

# Pinning Synchronization of Multiplex Delayed Networks With Stochastic Perturbations

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**Abstract**—As the monoplex network has its limitations in describing the real world, a new framework called the multiplex network is put forward and has received much attention in recent years. This paper focuses on synchronization of the multiplex network with multiple delays and stochastic perturbations. Due to the complexity, pinning control of the multiplex network is of particular interest. Based on the LaSalle-type invariance principle for stochastic differential delay equations and the Lyapunov stability theory, some control schemes and synchronization criteria are obtained. It is concluded that under some mild conditions, one can determine which nodes should be pinned in a multiplex network. In addition, it is found that the number of pinned nodes increases with the varying interval of noise and time delay, and decreases with the varying interval of intralayer coupling strength. Some two-layer and three-layer networks are employed to validate the effectiveness of the proposed algorithm.

**Index Terms**—Delay, multiplex network, pinning control, stochastic perturbation.

## I. INTRODUCTION

VARIOUS complex networks exist in the real world, such as the Internet, the social networks, the neural networks, the electrical power grids, etc. Although the study on monoplex complex networks has experienced rapid development in the past few years [1]–[5], many real-world networks cannot be easily abstracted by monoplex ones with the development of network science. People in modern society, for example, connect with each other via multiple relationships, say, kindred, friendship, or working relationships [6]. To describe this kind of complex social connection, the multiplex network is put forward. The study of the multiplex networks has become one of the most important branches of network science very recently, especially synchronization [7]–[14]. Sorrentino [7] and Irving and Sorrentino [8] studied the synchronization of a multiplex network of coupled dynamical systems, based on

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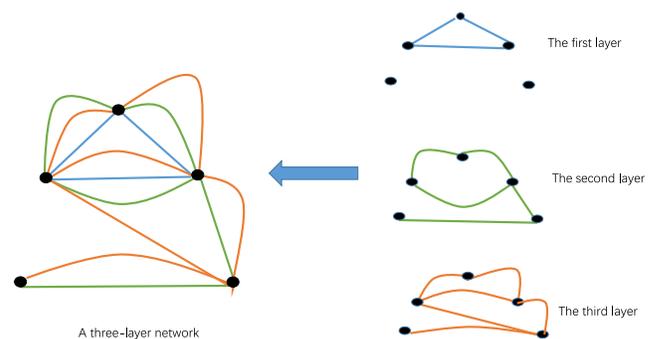


Fig. 1. Diagram of a three-layer network. Different colors represent the intralayer couplings of different layers. Each layer has its own time delay and noise perturbation.

master stability function. Aguirre *et al.* [9], Xu *et al.* [10], and Li *et al.* [11] investigated the synchronizability of two-layer star networks with different numbers of interlayer connections. Xu *et al.* [12] investigated the synchronizability of two interconnected BA scale-free networks. Recently, He *et al.* [13] further focused on the global synchronization of the multilayer network with additive couplings and Markovian switching couplings. Different from these models, our model presented here is a multiplex network where all layers consist of the same set of nodes and each layer has different time delays and noise perturbations. A diagram of a multiplex network discussed is depicted in Fig. 1.

As an efficient mechanism, pinning control is widely used in the traditional monoplex networks. By controlling only a small fraction of nodes, the synchronization can be realized in large-scale monoplex networks. Due to the complexity of multiplex networks, the pinning mechanism is expected to be particularly effective [15]–[25]. When designing a pinning control strategy, appropriately selecting the pinned nodes is of great importance. This problem was originally studied by using local stability analysis [16] and then by using global methods [17]. Song and Cao [21] presented some low-dimensional pinning criteria for global synchronization of both directed and undirected complex networks. DeLellis *et al.* [22] proposed a hierarchy of novel decentralized adaptive pinning strategies for controlled synchronization of complex networks based on edge snapping. In [23], a decentralized adaptive strategy was investigated for cluster synchronization of undirected complex dynamical networks via pinning control.

Actually, besides information delay, noise exists everywhere because of the environmental disturbances or

uncertainties [26]–[30]. Specific to a multiplex social communication network, communication links include Skype, Facebook, Wechat, etc. In the process of information dissemination, there are both time delays and noise perturbations in the Skype layer, the Facebook layer, and the Wechat layer. So it is crucial to propose a multiplex network model in which each layer has its own delay and noise. However, in most of the existing literature on pinning synchronization, it is often assumed that the node dynamics and network links are noise-free. How to synchronize information in this type of multiplex network is the thesis of this paper.

In this paper, pinning synchronization of the multiplex network is investigated, where the specific delay and stochastic perturbation exist in each layer. Based on the LaSalle-type invariance principle for stochastic differential delay equations and Lyapunov stability theory, it is concluded that the simple pinning adaptive feedback control is valid for the multiplex networks with delay and noise. In addition, we further discuss the impact of the difference of noise perturbations, time delays, and coupling strengths among each layer on the number of pinned nodes. We find that the number of pinned nodes increases with the interval of noise and the interval of time delay, and decreases with the interval of intralayer coupling strength.

The rest of this paper is organized as follows. Some mathematical preliminaries are introduced in Section II. The multiplex network model with time delays and stochastic perturbations in layers is proposed in Section III. Section IV presents our main results on synchronizing multiplex networks by a pinning strategy. In Section V, numerical examples and further discussions are shown to validate the main results. Finally, some conclusions are drawn in Section VI.

## II. MATHEMATICAL PRELIMINARIES

Throughout this paper,  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$  are introduced to denote the  $n$ -dimensional Euclidean space and the set of all  $n \times m$ -dimensional real matrices, respectively;  $\mathbb{R}_+$  is the set of positive real numbers;  $E[\cdot]$  is the mathematical expectation;  $\|\cdot\|$  represents the Euclidean norm defined by  $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$ , where  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{T}$  denotes the transpose of a vector or a matrix;  $\mathbf{I}_n \in \mathbb{R}^{n \times n}$  is an  $n \times n$  identity matrix;  $\text{diag}\{q_1, q_2, \dots, q_n\} \in \mathbb{R}^{n \times n}$  denotes a diagonal matrix whose diagonal elements are  $q_1, q_2, \dots, q_n$ ;  $\mathbf{G} = (g_{ij})_{n \times n}$  is an  $n$ -dimensional matrix, whose largest and smallest eigenvalues are  $\lambda_{\max}(\mathbf{G})$  and  $\lambda_{\min}(\mathbf{G})$ , respectively. The denotations  $\mathbf{G} > 0$ ,  $\mathbf{G} < 0$  and  $\mathbf{G}_l$  (or  $\lceil \mathbf{G} \rceil_l$ ) represent a positive definite matrix, a negative definite matrix and the minor matrix of  $\mathbf{G}$  by removing the first  $l$  row–column pairs. Let  $\mathbf{C}^{2,1}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$  be the family of all the non-negative functions  $\mathbf{V}(t, \mathbf{x})$  on  $\mathbb{R}_+ \times \mathbb{R}^n$ , being  $\mathbf{V}(t, \mathbf{x})$  continuously once differentiable in  $t$  and twice differentiable in  $\mathbf{x}$ ;  $\mathbf{C}_{\mathcal{F}_0}^b([-\tau, 0], \mathbb{R}^n)$  be the family of all the  $\mathcal{F}_0$ -measurable bounded  $\mathbf{C}([-\tau, 0], \mathbb{R}^n)$ -valued random variables;  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$ , where  $\{\mathcal{F}_t\}_{t \geq 0}$  is right continuous in  $t$  and  $\mathcal{F}_0$  contains all  $\mathcal{P}$ -null sets;  $\mathbf{w}(t) = (\mathbf{w}_1(t), \dots, \mathbf{w}_n(t))^T$  be an  $n$ -dimensional Brownian motion defined on the probability space.

Consider a nonautonomous  $n$ -dimensional stochastic differential delay equation

$$d\mathbf{x}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t - \tau))dt + \varphi(t, \mathbf{x}(t), \mathbf{x}(t - \tau))d\mathbf{w}(t) \quad (1)$$

on  $t \geq 0$  with initial value  $\xi \in \mathbf{C}_{\mathcal{F}_0}^b([-\tau, 0], \mathbb{R}^n)$ , where the measurable functions  $\mathbf{f} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfy the local Lipschitz condition and the linear growth condition [26]. It is known that for any initial value  $\xi$ , (1) has a unique solution on  $t \geq -\tau$ , denoted as  $\mathbf{x}(t; \xi)$ . Provided that  $\mathbf{f}(t, 0, 0) = \varphi(t, 0, 0) = 0$ , (1) has a trivial solution  $\mathbf{x}(t, 0) \equiv 0$ .

Defining an operator  $\mathcal{L}$  acting on  $\mathbf{C}^{2,1}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$  by

$$\begin{aligned} \mathcal{L}V &= V_t(t, \mathbf{x}) + V_x(t, \mathbf{x})\mathbf{f}(t, \mathbf{x}) \\ &\quad + \frac{1}{2}\text{trace}[\varphi^T(t, \mathbf{x})V_{xx}(t, \mathbf{x})\varphi(t, \mathbf{x})] \end{aligned} \quad (2)$$

where  $V_t(t, \mathbf{x}) = ([\partial V(\mathbf{x}, t)]/\partial t)$ ,  $V_x(t, \mathbf{x}) = ([\partial V(\mathbf{x}, t)]/\partial \mathbf{x}_1, \dots, [\partial V(\mathbf{x}, t)]/\partial \mathbf{x}_n)$ ,  $V_{xx}(t, \mathbf{x}) = ([\partial^2 V(\mathbf{x}, t)]/[\partial \mathbf{x}_i \partial \mathbf{x}_j])_{n \times n}$ , one gains

*Lemma 1* [30]: Suppose that:

- 1) there are functions  $V \in \mathbf{C}^{2,1}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ ,  $\gamma \in \mathbf{L}^1(\mathbb{R}_+, \mathbb{R}_+)$ , and  $\omega \in \mathbf{C}(\mathbb{R}^n, \mathbb{R}_+)$  such that  $\lim_{\|\mathbf{x}\| \rightarrow \infty} \inf_{0 \leq t < \infty} V(\mathbf{x}, t) = \infty$  and

$$\mathcal{L}V(\mathbf{x}, t) \leq \gamma(t) - \omega(\mathbf{x}), \quad (\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}_+.$$

- 2) for each initial value  $x_0 \in \mathbb{R}^n$  there is a constant  $p > 2$  such that

$$\sup_{0 \leq t < \infty} E|\mathbf{x}(t; x_0)|^p < \infty.$$

Then for every  $x_0 \in \mathbb{R}^n$ , the limit  $\lim_{t \rightarrow \infty} V(\mathbf{x}(t; x_0), t)$  is finite almost surely (a.s.), moreover

$$\lim_{t \rightarrow \infty} \omega(\mathbf{x}(t; x_0)) = 0 \text{ a.s.} \quad (3)$$

To obtain the main results, the following lemmas are necessary.

*Lemma 2* [7]: Suppose that  $\mathbf{G} = (G_{ij})_{n \times n}$  is an irreducible matrix satisfying  $G_{ij} = G_{ji} \geq 0$ ,  $i \neq j$ ,  $G_{ii} = -\sum_{j=1, j \neq i}^n G_{ij}$  for  $i = 1, 2, \dots, n$ . Then  $\mathbf{G}$  has a single eigenvalue 0 associated with the eigenvector  $(1, 1, \dots, 1)^T$ , and  $n - 1$  strictly negative eigenvalues that are real numbers.

*Lemma 3* (Schur Complement [31]): The linear matrix inequality

$$\begin{pmatrix} \mathcal{A}(x), \mathcal{B}(x) \\ \mathcal{B}^T(x), \mathcal{C}(x) \end{pmatrix} < 0$$

where  $\mathcal{A}^T(x) = \mathcal{A}(x)$ ,  $\mathcal{C}^T(x) = \mathcal{C}(x)$ , is equivalent to either of the following conditions.

- 1)  $\mathcal{A}(x) < 0$  and  $\mathcal{C}(x) - \mathcal{B}^T(x)\mathcal{A}(x)^{-1}\mathcal{B}(x) < 0$ .
- 2)  $\mathcal{C}(x) < 0$  and  $\mathcal{A}(x) - \mathcal{B}^T(x)\mathcal{C}(x)^{-1}\mathcal{B}(x) < 0$ .

*Lemma 4* [32]: Assume that  $\mathbf{P}$  is a diagonal matrix whose  $k$ th ( $k = 1, 2, \dots, l$ ) diagonal elements are  $p$  and the others are 0, where  $p > 0$  is a proper constant which is sufficiently large. Then,  $\mathbf{G} - \mathbf{P} < 0$  is equivalent to  $\mathbf{G}_l < 0$ .

### III. MULTIPLEX NETWORK MODEL

A general  $m$ -layer complex network consisting of  $N$  nonlinear systems is described by

$$d\mathbf{x}_i(t) = \left( \mathbf{F}(\mathbf{x}_i(t)) + \sum_{k=0}^{m-1} \sum_{j=1}^N c_k G_{ij}^{(k)} \mathbf{\Gamma}_k \mathbf{x}_j(t - \tau_k) + \mathbf{u}_i(t) \right) dt + \sum_{k=1}^{m-1} \varphi_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t - \tau_k)) d\mathbf{w}_i(t) \quad (4)$$

where  $i = 1, 2, \dots, N$ ,  $\mathbf{u}_i(t)$  is the controller imposed on the  $i$ th node. The state of the  $i$ th node  $\mathbf{x}_i(t) = [x_i^1(t), x_i^2(t), \dots, x_i^n(t)]^T$ ,  $\mathbf{F}(\cdot)$  is a nonlinear function. The coupling matrix of the  $k$ th layer  $\mathbf{G}^{(k)} = (G_{ij}^{(k)})_{N \times N}$  ( $k = 0, 1, 2, \dots, m-1$ ) is irreducible and satisfies  $G_{ij}^{(k)} = G_{ji}^{(k)} \geq 0$ ,  $i \neq j$ ,  $G_{ii}^{(k)} = -\sum_{j=1, j \neq i}^N G_{ij}^{(k)}$  for  $i = 1, 2, \dots, n$ . The  $k$ th layer's inner coupling matrix  $\mathbf{\Gamma}_k : \mathbb{R}^{n \times n}$ , the delay  $\tau_0 = 0$ ,  $\tau_k > 0$ ,  $k = 1, 2, \dots, m-1$ , and the intralayer coupling strength  $c_k$  vary with the layer generally. The denotations  $\varphi_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t - \tau_k))$  and  $\mathbf{w}_i(t)$  represents, respectively, the noise intensity matrix and bounded vector-form Brownian motion defined on a complete probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$  in which every two elements is statistically independent, then  $E\{\mathbf{w}_i(t)\} = 0$ ,  $E\{\mathbf{w}_i^T(t)\} = 1$ ,  $E\{\mathbf{w}_i(t)\mathbf{w}_i^T(s)\} = 0$ ,  $s \neq t$ .

Note that if a solution belongs to  $\mathbb{S} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}_1(t) = \mathbf{x}_2(t) = \dots = \mathbf{x}_N(t)\}$  over a time interval  $[t_0, t_0 + \tau_{\max}]$ , where  $\tau_{\max} = \max(\tau_1, \tau_2, \dots, \tau_{m-1})$ , then the solution will belong to  $\mathbb{S}$  for any time  $t > t_0 + \tau_{\max}$ . In this case, the synchronized solution  $\mathbf{x}_1(t) = \mathbf{x}_2(t) = \dots = \mathbf{x}_N(t)$ , denoted as  $\mathbf{s}(t)$ , is characterized by

$$\dot{\mathbf{s}}(t) = \mathbf{F}(\mathbf{s}(t)) \quad (5)$$

where  $\mathbf{s}(t)$  is an equilibrium point, a periodic orbit, or even a chaotic orbit.

Defining  $\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t)$ ,  $\sigma_i^{(k)}(t) = \varphi_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t - \tau_k)) - \varphi_i(t, \mathbf{s}(t), \mathbf{s}(t - \tau_k))$ ,  $1 \leq i \leq N$ ,  $k = 1, 2, \dots, m-1$ , one has

$$d\mathbf{e}_i(t) = (\mathbf{F}(\mathbf{e}_i(t) + \mathbf{s}(t)) - \mathbf{F}(\mathbf{s}(t)) + \sum_{k=0}^{m-1} \sum_{j=1}^N c_k G_{ij}^{(k)} \mathbf{\Gamma}_k \mathbf{e}_j(t - \tau_k) + \mathbf{u}_i(t)) dt + \sum_{k=1}^{m-1} \sigma_i^{(k)}(t) d\mathbf{w}_i(t). \quad (6)$$

To be cost-effective, only a fraction of nodes are controlled to realize synchronization of the multiplex network. Without loss of generality, let the first  $l$  nodes be controlled (if it is not the case, one can rearrange the ordinal of the nodes [19], [33]). Thus the pinning controlled network can be written as

$$d\mathbf{e}_i(t) = \left( \mathbf{F}(\mathbf{e}_i(t) + \mathbf{s}(t)) - \mathbf{F}(\mathbf{s}(t)) + \sum_{k=0}^{m-1} \sum_{j=1}^N c_k G_{ij}^{(k)} \mathbf{\Gamma}_k \mathbf{e}_j(t - \tau_k) + \mathbf{u}_i(t) \right) dt + \sum_{k=1}^{m-1} \sigma_i^{(k)}(t) d\mathbf{w}_i(t), \quad 1 \leq i \leq l$$

$$d\mathbf{e}_i(t) = \left( \mathbf{F}(\mathbf{e}_i(t) + \mathbf{s}(t)) - \mathbf{F}(\mathbf{s}(t)) + \sum_{k=0}^{m-1} \sum_{j=1}^N c_k G_{ij}^{(k)} \mathbf{\Gamma}_k \mathbf{e}_j(t - \tau_k) \right) dt + \sum_{k=1}^{m-1} \sigma_i^{(k)}(t) d\mathbf{w}_i(t), \quad l+1 \leq i \leq N. \quad (7)$$

The objective is to put forward proper pinning controllers  $\mathbf{u}_i$  ( $1 \leq i \leq l$ ) to asymptotically stabilize the trivial solution of error system (7), namely  $\lim_{t \rightarrow \infty} \|\mathbf{e}_i(t)\| = 0$  ( $1 \leq i \leq N$ ).

### IV. MAIN RESULTS

In this section, the synchronization criteria of multiplex network with stochastic noise and transmission delay is presented. To begin with, some hypotheses are introduced.

*Hypothesis 1:* Suppose that there exists a non-negative constant  $\alpha$  such that

$$\|\mathbf{F}(\mathbf{e}_i(t) + \mathbf{s}(t)) - \mathbf{F}(\mathbf{s}(t))\| \leq \alpha \|\mathbf{e}_i(t)\|.$$

*Hypothesis 2:* Suppose that there exist some non-negative constants  $\mu_i^{(k)}$ ,  $\nu_i^{(k)}$  such that for  $i = 1, 2, \dots, N$

$$\text{trace} \left( \left( \sigma_i^{(k)}(t) \right)^T \sigma_i^{(k)}(t) \right) \leq 2\mu_i^{(k)} \mathbf{e}_i^T(t) \mathbf{e}_i(t) + 2\nu_i^{(k)} \mathbf{e}_i^T(t - \tau_k) \mathbf{e}_i(t - \tau_k)$$

hold if  $\sigma_i^{(k)}(t)$ ,  $k = 1, 2, \dots, m-1$  are bounded.

*Hypothesis 3:* Suppose that  $\tau_k(t)$ ,  $k = 1, 2, \dots, m-1$  is a differentiable function with

$$\dot{\tau}_k(t) \leq \delta_k, \quad 0 \leq \delta_k < 1.$$

Under these assumptions, the main results can be deduced.

*Theorem 1:* Assume that Hypotheses 1–3 hold. If there exist positive constants  $l$  and  $r_k$ ,  $k = 1, 2, \dots, m-1$  satisfying

$$(i) \lambda_{\max} \left( \lceil c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} + \sum_{k=1}^{m-1} \kappa_k \left( \hat{\mathbf{G}}^{(k)} \right)^2 \rceil_l \right) < - \left[ \alpha + \sum_{k=1}^{m-1} \left( \mu_k + \frac{1}{2} r_k \right) \right] \\ (ii) \nu_k - \frac{1}{2} r_k (1 - \delta_k) < 0, \quad k = 1, 2, \dots, m-1$$

where  $\nu_k = \max_{1 \leq i \leq N} \{\nu_i^{(k)}\}$ ,  $\mu_k = \max_{1 \leq i \leq N} \{\mu_i^{(k)}\}$ ,  $\kappa_k = -(1/4)c_k^2 \gamma_k [v_k - (1/2)r_k(1 - \delta_k)]^{-1}$ ,  $\hat{\mathbf{G}}^{(k)}$  is the modified matrix of  $\mathbf{G}^{(k)}$  by replacing the diagonal elements  $G_{ii}^{(k)}$  with  $(\gamma'_k / \gamma_k) G_{ii}^{(k)}$ ,  $\gamma_k = \|\mathbf{\Gamma}_k\|$ ,  $\gamma'_k = \lambda_{\min}([\mathbf{\Gamma}_k + \mathbf{\Gamma}_k^T]/2)$ , the zero solution of error system (7) is asymptotically stable via the following pinning controller:

$$\mathbf{u}_i(t) = -d_i(t) \mathbf{e}_i(t) \\ \dot{d}_i(t) = h_i \mathbf{e}_i^T(t) \mathbf{e}_i(t) \quad (8)$$

where  $h_i$  are positive constants,  $i = 1, 2, \dots, l$ .

*Proof:* Consider the following Lyapunov candidate as:

$$V(t) = \frac{1}{2} \sum_{i=1}^N \mathbf{e}_i^T(t) \mathbf{e}_i(t) + \frac{1}{2} \sum_{i=1}^l \frac{1}{h_i} (d_i(t) - d)^2 + \frac{1}{2} \sum_{k=1}^{m-1} r_k \int_{t-\tau_k}^t \sum_{i=1}^N \mathbf{e}_i^T(\theta) \mathbf{e}_i(\theta) d\theta$$

where  $d > 0$  and  $r_k > 0, k = 1, 2, \dots, m - 1$  are some constants. One gains

$$\begin{aligned} \mathcal{L}V &= \sum_{i=1}^N \mathbf{e}_i^T(t) [\mathbf{F}(\mathbf{e}_i(t) + \mathbf{s}(t)) - \mathbf{F}(\mathbf{s}(t))] \\ &+ c_0 \sum_{j=1}^N G_{ij}^{(0)} \Gamma_0 \mathbf{e}_j(t) + \sum_{k=1}^{m-1} \sum_{j=1}^N c_k G_{ij}^{(k)} \Gamma_k \mathbf{e}_j(t - \tau_k) \\ &- d_i(t) \mathbf{e}_i(t) + \sum_{i=1}^l \frac{1}{h_i} (d_i(t) - d) \dot{d}_i \\ &+ \frac{1}{2} \sum_{k=1}^{m-1} \sum_{i=1}^N \text{trace} \left( \left( \sigma_i^{(k)} \right)^T \sigma_i^{(k)} \right) \\ &+ \frac{1}{2} \sum_{k=1}^{m-1} \sum_{i=1}^N r_k \mathbf{e}_i^T(t) \mathbf{e}_i(t) \\ &- \frac{1}{2} \sum_{k=1}^{m-1} \sum_{i=1}^N r_k (1 - \dot{\tau}_k(t)) \mathbf{e}_i^T(t - \tau_k) \mathbf{e}_i(t - \tau_k) \\ &= \sum_{i=1}^N \mathbf{e}_i^T(t) [\mathbf{F}(\mathbf{e}_i(t) + \mathbf{s}(t)) - \mathbf{F}(\mathbf{s}(t))] \\ &+ c_0 \sum_{j=1}^N \mathbf{e}_i^T(t) G_{ij}^{(0)} \Gamma_0 \mathbf{e}_j(t) \\ &+ \sum_{k=1}^{m-1} \sum_{j=1}^N c_k \mathbf{e}_i^T(t) G_{ij}^{(k)} \Gamma_k \mathbf{e}_j(t - \tau_k) \\ &- \sum_{i=1}^l d \mathbf{e}_i^T(t) \mathbf{e}_i(t) + \frac{1}{2} \sum_{k=1}^{m-1} \sum_{i=1}^N \text{trace} \left( \left( \sigma_i^{(k)} \right)^T \sigma_i^{(k)} \right) \\ &+ \frac{1}{2} \sum_{k=1}^{m-1} \sum_{i=1}^N r_k \mathbf{e}_i^T(t) \mathbf{e}_i(t) \\ &- \frac{1}{2} \sum_{k=1}^{m-1} \sum_{i=1}^N r_k (1 - \dot{\tau}_k(t)) \mathbf{e}_i^T(t - \tau_k) \mathbf{e}_i(t - \tau_k) \\ &\leq \sum_{i=1}^N \alpha \mathbf{e}_i^T(t) \mathbf{e}_i(t) - \sum_{i=1}^l d \mathbf{e}_i^T(t) \mathbf{e}_i(t) \\ &+ c_0 \sum_{i=1}^N \sum_{j=1, j \neq i}^N \gamma_0 G_{ij}^{(0)} \|\mathbf{e}_i(t)\| \|\mathbf{e}_j(t)\| \\ &+ c_0 \sum_{i=1}^N \gamma_0' G_{ii}^{(0)} \mathbf{e}_i^T(t) \mathbf{e}_i(t) \\ &+ \sum_{k=1}^{m-1} \sum_{i=1}^N \sum_{j=1, j \neq i}^N c_k \gamma_k G_{ij}^{(k)} \|\mathbf{e}_i(t)\| \|\mathbf{e}_j(t - \tau_k)\| \end{aligned}$$

$$\begin{aligned} &+ \sum_{k=1}^{m-1} \sum_{i=1}^N c_k \gamma_k' G_{ii}^{(k)} \mathbf{e}_i^T(t) \mathbf{e}_i(t - \tau_k) \\ &+ \sum_{k=1}^{m-1} \sum_{i=1}^N \mu_i^{(k)} \mathbf{e}_i^T(t) \mathbf{e}_i(t) \\ &+ \sum_{k=1}^{m-1} \sum_{i=1}^N \nu_i^{(k)} \mathbf{e}_i^T(t - \tau_k) \mathbf{e}_i(t - \tau_k) \\ &+ \frac{1}{2} \sum_{k=1}^{m-1} \sum_{i=1}^N r_k \mathbf{e}_i^T(t) \mathbf{e}_i(t) \\ &- \frac{1}{2} \sum_{k=1}^{m-1} \sum_{i=1}^N r_k (1 - \delta_k) \mathbf{e}_i^T(t - \tau_k) \mathbf{e}_i(t - \tau_k). \end{aligned}$$

For simplicity, denote

$$\mathbf{e}(t) = (\|\mathbf{e}_1(t)\|, \|\mathbf{e}_2(t)\|, \dots, \|\mathbf{e}_N(t)\|)^T$$

$$\mathbf{D} = \text{diag} \left\{ \underbrace{d, d, \dots, d}_l, \underbrace{0, \dots, 0}_{N-l} \right\}$$

and  $\tilde{\mathbf{x}} = (\mathbf{e}^T(t), \mathbf{e}^T(t - \tau_1), \dots, \mathbf{e}^T(t - \tau_{m-1}))^T$ .

The inequality is transformed into

$$\begin{aligned} \mathcal{L}V &\leq \mathbf{e}^T(t) \left[ \left( \alpha + \sum_{k=1}^{m-1} \mu_k + \frac{1}{2} \sum_{k=1}^{m-1} r_k \right) \mathbf{I}_N \right. \\ &\quad \left. + c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} - \mathbf{D} \right] \mathbf{e}(t) \\ &+ \sum_{k=1}^{m-1} c_k \gamma_k \mathbf{e}^T(t) \hat{\mathbf{G}}^{(k)} \mathbf{e}(t - \tau_k) \\ &+ \sum_{k=1}^{m-1} \left[ \nu_k - \frac{1}{2} r_k (1 - \delta_k) \right] \mathbf{e}^T(t - \tau_k) \mathbf{e}(t - \tau_k) \\ &= (\mathbf{e}^T(t), \mathbf{e}^T(t - \tau_1), \dots, \mathbf{e}^T(t - \tau_{m-1})) \Xi \\ &\quad (\mathbf{e}^T(t), \mathbf{e}^T(t - \tau_1), \dots, \mathbf{e}^T(t - \tau_{m-1}))^T \\ &= \tilde{\mathbf{x}}^T \Xi \tilde{\mathbf{x}} \end{aligned}$$

where

$$\Xi = \begin{pmatrix} \mathcal{A}(x), \mathcal{B}(x) \\ \mathcal{B}^T(x), \mathcal{C}(x) \end{pmatrix}$$

$$\mathcal{A}(x) = \left( \alpha + \sum_{k=1}^{m-1} \mu_k + \frac{1}{2} \sum_{k=1}^{m-1} r_k \right) \mathbf{I}_N + c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} - \mathbf{D}$$

$$\mathcal{B}(x) = \left( \frac{1}{2} c_1 \gamma_1 \hat{\mathbf{G}}^{(1)}, \frac{1}{2} c_2 \gamma_2 \hat{\mathbf{G}}^{(2)}, \dots, \frac{1}{2} c_{m-1} \gamma_{m-1} \hat{\mathbf{G}}^{(m-1)} \right)$$

and

$$\mathcal{C}(x) = \text{diag} \left\{ \nu_1 - \frac{1}{2} r_1 (1 - \delta_1), \dots, \nu_{m-1} - \frac{1}{2} r_{m-1} (1 - \delta_{m-1}) \right\}.$$

Since  $\lambda_{\max}(\lceil (c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} + \sum_{k=1}^{m-1} \kappa_k (\hat{\mathbf{G}}^{(k)})^2 \rceil) \rceil) < -[\alpha + \sum_{k=1}^{m-1} (\mu_k + (1/2)r_k)]$ ,

according to Lemma 4, one has

$$\begin{aligned} & A(x) - \mathcal{B}^T(x)\mathcal{C}(x)^{-1}\mathcal{B}(x) \\ &= \left( \alpha + \sum_{k=1}^{m-1} \left( \mu_k + \frac{1}{2} r_k \right) \right) \mathbf{I}_N \\ &+ c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} + \sum_{k=1}^{m-1} \kappa_k \left( \hat{\mathbf{G}}^{(k)} \right)^2 - \mathbf{D} < 0 \end{aligned}$$

when  $d$  is large enough. On the other hand,  $v_k - (1/2) r(1 - \delta) < 0$  brings about  $\mathcal{C}(x) < 0$ . According to Lemma 3, one gets  $\Xi < 0$ . And then  $\mathcal{L}V \leq \lambda_{\max}(\Xi) \tilde{\mathbf{x}}^T \tilde{\mathbf{x}} \triangleq -\omega(\tilde{\mathbf{x}})$ . The bounded  $\sigma_i$  as well as  $\lim_{\|\mathbf{x}\| \rightarrow \infty} \inf_{0 \leq t < \infty} V = \infty$  lead to the finite  $\lim_{t \rightarrow \infty} V(\mathbf{x}(t; x_0), t)$  a.s. and  $\lim_{t \rightarrow \infty} \omega(\tilde{\mathbf{x}}) = 0$  a.s. based on Lemma 1. This means that the synchronized solutions of multiplex network (4) is asymptotically stable. This completes the proof. ■

*Remark 1:* The matrices  $\mathbf{G}^{(k)}$ ,  $k = 0, 1, \dots, m$  discussed in Theorem 1 are symmetric. Actually if  $\mathbf{G}^{(k)}$  is asymmetric, one can replace  $\mathbf{G}^{(k)}$  with its symmetric part  $([\mathbf{G}^{(k)}]^T + \mathbf{G}^{(k)})/2$ .

*Remark 2:* We use  $\gamma_k = \|\Gamma_k\|$  and  $\gamma'_k = \lambda_{\min}([\Gamma_k + \Gamma_k^T]/2)$  instead of  $\gamma_k = \lambda_{\max}(\Gamma_k)$  and  $\gamma'_k = \lambda_{\min}(\Gamma_k)$  for considering the asymmetry of matrix  $\Gamma_k$ .

*Remark 3:* Theorem 1 reveals that the number of pinned nodes depends on  $\mu_k$ ,  $v_k$ ,  $\delta_k$ , and  $c_k$ ,  $k = 0, 1, \dots, m-1$ . That is, each layer's noise perturbation, time delay, and coupling strength influence the number of pinned nodes.

Theorem 1 tells that when there are noise perturbations and transmission delays in all layers, the whole multiplex network will achieve synchronization with some rather simple pinning strategies if Hypotheses 1–3 hold.

In some cases, not all the layers have stochastic perturbations in a multiplex network. Say, there exist stochastic perturbations only in layer 1 to layer  $m_0$  ( $1 \leq m_0 \leq m$ ) in a  $m$ -layer complex network. By selecting the Lyapunov candidate as

$$\begin{aligned} V(t) &= \frac{1}{2} \sum_{i=1}^N \mathbf{e}_i^T(t) \mathbf{e}_i(t) + \frac{1}{2} \sum_{k'=m_0+1}^m \sum_{i=1}^N \sum_{j=1}^N \frac{1}{\delta_{ij}^{(k')}} \left( G_{ij}^{(k')} \right)^2 \\ &+ \frac{1}{2} \sum_{i=1}^l \frac{1}{h_i} (d_i(t) - d^*)^2 \\ &+ \frac{1}{2} \sum_{k=1}^{m_0-1} r_k \int_{t-\tau_k(t)}^t \sum_{i=1}^N \mathbf{e}_i^T(\theta) \mathbf{e}_i(\theta) d\theta \end{aligned}$$

where  $d^* > 0$ ,  $r_k > 0$ ,  $k = 1, 2, \dots, m_0 - 1$ , the synchronization of the multiplex network will be realized with the following controller (9).

*Corollary 1:* Assume that layer 1 to layer  $m_0$  ( $1 \leq m_0 \leq m$ ) in an  $m$ -layer complex network have stochastic perturbations. Provided with Hypotheses 1–3, if there exist positive constants  $l$  and  $r_k$ ,  $k = 1, 2, \dots, m_0 - 1$  such that

$$(i) \lambda_{\max} \left( \lceil c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} + \sum_{k=1}^{m_0-1} \kappa_k \left( \hat{\mathbf{G}}^{(k)} \right)^2 \rceil_l \right)$$

$$< - \left[ \alpha + \sum_{k=1}^{m_0-1} \left( \mu_k + \frac{1}{2} r_k \right) \right]$$

$$(ii) v_k - \frac{1}{2} r_k (1 - \delta_k) < 0, k = 1, 2, \dots, m_0 - 1$$

synchronization manifold  $\mathbb{S}$  is asymptotically stable via the following pinning controller and updating laws:

$$\begin{aligned} \mathbf{u}_i(t) &= -d_i(t) \mathbf{e}_i(t) \\ \dot{d}_i(t) &= h_i \mathbf{e}_i^T(t) \mathbf{e}_i(t), \quad i = 1, 2, \dots, l \\ \dot{G}_{ij}^{(k')} &= -\delta_{ij}^{(k')} c_{k'} \mathbf{e}_i^T \Gamma_{k'} \mathbf{e}_j(t - \tau_{k'}), \quad i, j = 1, 2, \dots, N \\ k' &= m_0 + 1, \dots, m \end{aligned} \quad (9)$$

where  $h_i > 0$ ,  $\delta_{ij}^{(k')} > 0$  are arbitrary constants.

In the case that no perturbation exists in the multiplex network, applying the Lyapunov candidate which is the same as that in Theorem 1, one gains the following corollary.

*Corollary 2:* Suppose that Hypotheses 1 and 3 hold. For a multiplex network with time delay, if there exist positive constants  $l$  and  $r_k$ ,  $k = 1, 2, \dots, m-1$  such that  $\lambda_{\max}(\lceil c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} + \sum_{k=1}^{m-1} \kappa'_k \left( \hat{\mathbf{G}}^{(k)} \right)^2 \rceil_l) < -(\alpha + [1/2] \sum_{k=1}^{m-1} r_k)$ , where  $\kappa'_k = (1/4) c_k^2 \gamma_k [(1/2) r_k (1 - \delta_k)]^{-1}$ ,  $k = 1, 2, \dots, m-1$ , synchronization manifold  $\mathbb{S}$  is asymptotically stable via pinning controller (8).

Ideally, a multiplex network is in the absence of stochastic perturbation or transmission delay. One gains Corollary 3 by constructing Lyapunov candidate as

$$V(t) = \frac{1}{2} \sum_{i=1}^N \mathbf{e}_i^T(t) \mathbf{e}_i(t) + \frac{1}{2} \sum_{i=1}^l \frac{1}{h_i} (d_i(t) - d')^2$$

where  $d' > 0$ .

*Corollary 3:* Suppose that Hypothesis 1 hold. For a multiplex network without perturbation or time delay, if there exists a positive constant  $l$  such that  $\lambda_{\max}(\lceil c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} + \sum_{k=1}^{m-1} \kappa''_k \left( \hat{\mathbf{G}}^{(k)} \right)^2 \rceil_l) < -\alpha$  holds, where  $\kappa''_k = (1/4) c_k^2 \gamma_k$ ,  $k = 1, 2, \dots, m-1$ , synchronization manifold  $\mathbb{S}$  is asymptotically stable via pinning controller (8).

## V. FURTHER DISCUSSION

According to our main results, if there exist proper constants  $l$  and  $r_k$ ,  $k = 1, 2, \dots, m-1$  such that equation (i), (ii) in Theorem 1 hold, the  $m$ -layer network (4) will achieve pinning synchronization by applying our adaptive controller. Consider a multiplex network consisting of 100 Rössler chaotic oscillators. The Rössler oscillator is described by

$$\begin{aligned} F(\mathbf{x}) &= \begin{pmatrix} -x_2 - x_3 \\ x_1 + 0.2x_2 \\ 0.2 + (x_1 - 7)x_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0.2 + x_1 x_3 \end{pmatrix} \end{aligned} \quad (10)$$

with the phase depicted in Fig. 2. By simple calculation, one gets  $\alpha = 43.4730$  in Hypothesis 1.

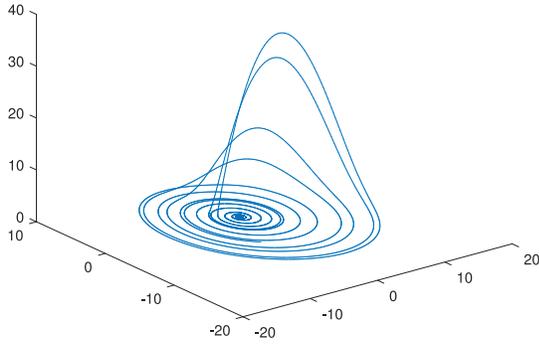


Fig. 2. Phase of the Rössler chaotic oscillator (10).

Let  $\sigma_i(t, \mathbf{e}_i(t), \mathbf{e}_i(t - \tau_k)) = \sigma_0^{(k)} \text{diag}\{e_{i1}(t) - e_{i1}(t - \tau_k), e_{i2}(t) - e_{i2}(t - \tau_k), e_{i3}(t) - e_{i3}(t - \tau_k)\}$  that satisfies the Lipschitz condition and the linear growth condition, where  $k = 1, 2, \dots, m - 1$ . Then  $\text{trace}(\sigma_i^T \sigma_i) \leq 2(\sigma_0^{(k)})^2 \mathbf{e}_i^T(t) \mathbf{e}_i(t) + 2(\sigma_0^{(k)})^2 \mathbf{e}_i^T(t - \tau_k) \mathbf{e}_i(t - \tau_k)$  and  $\sigma_i$  satisfies Hypothesis 2. It is obvious that  $\mu_k = \nu_k = (\sigma_0^{(k)})^2$ . In the simulations, the initial values of all the variables and the stochastic perturbations are randomly assigned from  $[0, 1]$ .

In the following section, numerical simulations are run in order to assess the impact of the three major factors on the number of pinned nodes  $l$ : the noise perturbations' related factor  $\mu_k = \nu_k$ , the time delays' related factor  $\delta_k$ , and the intralayer coupling strength  $c_k$ . Suppose there exist proper parameters  $r_k, k = 1, 2, \dots, m - 1$  that satisfy (ii). Let

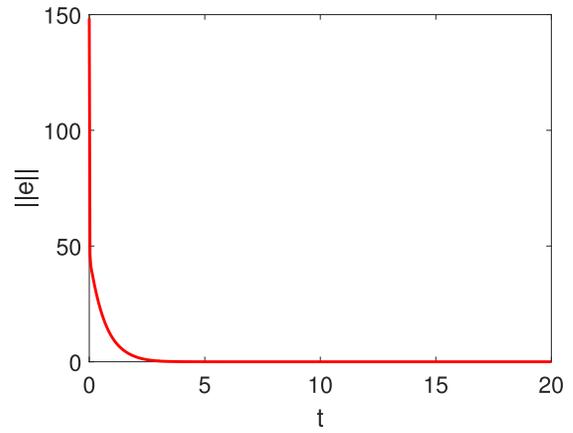
$$H = \lambda_{\max} \left( \lceil c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} + \sum_{k=1}^{m-1} \kappa_k (\hat{\mathbf{G}}^{(k)})^2 \rceil_l \right) + \left[ \alpha + \sum_{k=1}^{m-1} \left( \mu_k + \frac{1}{2} r_k \right) \right]$$

where the parameters have the same meanings as those in Theorem 1. Since  $H < 0$  is equivalent to (i), the following discussion focuses on  $H < 0$ .

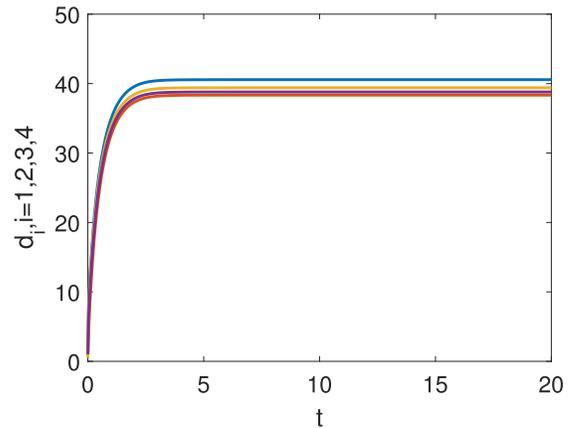
A. Further Discussion About Two-Layer NW-BA Network

In this section, a two-layer network which is coupled with a Newman–Watts (NW) small world network [34] and a Barabasi–Albert (BA) scale-free network [35]. In the NW network, the original  $K$ -nearest-neighbor network and the adding connections probability  $p$  are taken as  $K = 28$  and  $p = 0.1$ . In the BA network, the initial network size  $b_0$  and the average degree  $b$  are chosen as  $b_0 = b = 6$ . Assume that the parameters are selected as  $\mathbf{\Gamma}_0 = \text{diag}\{1, 1, 1\}$ ,  $\mathbf{\Gamma}_1 = \text{diag}\{1, 1, 1\}$ ,  $\tau_0 = 0$ ,  $\tau_1 = 0.1$ ,  $c_0 = 0.85$ ,  $c_1 = 0.3$  and  $\sigma_0^{(0)} = 0, \sigma_0^{(1)} = \sqrt{0.2}$ . It is seen that  $\mu_0 = \nu_0 = 0, \mu_1 = \nu_1 = 0.2, \delta_0 = \delta_1 = 0$ .

Since there exist constants  $l = 4$  and  $r_1 = 10$  such that  $\lambda_{\max}(\lceil c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} + \kappa_1 (\hat{\mathbf{G}}^{(1)})^2 \rceil_4) = -49.0545 < -48.6730 = -(\alpha + \mu_1 + [1/2]r_1)$  and  $\nu_1 - (1/2)r_1(1 - \delta_1) = -4.8 < 0$ , the four nodes with the highest degree should be pinned to



(a)



(b)

Fig. 3. (a) Synchronization error  $\|\mathbf{e}\|$  versus time  $t$ . It is seen that by applying the proposed pinning controller (8), the NW-BA network achieves synchronization briefly. (b) Adaptive feedback gains  $d_i, i = 1, 2, 3, 4$  for pinning the first four nodes. It is clear that there exists an upper bound of the feedback gains.

reach synchronization by adaptive controller (8) according to Theorem 1.

Fig. 3(a) shows the effectiveness of the proposed algorithm, where the synchronization error tends to zero briefly. Fig. 3(b) presents the adaptive feedback gains for pinning the first four nodes, which has an upper bound. Besides, we also get that the smaller the number of the pinned nodes, the larger the value of the feedback gains. It is seen that the main result is effective in a two-layer NW-BA network.

Fig. 4 displays the number of pinned nodes  $l$  versus the varying interval of the three parameters. The color depth in figure represents the value of  $H$ . The vertical axis shows the varying interval of the three parameters. The interval of noise factors, delay factors, and intralayer coupling strengths increases with  $\epsilon_\nu, \epsilon_\delta$ , and  $\epsilon_c$ , respectively. Fig. 4(a) shows the interval of the two layers' noise factors  $\nu_1 - \nu_0$  versus the number of pinned nodes  $l$ . The varying interval of noise factors are  $\nu_0 = 0, \nu_1 = 0 + 0.2 \cdot \epsilon_\nu, \epsilon_\nu = 1, 2, \dots, 20$ . It is seen that the noise difference between the two layers increases with  $\epsilon_\nu$ . From Fig. 4(a), it can be seen that the larger the interval of noise, the larger the number of

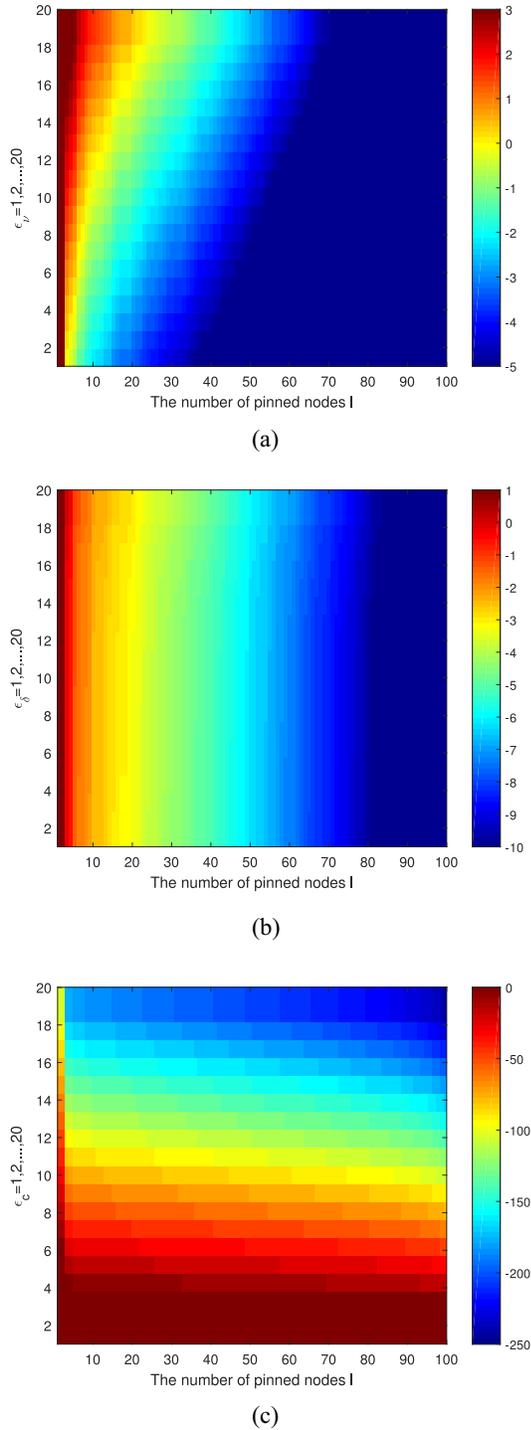


Fig. 4. (a) Vertical axis shows the varying interval of noise factors increases with  $\epsilon_v$  versus the number of pinned nodes  $l$ . The interval of the two layers' noise factors is  $v_1 - v_0$ , where  $v_0 = 0$ ,  $v_1 = 0 + 0.2 \cdot \epsilon_v$ ,  $\epsilon_v = 1, 2, \dots, 20$ . (b) Vertical axis shows the varying interval of delay factors increases with  $\epsilon_\delta$  versus the number of pinned nodes  $l$ . The interval of the two layers' time delay factors is  $\delta_1 - \delta_0$ , where  $\delta_0 = 0$ ,  $\delta_1 = 0 + 0.045 \cdot \epsilon_\delta$ ,  $\epsilon_\delta = 1, 2, \dots, 20$ . (c) Vertical axis shows the varying interval of intralayer coupling strengths increases with  $\epsilon_c$  versus the number of pinned nodes  $l$ . The interval of the number of two layers' intralayer coupling strengths is  $c_0 - c_1$ , where  $c_1 = 0.1$ ,  $c_0 = 0.1 + 0.2 \cdot \epsilon_c$ ,  $\epsilon_c = 1, 2, \dots, 20$ .

pinned nodes. Fig. 4(b) indicates the interval of the two layers' time delay factors  $\delta_1 - \delta_0$  versus the number of pinned nodes  $l$ . The varying interval of delay factors are set as

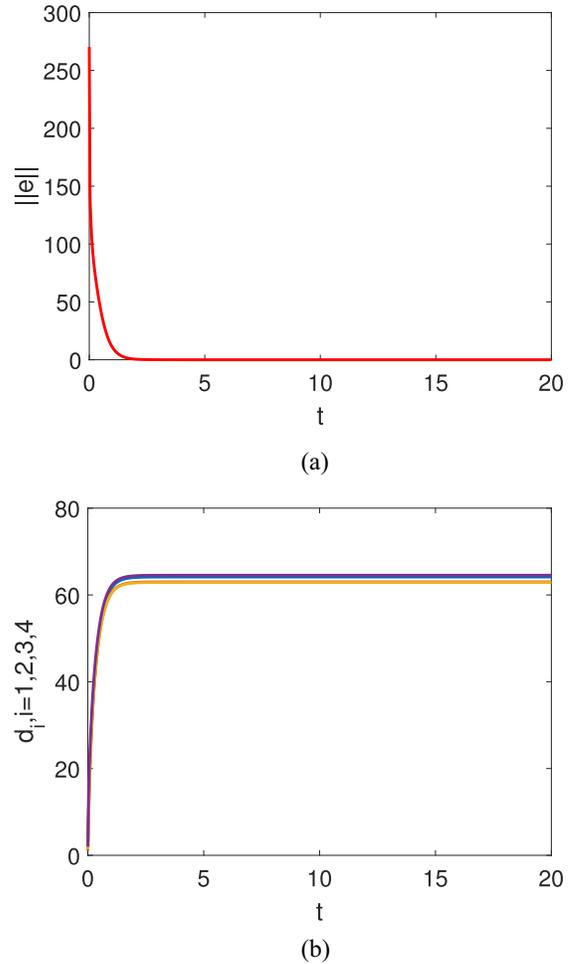


Fig. 5. (a) Synchronization error  $\|e\|$  versus time  $t$ . It is shown that by applying the proposed pinning controller (8), the NW-BA-FC network reaches adaptive synchronization. (b) Adaptive feedback gains  $d_i$ ,  $i = 1, 2, 3, 4$  for pinning the first four nodes. It is clear that there exists an upper bound of the feedback gains.

$\delta_0 = 0$ ,  $\delta_1 = 0 + 0.045 \cdot \epsilon_\delta$ ,  $\epsilon_\delta = 1, 2, \dots, 20$ . The larger the  $\delta_k$ , the faster the time delay changes. It is clear that when the difference of  $\delta_k$  between the two layers is larger, more nodes need to be pinned. Fig. 4(c) represents the interval of the two layers' intralayer coupling strengths  $c_0 - c_1$  versus the number of pinned nodes  $l$ . The varying interval of intralayer coupling strengths are  $c_1 = 0.1$ ,  $c_0 = 0.1 + 0.2 \cdot \epsilon_c$ ,  $\epsilon_c = 1, 2, \dots, 20$ . From Fig. 4(c), the larger the interval of the two layers' intralayer coupling strength, the smaller the number of pinned nodes.

In a word, the smaller the interval of intralayer coupling strength, the larger the interval of noise and the interval of time delay changes, the larger the number of pinned nodes.

### B. Further Discussion About Three-Layer NW-BA-FC Network

Here, we consider a three-layer network that is consisting of an NW small world network, a BA network, and a fully connected (FC) network. The parameters in the NW-BA-FC

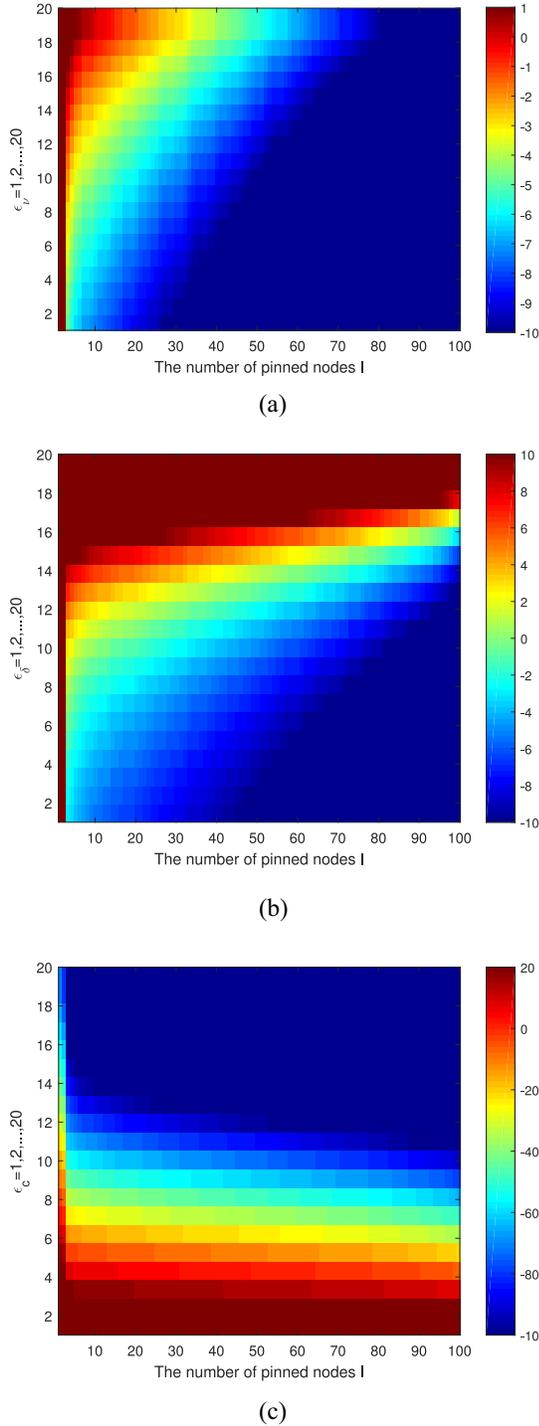


Fig. 6. (a) Vertical axis shows the varying interval of noise factors increases with  $\epsilon_v$  versus the number of pinned nodes  $l$ . The interval of the three layers' noise factors is  $v_2 - v_1 - v_0$ , where  $v_0 = 0, v_1 = 1, v_2 = 1 + 0.1 \cdot \epsilon_v, \epsilon_v = 1, 2, \dots, 20$ . (b) Vertical axis shows the varying interval of delay factors increases with  $\epsilon_\delta$  versus the number of pinned nodes  $l$ . The interval of the three layers' time delay factors is  $\delta_2 - \delta_1 - \delta_0$ , where  $\delta_0 = 0, \delta_1 = 0.1, \delta_2 = 0.1 + 0.03 \cdot \epsilon_\delta, \epsilon_\delta = 1, 2, \dots, 20$ . (c) Vertical axis shows the varying interval of intralayer coupling strengths increases with  $\epsilon_c$  versus the number of pinned nodes  $l$ . The interval of the number of three layers' intralayer coupling strengths is  $c_0 - c_1 - c_2$ , where  $c_2 = 0.1, c_1 = 0.1 + 0.1 \cdot \epsilon_c, c_0 = 0.1 + 0.2 \cdot \epsilon_c, \epsilon_c = 1, 2, \dots, 20$ .

network are set as  $\Gamma_0 = \text{diag}\{1, 1, 1\}$ ,  $\Gamma_1 = \text{diag}\{1, 1, 1\}$ ,  $\Gamma_2 = \text{diag}\{1, 1, 1\}$ ,  $\tau_0 = 0$ ,  $\tau_1 = 0.1$ ,  $\tau_2 = 0.01$ ,  $c_0 = 1.1$ ,  $c_1 = 0.3$ ,  $c_2 = 0.1$  and  $\sigma_0^{(0)} = 0$ ,  $\sigma_0^{(1)} = 1$ ,  $\sigma_0^{(2)} = \sqrt{1.1}$ .

By simple calculation, we have  $\delta_0 = \delta_1 = \delta_2 = 0$ ,  $\mu_0 = v_0 = 0$ ,  $\mu_1 = v_1 = 1$ ,  $\mu_2 = v_2 = 1.1$ . Letting  $r_1 = 3$ ,  $r_2 = 10$ , we obtain  $\lambda_{\max}(\lceil c_0 \gamma_0 \hat{\mathbf{G}}^{(0)} + \sum_{k=1}^2 \kappa_k (\hat{\mathbf{G}}^{(k)})^2 \rceil_4) = -56.2309 < -52.0730 = (-\alpha + \sum_{k=1}^2 (\mu_k + [1/2]r_k))$  and  $v_1 = 1 < 1.5 = (1/2)r_1(1 - \delta_1)$ ,  $v_2 = 1.1 < 5 = (1/2)r_2(1 - \delta_2)$ . Based on Theorem 1, the four nodes with the highest degree should be pinned.

Fig. 5 shows the synchronization error of the NW-BA-FC network and the adaptive feedback gain. Similar to the previous section, it is seen that the proposed controller is effective to the NW-BA-FC network, where the adaptive feedback gains have an upper bound.

Fig. 6 displays the number of pinned nodes  $l$  versus the varying interval of the three layers' network parameters. In Fig. 6(a), the varying interval of noise factors are  $v_0 = 0, v_1 = 1, v_2 = 1 + 0.1 \cdot \epsilon_v, \epsilon_v = 1, 2, \dots, 20$ . It is noticed that the larger the interval of noise, the larger the number of pinned nodes. In Fig. 6(b), the varying interval of delay factors are set as  $\delta_0 = 0, \delta_1 = 0.1, \delta_2 = 0.1 + 0.03 \cdot \epsilon_\delta, \epsilon_\delta = 1, 2, \dots, 20$ . When the interval of  $\delta_k$  is larger, more nodes need to be pinned. In Fig. 6(c), the interval of intralayer coupling strengths are  $c_2 = 0.1, c_1 = 0.1 + 0.1 \cdot \epsilon_c, c_0 = 0.1 + 0.2 \cdot \epsilon_c, \epsilon_c = 1, 2, \dots, 20$ . One see that the smaller the interval of intralayer coupling strength, the larger the number of pinned nodes.

Similar to the results in Section V-A, the number of pinned nodes increases with the interval of noise and the interval of time delay, decreases with the interval of intralayer coupling strength.

## VI. CONCLUSION

In this paper, pinning synchronization of a multiplex network with multiple intralayer coupling delays and stochastic perturbations has been investigated. Based on the LaSalle-type invariance principle for stochastic differential equations and the Lyapunov stability theory, synchronization criteria with controlling only a fraction of nodes are proposed. It is concluded that if there exist proper constants  $r_k$  and  $d$  which satisfy a mild condition, the corresponding  $l$  nodes can be controlled for synchronization. Moreover, the smaller the interval of intralayer coupling strength, the larger the interval of noise and the interval of time delay changes, the larger the number of pinned nodes. Some two-layer and three-layer networks have been employed to validate the effectiveness of the proposed algorithm. Future investigation includes identifying the most efficient nodes in a multiplex network, topology estimation of a multiplex network, and so on.

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