A New Method for Topology Identification of Complex Dynamical Networks

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***Abstract*—Topology identification of complex dynamical networks received extensive attention in the past decade. Most existing studies rely heavily on the linear independence condition (LIC). We find that a critical step in using this condition is not rigorous. Besides, it is difficult to verify this condition. Without regulating the original network, possible identification failure caused by network synchronization cannot be avoided. In this paper, we propose a new method to overcome these shortcomings. We add a regulation mechanism to the original network and construct an auxiliary network consisting of isolated nodes. Along with the outer synchronization between the regulated network and the auxiliary network, we show that the original network can be identified. Our method can avoid identification failure caused by network synchronization. Moreover, we show that there is no need to check the LIC. We finally provide some examples to demonstrate that our method is reliable and has good performances.**

***Index Terms*—Complex network, identification failure, linear independence, topology identification.**

I. Introduction

COMPLEX networks are ubiquitous in the real world, including social networks, information networks, technological networks, and biological networks [1]. In the past two decades, a lot of efforts have been devoted to the study of complex networks [2]–[6]. Synchronization is one of the most important topics in the field of complex networks, and fruitful results are obtained [7]–[14]. Apart from synchronization, topology identification has drawn increasing attention in recent years [15]–[28].

The topology of a complex dynamical network plays an important role in its dynamics. However, the topology of a network is usually not accessible or uncertain. In the past decade, lots of studies focused on the topology identification of various complex networks, in which the synchronization-based method is our present concern. The existing studies cover various network models [15]–[18], various types of outer synchronization [19]–[23] between the drive network and the response network, and various control schemes [24]–[28]. Many existing studies rely heavily on an assumption of the linear independence condition (LIC). Three problems related to the LIC should be highlighted. First, we find that a critical step in using the LIC is lack of rigorous derivation in many studies, which will be explained in Section II-A. Second, the LIC is difficult to verify. Some studies claim that the LIC naturally holds for most networks consisting of nonidentical nodes [17] or for networks subjected to stochastic perturbations [18], but theoretical analysis is missing. Until now, there is still no effective method for the verification of the LIC. Third, the topology identification will probably fail if the LIC does not hold. Chen *et al.* [29] showed that synchronization is an obstacle to topology identification. If the drive network reaches synchronization, the LIC will not hold any more, and the topology identification will probably fail.

Since the synchronization within the drive network is an obstacle to topology identification, one strategy is to add an appropriate regulation mechanism to the original network (original drive network) so that the regulated network is not able to reach synchronization. Based on this idea, this paper proposes a new method for topology identification of complex dynamical networks, which overcomes the above-mentioned three problems. We construct an auxiliary network consisting of isolated nodes. By adding an appropriate regulation mechanism to the original network, the couplings of the original network can be identified along with the outer synchronization between the regulated network and the auxiliary network. Our method can avoid identification failure caused by network synchronization, whereas existing methods might fail. Besides, the assumptions in our method are easy to verify, and there is no need to assume and check the LIC.

The rest of this paper is organized as follows. Section II presents some preliminaries. Section III presents the new method for topology identification. Section IV provides the numerical examples to verify the effectiveness of the proposed method. Section V concludes the investigation.

II. Preliminaries

Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all the $n \times m$-dimensional real matrices, respectively, $\mathbb{R}_+$ is the set of non-negative real numbers, $\|\cdot\|$ denotes the Euclidean norm of a vector, and I_N is the identity matrix of dimension N.

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A. Review of the Existing Studies

Consider a general complex network, with or without coupling delay, consisting of N systems of dimension n

$$\dot{x}_i(t) = f_i(t, x_i(t)) + \sum_{j=1}^{N} c_{ij}\Gamma x_j(t-\tau(t)) \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state vector of the ith node, $f_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function, $\Gamma \neq 0$ is the inner coupling matrix, $\tau(t)$ is the coupling delay (which might be zero), and $C = (c_{ij})_{N\times N}$ is the unknown weight configuration matrix. If there is a link from node i to node j ($j \neq i$), then the weight $c_{ij} > 0$; otherwise, $c_{ij} = 0$; the diagonal elements of C are defined by $c_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij}$, $1 \leq i \leq N$. Here, the configuration matrix C need not be symmetric or irreducible.

Remark 1: When $\tau(t) = 0$, network (1) reduces to a network without coupling delay. When $\tau(t) = \tau_0$ is a positive constant, network (1) is a network with constant coupling delay.

We first point out a critical but not rigorous step in most existing studies, where network (1) is referred to as the drive network. In [15]–[18], [22], [23], [27], and [30], the form of the drive network might differ slightly. For example, all f_i ($1 \leq i \leq N$) might be identical, $\tau(t)$ might be 0, a positive constant, or time varying, and some other factors, such as stochastic perturbations or system parameters might be involved, which do not have any essential influence on the following discussions.

In order to identify the unknown topology described by C, another network referred to as the response network is constructed

$$\dot{y}_i(t) = f_i(t, y_i(t)) + \sum_{j=1}^{N} \hat{c}_{ij}\Gamma y_j(t-\tau(t)) - d_i e_i(t) \tag{2}$$

with updating laws $\dot{\hat{c}}_{ij} = -e_i^T(t)\Gamma y_j(t-\tau(t))$ and $d_i = k_i\|e_i\|^2$, where $1 \leq i, j \leq N$, $e_i(t) = y_i(t) - x_i(t)$, and k_i is a positive constant. A cornerstone of the theory therein is the LIC, stated as follows.

Assumption 1 (A1): Suppose that $\{\Gamma x_i(t-\tau(t))\}_{i=1}^N$ are linearly independent on the orbit $\{x_i(t)\}_{i=1}^N$ of the outer synchronization manifold $\{x_i = y_i\}_{i=1}^N$.

From (1) and (2), it follows that:

$$\dot{e}_i(t) = f_i(t, y_i(t)) - f_i(t, x_i(t)) + \sum_{j=1}^{N} c_{ij}\Gamma e_j(t-\tau(t)) + \sum_{j=1}^{N} \tilde{c}_{ij}\Gamma y_j(t-\tau(t)) - d_i e_i(t) \tag{3}$$

where $\tilde{c}_{ij} = \hat{c}_{ij} - c_{ij}$, $1 \leq i, j \leq N$. It can be proved that $\lim_{t\to+\infty} e_i(t) = 0$ (see [16]). It then follows that:

$$\lim_{t\to+\infty} \sum_{j=1}^{N} \tilde{c}_{ij}(t)\Gamma y_j(t-\tau(t)) = 0, \ 1 \leq i, j \leq N. \tag{4}$$

In [15], [16], [23], and [30], (A1) is neglected and it is claimed that

$$\lim_{t\to+\infty} \tilde{c}_{ij}(t) = 0, 1 \leq i, j \leq N. \tag{5}$$

We now point out that this is not rigorous. Likewise, [17], [18], [22], and [27] deduced (5) from (A1), which is also not rigorous. Consider a simple example. Let $N = 2$, $n = 1$, $\tau(t) = 1$, $\Gamma = 1$, $y_1(t) = \cos(t)$, $y_2(t) = \cos(2t)$, $\tilde{c}_{11}(t) = \tilde{c}_{21}(t) = -\cos(2(t-1))$, and $\tilde{c}_{12}(t) = \tilde{c}_{22}(t) = \cos(t-1)$. It can be easily verified that (A1) and (4) hold, but (5) does not hold. This example indicates that the derivation of (5) from (A1) and (4) is not always possible. Thus, additional condition is necessary. Without restriction on $x_i(t)$ ($1 \leq i \leq N$), i.e., the solution of (1), it might be difficult to prove (5), which remains an interesting and challenging open problem.

B. Problem Formulation

In this paper, network (1) is to be identified. Different from the aforementioned methods, we construct an auxiliary network composed of N isolated nodes, described by

$$z_i(t) = v g_i(t) \tag{6}$$

where $1 \leq i \leq N$, z_i is the state vector of the ith node, $v \in \mathbb{R}^n$ is a constant vector satisfying $\Gamma v \neq 0$, and $g_i : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Moreover, we add a regulation mechanism to network (1), and the regulated network is described by

$$\dot{x}_i(t) = f_i(t, x_i(t)) + \sum_{j=1}^{N} c_{ij}\Gamma x_j(t-\tau(t)) + u_i(t) \tag{7}$$

where $u_i(t)$ is the regulation protocol to be designed, $1 \leq i \leq N$.

C. Mathematical Preliminaries

Here, we present several assumptions and lemmas, with the proofs for the lemmas given in the Appendix.

Assumption 2 (A2): Suppose that $\tau(t)$ is differentiable and $\dot{\tau}$ is bounded with $\overline{\lim}_{t\to+\infty}\dot{\tau}(t) \leq \eta < 1$, where η is a constant.

Remark 2: Since $\lim_{t\to+\infty}\dot{\tau}(t)$ might not exist, we use $\overline{\lim}_{t\to+\infty}\dot{\tau}(t)$. In other studies such as [16] and [17], it is supposed that $\dot{\tau}(t) \leq \eta$ or $0 \leq \dot{\tau}(t) \leq \eta$, which is stricter than $\overline{\lim}_{t\to+\infty}\dot{\tau}(t) \leq \eta$. Though $\tau(t)$ usually varies slowly, there may exist some moments where $\tau(t)$ increases so fast that $\dot{\tau}(t) \geq 1$ on some interval. Therefore, (A2) covers more general situations.

Assumption 3 (A3): Suppose that there exists a positive constant T satisfying $g_k(t) = g_k(t+T)$, $1 \leq k \leq N$, where g_k is defined in (6). Suppose also that there exists a sequence of time points $t_1^{(p)}, \ldots, t_p^{(p)} \in [0, T)$ satisfying

$$\sum_{l=1}^{p} g_k\left(t_l^{(p)}\right) = \begin{cases} 0, & 1 \leq k < p \\ \delta_p \neq 0, & k = p \end{cases}$$

for $p = 1, 2, \ldots, N$.

Assumption 4 (A4): Suppose that there exists a non-negative constant γ satisfying

$$\|f_i(t,x) - f_i(t,y)\| \leq \gamma\|x-y\| \tag{8}$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^n$, $1 \leq i \leq N$.

In the following four lemmas, Ξ is an interval of the form $(\alpha, +\infty)$ or $[\alpha, +\infty)$, where α is a constant or $-\infty$.

Lemma 1: Let $T > 0$ and c be constants. Let $a : \Xi \to \mathbb{R}$ be a differentiable function satisfying $\lim_{t\to+\infty} \dot{a}(t) = 0$. If

$$\lim_{m\to+\infty} a(mT + c) = 0,\ \ m \in \mathbb{N}$$

then $\lim_{t\to+\infty} a(t) = 0$.

Lemma 2: Suppose (A3) holds. Let $a_k : \Xi \to \mathbb{R}$ $(1 \le k \le N)$ be a sequence of differentiable functions satisfying $\lim_{t\to+\infty} \dot{a}_k(t) = 0$. If

$$\lim_{t\to+\infty} \sum_{k=1}^{N} a_k(t) g_k(t) = 0 \tag{9}$$

with g_k defined in (6), then $\lim_{t\to+\infty} a_k(t) = 0,\ 1 \le k \le N$.

Lemma 3: Suppose (A2) and (A3) hold. Let $a_k : \Xi \to \mathbb{R}$ $(1 \le k \le N)$ be a sequence of differentiable functions satisfying $\lim_{t\to+\infty} \dot{a}_k(t) = 0$. If

$$\lim_{t\to+\infty} \sum_{k=1}^{N} a_k(t) g_k(t - \tau(t)) = 0$$

with g_k defined in (6), then $\lim_{t\to+\infty} a_k(t) = 0,\ 1 \le k \le N$.

Lemma 4: Let m be a positive integer, and

$$\dot{a}(t) = h_1(t) + h_2(t) \tag{10}$$

where $a, h_2 : \Xi \to \mathbb{R}^m$ are the differentiable functions and $h_1 : \Xi \to \mathbb{R}^m$ is a function. If $\lim_{t\to+\infty} a(t)$ exists, $\lim_{t\to+\infty} h_1(t) = 0$, and $\dot{h}_2$ is bounded, then $\lim_{t\to+\infty} \dot{a}(t) = 0$.

Remark 3: It is noteworthy that the existence of $\lim_{t\to+\infty} a(t)$ does not necessarily imply that $\lim_{t\to+\infty} \dot{a}(t) = 0$. Consider for example the function

$$a(t) = \frac{1}{t}\cos\left(t^2\right),\ \ t \in (1, +\infty).$$

One has $\lim_{t\to+\infty} a(t) = 0$, but $\lim_{t\to+\infty} \dot{a}(t)$ does not exist.

III. Main Results

Denote $\mu_i(t) = x_i(t) - z_i(t),\ 1 \le i \le N$. Then, design the regulation protocol as

$$u_i(t) = \dot{z}_i(t) - f_i(t, z_i(t)) - \sum_{j=1}^{N} \hat{c}_{ij}(t)\Gamma x_j(t - \tau(t)) - d_i(t)\mu_i(t) \tag{11}$$

with updating laws

$$\begin{aligned} \dot{\hat{c}}_{ij}(t) &= \mu_i^T(t)\Gamma x_j(t - \tau(t)) \\ \dot{d}_i(t) &= k_i\|\mu_i(t)\|^2 \end{aligned} \tag{12}$$

where k_i is a positive constant, $1 \le i, j \le N$. Now, we present our main results on how to identify the topology of network (1).

Theorem 1: Suppose (A2)–(A4) hold. Then, the topology of network (1) can be identified by the estimation matrix $\hat{C}(t) = (\hat{c}_{ij}(t))_{N\times N}$ with auxiliary network (6), regulation protocol (11), and updating laws (12). Precisely, one has

$$c_{ij} = \lim_{t\to+\infty} \hat{c}_{ij}(t),\ \ 1 \le i, j \le N.$$

Proof: Denoting $\tilde{c}_{ij}(t) = c_{ij} - \hat{c}_{ij}(t)$ and combining (6), (7), and (11), one has the following error system:

$$\dot{\mu}_i(t) = f_i(t, x_i(t)) - f_i(t, z_i(t)) + \sum_{j=1}^{N} \tilde{c}_{ij}(t)\Gamma x_j(t - \tau(t)) - d_i(t)\mu_i(t) \tag{13}$$

where $1 \le i \le N$. Since (A4) holds, one has

$$\|f_i(t, x_i(t)) - f_i(t, z_i(t))\| \le \gamma\|\mu_i(t)\|$$

where $1 \le i \le N$. Choose a Lyapunov function as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} \mu_i^T(t)\mu_i(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{c}_{ij}^2(t) + \frac{1}{2}\sum_{i=1}^{N} \frac{(d_i(t) - d^*)^2}{k_i} \tag{14}$$

where d^* is a positive constant to be determined. Then, one has

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{N} \mu_i^T(t)\dot{\mu}_i(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{c}_{ij}(t)\dot{\hat{c}}_{ij}(t) + \sum_{i=1}^{N} \frac{(d_i(t) - d^*)\dot{d}_i(t)}{k_i} \\ &\le \sum_{i=1}^{N} \gamma\|\mu_i(t)\|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{c}_{ij}(t)\mu_i^T(t)\Gamma x_j(t - \tau(t)) \\ &\quad - \sum_{i=1}^{N} d_i(t)\|\mu_i(t)\|^2 \\ &\quad - \sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{c}_{ij}(t)\mu_i^T(t)\Gamma x_j(t - \tau(t)) \\ &\quad + \sum_{i=1}^{N} \left(d_i(t) - d^*\right)\|\mu_i(t)\|^2 \\ &= \left(\gamma - d^*\right)\|\mu(t)\|^2 \end{aligned}$$

where $\mu(t) = (\mu_1^T(t), \mu_2^T(t), \ldots, \mu_N^T(t))^T \in \mathbb{R}^{nN}$. Taking $d^* = \gamma + 1$, one has $\dot{V}(t) = -\|\mu(t)\|^2$, from which it can be deduced that

$$\int_0^t \|\mu(\alpha)\|^2 \mathrm{d}\alpha = -\int_0^t \dot{V}(\alpha)\mathrm{d}\alpha = V(0) - V(t) \le V(0)$$

for $t \ge 0$. Therefore, one has $\mu \in L^2$. As $\dot{V}(t) \le 0$, V is bounded. Then, from (14), it can be deduced that μ_i, $\tilde{c}_{ij}$, and d_i are bounded. As g_i is continuous and periodic, one deduces that g_i is bounded, indicating that z_i and $x_i = z_i + \mu_i$ are bounded. From (13), it follows that $\dot{\mu}$ is bounded. Since $\mu \in L^2$ and $\dot{\mu}$ is bounded, it can be deduced from Barbalat lemma [31] that

$$\lim_{t\to+\infty} \mu(t) = 0 \tag{15}$$

yielding that

$$\lim_{t\to+\infty} \dot{\hat{c}}_{ij}(t) = -\lim_{t\to+\infty} \mu_i^T(t)\Gamma x_j(t - \tau(t)) = 0. \tag{16}$$

Rewrite (13) as $\dot{\mu}_i(t) = h_{i1}(t) + h_{i2}(t)$, where

$$h_{i1}(t) = f_i(t, x_i(t)) - f_i(t, z_i(t)) \tag{17}$$

and

$$h_{i2}(t) = \sum_{j=1}^{N} \tilde{c}_{ij}(t)\Gamma x_j(t - \tau(t)) - d_i(t)\mu_i(t). \tag{18}$$

Since

$$\lim_{t\to+\infty} \|h_{i1}(t)\| \le \lim_{t\to+\infty} \gamma\|\mu_i(t)\| = 0$$

one has $\lim_{t\to+\infty} h_{i1}(t) = 0$. It is now to prove that $\dot{h}_{i2}$ is bounded. From (18), one gets

$$\begin{aligned}\dot{h}_{i2}(t) = &\sum_{j=1}^{N} \dot{\tilde{c}}_{ij}(t)\Gamma x_j(t - \tau(t)) \\ &+ (1 - \dot{\tau}(t))\sum_{j=1}^{N} \tilde{c}_{ij}(t)\Gamma \dot{x}_j(t - \tau(t)) \\ &- \dot{d}_i(t)\mu_i(t) - d_i(t)\dot{\mu}_i(t).\end{aligned} \tag{19}$$

It has been proved that $x_j, \tilde{c}_{ij}, d_i$, and $\dot{\mu}_i$ are bounded. From (12), (15), (16), and (A2), it follows that μ_i, $\dot{\tilde{c}}_{ij}$, $\dot{d}_i$, and $\dot{\tau}$ are all bounded. Since g_j is continuously differentiable and periodic, $\dot{z}_j$ is continuous and periodic, indicating that $\dot{z}_j$ is bounded. Thus, $\dot{x}_j = \dot{z}_j + \dot{\mu}_j$ is bounded. Therefore, $\dot{h}_{i2}$ is bounded. According to Lemma 4, one has

$$\lim_{t\to+\infty} \dot{\mu}_i(t) = 0. \tag{20}$$

It then follows from (13) that:

$$\lim_{t\to+\infty} \sum_{j=1}^{N} \tilde{c}_{ij}(t)\Gamma x_j(t - \tau(t)) = 0. \tag{21}$$

In the proof of Lemma 3, it has been proved that

$$\lim_{t\to+\infty} (t - \tau(t)) = +\infty$$

which leads to

$$\lim_{t\to+\infty} \mu(t - \tau(t)) = \lim_{(t-\tau(t))\to+\infty} \mu(t - \tau(t)) = 0. \tag{22}$$

Combining (21), (22), and the boundedness of $\tilde{c}_{ij}$, one has

$$\lim_{t\to+\infty} \sum_{j=1}^{N} \tilde{c}_{ij}(t)\Gamma z_j(t - \tau(t)) = 0. \tag{23}$$

It follows from $\Gamma v \neq 0$ that:

$$\lim_{t\to+\infty} \sum_{j=1}^{N} \tilde{c}_{ij}(t) g_j(t - \tau(t)) = 0. \tag{24}$$

Combining (A2), (A3), and $\lim_{t\to+\infty} \dot{\tilde{c}}_{ij}(t) = 0$, one deduces from Lemma 3 that $\lim_{t\to+\infty} \tilde{c}_{ij}(t) = 0$, implying that

$$c_{ij} = \lim_{t\to+\infty} \hat{c}_{ij}(t), \ 1 \le i, j \le N.$$

This completes the proof. ∎

In Theorem 1, there is no need to check (A1), which is difficult to verify. (A3) may seem to be difficult to verify, but we can introduce a class of function sequences that satisfy (A3).

Theorem 2: Suppose (A2) and (A4) hold, with

$$g_k(t) = \rho\cos(k(t + \theta)), \ 1 \le k \le N$$

where $\rho > 0$ and θ are the constants. Then, the topology of network (1) can be identified by the estimation matrix $\hat{C}(t) = (\hat{c}_{ij}(t))_{N\times N}$ with auxiliary network (6), regulation protocol (11), and updating laws (12). Precisely, one has

$$c_{ij} = \lim_{t\to+\infty} \hat{c}_{ij}(t), \ 1 \le i, j \le N.$$

Proof: Denote $s = (2\pi/p)$, where p is a positive integer. Let $z = e^{is} = \cos(s) + i\sin(s)$, where i is the imaginary unit. Then, one has $z^p = 1$ and $x = p$ is the minimal positive integer satisfying $z^x = 1$. When $k = p$, one has

$$\sum_{l=0}^{p-1} z^{lk} = \sum_{l=0}^{p-1} 1 = p.$$

When $1 \le k < p$, one has

$$\sum_{l=0}^{p-1} z^{lk} = \frac{1 - z^{pk}}{1 - z^k} = 0.$$

Therefore, one gets

$$\sum_{l=0}^{p-1} \cos(lks) = \Re\left\{\sum_{l=0}^{p-1} z^{lk}\right\} = \begin{cases} 0, & 1 \le k < p \\ p, & k = p \end{cases}$$

where $\Re\{z\}$ denotes the real part of the complex number z. Therefore, (A3) holds by setting $T = 2\pi$, $\delta_p = \rho p$, and

$$t_l^{(p)} = \frac{2(l-1)\pi}{p} - \theta \ (\mathrm{mod}\ 2\pi)$$

for $p = 1, 2, \ldots, N$, where $1 \le l \le p$. It then follows from Theorem 1 that $c_{ij} = \lim_{t\to+\infty} \hat{c}_{ij}(t)$, $1 \le i, j \le N$. This completes the proof. ∎

IV. Numerical Simulations

Example 1: Consider a network consisting of six nodes, where the first three nodes are Lorenz systems [32] and the other three nodes are Lü systems [33], described by

$$\dot{x}_i(t) = f_i(t, x_i(t)) + \sum_{j=1}^{6} c_{ij}\Gamma x_j(t - \tau(t)) \tag{25}$$

where $1 \le i \le 6$, $\tau(t) = 0.2$, $\Gamma = I_3$, and

$$C = \varepsilon\begin{pmatrix} -4 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 3 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 & 0 \\ 2 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

with $\varepsilon = 1$. It is obvious that (A2) holds, and (A4) holds due to the boundedness of Lorenz system and Lü system (see [15] for details). According to Theorem 2, C can be identified, as

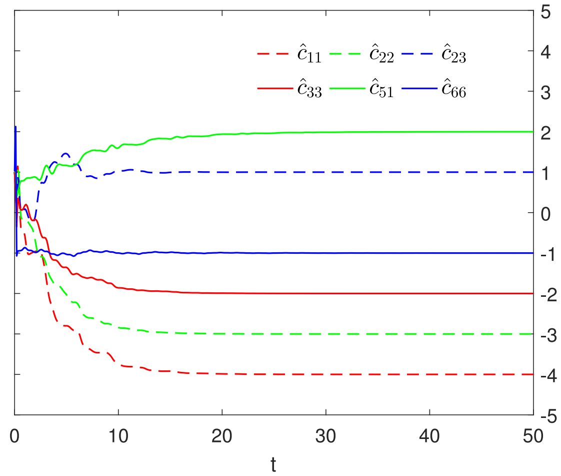

Fig. 1. Topology identification of network (25) by using Theorem 2: the evolution of the coupling set $\{\hat{c}_{11}, \hat{c}_{22}, \hat{c}_{23}, \hat{c}_{33}, \hat{c}_{51}, \hat{c}_{66}\}$. It is shown that these couplings converge to the real values.

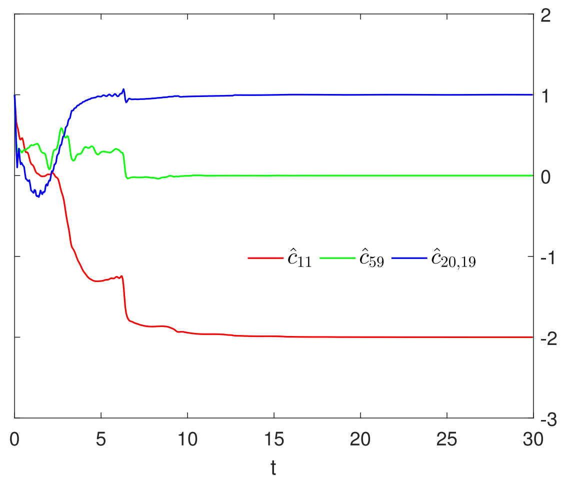

Fig. 2. Topology identification of network (26) by using Theorem 2: the evolution of the coupling set $\{\hat{c}_{11}, \hat{c}_{59}, \hat{c}_{20,19}\}$. It is shown that these couplings converge to the real values.

verified by the numerical result shown in Fig. 1, where only a part of $\hat{C}(t)$ is presented for clarity.

The parameters and initial values used are as follows. $x_i(0) = -0.5 + 1.5i + 0.5(1, 2, 3)^T$, $\hat{c}_{ij}(0) = 1$, $d_i(0) = 1$, $k_i = 10$, $\rho = 2$, $\theta = 0$, and $v = (1, 1, 1)^T$, $1 \leq i, j \leq 6$.

Example 2: Consider a ring network consisting of $N = 20$ identical nodes, described by

$$\dot{x}_i(t) = f(t, x_i(t)) + \varepsilon\Gamma x_{i-1}(t - \tau(t)) - 2\varepsilon\Gamma x_i(t - \tau(t)) + \varepsilon\Gamma x_{i+1}(t - \tau(t)) \quad (26)$$

where $1 \leq i \leq 20$, $f(t, x(t)) = -x(t) \in \mathbb{R}$, $\varepsilon = 1$, $\tau(t) = 0.1$, $\Gamma = 1$, $x_0 \equiv x_{20}$, and $x_{21} \equiv x_1$. It is obvious that (A2) holds, and (A4) holds with $\gamma = 1$. According to Theorem 2, C can be identified. Fig. 2 presents the evolution of the coupling set $\{\hat{c}_{11}, \hat{c}_{59}, \hat{c}_{20,19}\}$. Fig. 3 shows that the identification by using the method in [16] fails. This demonstrates the advantage of our new method.

In reality, external disturbances generally exist. In order to test the robustness of our method, the observed dynamics of

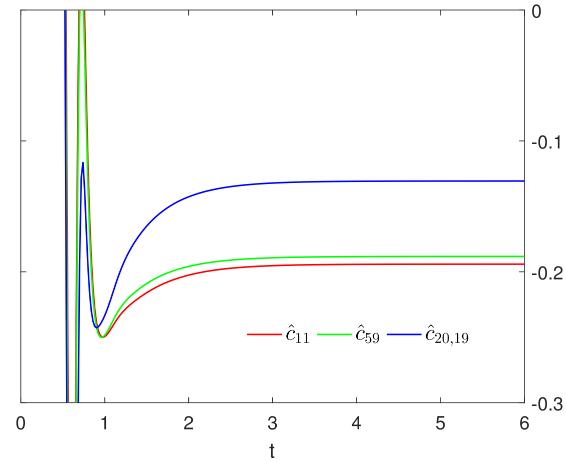

Fig. 3. Topology identification of network (26) by using the method in [16]: the evolution of the coupling set $\{\hat{c}_{11}, \hat{c}_{59}, \hat{c}_{20,19}\}$. It is shown that these couplings does not converge to the real values $-2, 0, 1$.

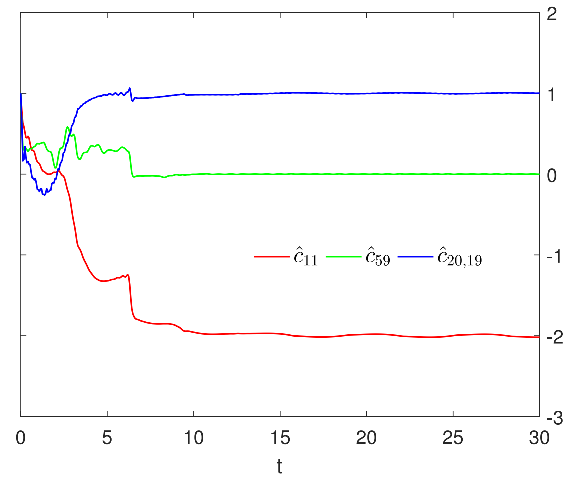

Fig. 4. Topology identification of network (26) subjected to white noise by using Theorem 2: the evolution of the coupling set $\{\hat{c}_{11}, \hat{c}_{59}, \hat{c}_{20,19}\}$. The noise is uniformly distributed on the interval $[0, 0.2]$. It is shown that these couplings nearly converge to the real values.

network (26) are now assumed to be disturbed by white noise uniformly distributed on the interval $[0, 0.2]$. Fig. 4 shows that the identified topology is very close to the real topology. This demonstrates the robustness of our new method.

The parameters and initial values used are as follows. $x_i(0) = 0.9 + 0.1i$, $\hat{c}_{ij}(0) = 1$, $d_i(0) = 1$, $k_i = 10$, $\rho = 4$, $\theta = 0$, and $v = 1$, $1 \leq i, j \leq 20$.

V. Conclusion

This paper shows clearly that a critical step in using the LIC is lack of rigorous derivation in most existing studies and the topology identification by using existing methods might fail due to network synchronization. In this paper, we proposed a new method, which overcomes the shortcomings. Our method can avoid identification failure caused by network synchronization, and the assumptions are easy to verify. Examples show that the topology of a complex network can be precisely identified by using our method, even when some existing methods fail. It is also shown that our method is robust against disturbances. The verification of (A1) and a more rigorous deduction of (5) remain to be interesting open problems.

APPENDIX

Proof of Lemma 1: Denote $m_t = [t/T]$, where $[x]$ means the integer part of a real number x. If $t > m_t T$, there exists $\xi_t \in (m_t T, t)$ such that

$$\begin{aligned}|a(t+c) - a(m_t T + c)| &= |\dot{a}(\xi_t + c)(t - m_t T)| \\ &\le T|\dot{a}(\xi_t + c)|.\end{aligned}$$

If $t = m_t T$, one also has $|a(t+c) - a(m_t T + c)| \le T|\dot{a}(\xi_t + c)|$ with $\xi_t = t$. Finally, it can be deduced that

$$\begin{aligned}&\overline{\lim}_{t\to+\infty}|a(t)| \\ &= \overline{\lim}_{t\to+\infty}|a(t+c)| \\ &\le \overline{\lim}_{t\to+\infty}|a(m_t T + c)| + \overline{\lim}_{t\to+\infty}T|\dot{a}(\xi_t + c)| \\ &= 0.\end{aligned}$$

It follows that $\lim_{t\to+\infty} a(t) = 0$, which completes the proof.

Proof of Lemma 2: The mathematical induction is used to prove that $\lim_{t\to+\infty} a_k(t) = 0,\ 1 \le k \le p$, if

$$\lim_{t\to+\infty}\sum_{k=1}^{p} a_k(t)g_k(t) = 0$$

where $1 \le p \le N$. Since

$$\lim_{t\to+\infty}\sum_{k=1}^{N} a_k(t)g_k(t) = 0$$

one gets $\lim_{t\to+\infty} a_k(t) = 0$ $(1 \le k \le N)$ by taking $p = N$.

Let ϵ be any positive real number.

First, consider the case of $p = 1$. According to (A3), one has $g_1(t_1^{(1)}) = \delta_1 \neq 0$. Since $\lim_{t\to+\infty} a_1(t)g_1(t) = 0$, there exists t_0 such that $|a_1(t)g_1(t)| < |\delta_1|\epsilon$ for $t \ge t_0$. Taking $t = mT + t_1^{(1)} \ge t_0$ with $m \in \mathbb{N}$, one gets $|a_1(mT + t_1^{(1)})| < \epsilon$, implying that $\lim_{m\to+\infty} a_1(mT + t_1^{(1)}) = 0$. From Lemma 1, one obtains $\lim_{t\to+\infty} a_1(t) = 0$, implying that the conclusion holds for $p = 1$.

Suppose the conclusion holds for $p = q-1$ with $2 \le q \le N$. Then, it is to prove that it holds for $p = q$.

Since g_k is continuous and periodic, g_k is bounded, that is, there exists a positive constant ρ satisfying that $|g_k| \le \rho$, $1 \le k \le N$. Since $\lim_{t\to+\infty} \dot{a}_k(t) = 0$, there exists an integer M_1 such that $|\dot{a}_k(t)| < ([|\delta_q|\epsilon]/[2\rho q^2 T])$ for all $t \ge M_1 T$ and $1 \le k \le q$. Since

$$\lim_{t\to+\infty}\sum_{k=1}^{q} a_k(t)g_k(t) = 0$$

there exists an integer M_2 such that

$$\left|\sum_{k=1}^{q} a_k(t)g_k(t)\right| < \frac{|\delta_q|\epsilon}{2q}$$

for all $t \ge M_2 T$. Denote $M = \max\{M_1, M_2\}$. For any $t \in (mT, mT+T)$ with any integer $m \ge M$, there exists $\xi_t \in (mT, t)$ such that

$$|a_k(t) - a_k(mT)| = |\dot{a}_k(\xi_t)(t - mT)| < \frac{|\delta_q|\epsilon}{2\rho q^2}.$$

When $t = mT$, one also has

$$|a_k(t) - a_k(mT)| < \frac{|\delta_q|\epsilon}{2\rho q^2}.$$

According to (A3), one has

$$\sum_{l=1}^{q} g_k\left(t_l^{(q)}\right) = \begin{cases} 0, & 1 \le k < q \\ \delta_q \neq 0, & k = q. \end{cases} \tag{27}$$

Moreover, one has

$$\begin{aligned}&\left|\sum_{l=1}^{q}\sum_{k=1}^{q} a_k\left(mT + t_l^{(q)}\right)g_k\left(mT + t_l^{(q)}\right)\right| \\ &\quad < q \times \frac{|\delta_q|\epsilon}{2q} = \frac{|\delta_q|\epsilon}{2}.\end{aligned} \tag{28}$$

On the other hand, one has

$$\begin{aligned}&\left|\sum_{l=1}^{q}\sum_{k=1}^{q} a_k\left(mT + t_l^{(q)}\right)g_k\left(mT + t_l^{(q)}\right)\right| \\ &\ge \left|\sum_{l=1}^{q}\sum_{k=1}^{q} a_k(mT)g_k\left(mT + t_l^{(q)}\right)\right| \\ &\quad - \left|\sum_{l=1}^{q}\sum_{k=1}^{q}\left[a_k\left(mT + t_l^{(q)}\right) - a_k(mT)\right]g_k\left(mT + t_l^{(q)}\right)\right|.\end{aligned}$$

From (27), one gets

$$\begin{aligned}&\left|\sum_{l=1}^{q}\sum_{k=1}^{q} a_k(mT)g_k\left(mT + t_l^{(q)}\right)\right| \\ &= \left|\sum_{k=1}^{q} a_k(mT)\sum_{l=1}^{q} g_k\left(t_l^{(q)}\right)\right| = |\delta_q|\left|a_q(mT)\right|\end{aligned}$$

and

$$\begin{aligned}&\left|\sum_{l=1}^{q}\sum_{k=1}^{q}\left[a_k\left(mT + t_l^{(q)}\right) - a_k(mT)\right]g_k\left(mT + t_l^{(q)}\right)\right| \\ &\le \rho\sum_{l=1}^{q}\sum_{k=1}^{q}\left|a_k\left(mT + t_l^{(q)}\right) - a_k(mT)\right| \\ &\le \rho\left|\sum_{l=1}^{q}\sum_{k=1}^{q}\frac{|\delta_q|\epsilon}{2\rho q^2}\right| = \frac{|\delta_q|\epsilon}{2}.\end{aligned}$$

It then follows that:

$$\begin{aligned}&\left|\sum_{l=1}^{q}\sum_{k=1}^{q} a_k\left(mT + t_l^{(q)}\right)g_k\left(mT + t_l^{(q)}\right)\right| \\ &\quad > |\delta_q|\left|a_q(mT)\right| - \frac{|\delta_q|\epsilon}{2}.\end{aligned} \tag{29}$$

Combining (28) and (29), it can be deduced that $|a_q(mT)| < \epsilon$, indicating that $\lim_{m\to+\infty} a_q(mT) = 0$. From Lemma 1, one has $\lim_{t\to+\infty} a_q(t) = 0$. Thus,

$$\begin{aligned}&\lim_{t\to+\infty}\sum_{k=1}^{q-1} a_k(t)g_k(t) \\ &= \lim_{t\to+\infty}\sum_{k=1}^{q} a_k(t)g_k(t) - \lim_{t\to+\infty} a_q(t)g_q(t) = 0.\end{aligned}$$

Since the conclusion holds for $p = q - 1$, one has

$$\lim_{t\to+\infty} a_k(t) = 0,\ \ 1 \le k \le q-1.$$

Consequently, the conclusion holds for $p = q$. Finally, the conclusion holds for all positive integers $p = 1, 2, \ldots, N$. This completes the proof.

Proof of Lemma 3: According to (A2), there exists a time instant t_0 such that

$$\dot{\tau}(t) \le \eta + \frac{1-\eta}{2} = \frac{1+\eta}{2} < 1$$

for $t > t_0$. Denoting $r(t) = t - \tau(t)$, one can show that $r(t)$ is monotonically increasing on the interval $(t_0, +\infty)$. Since the inequality

$$\int_{t_0}^{t} \dot{r}(\alpha)\mathrm{d}\alpha \ge \int_{t_0}^{t}\left(1 - \frac{1+\eta}{2}\right)\mathrm{d}\alpha = \frac{1-\eta}{2}(t - t_0)$$

holds for $t > t_0$, one has $\lim_{t\to+\infty} r(t) = +\infty$. Therefore, there exists an inverse function $t = t(r)$ for the function $r(t)|_{(t_0,+\infty)}$, where $t(r)$ is defined on the interval $(t_0 - \tau(t_0), +\infty)$. It is obvious that $\lim_{r\to+\infty} t(r) = +\infty$. Denote $\varphi_k(r) = a_k(t(r)),\ 1 \le k \le N$. From

$$\lim_{t\to+\infty} \sum_{k=1}^{N} a_k(t) g_k(t - \tau(t)) = 0$$

it follows that:

$$\lim_{r\to+\infty} \sum_{k=1}^{N} \varphi_k(r) g_k(r) = \lim_{t\to+\infty} \sum_{k=1}^{N} \varphi_k(r) g_k(r) = 0.$$

Since $\mathrm{d}r = (1 - \dot{\tau}(t))\mathrm{d}t$, one has

$$\begin{aligned} &\overline{\lim}_{r\to+\infty}\left|\frac{\mathrm{d}\varphi_k(r)}{\mathrm{d}r}\right| \\ &= \overline{\lim}_{r\to+\infty}\left|\frac{\mathrm{d}a_k(t(r))}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}\right| \\ &\le \overline{\lim}_{r\to+\infty}\left|\frac{\mathrm{d}a_k(t(r))}{\mathrm{d}t}\right| \overline{\lim}_{r\to+\infty}\left|\frac{1}{1-\dot{\tau}(t)}\right| \\ &\le \frac{2}{1-\eta}\lim_{t\to+\infty}\left|\frac{\mathrm{d}a_k(t)}{\mathrm{d}t}\right| = 0 \end{aligned}$$

where $1 \le k \le N$, thus $\lim_{r\to+\infty}(\mathrm{d}\varphi_k(r)/\mathrm{d}r) = 0$. Since (A3) holds, one has $\lim_{r\to+\infty} \varphi_k(r) = 0$ according to Lemma 2. Thus, one gets $\lim_{t\to+\infty} a_k(t) = \lim_{r\to+\infty} a_k(t(r)) = 0$. This completes the proof.

Proof of Lemma 4: First, consider the case of $m = 1$. Denote $\overline{\lim}_{t\to+\infty}|\dot{a}(t)| = 4A$. In the following, it is to prove $A = 0$ by contradiction. Suppose $A > 0$.

Since $\dot{h}_2$ is bounded, there exists a positive constant M satisfying that $|\dot{h}_2| < M$. Since $\lim_{t\to+\infty} a(t)$ exists, there exists a T_1 such that

$$|a(t_1) - a(t_2)| < \frac{A^2}{2M},\ \ \forall t_1, t_2 > T_1. \tag{30}$$

Since $\overline{\lim}_{t\to+\infty}|\dot{a}(t)| = 4A$, there exists a sequence $\{\beta_n\}$ in strictly increasing order satisfying that $\lim_{n\to+\infty} \beta_n = +\infty$ and $\lim_{n\to+\infty} |\dot{a}(\beta_n)| = 4A$. It follows that there exists a T_2 such that:

$$|\dot{a}(\beta_n)| > 2A,\ \ \forall \beta_n > T_2.$$

Since $\lim_{t\to+\infty} h_1(t) = 0$, there exists a T_3 such that

$$|h_1(t)| < \frac{A}{4},\ \ \forall t > T_3.$$

Let $T = \max\{T_1, T_2, T_3\}$. For a large enough n, one has $\beta_n > T$, thus $|\dot{a}(\beta_n)| > 2A$. For $t \in (0, (A/2M)]$, one has

$$\begin{aligned} &|\dot{a}(\beta_n) - \dot{a}(\beta_n + t)| \\ &= |h_1(\beta_n) + h_2(\beta_n) - h_1(\beta_n + t) - h_2(\beta_n + t)| \\ &\le |h_1(\beta_n)| + |h_1(\beta_n + t)| + |h_2(\beta_n) - h_2(\beta_n + t)| \\ &< \frac{A}{4} + \frac{A}{4} + |t \times \dot{h}_2(\xi_1)| \\ &\le \frac{A}{2} + \frac{A}{2M} \times M = A \end{aligned}$$

where $\xi_1 \in (\beta_n, \beta_n + t)$. This implies that $|\dot{a}(\beta_n + t)| > A,\ \forall t \in (0, (A/2M)]$. Then, one has

$$\left|a(\beta_n) - a\left(\beta_n + \frac{A}{2M}\right)\right| = \left|\frac{A}{2M} \times \dot{a}(\xi_2)\right| > \frac{A^2}{2M}$$

where $\xi_2 \in (\beta_n, \beta_n + [A/2M])$. This contradicts with (30). Therefore, one has $A = 0$, which implies that $\lim_{t\to+\infty} \dot{a}(t) = 0$ for $m = 1$.

Now, consider the case of $m > 1$. Rewrite (10) as

$$\dot{a}_i(t) = h_{i1}(t) + h_{i2}(t),\ \ 1 \le i \le m$$

where $\dot{a}_i, h_{i1}$, and h_{i2} are the components of $\dot{a}, h_1$, and h_2, respectively. It can be easily verified that $\lim_{t\to+\infty} a_i(t)$ exists, $\lim_{t\to+\infty} h_{i1}(t) = 0$, and $\dot{h}_{i2}$ is bounded. Therefore, one has $\lim_{t\to+\infty} \dot{a}_i(t) = 0,\ 1 \le i \le m$, which yields that $\lim_{t\to+\infty} \dot{a}(t) = 0$ for $m > 1$. This completes the proof.

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