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ESTIMATING THE REGION OF ATTRACTION ON A COMPLEX DYNAMICAL NETWORK*

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Abstract. Despite the prevalence of synchronization analysis on complex dynamical networks, little attention was paid to the problem of estimating their regions of attraction. This paper addresses the issue of estimating the region of attraction of an equilibrium point of a complex dynamical network, and briefly analyzes the network stability. A sufficient condition and a necessary condition are first established for the asymptotical stability of the network equilibrium point. Then, a general technique for region-of-attraction estimation is developed by combining the network structure and the node dynamics. In order to avoid the troublesome parameter selection in general region-of-attraction estimation, second-order estimation is solved under a mild additional condition. Examples are provided to verify the theoretical estimations.

Key words. complex network, region of attraction, stability, equilibrium point, synchronization

AMS subject classifications. 70K20, 70K42, 93A15, 93C10

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1. Introduction.

A complex network is a set of nodes connected by edges, both of which have certain physical meanings [21]. Complex networks have received much attention over the past two decades due to their wide applications in science and engineering [4, 26, 1, 22, 13, 6].

Synchronization [23, 2, 34, 3, 18, 37, 7, 38, 10, 24] is a fundamental problem for complex dynamical networks, and accordingly is one of the most important topics in the field. Synchronization can be classified as global synchronization and local synchronization. Usually, it is difficult to reach global synchronization; rather, local synchronization may be much easier. Various local synchronization criteria were derived [32, 15, 31, 33, 30]. It is noteworthy that network synchronization also depends on initial conditions due to the nonlinearity that may lead to multiple attractors. Consider, for example, a network of two nodes:

$$\begin{aligned} \dot{x}_1(t) &= f(x_1(t)) + x_2(t) - x_1(t), \\ \dot{x}_2(t) &= f(x_2(t)) + x_1(t) - x_2(t), \end{aligned} \tag{1.1}$$

where $f(x(t)) = -x(t) + x^2(t) \in \mathbb{R}$. If the initial values are set as $x_1(0) = 2$ and $x_2(0) = 3$, network (1.1) cannot synchronize to $x_1 = x_2 = 0$, although the equilibrium states $x_1 = x_2 = 0$ is locally asymptotically stable; if $x_1(0) = 0.2$ and $x_2(0) = 0.3$, however, it can be verified via simulation that network (1.1) synchronizes to $x_1 = x_2 = 0$. Hence, the initial values should be well selected so that network (1.1)

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can synchronize to $x_1 = x_2 = 0$. Since the existing criteria do not show how to select appropriate initial values, it is helpful and indeed necessary to know the region of attraction (ROA) of a complex dynamical network, which includes all the initial values starting from which the trajectories converge (or synchronize) to its associated equilibrium point. As is known, the shape of the ROA is very complicated in general, which can only be estimated by finding its subsets.

Despite the prevalence of general synchronization analysis, little attention was paid to the ROA estimation of a complex dynamical network. For estimating the ROA, a proper Lyapunov function should be searched in advance. A lot of methods for solving a Lyapunov function have been developed in the literature, e.g., [35, 9, 16]. In general, the ROA estimation focuses on two kinds of systems: linear systems with various constraints [19, 25, 36] and some nonlinear systems of low dimensions [12, 29, 8, 27, 28, 14, 20]. Since a network is usually of high dimension and its node dynamics are often nonlinear, the ROA estimation methods for the two kinds of systems cannot be directly applied to complex dynamical networks. Therefore, it is necessary to elaborately study the problem of the ROA estimation for complex dynamical networks.

Motivated by the above discussions, this paper aims to address the estimation of the ROA of an equilibrium point of a complex dynamical network. Since the ROA is a nonempty open set only if the equilibrium point is asymptotically stable, this paper also briefly analyzes the asymptotical stability issue. The main contributions of the paper are as follows.

(1) A sufficient condition and a necessary condition are established for the asymptotical stability of an equilibrium point of a complex dynamical network. Previous studies (for instance, [15] and [31]) have proposed some similar sufficient conditions for the asymptotical stability, but few studies developed necessary conditions.
(2) General ROA estimation is performed by combining the network structure and the node dynamics. As complex dynamical networks are usually of high dimensions, a decomposition technique inspired by the master stability function method [23] is developed to realize dimensionality reduction.
(3) To avoid the troublesome parameter selection in a general ROA estimation, second-order ROA estimation is developed under a mild additional condition.

The network model does not require a symmetric outer coupling matrix or a linear inner coupling function, so the stability analysis and ROA estimations proposed in this paper can be widely applied.

The rest of this paper is organized as follows. Section 2 introduces the network model and some mathematical preliminaries. Section 3 first analyzes the stability of an equilibrium point of a complex dynamical network, and then establishes general estimation and second-order estimation of the ROA of the equilibrium point. Section 4 provides two examples to verify the estimations. Section 5 concludes the paper.

2. Model and preliminaries.

2.1. Model and problem statement. First, some notation is introduced. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) denote the n-dimensional Euclidean space and the set of all $(n\times m)$-dimensional real (complex) matrices, respectively, $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ denote the maximal and minimal eigenvalues of a symmetric matrix, respectively, $\|\cdot\|$ denotes the 2-norm of a vector or a matrix, $|\cdot|$ denotes the modulus of a complex number, $\kappa_2(A) = \|A\|\|A^{-1}\|$ is the condition number of square matrix A, the superscript T (H) represents the transpose (conjugate transpose) of a vector or a matrix, $\otimes$ represents the Kronecker product, and I_N is the identity matrix of dimension N. A function $\alpha : [0, +\infty) \to [0, +\infty)$ is a class-$\mathcal{K}$ function if it is continuous, strictly increasing, and satisfying $\alpha(0) = 0$.

Consider a complex network consisting of N identical node systems of dimension n, described by

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} a_{ij}h(x_j(t)), \tag{2.1}$$

where $1 \le i \le N$, $x_i \in \mathbb{R}^n$ is the state vector of the ith node, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable functions, constant $c > 0$ is the coupling strength, and $A = (a_{ij})_{N\times N}$ is the outer coupling matrix: If there is an edge from node i to node j ($j \ne i$), then $a_{ij} > 0$; otherwise, $a_{ij} = 0$, and the diagonal elements are defined by $a_{ii} = -\sum_{j=1, j\ne i}^{N} a_{ij}$, $1 \le i \le N$.

Let s be an equilibrium point of the node system $\dot{x}(t) = f(x(t))$. Then, the homogeneous stationary state

$$x_1 = x_2 = \cdots = x_N = s$$

is an equilibrium point of the entire network (2.1). Denote

$$\begin{aligned} X(t) &= [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T \in \mathbb{R}^{nN}, \\ S &= [s^T, s^T, \ldots, s^T]^T \in \mathbb{R}^{nN}. \end{aligned}$$

The ROA of the equilibrium point S is defined by

$$R_A = \left\{ X_0 \in \mathbb{R}^{nN} \middle| \lim_{t\to+\infty} X(t; X_0) = S \right\}. \tag{2.2}$$

The objective of this paper is to estimate R_A by finding an open subset of R_A as large as possible.

Defining the error $e_i = x_i - s$, network (2.1) can be written as

$$\dot{e}_i = f(x_i) - f(s) + c\sum_{j=1}^{N} a_{ij}(h(x_j(t)) - h(s)), \tag{2.3}$$

where $1 \le i \le N$. Since f and h are continuously differentiable, (2.3) can be transformed to

$$\dot{e}_i = F_1 e_i + c\sum_{j=1}^{N} a_{ij}F_2 e_j + R_i^{(1)} + c\sum_{j=1}^{N} a_{ij}R_j^{(2)}, \tag{2.4}$$

where $F_1 = Df(s)$ and $F_2 = Dh(s)$ are the Jacobians of f and h, respectively, and

$$\begin{aligned} R_i^{(1)} &= f(x_i) - f(s) - F_1 e_i, \\ R_i^{(2)} &= h(x_i) - h(s) - F_2 e_i. \end{aligned} \tag{2.5}$$

Then, it follows that

$$\|R_i^{(k)}\| = o(\|e_i\|), \quad 1 \le i \le N, \quad 1 \le k \le 2.$$

Denote

$$\begin{aligned} \varphi_1(x) &= \tfrac{\|f(s+x)-f(s)-F_1x\|}{\|x\|}, \\ \varphi_2(x) &= \tfrac{\|h(s+x)-h(s)-F_2x\|}{\|x\|}, \end{aligned} \tag{2.6}$$

where $x \in \mathbb{R}^n\backslash\{0\}$. Then, one has

$$\|\varphi_k(x)\| = o(1), \quad 1 \le k \le 2. \tag{2.7}$$

Let

$$\begin{aligned} &e = [e_1^T, e_2^T, \dots, e_N^T]^T, \\ &R^{(k)} = \left[R_1^{(k)^T}, R_2^{(k)^T}, \dots, R_N^{(k)^T}\right]^T, \quad 1 \le k \le 2. \end{aligned} \tag{2.8}$$

Then, (2.4) can be rewritten in a compact form as

$$\dot{e} = Ze + R^{(1)} + c(A \otimes I_n)R^{(2)}, \tag{2.9}$$

where

$$Z = I_N \otimes F_1 + cA \otimes F_2. \tag{2.10}$$

Let the eigenvalues of A be $\lambda_1, \lambda_2, \dots, \lambda_N$, with

$$|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_N|. \tag{2.11}$$

Note that λ_i $(1 \le i \le N)$ may be a complex number when A is not symmetric.

2.2. Mathematical preliminaries. Here, several lemmas are presented.

LEMMA 2.1. *Let $x : \mathbb{R} \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to [0, +\infty)$ be differentiable functions satisfying*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \ \forall x \in \mathbb{R}^n,$$

where α_1 and α_2 are class-$\mathcal{K}$ functions. Let δ_0 be a positive constant. For any positive constant $\delta < \delta_0$, suppose that there exists a class-$\mathcal{K}$ function α_3 satisfying

$$\dot{g}(t) \le -\alpha_3(\|x(t)\|) \ \forall t \in \{t \mid \|x(t)\| \le \delta\}, \tag{2.12}$$

where $g(t) = V(x(t)),\ t \in \mathbb{R}$. If

$$x(t_0) \in \{x \in \mathbb{R}^n \mid \|x\| < \alpha_2^{-1}(\alpha_1(\delta_0))\},$$

then $\lim_{t\to+\infty} \|x(t)\| = 0$.

Proof. Since α_1 and α_2 are class-$\mathcal{K}$ functions, $\alpha_2^{-1} \circ \alpha_1$ is strictly monotonically increasing. It follows from $\|x(t_0)\| < \alpha_2^{-1}(\alpha_1(\delta_0))$ that there exists a positive constant $\delta_1 < \delta_0$ such that $\|x(t_0)\| < \alpha_2^{-1}(\alpha_1(\delta_1))$. Obviously, $\|x(t_0)\| < \alpha_2^{-1}(\alpha_2(\delta_1)) = \delta_1$.

Next, we prove by contradiction that $\|x(t)\| < \delta_1 \ \forall t > t_0$. Suppose that there exists a time instant $t_1 > t_0$ such that $\|x(t_1)\| \ge \delta_1$. Without loss of generality, let $t = t_1 > t_0$ be the earliest time satisfying that $\|x(t_1)\| = \delta_1$. Then, one has $\|x(t)\| \le \delta_1 \ \forall t \in [t_0, t_1]$. It follows that

$$\dot{g}(t) \le 0 \ \forall t \in [t_0, t_1].$$

Further, one has $V(x(t_1)) \leq V(x(t_0))$. However,

$$V(x(t_0)) \leq \alpha_2(\|x(t_0)\|) < \alpha_1(\delta_1)$$

and

$$V(x(t_1)) \geq \alpha_1(\|x(t_1)\|) = \alpha_1(\delta_1),$$

which contradict each other. Therefore, $\|x(t)\| < \delta_1 \ \forall t > t_0$.

Now, we prove that $\lim_{t\to+\infty} \|x(t)\| = 0$. Since $g(t) \geq 0$ is nonincreasing, $\lim_{t\to+\infty} g(t)$ exists, and $a = \lim_{t\to+\infty} g(t) \geq 0$. Then, one has $\|x(t)\| \geq \alpha_2^{-1}(a)$ and $\dot{g}(t) \leq -\alpha_3(\alpha_2^{-1}(a))$, yielding

$$g(t) = g(t_0) + \int_{t_0}^{t} \dot{g}(t)\mathrm{d}t \leq g(t_0) - \alpha_3(\alpha_2^{-1}(a))(t - t_0).$$

If $a > 0$, one has $\lim_{t\to+\infty} g(t) = -\infty$, which contradicts the fact that $g \geq 0$. Therefore, one has $a = 0$ and

$$\lim_{t\to+\infty} \|x(t)\| \leq \lim_{t\to+\infty} \alpha_1^{-1}(g(t)) = 0,$$

which is equivalent to $\lim_{t\to+\infty} \|x(t)\| = 0$. ∎

Remark 2.1. From (2.12), it is clear that

$$\dot{g}(t) < 0 \ \forall t \in \{t \,|\, 0 < \|x(t)\| < \delta_0\}.$$

But $\dot{g}(t)$ might be zero when $\|x(t)\| = \delta_0$, i.e., it is not required that there exists a class-$\mathcal{K}$ function satisfying (2.12) for $\delta = \delta_0$. Consider the following example:

$$\dot{g}(t) = -\|x(t)\|^2 + \|x(t)\|^3.$$

For $\delta \in (0, 1)$, there exists a class-$\mathcal{K}$ function $\alpha_3(\|x\|) = -(1 - \delta)\|x\|^2$ which satisfies (2.12); for $\delta = 1$, $\dot{g}(t)$ might be zero, thus no such α_3 exists.

LEMMA 2.2. *Let $x : \mathbb{R} \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to [0, +\infty)$ be differentiable functions satisfying*

$$a\|x\|^2 \leq V(x) \leq b\|x\|^2 \ \forall x \in \mathbb{R}^n,$$

where a and b are positive constants. Let δ_0 be a positive constant. For any positive constant $\delta < \delta_0$, suppose that there exists a positive constant β such that

$$\dot{g}(t) \leq -\beta\|x(t)\|^2 \ \forall t \in \{t \,|\, \|x(t)\| \leq \delta\},$$

where $g(t) = V(x(t)),\ t \in \mathbb{R}$. If

$$x(t_0) \in \{x \in \mathbb{R}^n \mid \|x\| < r_0\},$$

where $r_0 = \sqrt{\frac{a}{b}}\delta_0$, then $\lim_{t\to+\infty} \|x(t)\| = 0$.

Proof. Applying Lemma 2.1 completes the proof. ∎

LEMMA 2.3 (see [17]). *Let $\Psi \in \mathbb{R}^{N\times N}$ be a matrix with eigenvalues $\psi_1, \psi_2, \ldots, \psi_N$. For any positive constant γ, there exists a nonsingular matrix $U \in \mathbb{C}^{N\times N}$ such that $\Psi = U^{-1}JU$, where*

$$J = \begin{bmatrix} \psi_1 & \gamma_1 & & & \\ & \psi_2 & \ddots & & \\ & & \ddots & \gamma_{N-1} \\ & & & \psi_N \end{bmatrix}, \tag{2.13}$$

with $\gamma_i = 0$ or γ.

Remark 2.2. It is noted that J is not necessarily in the Jordan canonical form, and U varies with the change of γ.

LEMMA 2.4 (see [5]). *Given matrices A, B, C, and D of appropriate dimensions, the Kronecker product $\otimes$ satisfies the following properties:*

$$
\begin{aligned}
&A \otimes B + A \otimes C = A \otimes (B + C),\\
&(A \otimes B)(C \otimes D) = (AC) \otimes (BD),\\
&\|A \otimes B\| = \|A\|\|B\|.
\end{aligned}
$$

3. Main results. In this section, a sufficient condition and a necessary condition for the asymptotical stability of the equilibrium point S of network (2.1) are first established. Then, general estimation and second-order estimation of the ROA of the equilibrium point S are established. The ROA is estimated in the shape of a spheroid, where the center is the equilibrium point S and the radius r_0 describes the size of the estimated ROA. Obviously, the estimation is better with a larger r_0.

Assumption 3.1. (A1) Suppose there exist a positive definite matrix $P \in \mathbb{R}^{nN \times nN}$ and a positive constant ε satisfying

$$PZ + Z^T P + \varepsilon I_{nN} \leq 0, \tag{3.1}$$

where Z is defined in (2.10).

Assumption 3.2. (A2) Suppose there exist a positive definite matrix $P = \mathrm{diag}\{P_1, P_2, \ldots, P_N\}$ with $P_i \in \mathbb{R}^{n \times n}$ and a positive constant ε satisfying

$$P_i(F_1 + c\lambda_i F_2) + (F_1 + c\lambda_i F_2)^H P_i + \varepsilon I_n \leq 0, \tag{3.2}$$

where $1 \leq i \leq N$, λ_i is defined in (2.11), and F_1, F_2 are defined in (2.4).

Hereafter, when (A1) or (A2) is imposed, the notation therein is also declared implicitly, which will not be explained once again for simplicity.

3.1. Stability analysis.

THEOREM 3.1. *If* (A1) *holds, then the equilibrium point S of network* (2.1) *is asymptotically stable.*

Proof. Consider a linear system, $\dot{\omega} = Z\omega$. Defining a Lyapunov function as $V = \omega^T P \omega$, one has

$$\dot{V} = \omega^T (PZ + Z^T P)\omega \leq -\varepsilon \|\omega\|^2,$$

which implies that $\dot{\omega} = Z\omega$ is asymptotically stable about $\omega = 0$. Then, (2.9) is asymptotically stable about $e = 0$, and equivalently, the equilibrium point S of network (2.1) is asymptotically stable. ∎

Remark 3.1. Theorem 3.1 establishes a sufficient condition for the asymptotical stability of the equilibrium point S of network (2.1). The ROA of S is a nonempty open set only if S is asymptotically stable. Similar conditions (not only for equilibrium points) for various network models can be found in other studies, e.g., [15, 31].

THEOREM 3.2. *If the equilibrium point S of network* (2.1) *is asymptotically stable, then the equilibrium point s of the node system $\dot{x}(t) = f(x(t))$ is necessarily asymptotically stable.*

Proof. Since S is asymptotically stable, there exists a positive constant δ_0 such that

$$\lim_{t\to+\infty} \|X(t) - S\| = 0 \tag{3.3}$$

for any initial value $X(t_0) \in \mathbb{R}^{nN}$ of network (2.1) satisfying $\|X(t_0) - S\| < \delta_0$. Select

$$X(t_0) = [x_0, x_0, \ldots, x_0]^T \in \mathbb{R}^{nN},$$

where $x_0 \in \mathbb{R}^n$ satisfies $\|x_0 - s\| < \frac{\delta_0}{\sqrt{N}}$. Then,

$$X(t) = [x(t), x(t), \ldots, x(t)]^T \in \mathbb{R}^{nN}$$

is a solution trajectory of (2.1), where $x(t)$ is a solution trajectory of the system $\dot{x}(t) = f(x(t))$ satisfying $x(t_0) = x_0$. From (3.3), it follows that $\lim_{t\to+\infty} \|x(t) - s\| = 0$. ◻

Remark 3.2. Theorem 3.2 establishes a necessary condition for the asymptotical stability of the equilibrium point S of network (2.1). From Theorem 3.2, one can see that the equilibrium point S of network (2.1) is not asymptotically stable if the equilibrium point s of the node system is not asymptotically stable.

3.2. General ROA estimation. Define

$$\mathrm{B}_r = \{x \in \mathbb{R}^n \mid \|x\| < r\},$$
$$\overline{\mathrm{B}}_r = \{x \in \mathbb{R}^n \mid \|x\| \le r\}.$$

THEOREM 3.3. *Consider network* (2.1). *If* (A1) *holds, then*

(a) *for $\rho \in (0, 1)$, there exists a positive constant $\delta_0 = \delta_0(\rho)$ satisfying*

$$\varphi_k(x) < \nu_k \ \forall x \in \mathrm{B}_{\delta_0} \backslash \{0\}, \tag{3.4}$$

where $1 \le k \le 2$, φ_k is defined in (2.6), $\nu_1 = \frac{\varepsilon\rho}{2\|P\|}$, *and* $\nu_2 = \frac{\varepsilon(1-\rho)}{2c\|P\|\|A\|}$;

(b) *for* $r_0 = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}\delta_0$,

$$D = \left\{S + Y \in \mathbb{R}^{nN} \mid \|Y\| < r_0\right\}$$

is a subset of R_A, where S is the equilibrium point of (2.1).

Proof. (a) Choose a Lyapunov function as $V = e^T P e$, where e is defined in (2.8). From (2.9), one has

$$\begin{aligned}\dot{V} &= e^T(PZ + Z^T P)e + 2e^T PR^{(1)} + 2ce^T P(A \otimes I_n)R^{(2)} \\ &\le -\varepsilon\|e\|^2 + 2\left(\|R^{(1)}\| + c\|A\|\|R^{(2)}\|\right)\|P\|\|e\|.\end{aligned}$$

According to (2.7), there exist a positive constant δ_1 satisfying

$$\varphi_1(x) < \nu_1 \ \forall x \in \mathrm{B}_{\delta_1} \backslash \{0\}, \tag{3.5}$$

and a positive constant δ_2 satisfying

$$\varphi_2(x) < \nu_2 \ \forall x \in \mathrm{B}_{\delta_2} \backslash \{0\}. \tag{3.6}$$

Then, (3.4) follows by choosing $\delta_0 = \min\{\delta_1, \delta_2\}$.

(b) Let $\delta \in (0, \delta_0)$. Since $\varphi_k(x)$ is continuous in x in the set $\overline{\mathrm{B}}_\delta \backslash \{0\}$ and $\lim_{\|x\| \to 0} \varphi_k(x) = 0$, there exists a positive constant β such that

$$\varphi_k(x) \le \nu_{k+2} \ \forall x \in \overline{\mathrm{B}}_\delta \backslash \{0\},$$

where $1 \le k \le 2$, $\nu_3 = \frac{\rho(\varepsilon-\beta)}{2\|P\|}$, and $\nu_4 = \frac{(1-\rho)(\varepsilon-\beta)}{2c\|P\|\|A\|}$. It follows that

$$\frac{\|R^{(k)}\|}{\|e\|} \le \nu_{k+2} \ \forall t \in \{t \mid 0 < \|e\| \le \delta\},$$

yielding

$$\dot{V} \le -\beta \|e\|^2 \ \forall t \in \{t \mid \|e\| \le \delta\}.$$

According to Lemma 2.2, $\lim_{t \to +\infty} \|e\| = 0$ if $\|e(0)\| \le r_0$, i.e., D is a subset of R_A.□

Remark 3.3. Since n is usually small comparing to N, it is generally easy to find a δ_0 satisfying (3.4). In Theorem 3.3, the value of δ_0, generally varying with the change of the parameter ρ, determines the size of the estimated ROA. Suppose that P and ε have already been determined according to (3.1). With the increasing of ρ, the largest δ_1 that satisfies (3.5) does not decrease, whereas the largest δ_2 that satisfies (3.6) does not increase. Hence, $\delta_0 = \min\{\delta_1, \delta_2\}$ first increases and then decreases (not necessarily strictly) with the increasing of ρ in most cases. In such cases, dichotomy can be used to determine the value of ρ so as to make δ_0 as large as possible. The problem of finding ρ (probably not unique) that maximizes δ_0 can also be formulated as the following optimization problem:

$$\begin{array}{ll} \min & -\delta_0 \\ \text{s.t.} & \varphi_1(x) < \nu_1 \ \forall x \in \mathrm{B}_{\delta_0} \backslash \{0\}, \\ & \varphi_2(x) < \nu_2 \ \forall x \in \mathrm{B}_{\delta_0} \backslash \{0\}, \\ & \rho \in (0,1), \end{array} \tag{3.7}$$

where ν_1 and ν_2 are as defined in Theorem 3.3. If f and h in (2.1) are both linear, it is obvious that $\varphi_1 = \varphi_2 = 0$. In this case, one has $\delta_0 = +\infty$, no matter what value the ρ takes. Hence, $D = \mathbb{R}^n$, namely the equilibrium point S is globally asymptotically stable.

Remark 3.4. Theorem 3.3 establishes a simple estimation of the ROA of network (2.1). However, it is not easy to find P and ε that satisfy (3.1) when N is large, because P is of dimension $nN \times nN$. For this reason, another method, which reduces (3.1) to N inequalities of lower dimension, is suggested.

THEOREM 3.4. *Consider network* (2.1). *If* (A2) *holds, then*

(a) *there exists a nonsingular matrix* $U \in \mathbb{C}^{N \times N}$ *such that* $A = U^{-1}JU$, *where* J *is in the form of* (2.13) *with* $J_{ii} = \lambda_i$ *and* $J_{i,i+1} = 0$ *or* $\frac{\varepsilon}{4c\|P\|\|F_2\|}$;

(b) *for* $\rho \in (0,1)$, *there exists a positive constant* $\delta_0 = \delta_0(\rho)$ *satisfying*

$$\varphi_k(x) < \nu_k \ \forall x \in \mathrm{B}_{\delta_0} \backslash \{0\},$$

where $1 \le k \le 2$, φ_k *is defined in* (2.6), $\nu_1 = \frac{\varepsilon\rho}{4\|P\|\kappa_2(U)}$, *and* $\nu_2 = \frac{\varepsilon(1-\rho)}{4c\|P\|\|A\|\kappa_2(U)}$;

(c) *for* $r_0 = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}} \frac{\delta_0}{\kappa_2(U)}$,

$$D = \left\{ S + Y \in \mathbb{R}^{nN} \mid \|Y\| < r_0 \right\} \tag{3.8}$$

is a subset of R_A, *where* S *is the equilibrium point of* (2.1).

Proof. (a) According to Lemma 2.3, there exists a nonsingular matrix $U \in \mathbb{C}^{N\times N}$ such that A can be written as $A = U^{-1}JU$.

(b) The proof of (b) is very similar to that of Theorem 3.3(a) and thus omitted.

(c) Define $\gamma = \frac{\varepsilon}{4c\|P\|\|F_2\|}$, and

$$\eta = [\eta_1^T, \eta_2^T, \dots, \eta_N^T]^T = (U \otimes I_n)e,$$
$$G^{(k)} = \left[{G_1^{(k)}}^T, {G_2^{(k)}}^T, \dots, {G_N^{(k)}}^T\right]^T = (U \otimes I_n)R^{(k)},$$

where $1 \le k \le 2$. By making use of Lemma 2.4, herewith (2.9) takes the following form:

$$\dot\eta = (I_N \otimes F_1 + cJ \otimes F_2)\eta + G^{(1)} + c(J \otimes I_n)G^{(2)},$$

or, equivalently,

$$\dot\eta_i = (F_1 + c\lambda_i F_2)\eta_i + cJ_{i,i+1}F_2\eta_{i+1} + G_i^{(1)} + c\lambda_i G_i^{(2)} + cJ_{i,i+1}G_{i+1}^{(2)},$$

where $1 \le i \le N$, $J_{N,N+1} = 0$, $\eta_{N+1} = G_{N+1}^{(2)} = 0$. Choose a Lyapunov function as

$$V = \eta^H P\eta = \sum_{i=1}^N \eta_i^H P_i\eta_i.$$

It is obvious that $V \in \mathbb{R}$ and V is positive except at $\eta = 0$. Since $V \in \mathbb{R}$, it follows that $\dot V \in \mathbb{R}$. Then,

$$\begin{aligned}
\dot V &= \sum_{i=1}^N \eta_i^H P_i\dot\eta_i + \sum_{i=1}^N \dot\eta_i^H P_i\eta_i\\
&= \sum_{i=1}^N \eta_i^H \left(P_i(F_1 + c\lambda_iF_2) + (F_1 + c\lambda_iF_2)^H P_i\right)\eta_i\\
&\quad + \sum_{i=1}^N cJ_{i,i+1}\left(\eta_i^H P_iF_2\eta_{i+1} + \eta_{i+1}^H F_2^H P_i\eta_i\right)\\
&\quad + \sum_{i=1}^N \left(\eta_i^H P_iG_i^{(1)} + {G_i^{(1)}}^H P_i\eta_i\right) + \sum_{i=1}^N c\left(\lambda_i\eta_i^H P_iG_i^{(2)} + \overline{\lambda}_i{G_i^{(2)}}^H P_i\eta_i\right)\\
&\quad + \sum_{i=1}^N cJ_{i,i+1}\left(\eta_i^H P_iG_{i+1}^{(2)} + {G_{i+1}^{(2)}}^H P_i\eta_i\right),
\end{aligned} \tag{3.9}$$

where $\overline{z}$ denotes the complex conjugate of z. Note that each of the five terms after the second equal sign in (3.9) is a real number. It follows that

$$\begin{aligned}
\dot V &\le \sum_{i=1}^N \eta_i^H \left(P_i(F_1 + c\lambda_iF_2) + (F_1 + c\lambda_iF_2)^H P_i\right)\eta_i + \left|\sum_{i=1}^N 2cJ_{i,i+1}\eta_i^H P_iF_2\eta_{i+1}\right|\\
&\quad + 2\left|\eta^H PG^{(1)} + c\eta^H P(J \otimes I_n)G^{(2)}\right|\\
&\le -\varepsilon\sum_{i=1}^N \|\eta_i\|^2 + 2c\gamma\sum_{i=1}^N \|P_i\|\|F_2\|\|\eta_i\|\|\eta_{i+1}\|
\end{aligned}$$

$$
\begin{aligned}
&+ 2\left|\eta^H P(U \otimes I_n)\left(R^{(1)} + c(A \otimes I_n)R^{(2)}\right)\right| \\
\leq &- \varepsilon\|\eta\|^2 + c\gamma\|P\|\|F_2\|\sum_{i=1}^{N}\left(\|\eta_i\|^2 + \|\eta_{i+1}\|^2\right) \\
&+ 2\|\eta\|\|P\|\|U\|\left(\|R^{(1)}\| + c\|A\|\|R^{(2)}\|\right) \\
\leq &- \frac{\varepsilon}{2}\|\eta\|^2 + 2\|\eta\|\|P\|\|U\|\left(\|R^{(1)}\| + c\|A\|\|R^{(2)}\|\right).
\end{aligned}
$$

Let $\delta \in (0, \delta_0)$. Since $\varphi_k(x)$ is continuous in x in $\overline{\mathrm{B}}_\delta\backslash\{0\}$ and $\lim_{\|x\|\to 0}\varphi_k(x) = 0$, there exists a positive constant β such that

$$\varphi_k(x) \leq \nu_{k+2}\ \forall x \in \overline{\mathrm{B}}_\delta\backslash\{0\},$$

where $1 \leq k \leq 2$, $\nu_3 = \frac{\rho(\varepsilon-\beta)}{4\|P\|\kappa_2(U)}$, and $\nu_4 = \frac{(1-\rho)(\varepsilon-\beta)}{4c\|P\|\|A\|\kappa_2(U)}$. It follows that

$$\frac{\|R^{(k)}\|}{\|e\|} \leq \nu_{k+2}\ \forall t \in \{t \mid 0 < \|e\| \leq \delta\}.$$

Note that $\|U^{-1}\|^{-1}\|e\| \leq \|\eta\| \leq \|U\|\|e\|$. Then,

$$
\begin{aligned}
2\|\eta\|\|P\|\|U\|\left(\|R^{(1)}\| + c\|A\|\|R^{(2)}\|\right) &\leq 2\|\eta\|\|P\|\|U\|\|e\|\left(\nu_3 + c\|A\|\nu_4\right) \\
&= \frac{(\varepsilon-\beta)\|\eta\|\|e\|}{2\|U^{-1}\|} \leq \frac{(\varepsilon-\beta)}{2}\|\eta\|^2
\end{aligned}
$$

for $t \in \{t \mid \|e\| \leq \delta\}$. It follows that

$$\dot{V} \leq -\frac{\beta}{2}\|\eta\|^2 \leq -\frac{\beta}{2}\|U\|^2\|e\|^2\ \forall t \in \{t \mid \|e\| \leq \delta\}.$$

It is clear that

$$V \leq \lambda_{max}(P)\|\eta\|^2 \leq \lambda_{max}(P)\|U\|^2\|e\|^2$$

and

$$V \geq \lambda_{min}(P)\|\eta\|^2 \geq \lambda_{min}(P)\|U^{-1}\|^{-2}\|e\|^2.$$

According to Lemma 2.2, $\lim_{t\to+\infty}\|e\| = 0$ if $\|e(0)\| \leq r_0$, i.e., D is a subset of R_A. □

Remark 3.5. To be specific, the procedure of calculating r_0 according to Theorem 3.4 is shown in Table 1.

Remark 3.6. If P and ε satisfy (3.2), then

$$\tilde{P} = \operatorname{diag}\left\{\frac{\|P\|}{\|P_1\|}P_1, \ldots, \frac{\|P\|}{\|P_N\|}P_N\right\}$$

and ε together also satisfy (3.2). Since $\|P\| = \|\tilde{P}\|$, replacing P with $\tilde{P}$ does not affect $\kappa_2(U)$ or δ_0 in Theorem 3.4. Since $\lambda_{min}(\tilde{P}) \geq \lambda_{min}(P)$ and $\lambda_{max}(\tilde{P}) = \lambda_{max}(P)$, the ROA estimated by (3.8) does not shrink but may be enlarged by replacing P with $\tilde{P}$.

Remark 3.7. Theorem 3.4 establishes a more complicated but more feasible estimation of the ROA for network (2.1), especially when N is large. For a directed network, its outer coupling matrix A is not symmetric, and thus its U is nonorthogonal. The estimation given by Theorem 3.4 is generally more conservative than that of Theorem 3.3, because a nonorthogonal U usually results in a large $\kappa(U)$.

TABLE 1
Calculation of r_0 by Theorem 3.4.

Step 1	Calculate the eigenvalues of A, i.e., $\lambda_1, \lambda_2, \ldots, \lambda_N$.
Step 2	Solve the matrix inequality (3.2) to get ε and P_i. Without loss of generality, ε can be set as $\varepsilon = 1$.
Step 3	Decompose $A = U^{-1}JU$ to obtain U, where J is as defined in Theorem 3.4.
Step 4	Set the parameter ρ, and then calculate $\delta_0 = \delta_0(\rho)$. Remark 3.3 gives a possible method of determining ρ so as to make δ_0 as large as possible.
Step 5	Calculate $r_0 = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}\frac{\delta_0}{\kappa_2(U)}$.

COROLLARY 3.5. *Consider network* (2.1), *and suppose that its outer coupling matrix A is symmetric. If* (A2) *holds, then*

(a) *for $\rho \in (0,1)$, there exists a positive constant $\delta_0 = \delta_0(\rho)$ satisfying*

$$\varphi_k(x) < \nu_k \ \forall x \in \mathrm{B}_{\delta_0}\backslash\{0\},$$

where $1 \le k \le 2$, φ_k is defined in (2.6), $\nu_1 = \frac{\varepsilon\rho}{2\|P\|}$, *and* $\nu_2 = \frac{\varepsilon(1-\rho)}{2c\|P\|\|A\|}$;

(b) *for $r_0 = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}\delta_0$,*

$$D = \left\{S + Y \in \mathbb{R}^{nN} \mid \|Y\| < r_0\right\}$$

is a subset of R_A, where S is the equilibrium point of (2.1).

Proof. (a) The proof of (a) is very similar to that of Theorem 3.3(a) and thus omitted.

(b) Since A is symmetric, there exists an orthogonal matrix U such that $A = U^{-1}\Lambda U$, where $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_N\}$. Define

$$\eta = [\eta_1^T, \eta_2^T, \ldots, \eta_N^T]^T = (U \otimes I_n)e,$$
$$G^{(k)} = \left[{G_1^{(k)}}^T, {G_2^{(k)}}^T, \ldots, {G_N^{(k)}}^T\right]^T = (U \otimes I_n)R^{(k)},$$

where $1 \le k \le 2$. By making use of Lemma 2.4, herewith (2.9) takes the following form:

$$\dot{\eta} = (I_N \otimes F_1 + c\Lambda \otimes F_2)\eta + G^{(1)} + c(\Lambda \otimes I_n)G^{(2)},$$

or, equivalently,

$$\dot{\eta}_i = (F_1 + c\lambda_i F_2)\eta_i + G_i^{(1)} + c\lambda_i G_i^{(2)},$$

where $1 \le i \le N$. Define a Lyapunov function as $V = \eta^H P \eta$. Then,

$$\begin{aligned}\dot{V} = &\sum_{i=1}^N \eta_i^H \left(P_i(F_1 + c\lambda_i F_2) + (F_1 + c\lambda_i F_2)^H P_i\right)\eta_i + \sum_{i=1}^N \left(\eta_i^H P_i G_i^{(1)} + {G_i^{(1)}}^H P_i \eta_i\right) \\ &+ \sum_{i=1}^N c\left(\lambda_i \eta_i^H P_i G_i^{(2)} + \overline{\lambda}_i {G_i^{(2)}}^H P_i \eta_i\right) \\ \le &-\varepsilon \sum_{i=1}^N \|\eta_i\|^2 + 2\left|\eta^H P G^{(1)} + c\eta^H P (J \otimes I_n) G^{(2)}\right|\end{aligned}$$

$$= -\varepsilon\|\eta\|^2 + 2\left|\eta^H P(U \otimes I_n)\left(R^{(1)} + c(A \otimes I_n)R^{(2)}\right)\right|$$
$$\leq -\varepsilon\|\eta\|^2 + 2\|\eta\|\|P\|\|U\|\left(\|R^{(1)}\| + c\|A\|\|R^{(2)}\|\right).$$

It is obvious that $\kappa_2(U) = 1$. Then, (b) can be derived by following the proof of Theorem 3.4. ∎

Remark 3.8. The procedure of calculating r_0 according to Corollary 3.5 is very similar to that in Table 1, except that it is not necessary to find U due to the symmetry of A. Moreover, $\kappa_2(U)$ is minimized when A is symmetric, and consequently, the ROA estimation is less conservative for undirected networks than for directed networks.

Remark 3.9. Corollary 3.5 looks almost the same as Theorem 3.3, but they are actually different. The matrix P and the constant ε in Theorem 3.3 are obtained by solving (3.1), while those in Corollary 3.5 are obtained by solving (3.2), which is more feasible when N is large.

3.3. Second-order ROA estimation. In Step 4 of the procedure shown in Table 1, it is usually not easy to determine the optimal ρ. By adding the mild condition that f and h of network (2.1) satisfy $f, h \in \mathcal{C}^2(\mathbb{R}^n)$, some second-order estimations of the ROA can be established, for which there is no need to determine the optimal ρ. In this part, suppose that $f, h \in \mathcal{C}^2(\mathbb{R}^n)$.

Denote

$$f = [f_1, f_2, \ldots, f_n]^T,$$
$$h = [h_1, h_2, \ldots, h_n]^T,$$
$$R_i^{(k)} = [R_{i1}^{(k)}, R_{i2}^{(k)}, \ldots, R_{in}^{(k)}]^T,$$

where $1 \leq k \leq 2$ and $R_i^{(k)}$ is defined in (2.5). Since $f, h \in \mathcal{C}^2(\Omega^N)$, one has

$$R_{ij}^{(k)} = \int_0^1 (1-\tau)e_i^T \Phi_j^{(k)}(s + \tau e_i)e_i \mathrm{d}\tau, \tag{3.10}$$

where $1 \leq i \leq N$, $1 \leq j \leq n$, $1 \leq k \leq 2$, e_i is the error of network (2.1), and $\Phi_j^{(1)}$ and $\Phi_j^{(2)}$ are the Hessian matrices of f_j and h_j, respectively. Suppose that

$$\|\Phi_j^{(k)}(s + x)\| \leq \phi_{kj}(r) \ \forall x \in \overline{\mathrm{B}}_r,$$

where $\phi_{kj} : [0, +\infty) \to [0, +\infty)$ is continuous and nondecreasing. Denote

$$\phi_k(r) = \sqrt{\phi_{k1}^2(r) + \phi_{k2}^2(r) + \cdots + \phi_{kn}^2(r)}, \tag{3.11}$$

where $1 \leq k \leq 2$.

THEOREM 3.6. *Consider network* (2.1), *and suppose that* (A1) *holds. Let* δ_0 *be a positive constant satisfying*

$$\delta_0 \leq \sup\left\{\delta \in \mathbb{R} \mid \delta\|P\|\left(\phi_1(\delta) + c\|A\|\phi_2(\delta)\right) \leq \varepsilon\right\},$$

where ϕ_1 *and* ϕ_2 *are defined in* (3.11). *Then,*

$$D = \left\{S + Y \in \mathbb{R}^{nN} \mid \|Y\| < r_0\right\}$$

is a subset of R_A, *where* S *is the equilibrium point of* (2.1) *and* $r_0 = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}\delta_0$.

Proof. Define a Lyapunov function as $V = e^T Pe$, where e is defined in (2.8). Similar to the proof of Theorem 3.3, one gets

$$\dot{V} \leq -\varepsilon\|e\|^2 + 2\left(\|R^{(1)}\| + c\|A\|\|R^{(2)}\|\right)\|P\|\|e\|.$$

Take $\delta \in (0, \delta_0)$. From (3.10), it follows that

$$\|R_{ij}^{(k)}\| \leq \frac{1}{2}\phi_{kj}(\delta)\|e_i\|^2 \ \forall t \in \{t \mid \|e\| \leq \delta\},$$

yielding

$$\|R^{(k)}\| \leq \frac{1}{2}\phi_k(\delta)\|e\|^2 \ \forall t \in \{t \mid \|e\| \leq \delta\}.$$

Since $\phi_k \geq 0$ is nondecreasing, there exists a positive constant β such that

$$\delta\|P\|\left(\phi_1(\delta) + c\|A\|\phi_2(\delta)\right) = \varepsilon - \beta. \tag{3.12}$$

Then, one has

$$\dot{V} \leq -\beta\|e\|^2 \ \forall t \in \{t \mid \|e\| \leq \delta\}.$$

From Lemma 2.2, $\lim_{t\to+\infty}\|e\| = 0$ if $\|e(0)\| \leq r_0$, i.e., D is a subset of R_A. ∎

Remark 3.10. Denote

$$\bar{\varphi}(\delta) = \delta\|P\|(\phi_1(\delta) + c\|A\|\phi_2(\delta)),$$
$$\delta_1 = \sup\left\{\delta \in \mathbb{R} \mid \bar{\varphi}(\delta) \leq \varepsilon\right\}.$$

If $\phi_1 = \phi_2 = 0$, then $\bar{\varphi} = 0$. In this case, $\delta_1 = +\infty$, and obviously there exists $\beta > 0$ satisfying (3.12). By taking $\delta_0 = +\infty$, one gets $D = \mathbb{R}^n$. If $\phi_1 \neq 0$ or $\phi_2 \neq 0$, then $\bar{\varphi}$ is continuous and increasing, δ_1 is bounded, and $\bar{\varphi}(\delta_1) = \varepsilon$. Since δ in (3.12) satisfies $\delta < \delta_1$, obviously there exists $\beta > 0$ satisfying (3.12). In both of the two cases, $\bar{\varphi}$ is continuous and nondecreasing, and thus it is easy to estimate δ_1. If δ_1 can be calculated precisely, then let $\delta_0 = \delta_1$; otherwise, get a δ_0 close to δ_1 with $\delta_0 \leq \delta_1$.

Remark 3.11. For the ROA estimation, one important step is to estimate the upper bound of $R^{(k)}$. The general ROA estimation method does not tell us how to estimate $R^{(k)}$, while the second-order method provides a method. In this sense, the second-order method is easier to use than the general method. Furthermore, the second-order method may be more conservative, because it does not make full use of $R^{(k)}$.

THEOREM 3.7. *Consider network* (2.1). *If* (A2) *holds, then*

(a) *there exists a nonsingular matrix* $U \in \mathbb{C}^{N\times N}$ *such that* $A = U^{-1}JU$, *where* J *is in the form of* (2.13) *with* $J_{ii} = \lambda_i$ *and* $J_{i,i+1} = 0$ *or* $\frac{\varepsilon}{4c\|P\|\|F_2\|}$;

(b) *let* δ_0 *be a positive constant satisfying*

$$\delta_0 \leq \sup\left\{\delta \in \mathbb{R} \,\middle|\, \delta\kappa_2(U)\|P\|(\phi_1(\delta) + c\|A\|\phi_2(\delta)) \leq \frac{\varepsilon}{2}\right\}, \tag{3.13}$$

where ϕ_1 *and* ϕ_2 *are defined in* (3.11). *Then,*

$$D = \left\{S + Y \in \mathbb{R}^{nN} \mid \|Y\| < r_0\right\}$$

is a subset of R_A, *where* S *is the equilibrium of* (2.1) *and* $r_0 = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}\frac{\delta_0}{\kappa_2(U)}$.

Proof. (a) According to Lemma 2.3, A can be written as $A = U^{-1}JU$.

(b) Define a Lyapunov function as $V = \eta^H P\eta$, where $\eta = (U \otimes I_n)e$ with e defined in (2.8). By following the proof of Theorem 3.4, one gets

$$\begin{aligned}\dot{V} \leq & -\frac{\varepsilon}{2}\|\eta\|^2 + 2\|\eta\|\|P\|\|U\|\left(\|R^{(1)}\| + c\|A\|\|R^{(2)}\|\right)\\ \leq & -\frac{\varepsilon}{2}\frac{\|\eta\|\|e\|}{\|U^{-1}\|} + 2\|\eta\|\|P\|\|U\|\left(\|R^{(1)}\| + c\|A\|\|R^{(2)}\|\right).\end{aligned}$$

Taking $\delta \in (0, \delta_0)$, one has

$$\|R^{(k)}\| \leq \frac{1}{2}\phi_k(\delta)\|e\|^2 \ \forall t \in \{t \mid \|e\| \leq \delta\},$$

leading to

$$\dot{V} \leq \frac{\|\eta\|\|e\|}{\|U^{-1}\|}\left(-\frac{\varepsilon}{2} + \kappa_2(U)\left(\phi_1(\delta) + c\|A\|\phi_2(\delta)\right)\|P\|\|e\|\right) \ \forall t \in \{t \mid \|e\| \leq \delta\}.$$

Since $\phi_k \geq 0$ is nondecreasing, there exists a positive constant β such that

$$\delta\kappa_2(U)\|P\|\left(\phi_1(\delta) + c\|A\|\phi_2(\delta)\right) = \frac{1}{2}(\varepsilon - \beta).$$

Then, one has

$$\dot{V} \leq -\frac{\beta\|\eta\|\|e\|}{2\|U^{-1}\|} \leq -\frac{\beta\|e\|^2}{2\|U^{-1}\|^2} \ \forall t \in \{t \mid \|e\| \leq \delta\}.$$

From Lemma 2.2, $\lim_{t\to+\infty}\|e\| = 0$ if $\|e(0)\| \leq r_0$, i.e., D is a subset of R_A. ∎

COROLLARY 3.8. *Consider network* (2.1)*, and suppose that its outer coupling matrix A is symmetric. Let* (A2) *hold, and let δ_0 be a positive constant satisfying*

$$\delta_0 \leq \sup\left\{\delta \in \mathbb{R} \mid \delta\|P\|\left(\phi_1(\delta) + c\|A\|\phi_2(\delta)\right) \leq \varepsilon\right\},$$

where ϕ_1 and ϕ_2 are defined in (3.11)*. Then,*

$$D = \left\{S + Y \in \mathbb{R}^{nN} \mid \|Y\| < r_0\right\}$$

is a subset of R_A, where S is the equilibrium point of (2.1) *and* $r_0 = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}\delta_0$.

Proof. Since A is symmetric, there exists an orthogonal matrix U such that $A = U^{-1}\Lambda U$, where $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_N\}$. Define a Lyapunov function as $V = \eta^H P\eta$, where $\eta = (U \otimes I_n)e$ with e defined in (2.8). Similar to the proof of Corollary 3.5, one has

$$\dot{V} \leq -\varepsilon\|\eta\|^2 + 2\|\eta\|\|P\|\|U\|\left(\|R^{(1)}\| + c\|A\|\|R^{(2)}\|\right).$$

It is obvious that $\kappa_2(U) = 1$. Thus, the conclusion can be derived by following the proof of Theorem 3.7. ∎

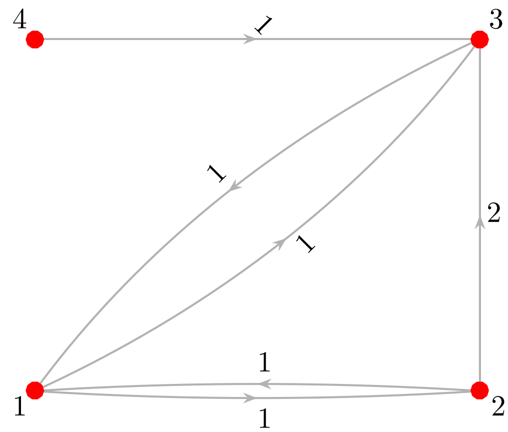

FIG. 1. *Topology of weighted and directed network* (4.1).

4. Numerical examples. Consider a laboratory device of coupled pendulums [11], where all the nodes are identical pendulums coupled via elastic links. The dynamics of a single pendulum are described by

$$f(z) = (z_2, -r_1 \sin z_1 - b z_2)^T,$$

where r_1 and b are system parameters. All the equilibrium points of a single pendulum are $(k\pi, 0)$ with $k \in \mathbb{Z}$. Take the inner coupling function as

$$h(z) = (z_1 + r_2 \sin z_2, z_2)^T,$$

where r_2 is a constant. Let $r_1 = 1$, $b = 2$, and $r_2 = 0.1$.

4.1. Example 1: A directed network. Consider a directed network consisting of four nodes, as illustrated by Figure 1. The network is described by

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^{4} a_{ij} h(x_j(t)), \tag{4.1}$$

where $c = 0.25$ and

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -3 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Since A is not symmetric, one should apply Theorem 3.4 and follow the procedure shown in Table 1 to estimate the ROA of the equilibrium point $S = 0$ of network (4.1).

Step 1. The eigenvalues of A are

$$\lambda_1 = 0, \quad \lambda_2 = -1, \quad \lambda_3 = -3, \quad \lambda_4 = -3.$$

Step 2. Solving (3.2) gives $\varepsilon = 1$, and

$$P_1 = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1.5704 & 0.4016 \\ 0.4016 & 0.5270 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1.6465 & 0.2557 \\ 0.2557 & 0.6280 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1.6465 & 0.2557 \\ 0.2557 & 0.6280 \end{bmatrix}.$$

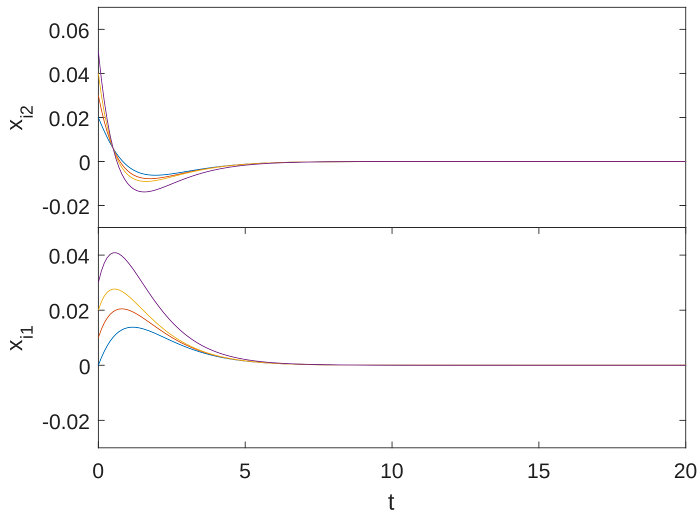

FIG. 2. *State variables* x_{ij} $(1 \le i \le 4, 1 \le j \le 2)$ *of network* (4.1).

Step 3. Decomposing A yields

$$U = \begin{bmatrix} -3 & -1 & -5 & 0 \\ -3.5892 & -1.7946 & -1.7946 & 7.1784 \\ -3.2206 & 0 & 3.2206 & 0 \\ 0 & -5.7796 & 5.7796 & 0 \end{bmatrix},$$

and $\kappa_2(U) = 2.5710$.

Step 4. Estimate φ_k by

$$\|\varphi_k(z)\| = \frac{1}{\|z\|}|r_k||z_k - \sin z_k| = \frac{r_k}{\|z\|}(|z_k| - \sin|z_k|) \le r_k\left(1 - \frac{\sin\|z\|}{\|z\|}\right),$$

where $k = 1, 2$. In order to make δ_0 as large as possible, one can solve the following optimization problem:

$$\begin{aligned} \min \quad & -\delta_0 \\ \text{s.t.} \quad & r_1\left(1 - \tfrac{\sin\|z\|}{\|z\|}\right) < 0.0570\rho \ \forall z \in \mathrm{B}_{\delta_0}\backslash\{0\}, \\ & r_2\left(1 - \tfrac{\sin\|z\|}{\|z\|}\right) < 0.0584(1-\rho) \ \forall z \in \mathrm{B}_{\delta_0}\backslash\{0\}, \\ & \rho \in (0, 1). \end{aligned}$$

It is easy to verify that δ_0 is maximized when $\frac{0.0570\rho}{r_1} = \frac{0.0584(1-\rho)}{r_2}$. Taking $\rho = 0.9112$, one obtains $\delta_0 = 0.5625$.

Step 5. Finally, one obtains $D = \left\{X \in \mathbb{R}^{nN} \mid \|X\| < r_0 = 0.0906\right\}$, which is a subset of R_A.

Let the initial values be $x_i(0) = 0.01[i-1, i+1]^T \in D$. Figure 2 shows that network (4.1) converges to $S = 0$, which verifies the theoretical result.

To show the influence of ρ on the size of the estimated ROA, calculate δ_0 and r_0 for different values of ρ. The results are given in Table 2, where ρ is distributed uniformly between $[0.0001, 0.9112]$ and $[0.9112, 0.9999]$, respectively. The results here

TABLE 2
δ_0 and r_0 with different values of ρ. ρ is distributed uniformly between $[0.0001, 0.9112]$ and $[0.9112, 0.9999]$, respectively.

ρ	0.0001	0.3038	0.6075	**0.9112**	0.9408	0.9703	0.9999
δ_0	0.0060	0.3231	0.4581	**0.5625**	0.4582	0.3234	0.0187
r_0	0.0010	0.0521	0.0738	**0.0906**	0.0738	0.0521	0.0030

suggest that ρ has significant influence on the size of the estimated ROA. Particularly, the radius r_0 of the estimated ROA moves closer to 0 when ρ changes gradually from the optimal value 0.9112 to 0 or to 1.

Since $f, h \in \mathcal{C}^2(\mathbb{R}^n)$, one can also use Theorem 3.7 to estimate the ROA. The first three steps are the same as above. Hence, only the last two steps are performed in the following.

Step 4'. It is easy to obtain $\Phi_1^{(1)} = 0$ and

$$\Phi_2^{(1)}(z) = \begin{bmatrix} r_1 \sin z_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, take $\phi_{11} = 0$ and

$$\phi_{12}(r) = \begin{cases} r_1 \sin r, & r \in [0, \frac{\pi}{2}], \\ r_1, & r \in (\frac{\pi}{2}, +\infty), \end{cases}$$

yielding $\phi_1 = \phi_{12}$. Similarly, take $\phi_{22} = 0$ and

$$\phi_2(r) = \phi_{21}(r) = \begin{cases} r_2 \sin r, & r \in [0, \frac{\pi}{2}], \\ r_2, & r \in (\frac{\pi}{2}, +\infty). \end{cases}$$

According to (3.13), one has

$$\begin{aligned} \delta_0 \leq & \sup\left\{\delta \in \mathbb{R} \middle| \, \delta\kappa_2(U)\|P\| \left(\phi_1(\delta) + c\|A\|\phi_2(\delta)\right) \leq \frac{\varepsilon}{2}\right\} \\ = & \sup\left\{\delta \in \left[0, \frac{\pi}{2}\right] \middle| \, \delta \sin\delta \leq 0.1038\right\} \\ = & \, 0.3251, \end{aligned}$$

from which one can take $\delta_0 = 0.3251$.

Step 5'. Finally, one obtains $D = \{X \in \mathbb{R}^{nN} \mid \|X\| < r_0 = 0.0524\}$, which is a subset of R_A.

It can be seen that the r_0 calculated by using Theorem 3.7 is smaller than that obtained by using Theorem 3.4. This can be explained by Remark 3.11. Moreover, there is no need to choose the value of ρ by using Theorem 3.7.

4.2. Example 2: An undirected network. Consider an undirected network consisting of four nodes, as illustrated by Figure 3. The network is described by

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{4} a_{ij} h(x_j(t)), \tag{4.2}$$

where $c = 0.1$ and

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -4 & 1 & 2 \\ 1 & 1 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{bmatrix}.$$

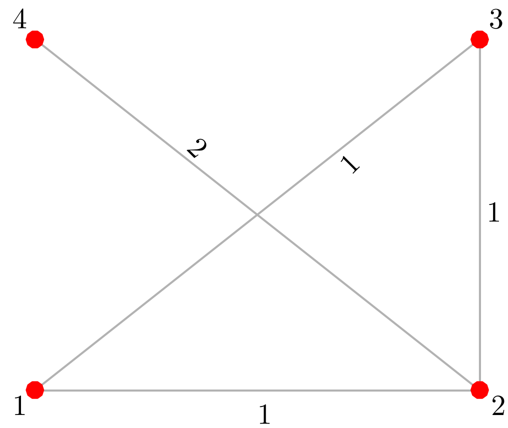

FIG. 3. *Topology of weighted and undirected network* (4.2).

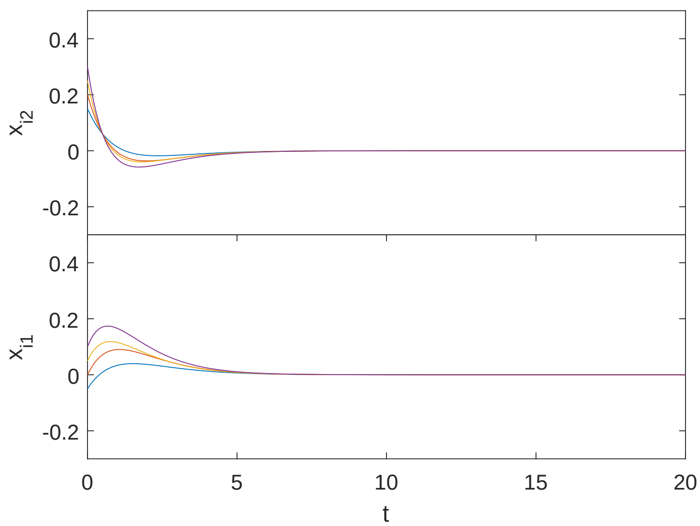

FIG. 4. *State variables* x_{ij} $(1 \leq i \leq 4, 1 \leq j \leq 2)$ *of network* (4.2).

Since A is symmetric, one can use Corollary 3.5 to estimate the ROA of the equilibrium point $S = 0$ of network (4.2). The procedure is as follows.

Step 1. The eigenvalues of A are

$$\lambda_1 = 0, \quad \lambda_2 = -1.4384, \quad \lambda_3 = -3, \quad \lambda_4 = -5.5616.$$

Step 2. Solving (3.2) gives $\varepsilon = 1$, and

$$P_1 = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1.5441 & 0.4413 \\ 0.4413 & 0.5123 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 1.5812 & 0.3841 \\ 0.3841 & 0.5352 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1.6243 & 0.3048 \\ 0.3048 & 0.5853 \end{bmatrix}.$$

Step 3. There is no need to calculate U due to the symmetry of A. Hence, this step can be skipped.

Step 4. Taking $\rho = 0.9473$, one obtains $\delta_0 = 1.3506$.

Step 5. Finally, one obtains $D = \{X \in \mathbb{R}^{nN} \mid \|X\| < r_0 = 0.5594\}$, which is a subset of R_A.

Let the initial values be $x_i(0) = 0.1[0.5i - 1, 0.5i + 1]^T \in D$. Figure 4 shows that network (4.2) converges to $S = 0$, which verifies the theoretical result.

It can be seen that r_0 in this example is much larger than the r_0 in the last example. It is partly because c and A are different in the two examples. Remark 3.8 also partly applies to this case.

5. Conclusion. In this paper, we have first briefly analyzed the asymptotical stability of the equilibrium point of a complex dynamical network, establishing a sufficient condition and a necessary condition. Thereafter, we have established general estimation and second-order estimation of the ROA for a complex dynamical network by combining the network structure and the node dynamics. We have finally provided two examples to verify the estimations, and meanwhile discussed the influence of the parameter ρ on the size of the estimated ROA.

Although our estimation of the ROA may still be conservative, it is fundamental and should be treated as a first step of elaborately exploring the exact ROA of a complex dynamical network. By taking advantage of the particularity of node dynamics, the ROA may be better estimated in other shapes such as ellipsoid. Future work also includes estimating the ROA of various network models, comparing the estimated ROA with the exact ROA if possible, and estimating the robust ROA of uncertain complex dynamical networks.

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