Let the eigenvalues of A be $\lambda_1, \lambda_2, \ldots, \lambda_N$ with

$$0 = |\lambda_1| < |\lambda_2| \le \cdots \le |\lambda_N|.$$

Let the left and right eigenvectors of A associated to the eigenvalue 0 be

$$\xi = (\xi_1, \ldots, \xi_N)^T, \;\; u = (1, \ldots, 1)^T \in \mathbb{R}^N. \tag{2}$$

Without loss of generality, suppose that $\sum_{i=1}^{N} \xi_i = 1$ with $\xi_i > 0$ (see Lemma 1, which will appear below).

Defining $f(t, x) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, x)$, (1) turns into

$$\dot{x}_i = f(t, x_i) + g_i(t, x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j \tag{3}$$

where $g_i = f_i - f$. Define the weighted average state and the synchronization error of the ith node as

$$\bar{x} = \sum_{i=1}^{N} \xi_i x_i, \; e_i = x_i - \bar{x}. \tag{4}$$

Letting $\bar{f}(t) = \sum_{j=1}^{N} \xi_j f(t, x_j)$ and $\bar{g}(t) = \sum_{j=1}^{N} \xi_j g_j(t, x_j)$, one has

$$\dot{\bar{x}} = \bar{f} + \bar{g} + c(\xi^T A) \otimes \Gamma \mathrm{vec}_N\{x_i\} = \bar{f} + \bar{g}. \tag{5}$$

From (3) and (5), it follows that:

$$\dot{e}_i = f(t, x_i) - \bar{f} + c\sum_{j=1}^{N} a_{ij}\Gamma e_j + g_i(t, x_i) - \bar{g}. \tag{6}$$

Let $F(t) = \frac{\partial f}{\partial x}(t, \bar{x}) \in \mathbb{R}^{n\times n}$ be the Jacobian of f and $R_i(t) = f(t, x_i) - f(t, \bar{x}) - F(t)e_i$. Considering

$$\begin{aligned} f(t, x_i) - \bar{f} &= f(t, x_i) - f(t, \bar{x}) + \sum_{j=1}^{N} \xi_j \left(f(t, x_j) - f(t, \bar{x})\right) \\ &= Fe_i + R_i - \sum_{j=1}^{N} \xi_j R_j \end{aligned}$$

it follows from (6) that:

$$\dot{e}_i = Fe_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j + R_i - \sum_{j=1}^{N} \xi_j R_j + g_i(t, x_i) - \bar{g}. \tag{7}$$

In virtue of the Kronecker product, (7) can be rewritten as

$$\dot{e} = (I_N \otimes F + cA \otimes \Gamma)e + (I_N - u\xi^T) \otimes I_n(R + G) \tag{8}$$

where $e = \mathrm{vec}_N\{e_i\}$, $R = \mathrm{vec}_N\{R_i\}$, and $G = \mathrm{vec}_N\{g_i(t, x_i)\}$. Defining

$$\varphi(t, x) = \begin{cases} \frac{\|f(t,\bar{x}(t)+x) - f(t,\bar{x}(t)) - F(t)x\|}{\|x\|}, & x \ne 0 \\ 0, & x = 0 \end{cases} \tag{9}$$

one has $\|R_i\| = \varphi(t, e_i)\|e_i\|$.

Definition 1: (See [21]). If there exists a nonempty open set D and a bounded set Q such that $e(t)$ starting from D converges to Q, i.e.,

$$e(t_0) \in D \Rightarrow \lim_{t\to+\infty} \inf_{v\in Q} \|e(t) - v\| = 0$$

then network (1) is said to achieve (local) BS. Moreover, network (1) is said to achieve global BS if $D = \mathbb{R}^{nN}$.

An objective of this article is to estimate a bounded set Q where the synchronization error e of network (1) dwells when the time evolves to infinity. Moreover, it is also to estimate D which includes all the admissible initial values.

B. Mathematical Preliminaries

Here, we present several assumptions and lemmas.

Assumption 1: (A1) Suppose that there exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\phi(0) = 0$ and

$$\varphi(t, x) \le \phi(r), \; \forall x \in \overline{\mathrm{B}}_r$$

for any $r \in \mathbb{R}_+$ and $t \in \mathbb{R}$, where φ is defined in (9).

Remark 1: In the literature, the Lipschitz condition is widely used to deal with the synchronization issue of networks. It is easy to find that (A1) is very similar to the local Lipschitz condition (weaker than the global one) if (A1) is rewritten as

$$\|f(t, \bar{x}(t) + x) - f(t, \bar{x}(t)) - F(t)x\| \le \phi(\|x\|)\|x\|.$$

The difference is that the linear term $F(t)x$ is subtracted, which is for better estimating the nonlinear part $R_i(t)$.

Assumption 2: (A2) Suppose that there exists a bounded function $\gamma : \mathbb{R} \to \mathbb{R}_+$ such that $\|G(t)\| \le \gamma(t)$, $t \in \mathbb{R}$.

Remark 2: (A2) is a necessary assumption for ensuring the BS of networks. Consider the following network of two nodes:

$$\begin{cases} \dot{x}_1(t) = f_1(t, x_1) + x_2 - x_1 \\ \dot{x}_2(t) = f_2(t, x_2) + x_1 - x_2 \end{cases} \tag{10}$$

where $f_1 = x_1 + 0.1\sin t$ and $f_2 = dx_2$ with d being a parameter. Let $x_1(0) = x_2(0) = 0$. When $d = 0.9$, (A2) does not hold, and simulation results (not presented due to space constraint) show that the error $x_1 - x_2$ goes to infinity even if the initial error is 0. When $d = 1$, (A2) holds, and simulation results show that this network achieves BS. This simple example accounts for the necessity of (A2). Additionally, much of the literature [21]–[30] uses (A2) or similar assumptions.

Assumption 3: (A3) Suppose that there exist a positive definite matrix $P = \mathrm{diag}\{P_1, P_2, \ldots, P_N\}$ with $P_i \in \mathbb{R}^{n\times n}$ and a positive constant ε satisfying

$$P_i(F(t) + c\lambda_i\Gamma) + (F(t) + c\lambda_i\Gamma)^H P_i + \varepsilon I_n \le 0 \tag{11}$$

where $2 \le i \le N$ and $P_1 = P_2$.

Remark 3: (A3) ensures that the linearized system of error system (8) is stable, but cannot ensure even the local stability of (8). Consider network (10) and let

$$f_1 = -2x_1 + x_1^2 + d\sin(0.1t),\ f_2 = -2x_2 + x_2^2$$

where d is a parameter. It is easy to verify that (A3) holds. Set $x_1(0) = x_2(0) = 0$. It is found via simulation that this network achieves BS when $d = 1$, while the BS is destroyed when increasing the value of d to 2. This simple example shows that (A3) alone cannot ensure even the local BS, though it is a necessary condition. In addition, P_1 is not used in what follows, so we take $P_1 = P_2$ for simplicity.

Lemma 1: (See [32] and [33]). Consider the outer coupling matrix A of network (1). Then,

(i) A has an eigenvalue 0 with multiplicity 1.

(ii) A has an eigenvector $\zeta = (\zeta_1, \ldots, \zeta_N)^T$ associated to 0 with $\zeta_1, \ldots, \zeta_N > 0$.

Lemma 2: For the outer coupling matrix A of network (1), there exists a matrix Φ satisfying $A = \Phi^{-1}J\Phi$ and

$$I_N - u\xi^T = \Phi^{-1}\begin{bmatrix} 1-\xi^T u & 0 \\ 0 & I_{N-1} \end{bmatrix}\Phi \tag{12}$$

where J is in the Jordan canonical form with $J_{11} = 0$, and ξ, u are defined in (2).

Proof: According to Lemma 1, A has a simple eigenvalue 0. Then, there exists a matrix $\Phi = (\Phi_1, \ldots, \Phi_N)^T \in \mathbb{C}^{N\times N}$ such that $A = \Phi^{-1}J\Phi$, where $J = \text{diag}\{0, J_{n_2}, \ldots, J_{n_k}\}$, J_{n_i} is the Jordan block associated to the eigenvalue $\lambda_i \neq 0$ of A, and $n_2 + \cdots + n_k = N - 1$. Since ξ and Φ_1 are both the left eigenvectors of A associated to 0, one has $\Phi_1 = \rho\xi$, where $\rho \in \mathbb{C}$.

Next, it is to prove that $\Phi_i^T u = 0$ for $2 \le i \le N$. Rewrite $A = \Phi^{-1}J\Phi$ as $\Phi A = J\Phi$. Letting $k = n_2 + 1$, one has $\Phi_k^T A = \lambda_2\Phi_k^T$. Then, $\lambda_2\Phi_k^T u = \Phi_k^T Au = 0$, yielding $\Phi_k^T u = 0$. Again, one has $\Phi_{k-1}^T A = \lambda_2\Phi_{k-1}^T + \Phi_k^T$. It follows that $\Phi_{k-1}^T Au = \lambda_2\Phi_{k-1}^T u + \Phi_k^T u$, and thus $\Phi_{k-1}^T u = 0$. In a similar way, one deduces that $\Phi_i^T u = 0$ for $2 \le i \le k$. Repeating the above process, one has $\Phi_i^T u = 0$ for $2 \le i \le N$.

Finally, it is easy to verify that $\Phi_i^T(I_N - u\xi^T) = \Phi_i^T$ for $2 \le i \le N$, and $\Phi_1^T(I_N - u\xi^T) = \rho\xi^T(I_N - u\xi^T) = (1-\xi^T u)\Phi_1^T$. Hence, (12) is verified. ■

Remark 4: Lemma 2 establishes a joint-diagonalization-like technique, where $I_n - u\xi^T$ is diagonalized and A is close to being diagonalized.

Lemma 3: If the outer coupling matrix A of network (1) is symmetric, there exists an orthogonal matrix Φ satisfying $A = \Phi^T\Lambda\Phi$ and

$$I_N - u\xi^T = \Phi^T\begin{bmatrix} 1-\xi^T u & 0 \\ 0 & I_{N-1} \end{bmatrix}\Phi \tag{13}$$

where Λ is a diagonal matrix with $\Lambda_{11} = 0$, and ξ, u are defined in (2).

Proof: Since symmetric A has an eigenvalue 0, there exists an orthogonal matrix $\Phi = (\Phi_1, \ldots, \Phi_N)^T \in \mathbb{R}^{N\times N}$ such that $A = \Phi^T\Lambda\Phi$. It is verified that $\Phi_1 = \rho_1\xi = \rho_2 u$, where $\rho_1, \rho_2 \in \mathbb{R}$. Then, $\Phi_1^T(I_N - u\xi^T) = (1-\xi^T u)\Phi_1^T$, and $\Phi_i^T(I_N - u\xi^T) = \Phi_i$ for $i > 1$, so (13) is verified. ■

III. Bounded Synchronization Analysis

In this section, a general theorem is established for analyzing the BS of network (1), from which several easy-to-use criteria are derived.

A. General Theorem

Theorem 1: Consider the error system (8) of network (1). Assume that there are functions $V: \mathbb{R}^{nN} \to \mathbb{R}_+$ and $\rho: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\alpha_1(\|e\|) \le V(e) \le \alpha_2(\|e\|) \tag{14}$$

$$\dot{V}|_{(8)} \le -\alpha_3(\|e\|),\ \text{if } \rho(t) \le \|e\| < \delta \tag{15}$$

where $\alpha_1, \alpha_2, \alpha_3$ are class-$\mathcal{K}_\infty$ functions, and δ is a positive constant satisfying $\alpha_2^{-1}(\alpha_1(\delta)) > \rho_0$ with $\rho_0 = \sup_{t\ge 0}\{\rho(t)\}$.

(i) If $\|e(t_0)\| < \alpha_2^{-1}(\alpha_1(\delta))$, then e converges to

$$Q = \{e \in \mathbb{R}^{nN} \mid \|e\| \le \alpha_1^{-1}(\alpha_2(\overline{\lim_{t\to+\infty}}\rho(t)))\}.$$

(ii) If $\delta = +\infty$, then e globally converges to Q.

Proof: (ii) can be easily derived from (i). Select a large enough $\delta_0 > 0$, then the given conditions hold by replacing δ with δ_0. From (i), if $\|e(t_0)\| < \alpha_2^{-1}(\alpha_1(\delta_0))$, then e converges to Q. Since $\alpha_2^{-1}(\alpha_1(\delta_0))$ can be arbitrarily large when δ_0 is large enough, e globally converges to Q. Hence, it is sufficient to only prove (i).

It is first to prove by contradiction that $\|e(t)\| < \delta$, $\forall t \ge t_0$. Suppose that there exists a $t_3 > t_0$ such that $\|e(t_3)\| \ge \delta$. Let

$$t_2 = \inf_{t>t_0}\{t \mid \|e(t)\| = \delta\}.$$

Obviously, $\|e(t_2)\| = \delta$ due to the continuity of e. Let δ_1 be a constant such that

$$\max\{\rho_0, \|e(t_0)\|\} \le \delta_1 < \alpha_2^{-1}(\alpha_1(\delta)).$$

Since $\delta_1 < \alpha_2^{-1}(\alpha_1(\delta)) \le \alpha_2^{-1}(\alpha_2(\delta)) = \delta$, there exists $t \in [t_0, t_2)$ such that $\|e(t)\| = \delta_1$. Let

$$t_1 = \sup_{t\in[t_0,t_2)}\{t \mid \|e(t)\| = \delta_1\}.$$

Then, $\|e(t_1)\| = \delta_1$, $t_1 < t_2$, and

$$\|e(t)\| \in [\delta_1, \delta),\ \forall t \in [t_1, t_2).$$

According to (15), one has $\dot{V}|_{(8)} < 0$, $\forall t \in [t_1, t_2)$, leading to $V(e(t_1)) > V(e(t_2))$. According to (14), one has $V(e(t_1)) \le \alpha_2(\delta_1) < \alpha_1(\delta) \le V(e(t_2))$, which contradicts. Hence, one has $\|e(t)\| < \delta$, $\forall t \ge t_0$.

Next, it is to prove by contradiction that

$$\nu_1 \triangleq \underline{\lim}_{t\to+\infty}\|e(t)\| \le \rho_1 \triangleq \overline{\lim_{t\to+\infty}}\rho(t).$$

Suppose that $\nu_1 > \rho_1$. Denote $\epsilon_1 = \frac{1}{3}(\nu_1 - \rho_1)$. There exist a constant T_1 such that $\|e(t)\| > \nu_1 - \epsilon_1$, $\forall t > T_1$, and a constant T_2 such that $\rho(t) \le \rho_1 + \epsilon_1$, $\forall t > T_2$. Taking $T_0 = \max\{T_1, T_2\}$ yields

$$\|e(t)\| > \nu_1 - \epsilon_1 > \rho_1 + \epsilon_1 \ge \rho(t),\ \forall t > T_0.$$

In view of (15)

$$\dot{V}|_{(8)} \le -\alpha_3(\|e(t)\|) < -\alpha_3(\nu_1 - \epsilon_1) < 0,\ \forall t > T_0$$

leading to $\lim_{t\to+\infty}V(e(t)) = -\infty$, a contradiction with $V \ge 0$. Hence, one has $\nu_1 \le \rho_1$.

Finally, it is to prove by contradiction that e converges to Q, i.e.,

$$\nu_2 \triangleq \overline{\lim_{t\to+\infty}}\|e(t)\| \le \rho_2 \triangleq \alpha_1^{-1}(\alpha_2(\rho_1)). \tag{16}$$

Suppose that $\nu_2 > \rho_2$. Denote $\epsilon_2 = \frac{1}{3}(\nu_2 - \rho_2)$. Let ϵ be any constant in $(0, \epsilon_2)$. There exists a constant T_3 such that $\rho(t) \le \rho_1 + \epsilon$, $\forall t > T_3$, a constant $t_4 > T_3$ such that $\|e(t_4)\| < \nu_1 + \epsilon$, and a constant

$t_6 > t_4$ such that $\|e(t_6)\| > \nu_2 - \epsilon_2$. Then, $\|e(t_4)\| < \rho_1 + \epsilon$ and $\|e(t_6)\| > \rho_2 + \epsilon_2 \geq \rho_1 + \epsilon_2 > \rho_1 + \epsilon$. Let

$$t_5 = \sup_{t \in (t_4, t_6)} \{t \mid \|e(t)\| = \rho_1 + \epsilon\} < t_6.$$

Then, $\|e(t)\| \geq \rho_1 + \epsilon,\ \forall t \in [t_5, t_6]$. Since $t_5 > t_4 > T_3$, it follows that $\|e(t)\| \geq \rho(t),\ \ \forall t \in [t_5, t_6]$. According to (15), one has $\dot{V}|_{(8)} < 0,\ \forall t \in [t_5, t_6]$, leading to $V(e(t_5)) > V(e(t_6))$. Moreover, one has $V(e(t_5)) \leq \alpha_2(\rho_1 + \epsilon)$ and $V(e(t_6)) \geq \alpha_1(\|e(t_6)\|) > \alpha_1(\rho_2 + \epsilon_2) > \alpha_1(\rho_2) = \alpha_2(\rho_1)$. Letting $\epsilon \to 0$ leads to $V(e(t_5)) \leq \alpha_2(\rho_1) < V(e(t_6))$, which contradicts. Hence, (16) is verified. ■

Remark 5: Theorem 1 establishes a general theorem for analyzing both the local and global BS of a class of heterogeneous networks in a unified approach. Sufficient conditions are given, under which the synchronization error are ultimately bounded in the compact set Q. Particularly, the initial values $e(t_0)$ enabling a heterogeneous network to achieve local BS are also estimated by a positive constant $\alpha_2^{-1}(\alpha_1(\delta))$. This is quite different from [21]–[29], which only addressed global BS.

Remark 6: It seems that Theorem 1 is only used for the error system (8). In fact, the specific dynamics described by (8) are not used, and we refer to (8) just for simplicity of notations. Therefore, Theorem 1 is actually very general.

Corollary 1: Consider the error system (8) of network (1). Assume that there is a function $V : \mathbb{R}^{nN} \to \mathbb{R}_+$ such that

$$a\|e\|^2 \leq V(e) \leq b\|e\|^2$$

$$\dot{V}|_{(8)} \leq -\varphi_1(t)\|e\|^2 + \varphi_2(t)\|e\|, \text{ if } \|e\| < \delta$$

where a, b are positive constants, φ_1, φ_2 are functions satisfying $\inf_{t\geq 0}\{\varphi_1\} > 0$ and $\varphi_2 \geq 0$, and δ is a positive constant such that $\delta > \sqrt{\frac{b}{a}} \sup_{t\geq 0}\{\frac{\varphi_2}{\varphi_1}\}$.

(i) If $\|e(t_0)\| < \sqrt{\frac{a}{b}}\delta$, then e converges to

$$Q = \left\{ e \in \mathbb{R}^{nN} \,\middle|\, \|e\| \leq \sqrt{\frac{b}{a}} \varlimsup_{t\to+\infty} \frac{\varphi_2(t)}{\varphi_1(t)} \right\}.$$

(ii) If $\delta = +\infty$, then e globally converges to Q.

Proof: As (ii) can be easily derived from (i), it is sufficient to only prove (i).

Denoting $\alpha_1(\|e\|) = a\|e\|^2$, $\alpha_2(\|e\|) = b\|e\|^2$, $\rho_1(t) = \frac{\varphi_2(t)}{\varphi_1(t)}$, $\nu_0 = \sup_{t\geq 0}\{\rho_1\}$, $\nu_1 = \inf_{t\geq 0}\{\varphi_1\}$, one obtains

$$-\varphi_1(t)\|e\|^2 + \varphi_2(t)\|e\| \leq -\alpha_3(\|e\|), \text{ if } \|e\| \geq \rho(t) \tag{17}$$

by setting $\rho(t) = \rho_1(t) + \epsilon$ and $\alpha_3(\|e\|) = \nu_1\epsilon\|e\|$, where ϵ can be any constant in $(0, \sqrt{\frac{a}{b}}\delta - \nu_0)$. Since $\alpha_2^{-1}(\alpha_1(\delta)) > \sup_{t\geq 0}\{\rho\}$ and $\|e(t_0)\| < \sqrt{\frac{a}{b}}\delta = \alpha_2^{-1}(\alpha_1(\delta))$, it follows from Theorem 1 that e converges to:

$$Q_\epsilon = \left\{ e \in \mathbb{R}^{nN} \,\middle|\, \|e\| \leq \sqrt{\frac{b}{a}} \varlimsup_{t\to+\infty} \frac{\varphi_2(t)}{\varphi_1(t)} + \epsilon\sqrt{\frac{b}{a}} \right\}.$$

Letting $\epsilon \to 0$ completes the proof. ■

Compared to Theorem 1, Corollary 1 is more convenient to use, but still not so straightforward. Next, some easy-to-use criteria are derived in light of Corollary 1.

B. Several Easy-to-Use Criteria

From Lemma 2, there exists a matrix

$$\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_N)^T \in \mathbb{C}^{N\times N}$$

satisfying $A = \Phi^{-1}J\Phi$ and

$$I_N - u\xi^T = \Phi^{-1}\begin{bmatrix} 1-\xi^T u & 0 \\ 0 & I_{N-1}\end{bmatrix}\Phi$$

where J is in the Jordan canonical form with $J_{ii} = \lambda_i$. Letting $\eta = \text{vec}_N\{\eta_i\} = (\Phi \otimes I_n)e$, $\hat{R} = \text{vec}_N\{\hat{R}_i\} = (\Phi \otimes I_n)R$, and $\hat{G} = \text{vec}_N\{\hat{G}_i\} = (\Phi \otimes I_n)G$, (8) is transformed to

$$\dot{\eta} = (I_N \otimes F + cJ \otimes \Gamma)\eta + \begin{bmatrix} 1-\xi^T u & 0 \\ 0 & I_{N-1}\end{bmatrix}(\hat{R} + \hat{G}). \tag{18}$$

It is verified that $\Phi_1 = \rho\xi$ with $\rho \in \mathbb{C}$. Then, $\eta_1 = (\Phi_1^T \otimes I_n)e = \rho(\xi^T \otimes I_n)e = 0$.

The following denotations will be extensively used:

$$a := \lambda_{\min}(P)$$

$$b := \lambda_{\max}(P)$$

$$\phi_0 := \sup_{r\geq 0}\{\phi(r)\}$$

$$\gamma_0 := \sqrt{\frac{b}{a}} \varlimsup_{t\to+\infty} \gamma(t) \tag{19}$$

where P, ϕ, and γ are defined in (A1)–(A3).

Theorem 2: Consider network (1), and assume that (A1)–(A3) hold.

(i) If there exists a constant $\delta > 0$ such that $\sup_{t\geq 0}\gamma(t) < \sqrt{\frac{a}{b}}\frac{\delta(\nu_0 - \phi(\delta))}{\kappa_0}$, then network (1) achieves BS. Specifically, if $\|e(t_0)\| < \sqrt{\frac{a}{b}}\frac{\delta}{\kappa_0}$, then e converges to $\overline{\mathrm{B}}_{r_1}$ with $r_1 = \frac{\kappa_0\gamma_0}{\nu_0 - \phi(\delta)}$;

(ii) if $\phi_0 < \nu_0$, then network (1) reaches global BS, and e converges to $\overline{\mathrm{B}}_{r_2}$ with $r_2 = \frac{\kappa_0\gamma_0}{\nu_0 - \phi_0}$,

where a, b, ϕ_0, γ_0 are defined in (19), $\epsilon = \frac{\varepsilon}{4bc\|\Gamma\|}$, $U = \text{diag}\{1, \epsilon^{-1}, \ldots, \epsilon^{1-N}\}$, $\kappa_0 = \|U\Phi\|\|(U\Phi)^{-1}\|$, and $\nu_0 = \frac{\varepsilon}{4b\kappa_0}$.

Proof: Denoting $\tilde{J} = UJU^{-1}$, it can be easily verified that $\tilde{J}_{ii} = \lambda_i$, $\tilde{J}_{i,i+1} = 0$ or ϵ, and all the other entries of $\tilde{J}$ are 0. Let $\Psi = U\Phi$, and

$$\tilde{\eta} = \text{vec}_N\{\tilde{\eta}_i\} = (U \otimes I_n)\eta = (\Psi \otimes I_n)e$$

$$\tilde{R} = \text{vec}_N\{\tilde{R}_i\} = (U \otimes I_n)\hat{R} = (\Psi \otimes I_n)R$$

$$\tilde{G} = \text{vec}_N\{\tilde{G}_i\} = (U \otimes I_n)\hat{G} = (\Psi \otimes I_n)G.$$

Considering $J = U^{-1}\tilde{J}U$ and

$$\begin{bmatrix} 1-\xi^T u & 0 \\ 0 & I_{N-1}\end{bmatrix} = U^{-1}\begin{bmatrix} 1-\xi^T u & 0 \\ 0 & I_{N-1}\end{bmatrix}U$$

(18) turns into

$$\dot{\tilde{\eta}} = (I_N \otimes F + c\tilde{J} \otimes \Gamma)\tilde{\eta} + \begin{bmatrix} 1-\xi^T u & 0 \\ 0 & I_{N-1}\end{bmatrix}(\tilde{R} + \tilde{G}). \tag{20}$$

Since $\tilde{\eta}_1 = \eta_1 = 0$, (20) can be equivalently rewritten as

$$\dot{\tilde{\eta}}_i = (F + c\lambda_i\Gamma)\tilde{\eta}_i + c\tilde{J}_{i,i+1}\Gamma\tilde{\eta}_{i+1} + \tilde{R}_i + \tilde{G}_i,\ i \geq 2 \tag{21}$$

where $\tilde{J}_{N,N+1} = 0$ and $\tilde{\eta}_{N+1} = 0$. Define $V(\tilde{\eta}(t)) = \tilde{\eta}^H P\tilde{\eta}$. In view of $\tilde{\eta}_1 = 0$, one gets

$$\dot{V}|_{(20)} = \sum_{i=2}^{N} \tilde{\eta}_i^H \left(P_i(F(t) + c\lambda_i\Gamma) + (F(t) + c\lambda_i\Gamma)^H P_i\right)\tilde{\eta}_i$$

$$+ \sum_{i=2}^{N} c\tilde{J}_{i,i+1}\left(\tilde{\eta}_i^H P_i\Gamma\tilde{\eta}_{i+1} + \tilde{\eta}_{i+1}^H\Gamma^T P_i\tilde{\eta}_i\right)$$

$$+ \sum_{i=2}^{N}\left(\tilde{\eta}_i^H P_i(\tilde{R}_i + \tilde{G}_i) + (\tilde{R}_i + \tilde{G}_i)^H P_i\tilde{\eta}_i\right)$$

$$\begin{aligned}
&\leq -\varepsilon\|\tilde{\eta}\|^2+2bc\epsilon\|\Gamma\|\sum_{i=2}^{N}\|\tilde{\eta}_i\|\|\tilde{\eta}_{i+1}\|+2\tilde{\eta}^TP(\tilde{R}+\tilde{G})\\
&\leq -\varepsilon\|\tilde{\eta}\|^2+bc\epsilon\|\Gamma\|\sum_{i=2}^{N}\left(\|\tilde{\eta}_i\|^2+\|\tilde{\eta}_{i+1}\|^2\right)\\
&\quad+2\tilde{\eta}^TP(\tilde{R}+\tilde{G})\\
&\leq -\frac{\varepsilon}{2}\|\tilde{\eta}\|^2+2b\|\tilde{\eta}\|(\|\tilde{R}\|+\|\tilde{G}\|). \qquad (22)
\end{aligned}$$

From (A1), $\|R_i\|\leq\phi(\|e_i\|)\|e_i\|\leq\phi(\|e\|)\|e_i\|$, yielding

$$\|R\|\leq\phi(\delta)\|e\|,\ \text{if}\ \|e\|<\delta. \qquad (23)$$

Note that $\|\Psi^{-1}\|^{-1}\|e\|\leq\|\tilde{\eta}\|\leq\|\Psi\|\|e\|$, $\|\tilde{R}\|\leq\|\Psi\|\|R\|$, and $\|\tilde{G}\|\leq\|\Psi\|\|G\|$. If $\|e\|<\delta$, then

$$\begin{aligned}
\dot{V}|_{(20)}\leq&-\frac{\varepsilon}{2}\|\tilde{\eta}\|\|\Psi^{-1}\|^{-1}\|e\|+2b\phi(\delta)\|\tilde{\eta}\|\|\Psi\|\|e\|\\
&+2b\gamma(t)\|\tilde{\eta}\|\|\Psi\|\\
=&\|\Psi^{-1}\|^{-1}\|\tilde{\eta}\|\left(-\frac{\varepsilon}{2}\|e\|+2b\kappa_0\phi(\delta)\|e\|+2b\kappa_0\gamma(t)\right).
\end{aligned}$$

Define

$$\rho(t)=\begin{cases}\|\Psi^{-1}\|^{-1}\frac{\|\tilde{\eta}(t)\|}{\|e(t)\|}, & e(t)\neq 0\\ 1, & e(t)=0\end{cases}$$

ρ is a bounded function satisfying $\inf_{t\geq 0}\rho(t)>0$, because $\|\Psi^{-1}\|^{-1}\leq\frac{\|\tilde{\eta}\|}{\|e\|}\leq\|\Psi\|$. Since $\tilde{\eta}(t)=0$ if $e(t)=0$, one gets $\|\Psi^{-1}\|^{-1}\|\tilde{\eta}\|=\rho(t)\|e\|$, $\forall t\geq 0$. Then

$$\dot{V}|_{(20)}\leq -p_1(t)\|e\|^2+p_2(t)\|e\|,\ \text{if}\ \|e\|<\delta \qquad (24)$$

where $p_1=2b\kappa_0\rho(t)(\nu_0-\phi(\delta))$ and $p_2=2b\kappa_0\rho(t)\gamma(t)$. From $\sup_{t\geq 0}\gamma(t)<\sqrt{\frac{a}{b}}\frac{\delta(\nu_0-\phi(\delta))}{\kappa_0}$, it follows that $\inf_{t\geq 0}\{p_1\}>0$, and $p_2\geq 0$ is bounded. Moreover

$$a_1\|e\|^2\leq a\|\tilde{\eta}\|^2\leq V\leq b\|\tilde{\eta}\|^2\leq b_1\|e\|^2$$

where $a_1=a\|\Psi^{-1}\|^{-2}$ and $b_1=b\|\Psi\|^2$.

It is first to prove (i). Since

$$\delta>\frac{\kappa_0}{\nu_0-\phi(\delta)}\sqrt{\frac{b}{a}}\sup_{t\geq 0}\{\gamma\}=\sqrt{\frac{b_1}{a_1}}\sup_{t\geq 0}\left\{\frac{p_2}{p_1}\right\}$$

it follows from Corollary 1(i) that network (1) reaches local BS, and e converges to $\overline{\mathrm{B}}_{r_1}$ if $\|e(t_0)\|<\sqrt{\frac{a}{b}}\frac{\delta}{\kappa_0}$.

It is now to prove (ii). From (23), one obtains $\|R\|\leq\phi_0\|e\|$. Similar to the proof of (24), one has

$$\dot{V}|_{(20)}\leq -\left(\frac{\varepsilon}{2}-2b\kappa_0\phi_0\right)\rho(t)\|e\|^2+2b\kappa_0\rho(t)\gamma(t).$$

Then, network (1) reaches global BS according to Corollary 1(ii), and e converges to $\overline{\mathrm{B}}_{r_2}$. ■

Remark 7: Theorem 2 establishes several easy-to-use BS criteria from Corollary 1. Analyzing Theorem 2, we have the following observations.

1) The result is better with a smaller compact set $\overline{\mathrm{B}}_{r_1}$ or $\overline{\mathrm{B}}_{r_2}$, where the synchronization error dwells when the time evolves to infinity. A smaller compact set means the BS is closer to complete synchronization.
2) ϕ, an upper bound of the nonlinear part of f, has significant influence on the BS. The BS is easier to be achieved if the nonlinearity is less significant. The supremum of ϕ determines whether the global BS can be achieved if (A1)–(A3) holds.
3) $\frac{\varepsilon}{b}$, which quantizes the stability of the system $\dot{x}=f(t,x)$ at $\bar{x}$, determines partly the size of the compact set. The compact set is smaller with a larger value of $\frac{\varepsilon}{b}$, i.e., a stronger stability.
4) γ, an upper bound of the node difference, also determines partly the size of the compact set. The compact set is smaller if the node difference is less significant.
5) If $\gamma=0$, network (1) reduces to a network of identical nodes, and the BS becomes complete synchronization, since $r_1=r_2=0$. Therefore, Theorem 2 is also effective in analyzing the complete synchronization of networks of identical nodes.

Remark 8: The number of flops needed to compute a ϱ-accurate solution for a LMI is bounded by $O(\mathcal{M}\mathcal{N}^3\log(\mathcal{V}/\varrho))$, where $\mathcal{M}$ is the total row size of the LMI, $\mathcal{N}$ is the total number of scalar decision variables, $\mathcal{V}$ is a data-dependent scaling factor, and ϱ is a relative accuracy [31]. If $\mathcal{V}$ and ϱ are given, the time complexity is $O(\mathcal{M}\mathcal{N}^3)$. In (A3) of Theorem 2, there are $(N-1)$ LMIs of dimension n. Hence, the time complexity of the LMIs in (A3) of Theorem 2 is $O(n\times(\frac{n^2+n}{2})^3)\times(N-1)=O(Nn^7)$. In [30], the LMI is of dimension $(N-1)n$, thus the time complexity is $O(N^7n^7)$, which is far larger than $O(Nn^7)$ and grows sharply with N. It is tested via the Matlab LMI Toolbox that solving one LMI of dimension 140 needs nearly 176 s, while solving seven LMIs of dimension 20 needs only $0.031\times 7=0.217$ s. This demonstrates the advantage of using the joint-diagonalization-like technique.

Remark 9: In the absence of external control, a heterogeneous network may not be able to reach BS. In this case, one can add control $u_i=-ke_i$ to the ith node of network (1). Network (1) under control can reach BS is k is large enough.

Corollary 2: Consider network (1), and assume that (A1)–(A3) hold. If the outer coupling matrix A is diagonalizable, then

(i) if there exists a constant $\delta>0$ such that $\sup_{t\geq 0}\gamma(t)<\sqrt{\frac{a}{b}}\frac{\delta(\nu_0-\phi(\delta))}{\kappa_0}$, then network (1) achieves BS. Specifically, if $\|e(t_0)\|<\sqrt{\frac{a}{b}}\frac{\delta}{\kappa_0}$, then e converges to $\overline{\mathrm{B}}_{r_1}$ with $r_1=\frac{\kappa_0\gamma_0}{\nu_0-\phi(\delta)}$;
(ii) if $\phi_0<\nu_0$, then network (1) reaches global BS, and e converges to $\overline{\mathrm{B}}_{r_2}$ with $r_2=\frac{\kappa_0\gamma_0}{\nu_0-\phi_0}$,

where a,b,ϕ_0,γ_0 are defined in (19), $\kappa_0=\|\Phi\|\|\Phi^{-1}\|$, and $\nu_0=\frac{\varepsilon}{2b\kappa_0}$.

Proof: In Theorem 2, take $U=I_N$. Since A is diagonalizable, one has $\tilde{J}_{i,i+1}=0$, where $\tilde{J}_{i,i+1}$ is defined in (21). According to (22), one has

$$\dot{V}|_{(20)}\leq -\varepsilon\|\tilde{\eta}\|^2+2b\|\tilde{\eta}\|(\|\tilde{R}\|+\|\tilde{G}\|)$$

where all the symbols are defined in the proof of Theorem 2. Then, the conclusion can be obtained by following the proof of Theorem 2. ■

Corollary 3: Consider network (1), and assume that (A1)–(A3) hold. If the outer coupling matrix A is symmetric, then

(i) If there exists a constant $\delta>0$ such that $\sup_{t\geq 0}\gamma(t)<\sqrt{\frac{a}{b}}\delta(\nu_0-\phi(\delta))$, then network (1) reaches BS. Specifically, if $\|e(t_0)\|<\sqrt{\frac{a}{b}}\delta$, then e converges to $\overline{\mathrm{B}}_{r_1}$ with $r_1=\frac{\gamma_0}{\nu_0-\phi(\delta)}$;
(ii) if $\phi_0<\nu_0$, then network (1) reaches global BS, and e converges to $\overline{\mathrm{B}}_{r_2}$ with $r_2=\frac{\gamma_0}{\nu_0-\phi_0}$,

where a,b,ϕ_0,γ_0 are defined in (19), and $\nu_0=\frac{\varepsilon}{2b}$.

Proof: From Lemma 3, one can require that Φ be orthogonal and J is diagonal. Then, the conclusion can be derived from Corollary 2 as $\kappa_0:=\|\Phi\|\|\Phi^{-1}\|=1$. ■

IV. Numerical Examples

Two numerical examples are given here to verify the theoretical results. The first one is an undirected network, which is exactly the same as [21, Example 5.1] for the sake of making comparison. The second one is a directed network.

A. Example 1: Global BS of an Undirected Network

Consider an undirected network consisting of $N = 5$ nonidentical pendulums [34]

$$\dot{x}_i(t) = f_i(t, x_i(t)) + c\sum_{j=1}^{5} a_{ij}\Gamma x_j(t),\ 1 \le i \le 5 \tag{25}$$

where $f_i(t,z) = (z_2, -r_i \sin z_1 - bz_2 + d_i \sin t)^T$, $r_i = 0.02i$, $(d_1, d_2, d_3, d_4, d_5)^T = (0.1, -0.15, -0.2, 0.15, 0.2)^T$, $c = 1$, and

$$\Gamma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ A = \begin{bmatrix} -3 & 1 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -3 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

Since A is symmetric, one can use Corollary 3 to analyze the BS of (25). Take $\gamma = 0.0632 + 0.3647|\sin t|$. Denoting $\bar{x} = [\bar{x}_1, \bar{x}_2]^T$ yields $F(t) = [\begin{smallmatrix} 0 & 1 \\ -0.06\cos\bar{x}_1 & -b \end{smallmatrix}]$, and

$$\begin{aligned}\varphi(t,z) &= \frac{\|f(t,\bar{x}+z) - f(t,\bar{x}) - F(t)z\|}{\|z\|} \\ &= \frac{0.06}{\|z\|}\left|\sin(\bar{x}_1 + z_1) - \sin\bar{x}_1 - z_1\cos\bar{x}_1\right| \\ &= \frac{0.06|z_1|}{\|z\|}\left|\cos(\bar{x}_1 + \zeta_1) - \cos\bar{x}_1\right| \\ &\le 0.06\left|\cos(\bar{x}_1 + \zeta_1) - \cos\bar{x}_1\right|\end{aligned}$$

where $0 \le |\zeta_1| \le |z_1|$. Hence, ϕ is taken as

$$\phi(r) = \begin{cases} 0.06r, & r \in [0,2] \\ 0.12, & r > 2. \end{cases}$$

See [21] for the value of ε and P. Since $\phi_0 = 0.12 < 0.3443 = \nu_0$, it follows from Corollary 3(ii) that e globally converges to $\overline{\mathrm{B}}_r$ with $r = 4.2653$, which is better than the $r = 13.2584$ in [21]. It is found that the synchronization error (see [21] for the state trajectories of network (25)) finally satisfies $\|e(t)\| < r = 4.2653$, i.e., e converges to $\overline{\mathrm{B}}_r$, which verifies the theoretical result.

B. Example 2: Local BS of a Directed Network

Consider a directed network consisting of $N = 10$ nonidentical nodes

$$\dot{x}_i(t) = f_i(t, x_i(t)) + c\sum_{j=1}^{10} a_{ij}\Gamma x_j(t) \tag{26}$$

where $f_i(t,z) = (-z_1 + (5 + d_i)\tanh(0.1z_2 - 1), -2z_2 + d_i\cos t)^T$, $d_i = 0.03(2i-1)$, $c = 0.2$, $\Gamma = [\begin{smallmatrix} 1 & 0.2 \\ 0 & 1 \end{smallmatrix}]$, and a_{ij} $(i \ne j)$ is defined by

$$a_{ij} = \begin{cases} 2, & 1 \le |j-i| \le 2 \text{ or } 1 \le |10+j-i| \le 2 \\ 1, & i = 1 \text{ and } j = 6 \\ 0, & \text{otherwise} \end{cases}$$

with $1 \le i, j \le 10$.

Since A is diagonalizable, we use Corollary 2 to analyze the BS of network (26). Take $\gamma(t) = 0.3\sqrt{3.3(1+\cos^2(t))}$. Similarly to Example 1, one gets

$$\begin{aligned}\varphi(t,z) &= \frac{5.3}{\|z\|}\Big|\tanh(0.1(\bar{x}_2 + z_2) - 1) - \tanh(0.1\bar{x}_2 - 1) \\ &\quad - \frac{0.1z_2}{\cosh^2(0.1\bar{x}_2 - 1)}\Big| \\ &= \frac{0.53|z_2|}{\|z\|}\left|h(\bar{x}_2 + \zeta_2) - h(\bar{x}_2)\right| \\ &\le 0.53\left|h(\bar{x}_2 + \zeta_2) - h(\bar{x}_2)\right|\end{aligned}$$

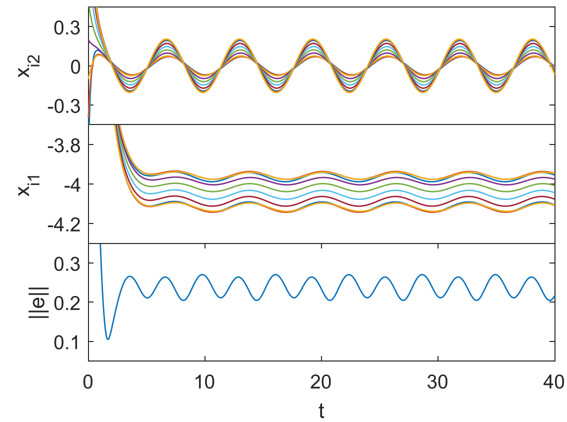

Fig. 1. State trajectories x_{i1}, x_{i2} and synchronization error $\|e\|$ of network (26), where $1 \le i \le 10$.

where $h(x) = \frac{1}{\cosh^2(0.1x-1)}$ and $0 \le |\zeta_2| \le |z_2|$. Considering $|h'(x)| \le \frac{2}{15\sqrt{3}}$ and $|h(x)| \le 1$, ϕ can be taken as

$$\phi(r) = \begin{cases} \frac{1.06}{15\sqrt{3}}r, & r \in [0, 15\sqrt{3}] \\ 1.06, & r > 15\sqrt{3}. \end{cases}$$

Next, we solve the LMIs in (A3). The definition of $F(t)$ gives $F(t) = [\begin{smallmatrix} -1 & 0.53\varrho(t) \\ 0 & -2 \end{smallmatrix}]$, where $\varrho(t) = h(\bar{x}_2(t)) \in (0,1]$. Since $F(t) = \varrho(t)F_1 + (1 - \varrho(t))F_2$, one has

$$\begin{aligned}&P_i(F(t) + c\lambda_i\Gamma) + (F(t) + c\lambda_i\Gamma)^H P_i \\ &= \varrho(t)[P_i(F_1 + c\lambda_i\Gamma) + (F_1 + c\lambda_i\Gamma)^H P_i] \\ &\quad + (1 - \varrho(t))[P_i(F_2 + c\lambda_i\Gamma) + (F_2 + c\lambda_i\Gamma)^H P_i]\end{aligned}$$

where $F_1 = [\begin{smallmatrix} -1 & 0.53 \\ 0 & -2 \end{smallmatrix}]$ and $F_2 = [\begin{smallmatrix} -1 & 0 \\ 0 & -2 \end{smallmatrix}]$. Then, (11) in (A3) holds if

$$P_i(F_j + c\lambda_i\Gamma) + (F_j + c\lambda_i\Gamma)^H P_i + \varepsilon I_n \le 0 \tag{27}$$

where $2 \le i \le 10$ and $1 \le j \le 2$. Using the Matlab LMI Toolbox to solve (27) gives $\varepsilon = 0.9027$ and

$$P_2 = \cdots = P_{10} = \begin{bmatrix} 0.2858 & 0.0080 \\ 0.0080 & 0.1827 \end{bmatrix}.$$

Solving the following inequality in Corollary 2(i):

$$\delta(\nu_0 - \phi(\delta)) > \kappa_0\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\sup_{t\ge 0}\gamma(t) = 1.4767$$

one obtains $\delta \in (1.5236, 23.7558)$. Select $\delta = 10$. According to Corollary 2(i), e converges to $\overline{\mathrm{B}}_r$ with $r = 2.3688$ if the initial values satisfy $\|e(0)\| < 5.2192$.

Let the initial values be $x_i(0) = (-2 + 0.3i, -1 + 0.3i)^T$ for $1 \leq i \leq 10$, which satisfy $\|e(0)\| < 5.2192$. Fig. 1 shows that the synchronization error finally satisfies $\|e(t)\| < r = 2.3688$, i.e., e converges to $\overline{\mathrm{B}}_r$, which verifies the theoretical result.

V. Conclusion

In this article, we have established a general theorem for analyzing both the local and global BS of a class of heterogeneous networks, and then derived several easy-to-use BS criteria with low-dimensional LMIs. Two numerical examples have been provided to verify the theoretical results, and the comparison with [21] in Example 1 shows that our method is better. Although our estimation of the compact set $\overline{\mathrm{B}}_r$ may still be conservative, it is fundamental and makes the first step in establishing local and global BS criteria in a unified approach. Estimating the compact set in other shapes such as ellipsoid may reduce the conservativeness, which deserves future study.

References

[1] A. L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

[2] S. H. Strogatz, "Exploring complex networks," *Nature*, vol. 410, no. 6825, pp. 268–276, 2001.

[3] R. Albert and A. L. Barabási, "Statistical mechanics of complex networks," *Rev. Mod. Phys.*, vol. 74, no. 1, pp. 47–97, 2002.

[4] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, "Synchronization in complex networks," *Phys. Rep.*, vol. 469, no. 3, pp. 93–153, 2008.

[5] J. Zhou, J. Chen, J. Lu, and J. Lü, "On applicability of auxiliary system approach to detect generalized synchronization in complex network," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3468–3473, Jul. 2017.

[6] J. D. Noh and H. Rieger, "Random walks on complex networks," *Phys. Rev. Lett.*, vol. 92, no. 11, 2004, Art. no. 118701.

[7] S. Zhu, J. Zhou, and J. Lu, "Identifying partial topology of complex dynamical networks via a pinning mechanism," *Chaos*, vol. 28, no. 4, 2018, Art. no. 043108.

[8] K. I. Goh, M. E. Cusick, D. Valle, B. Childs, M. Vidal, and A. L. Barabsi, "The human disease network," *Proc. Nat. Acad. Sci. U.S.A.*, vol. 104, no. 21, pp. 8685–8690, 2007.

[9] E. Bullmore and O. Sporns, "Complex brain networks: Graph theoretical analysis of structural and functional systems," *Nat. Rev. Neurosci.*, vol. 10, no. 3, pp. 186–198, 2009.

[10] J. Lu, H. Liu, and J. Chen, *Synchronization in Complex Dynamical Networks*. Beijing, China: Higher Educ. Press (in Chinese), 2016.

[11] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Int. J. Bifurc. Chaos*, vol. 9, no. 12, pp. 2315–2320, 1999.

[12] C. Li and G. Chen, "Synchronization in general complex dynamical networks with coupling delays," *Physica A*, vol. 343, pp. 263–278, 2004.

[13] M. Chavez, D. U. Hwang, A. Amann, H. G. Hentschel, and S. Boccaletti, "Synchronization is enhanced in weighted complex networks," *Phys. Rev. Lett.*, vol. 94, no. 21, 2005, Art. no. 218701.

[14] L. Huang, K. Park, Y. C. Lai, L. Yang, and K. Yang, "Abnormal synchronization in complex clustered networks," *Phys. Rev. Lett.*, vol. 97, no. 16, 2006, Art. no. 164101.

[15] X. Zhang, X. Hu, J. Kurths, and Z. Liu, "Explosive synchronization in a general complex network." *Phys. Rev. E*, vol. 88, no. 1, 2013, Art. no. 010802.

[16] J. Lü and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 841–846, Jun. 2005.

[17] J. Zhou, J. Lu, and J. Lü, "Adaptive synchronization of an uncertain complex dynamical network," *IEEE Trans. Autom. Control*, vol. 51, no. 4, pp. 652–656, Apr. 2006.

[18] W. Yu, G. Chen, J. Lü, and J. Kurths, "Synchronization via pinning control on general complex networks," *SIAM J. Control Optim.*, vol. 51, no. 2, pp. 1395–1416, 2013.

[19] J. Zhou, J. Lu, and J. Lü, "Pinning adaptive synchronization of a general complex dynamical network," *Automatica*, vol. 44, no. 4, pp. 996–1003, 2008.

[20] Y. Liang, X. Wang, and J. Eustace, "Adaptive synchronization in complex networks with non-delay and variable delay couplings via pinning control," *Neurocomputing*, vol. 123, no. 5, pp. 292–298, 2014.

[21] J. Zhao, D. J. Hill, and T. Liu, "Global bounded synchronization of general dynamical networks with nonidentical nodes," *IEEE Trans. Autom. Control*, vol. 57, no. 10, pp. 2656–2662, Oct. 2012.

[22] W. S. Zhong, G. P. Liu, and C. Thomas, "Global bounded consensus of multiagent systems with nonidentical nodes and time delays," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 5, pp. 1480–1488, Oct. 2012.

[23] L. Wang, S. L. Yan, and Q. G. Wang, "Bounded consensus in multi-agent systems of asymmetrically coupled nonidentical agents," in *Proc. IEEE Int. Conf. Control Autom.*, Hangzhou, China, 2013, pp. 1172–1177.

[24] L. Wang, W. Qian, and Q. G. Wang, "Bounded synchronisation of a time-varying dynamical network with nonidentical nodes," *Int. J. Syst. Sci.*, vol. 46, no. 7, pp. 1234–1245, 2015.

[25] L. Wang, W. J. Feng, M. Z. Q. Chen, and Q. G. Wang, "Global bounded consensus in heterogeneous multi-agent systems with directed communication graph," *IET Control Theory Appl.*, vol. 9, no. 1, pp. 147–152, 2015.

[26] W. He, F. Qian, J. Lam, G. Chen, Q. L. Han, and J. Kurths, "Quasi-synchronization of heterogeneous dynamic networks via distributed impulsive control: Error estimation, optimization and design," *Automatica*, vol. 62, pp. 249–262, 2015.

[27] P. Delellis, M. Di Bernardo, and D. Liuzza, "Convergence and synchronization in heterogeneous networks of smooth and piecewise smooth systems," *Automatica*, vol. 56, pp. 1–11, 2015.

[28] W. He, F. Qian, Q. L. Han, and J. Cao, "Synchronization error estimation and controller design for delayed Lur'e systems with parameter mismatches," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 10, pp. 1551–1563, Oct. 2012.

[29] Z. Tang, H. P. Ju, and J. Feng, "Impulsive effects on quasi-synchronization of neural networks with parameter mismatches and time-varying delay," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 4, pp. 908–919, Apr. 2018.

[30] L. Wang, M. Z. Q. Chen, and Q. G. Wang, "Bounded synchronization of a heterogeneous complex switched network," *Automatica*, vol. 56, no. 6, pp. 19–24, 2015.

[31] L. Wang, Z. Wang, G. Wei, and F. E. Alsaadi, "Finite-time state estimation for recurrent delayed neural networks with component-based event-triggering protocol," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 4, pp. 1046–1057, Apr. 2018.

[32] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

[33] J. Lu, D. W. Ho, and Z. Wang, "Pinning stabilization of linearly coupled stochastic neural networks via minimum number of controllers." *IEEE Trans. Neural Netw.*, vol. 20, no. 10, pp. 1617–29, Oct. 2009.

[34] A. Fradkov, B. Andrievsky, and K. Boykov, "Control of the coupled double pendulums system," *Mechatronics*, vol. 15, no. 10, pp. 1289–1303, 2005.