Synchronization of Complex Networks With Nondifferentiable Time-Varying Delay

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***Abstract*—In this article, we investigate the synchronization of complex networks with general time-varying delay, especially with nondifferentiable delay. In the literature, the time-varying delay is usually assumed to be differentiable. This assumption is strict and not easy to verify in engineering. Until now, the synchronization of networks with nondifferentiable delay through adaptive control remains a challenging problem. By analyzing the boundedness of the adaptive control gain and extending the well-known Halanay inequality, we solve this problem and establish several synchronization criteria for networks under the centralized adaptive control and networks under the decentralized adaptive control. Particularly, the boundedness of the centralized adaptive control gain is theoretically proved. Numerical simulations are provided to verify the theoretical results.**

***Index Terms*—Adaptive control, complex network, nondifferentiable time-varying delay, synchronization.**

I. Introduction

Complex networks, generally consisting of numerous interconnected systems, have drawn remarkable attention in the past two decades [1]–[10]. In network science, one of the most important problems is the synchronization among the interconnected systems of a network.

A great deal of networks cannot realize synchronization by themselves, so external control is generally inevitable. In the literature, two kinds of control are widely used for realizing synchronization, which are the *linear feedback control* [11]–[13] and the *adaptive feedback control* [14]–[16]. Despite easy implementation, the linear control has two major disadvantages. First, its control gain should be designed according to the network parameters. Therefore, the linear control is incapable when the network parameters are unknown. Second, the linear control is not reusable, that is, the control gain has to be redesigned when the network parameters change. Therefore, the adaptive control is usually a preferred choice as its control gain can be adapted by itself according to certain updating laws, especially for networks with unknown topology [16]–[18]. A basic problem for the adaptive control is to determine the boundedness of the control gain because controllers cannot provide infinitely large control gain in real situations. However, the basic problem was neglected in most existing studies, which is to be considered in this article.

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During the communications among the interconnected systems of a network, the coupling delay is hardly negligible due to the finite transmission speed of the signals. The synchronization problem of networks with differentiable time-varying delay has been extensively studied [19]–[28]. In [19]–[26], it is assumed that the time-varying delay $\tau(t)$ is differentiable and satisfies

$$\dot{\tau}(t) \leq \mu_1 < 1$$

which is one of the most widelyused assumptions. In [27] and [28], this assumption is relaxed to $\dot{\tau}(t) \leq \mu_2$. In reality, the delay may change in a random way that is unpredictable, so it is hardly possible to know the analytical expression of the delay, making it difficult to verify whether the delay is differentiable. When the control is linear, the synchronization problem of networks with nondifferentiable delay is generally solved in virtue of delay differential inequalities [29], [30]. However, dealing with the adaptive synchronization of networks with nondifferentiable delay is still a challenging problem. In 2008, the adaptive robust convergence of neural networks with nondifferentiable delays was addressed in [31], whereas the control therein is actually not adaptive. In 2013, the adaptive synchronization of networks with nondifferentiable delay was investigated in [32]. We find that the control therein is complicated and the *LaSalle invariance principle* used therein is, in fact, inapplicable as the network error system is nonautonomous. To this end, the main aim of this article is to solve this challenging problem.

Motivated by the above discussions, this article investigates the adaptive synchronization of complex networks with general time-varying delay, especially with nondifferentiable delay. The main contributions are as follows.

1) The synchronization criteria for networks under centralized adaptive control are established by analyzing the boundedness of the adaptive control gain. Compared with conventional techniques, the gain analysis technique here provides new perspectives on addressing the adaptive control and can well deal with the nondifferentiable delay.
2) The boundedness of the centralized adaptive control gain is theoretically proved so as to ensure the practicality of adaptive controllers.
3) The synchronization criteria for networks under decentralized adaptive control are established by analyzing the control gain and extending the well-known Halanay inequality.

The remainder of this article is organized as follows. Section II presents some preliminaries. Section III establishes the main results on the adaptive synchronization of networks with nondifferentiable delay. Section IV verifies the theoretical results via numerical simulations. Section V draws the conclusion.

II. Preliminaries

A. Notations and Network Model

First, some notations are introduced. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all the

$(n \times m)$-dimensional real matrices, respectively, the superscript T represents the transpose of a vector or a matrix, $\|\cdot\|$ denotes the 2-norm of a vector or a matrix; $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a symmetric matrix, and I_n is the identity matrix of dimension n. The Dini derivative is defined as $D^+V(t) := \limsup_{h\to 0^+}[(V(t+h)-V(t))/h]$. For a continuous function φ on $\mathbb{R}$, define $\|\varphi\|_{t,h} = \max_{s\in[t-h,t]}\|\varphi(s)\|$, where h is a positive constant.

Consider a controlled complex network consisting of N identical systems with time-varying coupling delay

$$\dot{x}_i(t) = f(x_i(t), t) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t-\tau(t)) + u_i(t) \tag{1}$$

where $1 \le i \le N$, $x_i \in \mathbb{R}^n$ represents the state vector of the ith node, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuous function, $c > 0$ is the coupling strength, $A = (a_{ij})_{N\times N}$ denotes the weighted outer coupling matrix, Γ denotes the inner coupling matrix, $\tau(t)$ is the continous time-varying coupling delay, and $u_i(t)$ is the control input to the ith node. If there is an edge from node i to node $j (j \ne i)$, then $a_{ij} > 0$; otherwise, $a_{ij} = 0$; the diagonal elements of A are defined by $a_{ii} = -\sum_{j=1, j\ne i}^{N} a_{ij}$, where $1 \le i \le N$.

The objective of this article is to synchronize the states of all the nodes to a desired state $s(t)$, which is a solution of the following isolated node system:

$$\dot{s}(t) = f(s(t), t). \tag{2}$$

Defining the synchronization error as $e_i(t) = x_i(t) - s(t)$, one has the following error system:

$$\dot{e}_i(t) = f(x_i(t), t) - f(s(t), t) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t-\tau(t)) + u_i(t) \tag{3}$$

where $1 \le i \le N$. Obviously, network (1) realizes synchronization if $e(t) = [e_1^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t)]^{\mathrm{T}}$ converges to 0.

B. Mathematical Preliminaries

Here, we present several assumptions and lemmas.

Assumption 1 (A1): Suppose that the time-varying delay $\tau(t)$ satisfies

$$0 \le \tau(t) \le \tau_{\max} \tag{4}$$

where $\tau_{\max}$ is a positive constant.

Remark 1: Compared with much of the literature, the assumption (A1) here is very general, since $\tau(t)$ can be constant, differentiable, or even nondifferentiable. In addition, the set of nondifferentiable points of $\tau(t)$ may be uncountable.

Assumption 2 (A2): Suppose that there exists a non-negative constant γ satisfying

$$\|f(x,t) - f(y,t)\| \le \gamma\|x - y\| \tag{5}$$

for any $x, y \in \mathbb{R}^n$ and $t \ge 0$.

Lemma 1 (Halanay Inequality [33]): Let $\alpha > \beta$ be positive constants. If $V(t)$ is a non-negative continuous function on $[t_0 - \hat{\tau}, t_0]$ and satisfies

$$D^+V(t) \le -\alpha V(t) + \beta\|V\|_{t,\hat{\tau}}, \ \ t \ge t_0$$

then

$$V(t) \le M_0 \exp(-\sigma(t-t_0)), \ \ t \ge t_0$$

where $\hat{\tau} > 0$, $M_0 = \|V\|_{t_0,\hat{\tau}}$, and σ is the unique positive solution of the equation $\sigma = \alpha - \beta\exp(\sigma\hat{\tau})$.

Lemma 2 (Barbălat Lemma [34, Lemma 8.2]): Let $h : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $[0, +\infty)$. Then, $\lim_{t\to+\infty} h(t) = 0$ if $\int_0^{+\infty} h(\omega)d\omega$ converges.

Lemma 3: Let $\alpha > \beta \ge 0$ be constants, and let $g : \mathbb{R} \to \mathbb{R}_+$ be a non-negative continuous function. If V is a non-negative continuous function on $[t_0 - \hat{\tau}, t_0]$ and satisfies

$$\dot{V}(t) \le -\alpha V(t) + \beta\|V\|_{t,\hat{\tau}} + g(t), \ \ t \ge t_0$$

then

$$V(t) \le M_0 \exp(-\sigma(t-t_0)) + \hat{g}(t), \ \ t \ge t_0 \tag{6}$$

where $\hat{\tau} > 0$, $M_0 = \|V\|_{t_0,\hat{\tau}}$, $\hat{g}(t) = \int_{t_0}^{t} g(\theta)\exp(-\sigma(t-\theta))d\theta$, and σ is the unique positive solution of the equation $\sigma = \alpha - \beta\exp(\sigma\hat{\tau})$.

Proof: Consider the following comparison system:

$$\begin{cases} \dot{W}(t) = -\alpha W(t) + \beta\|W\|_{t,\hat{\tau}} + g_1(t), & t \ge t_0 \\ W(t) = V(t), & t \le t_0 \end{cases} \tag{7}$$

where $g_1(t) = g(t) + \varepsilon$ and ε is a positive constant. Let M be any constant larger than M_0. We shall prove that

$$W(t) < M\exp(-\sigma(t-t_0)) + \hat{g}_1(t) := h(t), \ \ t \ge t_0 \tag{8}$$

where $\hat{g}_1(t) = \int_{t_0}^{t} g_1(\theta)\exp(-\sigma(t-\theta))d\theta$. Define $\hat{g}_1(t) = 0$ if $t < t_0$. When $t \in [t_0 - \hat{\tau}, t_0]$, one has $W(t) \le M_0 < M \le h(t)$. If (8) is not true, then

$$t_1 = \inf\{t > t_0 | W(t) \ge h(t)\}$$

is well defined. It follows that $W(t_1) = h(t_1)$, $t_1 > t_0$, and $W(t) < h(t)$ for $t \in [t_0 - \hat{\tau}, t_1)$. From (7), one has

$$\frac{d}{dt}\big[W(t)\exp(\alpha t)\big] = \exp(\alpha t)\big[\beta\|W\|_{t,\hat{\tau}} + g_1(t)\big]. \tag{9}$$

Integrating (9) from t_0 to t_1 gives

$$\begin{aligned} &W(t_1)\exp(\alpha t_1) - W(t_0)\exp(\alpha t_0) \\ &\quad = \int_{t_0}^{t_1} \exp(\alpha t)\big[\beta\|W\|_{t,\hat{\tau}} + g_1(t)\big]\mathrm{d}t. \end{aligned} \tag{10}$$

There exists $\hat{t} \in [t - \hat{\tau}, t]$ such that $\|\hat{g}_1\|_{t,\hat{\tau}} = \hat{g}_1(\hat{t})$, yielding

$$\begin{aligned} \|\hat{g}_1\|_{t,\hat{\tau}} &= \int_{t_0}^{\hat{t}} g_1(\theta)\exp\big(-\sigma\big(\hat{t} - \theta\big)\big)d\theta \\ &\le \int_{t_0}^{t} g_1(\theta)\exp\big(-\sigma\big(t - \hat{\tau} - \theta\big)\big)d\theta \\ &= \exp\big(\sigma\hat{\tau}\big)\hat{g}_1(t). \end{aligned} \tag{11}$$

Then, one has $\|W\|_{t,\hat{\tau}} < \|h\|_{t,\hat{\tau}} \le M\exp(\sigma\hat{\tau})\exp(-\sigma(t-t_0)) + \exp(\sigma\hat{\tau})\hat{g}_1(t)$ for $t \in [t_0, t_1)$.

Recalling (10), one has

$$W(t_1)\exp(\alpha t_1) < \Omega_1 + \Omega_2 + \Omega_3 \tag{12}$$

where $\Omega_1 = \int_{t_0}^{t_1}\exp(\alpha t)\beta M\exp(\sigma\hat{\tau})\exp(-\sigma(t-t_0))\mathrm{d}t + W(t_0)\exp(\alpha t_0)$, $\Omega_2 = \int_{t_0}^{t_1}\exp(\alpha t)\beta\exp(\sigma\hat{\tau})\hat{g}_1(t)\mathrm{d}t$, and $\Omega_3 = \int_{t_0}^{t_1}\exp(\alpha t)g_1(t)\mathrm{d}t$.

Considering $W(t_0) \le M$ and $\sigma = \alpha - \beta\exp(\sigma\hat{\tau})$, one has

$$\begin{aligned} \Omega_1 &\le M(\alpha - \sigma)\int_{t_0}^{t_1}\exp(\alpha t)\exp(-\sigma(t-t_0))\mathrm{d}t \\ &\quad + M\exp(\alpha t_0) \\ &= M\exp(\alpha t_1)\exp(-\sigma(t_1 - t_0)). \end{aligned} \tag{13}$$

Since g is continuous, one has

$$\begin{aligned}\Omega_2 &= \beta \exp\big(\sigma \hat{\tau}\big) \int_{t_0}^{t_1} \exp(\alpha t)\mathrm{d}t \int_{t_0}^{t} g_1(\theta)\exp(-\sigma(t-\theta))d\theta \\ &= (\alpha-\sigma)\int_{t_0}^{t_1} g_1(\theta)\exp(\sigma\theta)d\theta \int_{\theta}^{t_1}\exp((\alpha-\sigma)t)\mathrm{d}t \\ &= \int_{t_0}^{t_1} g_1(\theta)\big[\exp(\sigma\theta)\exp((\alpha-\sigma)t_1)-\exp(\alpha\theta)\big]d\theta \\ &= \exp(\alpha t_1)\hat{g}_1(t_1)-\Omega_3. \end{aligned} \tag{14}$$

Combining (12)–(14), it can be deduced that $W(t_1) < h(t_1)$, a contradiction with $W(t_1) = h(t_1)$. Therefore, $W(t) < h(t)$ for $t \geq t_0$.

Since $\hat{g}_1(t) = \hat{g}(t) + (\varepsilon/\sigma)[1 - \exp(-\sigma(t - t_0))] < \hat{g}(t) + (\varepsilon/\sigma)$, one obtains

$$V(t) < W(t) < h(t) < M\exp(-\sigma(t-t_0)) + \hat{g}(t) + \frac{\varepsilon}{\sigma}$$

where $t \geq t_0$. Letting $\varepsilon \to 0^+$ and $M \to M_0^+$ leads to (6). ■

Lemma 3 is an extension of Lemma 1, which is to deal with the case when disturbance $g(t)$ exists. Particularly, this lemma becomes the Halanay inequality if $g(t) \equiv 0$.

III. Main Results

In this section, two kinds of adaptive control, including the centralized adaptive control and the decentralized adaptive control, are used to realize the adaptive synchronization of network (1).

A. Centralized Adaptive Control

Consider the following centralized adaptive control

$$u_i(t) = -d(t)e_i(t),\ \ 1 \leq i \leq N \tag{15}$$

with updating laws

$$\dot{d}(t) = k\|e(t)\|^2 \tag{16}$$

where k is a positive constant and $d(0) \geq 0$.

In (15), the adaptive control gain $d(t)$ of each control u_i is identical, and the updating laws (16) needs to know the global information $e(t)$ of network (1).

The following theorem establishes adaptive synchronization criteria for network (1) under centralized adaptive control (15). In the meantime, the boundedness of the adaptive control gain $d(t)$ is guaranteed theoretically.

Theorem 1: Let assumptions (A1) and (A2) hold. Then, network (1) realizes synchronization under centralized adaptive control (15) with updating laws (16). In addition, the control gain of adaptive control (15) is necessarily bounded.

Proof: Consider the following Lyapunov function:

$$V(t) = e^{\mathrm{T}}(t)e(t) = \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t). \tag{17}$$

Differentiating V along the solution of (3) yields

$$\begin{aligned}\dot{V}(t) &= 2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\big[f(x_i(t),t) - f(s(t),t)\big] \\ &\quad + 2c\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}e_i^{\mathrm{T}}(t)\Gamma e_j(t-\tau(t)) \\ &\quad - 2d(t)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t).\end{aligned}$$

It then follows from (A2) that:

$$\begin{aligned}\dot{V}(t) &\leq 2(\gamma - d(t))\|e(t)\|^2 + 2ce^{\mathrm{T}}(t)(A\otimes\Gamma)e(t-\tau(t)) \\ &\leq 2(\gamma - d(t))\|e(t)\|^2 + 2c\|A\otimes\Gamma\|\|e(t)\|\|e(t-\tau(t))\| \\ &\leq -(2d(t) - 2\gamma - c\|A\|\|\Gamma\|)\|e(t)\|^2 \\ &\quad + c\|A\|\|\Gamma\|\|e(t-\tau(t))\|^2. \end{aligned} \tag{18}$$

The following proof is divided into three steps.

Step 1 [Boundedness of d(t)]: From (16), it is derived that $d(t)$ is nondecreasing. If

$$d(t) < d_0 := \max\{\gamma + c\|A\|\|\Gamma\|,\ d(0)\} + 1\ \ \forall t \geq 0$$

then $d(t)$ is bounded. Otherwise, there exists $t_0 > 0$ such that $d(t_0) = d_0$. Then

$$d(t) \geq \gamma + c\|A\|\|\Gamma\| + 1,\ \ t \geq t_0.$$

Combining with (18), one obtains

$$\dot{V}(t) \leq -\alpha V(t) + \beta\|V\|_{t,\tau_{\max}},\ \ t \geq t_0$$

where $\alpha = c\|A\|\|\Gamma\| + 2$ and $\beta = c\|A\|\|\Gamma\|$. Applying Lemma 1 gives

$$V(t) \leq M\exp(-\sigma(t-t_0)),\ \ t \geq t_0$$

where $M = \|V\|_{t_0,\tau_{\max}}$ and σ is the unique positive solution of the equation $\beta\exp(\sigma\tau_{\max}) + \sigma = \alpha$. From (16), it follows that:

$$d(t) = d(t_0) + k\int_{t_0}^{t} V(s)\mathrm{d}s \leq d_0 + \frac{kM}{\sigma}.$$

Hence, $d(t)$ is bounded.

Step 2 [Boundedness of e(t)]: Since $d(t)$ is nondecreasing and bounded, $d(+\infty)$ exists. Then

$$\int_0^{+\infty} V(t)\mathrm{d}t = \frac{d(+\infty) - d(0)}{k} \tag{19}$$

where $d(+\infty) = \lim_{t\to+\infty} d(t)$. If $e(t)$ is unbounded, then $V(t)$ is unbounded. There exists a positive constant ϵ and a sequence $\{\omega_m\}$ in strictly increasing order satisfying that $\lim_{m\to+\infty}\omega_m = +\infty$ and $V(\omega_m) \geq \epsilon$. Let

$$M_m = \max_{[\omega_m,\ \omega_m+2\tau_{\max}]} V(t) \geq \epsilon.$$

Due to the unboundedness of $V(t)$

$$\hat{\omega}_m = \inf\{t \geq \omega_m + 2\tau_{\max} |\ V(t) \geq M_m\}$$

is well defined. The continuity of $V(t)$ gives $V(\hat{\omega}_m) = M_m$ and $V(t) \leq M_m$ for $t \in [\omega_m, \hat{\omega}_m]$. When $t \in [\omega_m+\tau_{\max}, \hat{\omega}_m]$, it follows from (18) that:

$$\begin{aligned}\dot{V}(t) &\leq (2\gamma + c\|A\|\|\Gamma\|)V(t) + c\|A\|\|\Gamma\|V(t-\tau(t)) \\ &\leq 2(\gamma + c\|A\|\|\Gamma\|)M_m.\end{aligned}$$

In the case of $\tau_0 := (1/[2(\gamma + c\|A\|\|\Gamma\|)]) \geq \tau_{\max}$, one has

$$\begin{aligned}V(t) &\geq V\big(\hat{\omega}_m\big) - 2(\gamma + c\|A\|\|\Gamma\|)M_m\big(\hat{\omega}_m - t\big) \\ &= M_m\big[1 - 2(\gamma + c\|A\|\|\Gamma\|)\big(\hat{\omega}_m - t\big)\big]\end{aligned} \tag{20}$$

where $t \in [\hat{\omega}_m - \tau_{\max}, \hat{\omega}_m] \subset [\omega_m + \tau_{\max}, \hat{\omega}_m]$. Then

$$\begin{aligned}\int_{\hat{\omega}_m-\tau_{\max}}^{\hat{\omega}_m} V(t)\mathrm{d}t &\geq M_m\tau_{\max}\big[1 - (\gamma + c\|A\|\|\Gamma\|)\tau_{\max}\big] \\ &\geq \epsilon(\gamma + c\|A\|\|\Gamma\|)\tau_{\max}^2\end{aligned}$$

which contradicts (19). In the case of $\tau_0 < \tau_{\max}$, (20) holds for $t \in [\hat{\omega}_m - \tau_0, \hat{\omega}_m]$. Then

$$\int_{\hat{\omega}_m-\tau_0}^{\hat{\omega}_m} V(t)\mathrm{d}t \geq \frac{M_m}{4(\gamma + c\|A\|\|\Gamma\|)} \geq \frac{\epsilon}{4(\gamma + c\|A\|\|\Gamma\|)}$$

which contradicts (19). Therefore, $e(t)$ is bounded.

Step 3 [Convergence of e(t) to Zero]: Considering

$$\|f(x_i(t), t) - f(s(t), t)\| \leq \gamma\|e_i(t)\|$$

one obtains the boundedness of $f(x_i(t), t) - f(s(t), t)$. Since $d(t)$ is bounded, it follows from (3) that $\dot{e}(t)$ is bounded. Then, $\dot{V}(t) = 2e^{\mathrm{T}}(t)\dot{e}(t)$ is bounded, implying that $V(t)$ is uniformly continuous on $[0, +\infty)$. By using Lemma 2, it follows from (19) that

$$\lim_{t\to+\infty} V(t) = 0$$

leading to $\lim_{t\to+\infty} e(t) = 0$, that is, network (1) realizes synchronization. ■

Remark 2: Theorem 1 establishes synchronization criteria for network (1) under centralized adaptive control. It is noteworthy that the criteria here are independent of the network topology, and thereby are convenient to use. This benefits from using the adaptive control.

Remark 3: When the delay $\tau(t)$ is nondifferentiable, especially when $\tau(t)$ has an uncountable set of nondifferentiable points, conventional methods does not work for solving the adaptive synchronization problem of networks. In the literature, the following Lyapunov function:

$$\begin{aligned} V(t) &= e^{\mathrm{T}}(t)e(t) + \frac{1}{2(1-\mu)}\int_{t-\tau(t)}^{t} e^{\mathrm{T}}(\theta)e(\theta)\mathrm{d}\theta \\ &\quad + \frac{\left(d(t) - d^*\right)^2}{k} \\ &:= V_1(t) + V_2(t) + V_3(t) \end{aligned}$$

is usually used for analyzing the adaptive synchronization of networks with time-varying delay, where $\dot{\tau}(t) \leq \mu < 1$ and d^* is a large enough constant. When $\tau(t)$ is not differentiable, $\dot{V}_2(t)$ does not exist, and $D^+V_2(t)$ may be infinite, making it difficult to analyze $V_2(t)$. If the Lyapunov function is chosen as

$$V(t) = V_1(t) + V_3(t)$$

the Hanalay inequality (delay differential inequality that deals with delay, especially nondifferentiable delay) becomes difficult to use due to $V_3(t)$. If the control is linear rather than adaptive, the problem would be simple, since the Hanalay inequality can be used by choosing $V(t) = V_1(t)$.

Remark 4: Instead of using the term V_3 (defined in Remark 3) to deal with the adaptive control, we analyze the boundedness of the adaptive control gain $d(t)$, which provides new perspectives on addressing the adaptive control. Based on the gain analysis, we further derive the synchronization criteria of network (1). The gain analysis here also helps avoid using the information on the differentiability of the delay.

Remark 5: In much of the literature, the boundedness of the control gain $d(t)$ is neglected. However, this is an essential problem for the adaptive control as controllers cannot provide infinitely large control gain in engineering.

Remark 6: When the initial synchronization error $e(0)$ is large, the control gain $d(t)$ may be large, causing an overkill control effect. This kind of effect can be greatly alleviated by choosing a small value of the parameter k, which will be verified by simulations.

B. Decentralized Adaptive Control

Consider the following decentralized adaptive control:

$$u_i(t) = -d_i(t)e_i(t), \ 1 \leq i \leq N \tag{21}$$

with updating laws

$$\dot{d}_i(t) = k_i\|e_i(t)\|^2 \tag{22}$$

where k_i is a positive constant and $d_i(0) \geq 0$.

Different from the centralized adaptive control (15), each control here has its own control gain $d_i(t)$, and the updating laws (22) only depend on local information $e_i(t)$.

Centralized adaptive control may not be practical for large-scale networks. The adaptive synchronization criteria for network (1) under decentralized adaptive control (21) is therefore established in the following theorem.

Theorem 2: Let assumptions (A1) and (A2) hold. Then, network (1) realizes synchronization under decentralized adaptive control (21) with updating laws (22).

Proof: Define $\Lambda = \{i|\ d_i(t) < d_0\ \ \forall t \geq 0\}$, where

$$d_0 = \max\{\gamma + c\|A\|\|\Gamma\|,\ d_1(0),\ \ldots,\ d_N(0)\} + 1.$$

Denote $l = |\Lambda|$, where $|\Lambda|$ is the number of the elements in Λ. Without loss of generality, let $\Lambda = \{1, \ldots, l\}$.

If $l = N$, then all $d_i(t)$ are bounded, so $\int_0^{+\infty} \|e(t)\|^2\mathrm{d}t$ converges. Following steps 2 and 3 of the proof of Theorem 1, one has $\lim_{t\to+\infty} e(t) = 0$.

It is next to deal with the case of $0 \leq l < N$. Denote

$$\begin{aligned} \mu_1(t) &= \left[e_1^{\mathrm{T}}(t), \ldots, e_l^{\mathrm{T}}(t)\right]^{\mathrm{T}} \\ \mu_2(t) &= \left[e_{l+1}^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t)\right]^{\mathrm{T}} \\ A &= \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \end{aligned}$$

where $A_1 \in \mathbb{R}^{l\times N}$ and $A_2 \in \mathbb{R}^{(N-l)\times N}$. Particularly, one has $\mu_1 = 0$, $\mu_2 = e$, and $A_2 = A$ if $l = 0$.

Consider the following Lyapunov functions:

$$V_1(t) = \mu_1^T(t)\mu_1(t), V_2(t) = \mu_2^T(t)\mu_2(t).$$

Differentiating V_1 along the solution of (3) yields

$$\begin{aligned} \dot{V}_1(t) &= 2\sum_{i=1}^{l} e_i^{\mathrm{T}}(t)\left[f(x_i(t), t) - f(s(t), t)\right] \\ &\quad + 2c\sum_{i=1}^{l}\sum_{j=1}^{N} a_{ij}e_i^{\mathrm{T}}(t)\Gamma e_j(t - \tau(t)) \\ &\quad - 2\sum_{i=1}^{l} d_i(t)e_i^{\mathrm{T}}(t)e_i(t). \end{aligned}$$

Considering (A2) and $d_i(t) \geq d_i(0) \geq 0$, one has

$$\dot{V}_1(t) \leq 2\gamma V_1(t) + 2c\mu_1^{\mathrm{T}}(t)(A_1 \otimes \Gamma)e(t - \tau(t)). \tag{23}$$

Since $\|A_1\|^2 = \lambda_{\max}(A_1^{\mathrm{T}}A_1) \leq \lambda_{\max}(A_1^{\mathrm{T}}A_1 + A_2^{\mathrm{T}}A_2) = \lambda_{\max}(A^{\mathrm{T}}A) = \|A\|^2$, one has

$$\begin{aligned} &2c\mu_1^{\mathrm{T}}(t)(A_1 \otimes \Gamma)e(t - \tau(t)) \\ &\quad \leq 2c\|A_1\|\|\Gamma\|\|\mu_1(t)\|\|e(t - \tau(t))\| \\ &\quad \leq c\|A\|\|\Gamma\|\left[V_1(t) + V_1(t - \tau(t)) + V_2(t - \tau(t))\right] \\ &\quad \leq c\|A\|\|\Gamma\|\left[2\|V_1\|_{t,\tau_{\max}} + \|V_2\|_{t,\tau_{\max}}\right] \end{aligned} \tag{24}$$

which further gives

$$\dot{V}_1(t) \leq \alpha_1\|V_1\|_{t,\tau_{\max}} + \beta_1\|V_2\|_{t,\tau_{\max}} \tag{25}$$

where $\alpha_1 = 2\gamma + 2c\|A\|\|\Gamma\|$ and $\beta_1 = c\|A\|\|\Gamma\|$.

According to the definition of l, there exists $t_0 \geq 0$ such that $d_i(t) \geq d_0 \quad \forall t \geq t_0$ for $i > l$. Similar to (23), one obtains

$$\dot{V}_2(t) \leq 2(\gamma - d_0)V_2(t) + 2c\mu_2^{\mathrm{T}}(t)(A_2 \otimes \Gamma)e(t - \tau(t)) \tag{26}$$

for $t \geq t_0$. Similar to (24), one obtains

$$\begin{aligned}&2c\mu_2^{\mathrm{T}}(t)(A_2 \otimes \Gamma)e(t - \tau(t)) \\ &\quad \leq c\|A\|\|\Gamma\|\big[2\|V_2\|_{t,\tau_{\max}} + \|V_1\|_{t,\tau_{\max}}\big]\end{aligned}$$

which further gives

$$\dot{V}_2(t) \leq -\alpha_2 V_2(t) + \beta_2\|V_2\|_{t,\tau_{\max}} + g(t) \quad \forall t \geq t_0 \tag{27}$$

where $\alpha_2 = 2(d_0 - \gamma) = 2\beta_1 + 2$, $\beta_2 = 2\beta_1$, and $g(t) = \beta_1\|V_1\|_{t,\tau_{\max}}$.

The following proof is divided into two steps.

Step 1 [Boundedness of $V_1(t)$]: Suppose that V_1 is unbounded. According to the definition of l and the updating laws (22), $\int_0^{+\infty} \|e_i(t)\|^2 \mathrm{d}t$ converges for $1 \leq i \leq l$, thus

$$\chi_0 := \int_0^{+\infty} V_1(t)\mathrm{d}t \tag{28}$$

is finite.

Let $M_0 = \max_{s \in [t_0 - \tau_{\max}, t_0 + 2\tau_{\max}]} V_1(s)$, and let M be any positive constant larger than M_0 and $(2\chi_0/\tau_{\max})$. Then

$$t_1 = \inf\{t \geq t_0 |\ V_1(t) \geq M\}$$

is well defined. It is obvious that $t_1 > t_0 + 2\tau_{\max}$, $V_1(t_1) = M$, and

$$V_1(t) \leq M \quad \forall t \in \big[t_0 - \tau_{\max}, t_1\big]. \tag{29}$$

There exists $t_2 \in [t_1 - \delta, t_1]$ such that $V_1(t_2) = m$, where $\delta = (2\chi_0/M) < \tau_{\max}$ and $m = \min_{s \in [t_1 - \delta, t_1]} V_1(s)$. Since

$$m\delta \leq \int_{t_1 - \delta}^{t_1} V_1(t)\mathrm{d}t \leq \chi_0$$

one has $m \leq (M/2)$. Then, there exists $t_3 \in [t_2, t_1]$ such that

$$\dot{V}_1(t_3) = \frac{V_1(t_1) - V_1(t_2)}{t_1 - t_2} \geq \frac{M - m}{\delta} \geq \frac{M^2}{4\chi_0}. \tag{30}$$

When $t \in [t_0, t_1]$, it follows from (29) that $g(t) \leq \beta_1 M$. According to (27) and Lemma 3, one has

$$\begin{aligned}V_2(t) &\leq \hat{M}_0 \exp(-\sigma(t - t_0)) + \int_{t_0}^t g(\theta)\exp(-\sigma(t - \theta))d\theta \\ &\leq \hat{M}_0 + \int_{t_0}^t \beta_1 M \exp(-\sigma(t - \theta))d\theta \\ &\leq \hat{M}_0 + \frac{\beta_1}{\sigma}M \quad \forall t \in \big[t_0, t_1\big]\end{aligned} \tag{31}$$

where $\hat{M}_0 = \|V_2\|_{t_0,\tau_{\max}}$ and σ is the unique positive solution of the equation $\sigma = \alpha_2 - \beta_2\exp(\sigma\tau_{\max})$. Combining (25), (29), and (30), one has

$$\frac{M^2}{4\chi_0} \leq \dot{V}_1(t_3) \leq \beta_1\hat{M}_0 + \left(\alpha_1 + \frac{\beta_1^2}{\sigma}\right)M$$

leading to a contradiction if letting $M \to +\infty$. Hence, $V_1(t)$ is bounded.

Step 2 [Convergence of $e(t)$ to Zero]: Since $V_1(t)$ is bounded, $g(t)$ is also bounded. Similar to (31), it can be proved that $V_2(t)$ is bounded. Then, it follows from (25) that $\dot{V}_1(t)$ is bounded, so V_1 is uniformly continuous on $[0, +\infty)$. According to (28) and Lemma 2, one has $\lim_{t\to+\infty} V_1(t) = 0$ and thus $\lim_{t\to+\infty} g(t) = 0$.

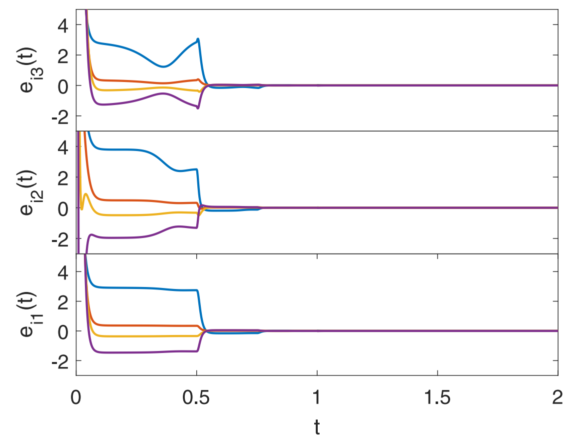

Fig. 1. Trajectories of the synchronization errors $e_{ij}(t)$ of network (34), where $1 \leq i \leq 4$ and $1 \leq j \leq 3$.

Let ε be any positive constant. There exists a constant $T \geq t_0$ such that $g(t) \leq \varepsilon$ for $t \geq T$. According to (27) and Lemma 3, one has

$$\begin{aligned}V_2(t) &\leq \hat{M}_T \exp(-\sigma(t - T)) + \int_T^t g(\theta)\exp(-\sigma(t - \theta))d\theta \\ &\leq \hat{M}_T \exp(-\sigma(t - T)) + \frac{\varepsilon}{\sigma} \quad \forall t \geq T\end{aligned} \tag{32}$$

where $\hat{M}_T = \|V_2\|_{T,\tau_{\max}}$ and σ is defined in (31). This gives $\overline{\lim}_{t\to+\infty} V_2(t) \leq (\varepsilon/\sigma)$. Letting $\varepsilon \to 0^+$ leads to

$$0 \leq \overline{\lim}_{t\to+\infty} V_2(t) \leq 0$$

that is, $\lim_{t\to+\infty} V_2(t) = 0$.

By $\lim_{t\to+\infty} V_1(t) = 0$ and $\lim_{t\to+\infty} V_2(t) = 0$, it can deduced that $\lim_{t\to+\infty} e(t) = 0$, that is, network (1) realizes synchronization. ■

Remark 7: Theorem 2 establishes synchronization criteria for network (1) under decentralized adaptive control. Compared with centralized control, decentralized control is easier to implement in engineering, especially for large-scale networks. It should be noted that for centralized control, it is easy to theoretically guarantee the boundedness of the control gain. However, it is not easy to prove the boundedness of the control gain for decentralized control, which deserves further study.

IV. Numerical Simulations

The well-known Lorenz system [35] is described by

$$\dot{s}(t) = f(s(t)) = Bs(t) + W(s(t)) \tag{33}$$

where

$$B = \begin{bmatrix} -c_1 & c_1 & 0 \\ c_3 & -1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix}, \quad W(s(t)) = \begin{bmatrix} 0 \\ -s_1(t)s_3(t) \\ s_1(t)s_2(t) \end{bmatrix}$$

with $c_1 = 10$, $c_2 = 8/3$, and $c_3 = 28$.

Example 1: Consider a network consisting of four Lorenz systems under centralized control

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^4 a_{ij}\Gamma x_j(t - \tau(t)) - d(t)e_i(t) \tag{34}$$

with updating laws

$$\dot{d}(t) = k\|e(t)\|^2$$

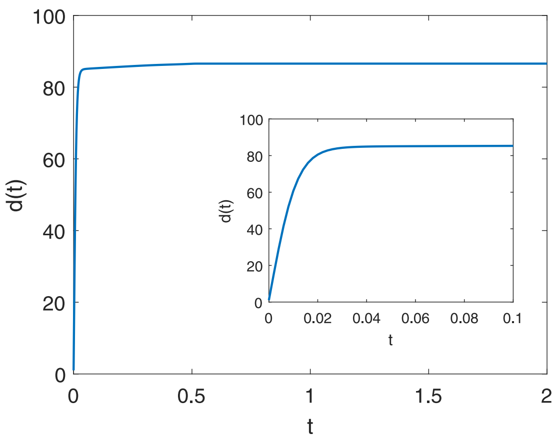

Fig. 2. Adaptive control gain $d(t)$ of network (34).

TABLE I
T_{SYNC} AND $d(+\infty)$ FOR DIFFERENT VALUES OF k, WHERE T_{SYNC} IS THE FIRST INSTANT AFTER WHICH $\|e(t)\| < 10^{-4}$ AND $d(+\infty)$ IS APPROXIMATED BY $d(T_{\text{SYNC}})$

k	1	0.3	0.1	0.03	0.01	0.006	0.003	0.002	0.001
T_{sync}	0.92	1.09	2.58	4.86	8.98	12.50	22.17	83.10	81.12
$d(+\infty)$	86.56	45.07	24.49	12.41	6.41	4.92	3.23	3.67	3.29

where $c = 1$, $\Gamma = I_3$, $k = 1$, and

$$A = \begin{bmatrix} -4 & 1 & 2 & 1 \\ 1 & -3 & 2 & 0 \\ 2 & 0 & -5 & 3 \\ 0 & 1 & 2 & -3 \end{bmatrix}$$
$$\tau(t) = |2m + 1 - t|, \text{ if } t \in [2m, 2m + 2)$$

with $m = 0, 1, 2, \ldots$

Most existing results on adaptive synchronization cannot deal with network (34) as the time-varying delay $\tau(t)$ is nondifferentiable, while ours can. It is verified that (A1) holds with $\tau_{\max} = 1$ and (A2) holds due to the boundedness of the Lorenz system [36]. According to Theorem 1, network (34) can realize synchronization.

Set $d(0) = 1$, $s(0) = [0, -1, -5]^{\mathrm{T}}$, and

$$x_i(t) = 30i + [1, -10, -15]^T, \ t \in [-1, 0]$$

where $1 \leq i \leq 4$. The synchronization errors $e_i(t)$ and the control gain $d(t)$ are presented in Figs. 1 and 2, respectively. It is shown that network (34) realizes synchronization and the control gain $d(t)$ is bounded, which demonstrates the effectiveness of Theorem 1.

From Fig. 2, we find that the control gain $d(t)$ is large when we set $k = 1$. We next show how to reduce the control gain by tuning the value of k. For simplicity, the synchronization is considered to be realized at T_0 if the synchronization error satisfies $\|e(t)\| < 10^{-4}$ for $t \geq T_0$. The lower bound of T_0 is denoted by T_{sync}, and the upper bound $d(+\infty)$ of the control gain $d(t)$ is approximated by $d(T_{\text{sync}})$. The values of T_{sync} and $d(+\infty)$ for several different values of k are given in Table I. Roughly speaking, $d(+\infty)$ decreases with the decrease of k when $k \geq 0.003$. Therefore, we can choose a small k to avoid the overkill control effect. On the other hand, k should not be too small, because T_{sync} increases quickly when k becomes very small.

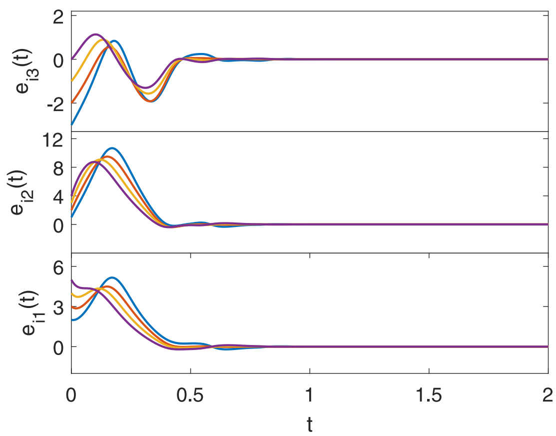

Fig. 3. Trajectories of the synchronization errors $e_{ij}(t)$ of network (35), where $1 \leq i \leq 4$ and $1 \leq j \leq 3$.

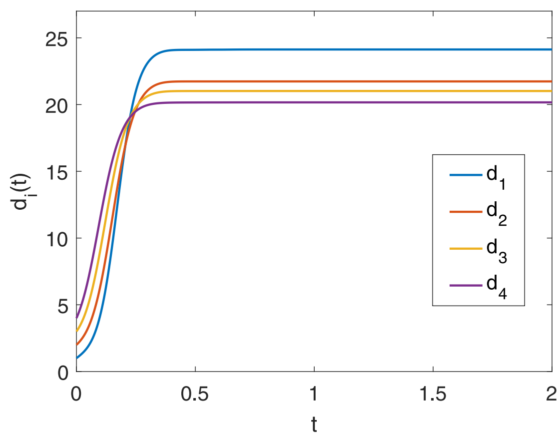

Fig. 4. Adaptive control gain $d_i(t)$ of network (35), where $1 \leq i \leq 4$.

Example 2: Consider another network of four Lorenz systems under decentralized control

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{4} a_{ij}\Gamma x_j(t - \tau(t)) - d_i(t)e_i(t) \qquad (35)$$

with updating laws

$$\dot{d}_i(t) = k_i\|e_i(t)\|^2$$

where $k_i = 1$ and the other parameters are as set in Example 1. According to Theorem 2, network (35) can realize synchronization, as (A1) and (A2) hold (verified in Example 1).

Set $d_i(0) = i$, $s(0) = [0, -1, 2]^{\mathrm{T}}$, and

$$x_i(t) = i + [1, -1, -2]^T, \ t \in [-1, 0]$$

where $1 \leq i \leq 4$. The synchronization errors $e_i(t)$ and the control gain $d_i(t)$ are presented in Figs. 3 and 4, respectively. Fig. 3 shows that network (35) realizes synchronization, which demonstrates the effectiveness of Theorem 2. Fig. 4 shows the control gain $d_i(t)$ is bounded, but the boundedness of $d_i(t)$ has not been theoretically proved.

V. CONCLUSION

In this article, the synchronization of complex networks with a time-varying delay has been investigated. Unlike most existing studies, the delay here could be nondifferentiable. The synchronization criteria for networks under the centralized adaptive control and networks under the decentralized adaptive control have been established by analyzing the control gain and extending the well-known Halanay inequality. For centralized control, the boundedness of the control gain has been theoretically proved. For decentralized control, however, determining the boundedness of the control gain is still an open problem, which deserves further study.

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