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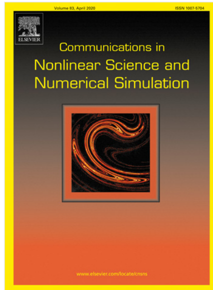

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Identifying partial topology of complex networks with stochastic perturbations and time delay

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Abstract

In various practical scenarios, the topology of a complex network is generally unknown or unavailable. Based on the adaptive feedback control strategy and the pining mechanism, some controllers and observers are designed in this manuscript to identify the partial topology of the noise-contaminated networks with coupling delay. The approach can be extended to infer the whole topology of the networks. For large-scale complex networks, our approach greatly reduces the calculation compared with the previous methods. Numerical simulations are employed to illustrate the effectiveness of the proposed theoretical results.

Keywords: Complex network, topology identification, stochastic perturbations, coupling delay, adaptive feedback control

1. Introduction

Complex networks have played an increasing role in scientific and engineering fields [1, 2], such as computer science, social sciences, and biological sciences. With abundant achievements in the synchronization of complex networks, many theoretical results have been proposed [3, 4], including cluster synchronization [5], generalized synchronization [6], projective synchronization [7]. The spontaneous coordinate behavior of the complex networks is dramatically affected by the network structure in coupling dynamical systems [8]. Most research on synchronization based on the known network topology. However, for the real networks, their topology structures are often unknown or partially unavailable. The network structure is a prerequisite for analyzing the evolution mechanism and functional behavior of the complex network. Therefore, identifying network topology has become a research topic of immense interest.

In recent years, the research of identifying topology of complex networks has grown steadily. For example, Wu *et al.* [9] employed the adaptive feedback controller to infer the whole structure of the noise-contaminated networks. Yan *et al.* [10] investigated the complex-variable network structures by using the technique of adaptive control scheme. The research of finite-time topological identification

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was considered in [11]. The network structure has been studied from different angles and a variety of network models were proposed, such as the weighted networks [12], the networks with uncertain system parameters [13], and the networks with time delay [14]. However, the above-mentioned researches mainly concentrated on constructing the auxiliary network that has the identical size of nodes with the original network. Recent work has tried to identify the partial configuration of the original network by using fewer nodes and observers. Zhu *et al.* [15] used a pining mechanism to infer the partial configuration of the network, and significantly reduced the size of nodes in the auxiliary network.

In some engineering fields, inferring the partial topology of networks has practical application background. For instance, when a massive power grid malfunctions, it is necessary to find out the breakdown location quickly. If the whole power grid structure is checked, the cost of time and economy will be greatly increased. According to the information acquired, one may need to inspect a part of the power network structure. Similarly, for other practical complex networks such as epidemic spreading networks, we only concern about the people contacted by the newly infected person rather than the entire interpersonal system. In fact, there are many factors that affect the structure of the complex network, of which noise is not negligible. Noise is ubiquitous because of external interference [16], frequently arising in biological system such as neural networks. In addition, due to the limitation of information processing and transmission speed [17], time delay is unavoidable. This is another inevitabl factor that affects the network structure. As a result, the study of complex dynamical networks with stochastic noise and time delay has critical practical significance and representativeness. To our knowledge, few studies focused on the identifying partial topology method and the existing works were not considered the impact of external environmental interference and coupling delay.

Motivated by the aforementioned discussions, a realistic and representative method is proposed. Based on the stochastic perturbations and coupling delay in the network, the adaptive feedback control and the pining approach are proposed to infer the partial topology of the complex network. There are two main contributions in our study. First, different from the previous work, to infer the partial structures of the original network, the proposed controllers and observers employ pinning mechanism and delay-based strategy. Second, for handling the same large-scale networks, our theoretical findings provide an approach to reduce the calculation and the computing memory space compared with the previous schemes [9, 14].

The manuscript is organized as follows. Notations and properties of the stochastic differential equations with delay are introduced in Section 2. In Section 3, the partial topology identification for noised-perturbed complex networks is explored. In Section 4, several numerical simulations are utilized to verify the effectiveness of the theoretical findings. Finally, Section 5 summarizes the discussions and conclusions.

2. Preliminaries

Let $\|\cdot\|$ be the Euclidean norm for a vector, and $\otimes$ the Kronecker product. The transpose of a matrix $\mathbf{A}$ is introduced by $\mathbf{A}^T$. $\lambda_{max}(\mathbf{A})$ stands for the maximum eigenvalue of matrix $\mathbf{A}$. $b_1 \vee b_2$ means the maximum of b_1 and b_2. $L^1(\mathbb{R}_+;\mathbb{R}_+)$ denotes the family of functions $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty \vartheta(t)dt < \infty$. $(\mathfrak{D}, \mathfrak{F}, \mathcal{P})$ is a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ possessing $\mathfrak{F}_t = \cap_{l>t}\mathfrak{F}_l$ for all $t \geq 0$ and $\mathfrak{F}_0$ contains all $\mathcal{P}$-null sets [18], and $\mathbb{W}(t) = (\mathbb{W}_1(t), \mathbb{W}_2(t), \ldots, \mathbb{W}_m(t))^T$ denotes an m-dimensional Brownian motion. $C^b_{\mathfrak{F}_0}([-\tau, 0]; \mathbb{R}^n)$ stands for the space of $\mathfrak{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-value bounded random variables z.

Consider an n-dimensional stochastic differential equation affected by time delay

$$\mathrm{d}\mathbf{z}(t) = \Phi(t, \mathbf{z}(t), \mathbf{z}(t-\tau))\mathrm{d}t + \varpi(t, \mathbf{z}(t), \mathbf{z}(t-\tau))\mathrm{d}\mathbb{W}(t) \tag{1}$$

on $t \geq 0$ with initial condition $\mathbf{z}(\theta) : -\tau < \theta < 0 = z_0 \in C^b_{\mathfrak{F}_0}([-\tau, 0]; \mathbb{R}^n)$, where $\mathrm{d}\mathbb{W}(t)$ represents an increment covariance of $I_{m\times m}\mathrm{d}t$, both $\Phi(t, \mathbf{z}) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\varpi(t, \mathbf{z}) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable functions. A hypothesis is proposed to ensure that there exists a unique solution of Eq. (1).

Hypothesis 1 (H1). Both Φ and ϖ are provided with the local Lipschitz condition and the linear growth condition. For each positive integer $k = 1, 2, \cdots$, there is a $b_k > 0$ such that

$$||\Phi(t, \mathbf{z}_1, \mathbf{z}_2) - \Phi(t, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2)|| \vee ||\varpi(t, \mathbf{z}_1, \mathbf{z}_2) - \varpi(t, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2)|| \leq b_k(||\mathbf{z}_1 - \bar{\mathbf{z}}_2|| + ||\mathbf{z}_1 - \bar{\mathbf{z}}_2||)$$

for all $t \geq 0$ and $\mathbf{z}_1, \mathbf{z}_2, \bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2 \in \mathbb{R}^n$ with $||\mathbf{z}_1|| \vee ||\mathbf{z}_2|| \vee ||\bar{\mathbf{z}}_1|| \vee ||\bar{\mathbf{z}}_2|| \leq k$, there exists $\bar{b} > 0$ such that

$$||\Phi(\mathbf{z}_1, \mathbf{z}_2, t)|| \vee ||\varpi(\mathbf{z}_1, \mathbf{z}_2, t)|| \leq \bar{b}(1 + ||\mathbf{z}_1|| + ||\mathbf{z}_2||)$$

for all $(t, \mathbf{z}_1, \mathbf{z}_1) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$.

Under the Hypothesis (H1), there exists a unique solution [18] to Eq.(1) , which is expressed as $\mathbf{z}(t, z_0)$ on $t > -\tau$ to Eq. (1), for any initial value $\{\mathbf{z}(\theta) : -\tau < \theta < 0 = z_0 \in C^b_{\mathfrak{F}_0}([-\tau, 0]; \mathbb{R}^n)\}$.

Denote $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ as the family of continuous functions $V(t, \mathbf{z})$ on $\mathbb{R}_+ \times \mathbb{R}^n$ that are continuously twice differentiable in $\mathbf{z}$ and once differentiable in t. Then, define a diffusion operator $\mathcal{L}$ acting on $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ along with Eq. (1) as

$$\mathcal{L}V = V_t(t, \mathbf{z}) + V_z(t, \mathbf{z})\Phi(t, \mathbf{z}) + \frac{1}{2}trace[\varpi^T(t, \mathbf{z})V_{zz}(t, \mathbf{z})\varpi(t, \mathbf{z})], \tag{2}$$

where $V_t(t, \mathbf{z}) = \frac{\partial V(t,\mathbf{z})}{\partial t}, V_z(t, \mathbf{z}) = \left(\frac{\partial V(t,\mathbf{z})}{\partial \mathbf{z}_1}, \dots, \frac{\partial V(t,\mathbf{z})}{\partial \mathbf{z}_n}\right), V_{zz}(t, \mathbf{z}) = \left(\frac{\partial^2 V(t,\mathbf{z})}{\partial \mathbf{z}_i \partial \mathbf{z}_j}\right)_{n\times n}$.

Lemma 1. [9] Suppose that there is a nonnegative function $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$, a function $\vartheta \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, and two continuous functions $\varrho_1, \varrho_2 : \mathbb{R}^n \to \mathbb{R}_+$ such that the differential operator $\mathcal{L}$ satisfies

$$\mathcal{L}V(t, \mathbf{z}, \mathbf{u}) \leq \vartheta(t) - \varrho_1(\mathbf{z}) + \varrho_2(\mathbf{u}),\ (t, \mathbf{z}, \mathbf{u}) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n,$$
$$\varrho_1(\mathbf{z}) > \varrho_2(\mathbf{z}), \forall \mathbf{z} \neq 0$$
$$\lim_{||\mathbf{z}||\to\infty} \inf_{0\leq t<\infty} V(t, \mathbf{z}) = \infty.$$

Then

$$\lim_{t\to\infty} \mathbf{z}(t; z_0) = 0 \quad a.s. \tag{3}$$

for each initial value $z_0 \in C^b_{\mathfrak{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

3. Main Results

In this section, under the pinning mechanism and the adaptive feedback controlling method, the main results of identifying partial topology in the noised-perturbed complex networks affected by coupling delay are obtained.

Consider an n-dimensional model for a noise-contaminated complex network affected by coupling delay. The state of each node is evolved by

$$\mathrm{d}\mathbf{x}_i(t) = \left(F_i\left(t, \mathbf{x}_i(t)\right) + \sum_{j=1}^{N} c_{ij}D\mathbf{x}_j(t-\tau)\right)\mathrm{d}t + \phi_i\left(t, \mathbf{x}_i(t), \mathbf{x}_i(t-\tau)\right)\mathrm{d}\mathbb{W}_i(t), \tag{4}$$

for $1 \leq i \leq N$. Here, the system (4) is referred to as the original network. $\mathbf{x}_i \in \mathbb{R}^n$ denotes the evolving state of the ith node, $F_i(t, \mathbf{x}(t)) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ represents the individual dynamics of the ith node, and $C = (c_{ij})_{N\times N}$ is the unknown weighted configuration matrix associated to the topology of the original network. The matrix C is not neccessarily subject to symmetrical and irreducible constraints. If a link between two nodes $i, j(i \neq j)$ exists, then c_{ij} is a non-zero weight $c_{ij} \neq 0$; otherwise $c_{ij} = 0$; and the diagonal elements are defined by $c_{ii} = -\sum_{j=1,j\neq i}^{N} c_{ij}$. D is the inner coupling matrix governing coupled variables, and it is not imposed to be symmetric. τ denotes the transmission speed delay between connected nodes, and $\phi_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t-\tau)) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ denotes the vector-valued noise intensity function of the ith node.

Without loss of generality, suppose that only the couplings among the first $s(1 \leq s \leq N)$ nodes and their adjacent nodes are concerned here. Then, construct the corresponding auxiliary network consisting of s nodes:

$$\begin{aligned}\mathrm{d}\mathbf{y}_i(t) = &\left(F_i(t, \mathbf{y}_i(t)) + \sum_{j=1}^{s} \hat{c}_{ij} D\mathbf{y}_j(t-\tau) + \sum_{j=s+1}^{N} \hat{c}_{ij} D\mathbf{x}_j(t-\tau) + \mathbf{v}_i\right)\mathrm{d}t \\ &+ \phi_i(t, \mathbf{y}_i(t), \mathbf{y}_i(t-\tau))\,\mathrm{d}\mathbb{W}_i(t),\end{aligned} \tag{5}$$

for $1 \leq i \leq s$, where $\mathbf{y}_i \in \mathbb{R}^n$ denotes the evolving state of the ith node in the auxiliary network, $\hat{c}_{ij}$ is the estimation of the uncertain weight c_{ij}, and $\mathbf{v}_i$ represents the adaptive controller to be determined.

Denote $\mathbf{e}_i(t) = \mathbf{y}_i(t) - \mathbf{x}_i(t)$, $\tilde{c}_{ij} = \hat{c}_{ij} - c_{ij}$ and $\kappa_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t-\tau), \mathbf{y}_i(t), \mathbf{y}_i(t-\tau)) = \phi_i(t, \mathbf{y}_i(t), \mathbf{y}_i(t-\tau)) - \phi_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t-\tau))$. If the estimating value $\hat{c}_{ij}$ tends to c_{ij} (i.e., $\tilde{c}_{ij}$ tends to 0) for $1 \leq i \leq s, 1 \leq j \leq N$, then partial uncertain weights are successfully inferred. Thus, one derives the following error system from (4) and (5):

$$\begin{aligned}\mathrm{d}\mathbf{e}_i(t) = &\left(F_i(t, \mathbf{y}_i(t)) - F_i(t, \mathbf{x}_i(t)) + \sum_{j=1}^{s} c_{ij} D\mathbf{e}_j(t-\tau) + \sum_{j=1}^{s} \tilde{c}_{ij} D\mathbf{y}_j(t-\tau)\right. \\ &\left. + \sum_{j=s+1}^{N} \tilde{c}_{ij} D\mathbf{x}_j(t-\tau) + \mathbf{v}_i\right)\mathrm{d}t + \kappa_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t-\tau), \mathbf{y}_i(t), \mathbf{y}_i(t-\tau))\,\mathrm{d}\mathbb{W}_i(t),\end{aligned} \tag{6}$$

where $1 \leq i \leq s$, $1 \leq j \leq N$. The following hypotheses are required for obtaining the principal results.

Hypothesis 2 (H2). For each $i = 1, 2, \cdots, s$, there exist a constant α_i and a neighborhood U satisfying

$$\|F_i(t, \mathbf{y}(t)) - F_i(t, \mathbf{x}(t))\| \leqslant \alpha_i \|\mathbf{y}(t) - \mathbf{x}(t)\|$$

for $\forall \mathbf{x}(t), \mathbf{y}(t) \in U \subset \mathbb{R}^n$.

Hypothesis 3 (H3). For each $i = 1, 2, \cdots, s$, assume that $\kappa_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t-\tau), \mathbf{y}_i(t), \mathbf{y}_i(t-\tau))$ is bounded, and there exist constants $\beta_i, \eta_i \geq 0$ satisfying

$$trace(\kappa_i^T \kappa_i) \leq 2\beta_i \mathbf{e}_i^T(t)\mathbf{e}_i(t) + 2\eta_i \mathbf{e}_i^T(t-\tau)\mathbf{e}_i(t-\tau).$$

Remark 1. Hypothesis (H3) indicates that the noise is controllable. If the perturbation is unbounded, the network behaviors are obscured by noise, resulting in the inability to identify the specific network topology.

Hypothesis 4 (H4). Suppose that $\{D\mathbf{x}_i(t)\}_{i=1}^{N}$ are linearly independent on the orbit of the synchronization manifold $\{\mathbf{x}_i(t) = \mathbf{y}_i(t)\}_{i=1}^{s}$.

Lemma 2. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be any two vectors and $G \in \mathbb{R}^{n\times n}$ any positive definite matrix. Then, it holds that

$$2\mathbf{x}^T\mathbf{y} \le \mathbf{x}^T G\mathbf{x} + \mathbf{y}^T G^{-1}\mathbf{y}.$$

Theorem 1. Under the Hypotheses (H2)-(H4), the unknown partial configuration $C_{s\times N}$ is inferred by the estimated matrix $\hat{C}_{s\times N}$ with probability one by designing the adaptive controllers and the updating laws:

$$\begin{cases} \mathbf{v}_i = -h_i\mathbf{e}_i(t), \\ \dot{h}_i = \omega_i \mathbf{e}_i^T(t)\mathbf{e}_i(t), \\ \dot{\hat{c}}_{ij} = -\mathbf{e}_i^T(t)D\mathbf{y}_j(t-\tau), \quad 1 \le j \le s, \\ \dot{\hat{c}}_{ij} = -\mathbf{e}_i^T(t)D\mathbf{x}_j(t-\tau), \quad s < j \le N, \end{cases} \tag{7}$$

where $1 \le i \le s$, $1 \le j \le N$, and $\omega_i (1 \le i \le s)$ are arbitrary positive constants.

Proof. Consider a nonnegative candidate as $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{ns+N^2+N}; \mathbb{R}_+)$

$$V = \frac{1}{2}\sum_{i=1}^{s}\mathbf{e}_i^T(t)\mathbf{e}_i(t) + \frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{N}\tilde{c}_{ij}^2 + \frac{1}{2}\sum_{i=1}^{s}\frac{(h_i - h^*)^2}{\omega_i}, \tag{8}$$

where h^* is a large enough positive constants to be designed. Then one derives

$$\begin{aligned}
\mathcal{L}V =& \sum_{i=1}^{s}\mathbf{e}_i^T(t)\dot{\mathbf{e}}_i(t) + \sum_{i=1}^{s}\sum_{j=1}^{N}\tilde{c}_{ij}\dot{\hat{c}}_{ij} + \sum_{i=1}^{s}\frac{(h_i - h^*)\dot{h}_i}{\omega_i} + \frac{1}{2}\,\mathrm{trace}\sum_{i=1}^{s}\left(\kappa_i^T\kappa_i\right) \\
=& \sum_{i=1}^{s}\mathbf{e}_i^T(t)\Bigg(F_i(t,\mathbf{y}_i(t)) - F_i(t,\mathbf{x}_i(t)) + \sum_{j=1}^{s}c_{ij}D\mathbf{e}_j(t-\tau) + \sum_{j=1}^{s}\tilde{c}_{ij}D\mathbf{y}_j(t-\tau) \\
&+ \sum_{j=s+1}^{N}\tilde{c}_{ij}D\mathbf{x}_j(t-\tau) + \mathbf{v}_i\Bigg) - \sum_{i=1}^{s}\sum_{j=1}^{s}\tilde{c}_{ij}\mathbf{e}_i^T(t)D\mathbf{y}_j(t-\tau) \\
&- \sum_{i=1}^{s}\sum_{j=s+1}^{N}\tilde{c}_{ij}\mathbf{e}_i^T(t)D\mathbf{x}_j(t-\tau) + \sum_{i=1}^{s}(h_i - h^*)\,\mathbf{e}_i^T(t)\mathbf{e}_i(t) + \frac{1}{2}\,\mathrm{trace}\sum_{i=1}^{s}\left(\kappa_i^T\kappa_i\right) \\
\le& \sum_{i=1}^{s}\alpha_i\mathbf{e}_i^T(t)\mathbf{e}_i(t) + \sum_{i=1}^{s}\sum_{j=1}^{s}c_{ij}\mathbf{e}_i^T(t)D\mathbf{e}_j(t-\tau) + \sum_{i=1}^{s}\beta_i\mathbf{e}_i^T(t)\mathbf{e}_i(t) \\
&+ \sum_{i=1}^{s}\eta_i\mathbf{e}_i^T(t-\tau)\mathbf{e}_i(t-\tau) - \sum_{i=1}^{s}h^*\mathbf{e}_i^T(t)\mathbf{e}_i(t) \\
=& \sum_{i=1}^{s}(\alpha_i + \beta_i - h^*)\mathbf{e}_i^T(t)\mathbf{e}_i(t) + \sum_{i=1}^{s}\eta_i\mathbf{e}_i^T(t-\tau)\mathbf{e}_i(t-\tau) + \sum_{i=1}^{s}\sum_{j=1}^{s}c_{ij}\mathbf{e}_i^T(t)D\mathbf{e}_i(t-\tau).
\end{aligned}$$

Let $\mathbf{e}(t) = \left(\mathbf{e}_1^T(t), \mathbf{e}_2^T(t), \dots, \mathbf{e}_s^T(t)\right)^T \in \mathbb{R}^{ns\times s}$, $\alpha = \max_{1\le i\le s}\{\alpha_i\}$, $\beta = \max_{1\le i\le s}\{\beta_i\}$, $\eta = \max_{1\le i\le s}\{\eta_i\}$ and $Q = C_{s\times s} \otimes D$. Then, one gets

$$\mathcal{L}V \le -(h^* - \alpha - \beta)\,\mathbf{e}^T(t)\mathbf{e}(t) + \mathbf{e}^T(t)Q\mathbf{e}(t-\tau) + \eta\mathbf{e}^T(t-\tau)\mathbf{e}(t-\tau).$$

Using Lemma 2, we obtain

$$\mathbf{e}^T(t)Q\mathbf{e}(t-\tau) = \left(Q^T\mathbf{e}(t)\right)^T \mathbf{e}(t-\tau) \leq \frac{1}{2}[\mathbf{e}^T(t)QQ^T\mathbf{e}(t) + \mathbf{e}^T(t-\tau)\mathbf{e}(t-\tau)].$$

Thus

$$\begin{aligned}\mathcal{L}V \leq & -\left(h^* - \alpha - \beta - \frac{1}{2}\lambda_{\max}\left(QQ^T\right)\right)\mathbf{e}^T(t)\mathbf{e}(t) + \left(\eta + \frac{1}{2}\right)\mathbf{e}^T(t-\tau)\mathbf{e}(t-\tau) \\ \triangleq & -\varrho_1(\mathbf{e}(t)) + \varrho_2(\mathbf{e}(t-\tau)),\end{aligned}$$

where $\varrho_1(\mathbf{x}) = \left(h^* - \alpha - \beta - \frac{1}{2}\lambda_{\max}\left(QQ^T\right)\right)\mathbf{x}^T\mathbf{x}$, $\varrho_2(\mathbf{x}) = \left(\eta + \frac{1}{2}\right)\mathbf{x}^T\mathbf{x}$. If $h^* > \alpha + \beta + \frac{1}{2}\lambda_{\max}(QQ^T)$, one gets $\varrho_1(\mathbf{x}) > \varrho_2(\mathbf{x})$, $\forall \mathbf{x} \neq 0$.

Furthermore, $\lim_{||\mathbf{e}|| \to \infty} \inf_{0 \leq t < \infty} V = \infty$ and κ_i is bounded. One obtains from Lemma 1 that $\lim_{t\to\infty} V(t; \mathbf{e}_i; \tilde{c}_{ij}; h_i)$ exists and $\lim_{t\to\infty} \mathbf{e}(t) = 0$ $a.s.$ By the error system (6) and the control strategy (7), the solution will be asymptotically stabilized at $M = \{(\mathbf{e}_i; \tilde{c}_{ij}; h_i) : \mathbf{e} = 0\}$ with probability one. According to (H4) and the error system (6), one gets $M = \{\mathbf{e}_i = 0; \tilde{c}_{ij} = 0; h_i = a_i : 1 \leq i \leq s, 1 \leq j \leq N\}$, where $a_i (1 \leq i \leq s)$ are constants. Therefore, we obtain $\lim_{t\to\infty} \sum_{i=1}^{s} \sum_{j=1}^{N} |\hat{c}_{ij}(t) - c_{ij}| = 0$ $a.s.$ That is to say, the unknown partial configuration $C_{s\times N}$ is successfully inferred by the estimation matrix $\hat{C}_{s\times N}$ almost surely. The proof is completed. ■

Remark 2. When $s = N$, our results are consistent with the theory in Ref. [9] since it can infer the whole network topology. Namely, the proposed method is more general.

Remark 3. Note that the linear independence condition in (H4) is the essential condition of successfully guaranteeing topology identification. If synchronization in the original network is achieved, then the linear independence condition will not hold anymore, making topology identification infeasible.

In the synchronization-based topology identification method, the linear independence condition of the coupling term is very important for network identification [19]. Synchronization is an obstacle to identify the network topology, because it weakens or even eliminates the differences between nodes, making the condition of linear independence no longer valid.

From the perspective of control theory, noise hinders the control and evolution behavior of networks. However, moderate noise or coupling delay has an unexpected effect on the network topology identification. This is because noise makes the trajectory of nodes more diverse, and coupling delay makes it impossible for nodes to obtain the real-time information of other nodes. This not only hinders the generation of synchronization, but also makes the node trajectories different, and the feedback information is more abundant, which speeds up the time of topology identification.

If there is no coupling delay or noise in the original network, Theorem 1 is simplified by the following corollaries.

Corollary 1. Assume that the Hypotheses H2-H4 hold. If the original network (4) is in the absence of the coupling delay, the unknown partial configuration $C_{s\times N}$ is successfully inferred by $\hat{C}_{s\times N}$ with probability one by using the adaptive controllers and updating laws:

$$\begin{cases} \mathbf{v}_i = -h_i\mathbf{e}_i(t), \\ \dot{h}_i = \omega_i\mathbf{e}_i^T(t)\mathbf{e}_i(t), \\ \dot{\hat{c}}_{ij} = -\mathbf{e}_i^T(t)D\mathbf{y}_j(t), \quad 1 \leq j \leq s, \\ \dot{\hat{c}}_{ij} = -\mathbf{e}_i^T(t)D\mathbf{x}_j(t), \quad s < j \leq N, \end{cases} \tag{9}$$

where $1 \leq i \leq s$, $1 \leq j \leq N$, and $\omega_i (1 \leq i \leq s)$ are arbitrary positive constants.

Corollary 2. Suppose that the Hypotheses H2 and H4 hold. In the original network (4) without the noise, the unknown partial configuration $C_{s\times N}$ is successfully inferred by $\hat{C}_{s\times N}$ via the controllers and updating laws (7).

Obviously, if the original network is in the absence of coupling delay and disturbances, Theorem 1 becomes a simpler form. In fact, this inference is the main result in the literature [15]. Although Corollary 2 is consistent with the controllers and updating laws of the Theorem 1, the original network of Corollary 2 is governed in a deterministic environment. Hence, the limitation that the noise is controllable is unnecessary anymore.

4. Examples

In what follows, the effectiveness of our results is verified. To this aim, *Lü* chaotic system is considered and described by

$$\dot{x} = \begin{pmatrix} -a_1 & a_1 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}, \tag{10}$$

where $a_1 = 36, a_2 = 3$ and $a_3 = 20$. It is obvious that *Lü* chaotic system satisfies (H1) because of its boundedness [12, 20].

In the following simulations, we are interested in the couplings between the first 3 nodes whether in a 20-node network or in large-scale networks.

4.1. A 20-node complex dynamical network

Consider a network consisting of $N = 20$ *Lü* systems, described by

$$\mathrm{d}\mathbf{x}_i(t) = \left(F_i\left(t, \mathbf{x}_i(t)\right) + \sigma \sum_{j=1}^{20} c_{ij} D \mathbf{x}_j(t-\tau) \right) \mathrm{d}t + \phi_i\left(t, \mathbf{x}_i(t), \mathbf{x}_i(t-\tau)\right) \mathrm{d}\mathbb{W}_i(t), \tag{11}$$

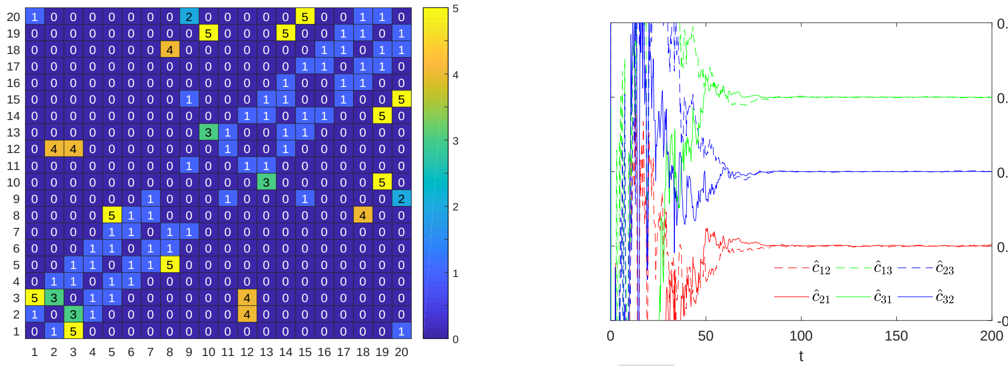
where $1 \le i \le 20$, $C = \sigma C_{in}$ with the coupling strength $\sigma > 0$. The non-diagonal elements of C_{in} are depicted by colormap, while the diagonal elements should be calculated according to the zero row sum.

First, a 20-node network generated by the Watts-Strogatz (W-S) algorithm [21] is employed. Each node is connected to its $K = 4$ nearest neighbors. Then each edge is rewired in a certain way with probability $p_r = 0.2$. The coupling weights are randomly set to be integers between 1 and 5. The inner coupling matrix D is set to be an identity matrix. The initial estimated values coupling configuration matrices are set to be zero. The stochastic perturbations are randomly taken from 1 to 10, and $h_i(0) = 1, \omega_i(0) = 1,\ x_j(0) = (1+j, 2+j, 1+j)^T, y_i(0) = -2.6 + 0.2(1+i, 2+i, 1+i)^T$, where $1 \le i \le 3$ and $1 \le j \le 20$.

The Euler-Maruyama method [22] is utilized to numerically solve the stochastic differential equations affected by delay with an equal time step of 0.01. For simplicity, set noise intensity function $\phi_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t-\tau)) = \mu\,\mathrm{diag}(0.1x_{i1}(t) + 0.1x_{i1}(t-\tau), 0.1x_{i2}(t) + 0.1x_{i2}(t-\tau), 0.1x_{i3}(t) + 0.1x_{i3}(t-\tau))$, then $\kappa_i = \phi_i(t, \mathbf{y}_i(t), \mathbf{y}_i\ (t-\tau)) - \phi_i(t, \mathbf{x}_i(t), \mathbf{x}_i(t-\tau)) = \mu\,\mathrm{diag}(0.1e_{i1}(t) + 0.1e_{i1}(t-\tau), 0.1e_{i2}(t) + 0.1e_{i2}(t-\tau), 0.1e_{i3}(t) + 0.1e_{i3}(t-\tau))$. Obviously $trace(\kappa_i^T\kappa)$ satisfies (H3). The constant μ is utilized to represent the variation intensities of system noise.

When the coupling is symmetric, the non-diagonal elements of C_{in} are depicted in Fig. 1(a), namely the coupling weight $c_{12} = c_{21} = \sigma, c_{13} = c_{31} = 5\sigma$ and $c_{23} = c_{32} = 3\sigma$ for network (11).

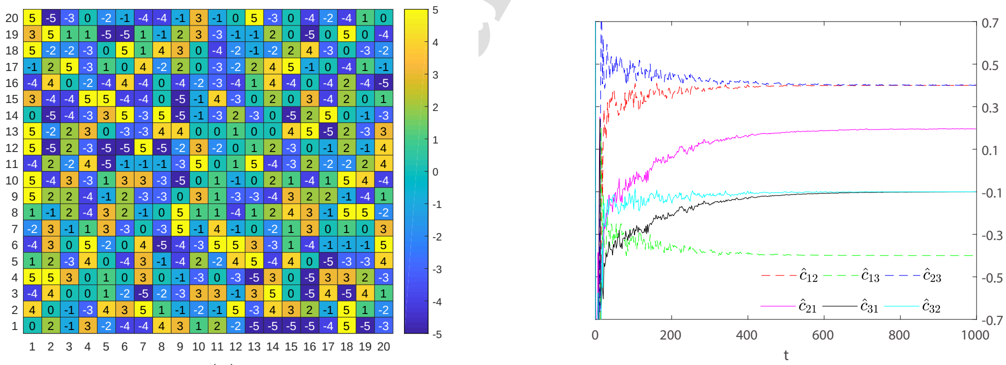
Fig. 1(b) illustrates the evolutionary process for the estimating coupling set $\{\hat{c}_{12}, \hat{c}_{13}, \hat{c}_{23}, \hat{c}_{21}, \hat{c}_{31}, \hat{c}_{32}\}$ with coupling strength $\sigma = 0.1$, coupling delay $\tau = 0.1$ and noise intensity $\mu = 0.2$. After a period of oscillation, the estimated value of the parameter remains stable at constant values. It is shown that $\hat{c}_{12}$ and $\hat{c}_{21}$ tend to $c_{12} = c_{21} = 0.1$. Similarly, $\hat{c}_{13}$ and $\hat{c}_{31}$ go to $c_{13} = c_{31} = 0.5$, and $\hat{c}_{23}, \hat{c}_{32}$ to $c_{23} = c_{32} = 0.3$. It means that the topology parameters are effectively tracked, the auxiliary network and the original network reach a synchronization state, and the identification of the partial topology succeeds.



(a) (b)

Figure 1: (a) The colormap of symmetric matrix C_{in}. The values of the coordinate axis represent the node order. (b) Identification of parameters $\{\hat{c}_{12}, \hat{c}_{13}, \hat{c}_{23}, \hat{c}_{21}, \hat{c}_{31}, \hat{c}_{32}\}$ for network (11) with $\sigma = 0.1, \tau = 0.1$ and $\mu = 0.2$.

When the coupling is asymmetric, the non-diagonal elements of C_{in} are depicted in Fig. 2(a). The evolutionary process for the estimating coupling set $\{\hat{c}_{12}, \hat{c}_{13}, \hat{c}_{23}, \hat{c}_{21}, \hat{c}_{31}, \hat{c}_{32}\}$ is shown in Fig. 2(b) with coupling strength $\sigma = 0.1$, coupling delay $\tau = 0.1$ and noise intensity $\mu = 0.2$.



(a) (b)

Figure 2: (a) The colormap of matrix asymmetric C_{in}. The values of the coordinate axis represent the node order. (b) Identification of parameters $\{\hat{c}_{12}, \hat{c}_{13}, \hat{c}_{23}, \hat{c}_{21}, \hat{c}_{31}, \hat{c}_{32}\}$ for network (11) with $\sigma = 0.1, \tau = 0.1$ and $\mu = 0.2$.

In Fig. 3, we compare the identification time between the traditional method [9, 14] and our proposed method. In Fig. 3(a) and (c), the number of nodes in the auxiliary network is the same as the original network. In Fig. 3(b) and (d), there are fewer nodes in the auxiliary network than in the original network. The results show that the traditional method takes more time than partial topology identification at $\tau = 0, \mu = 0.2$. The same is true for $\tau = 0.1, \mu = 0$.

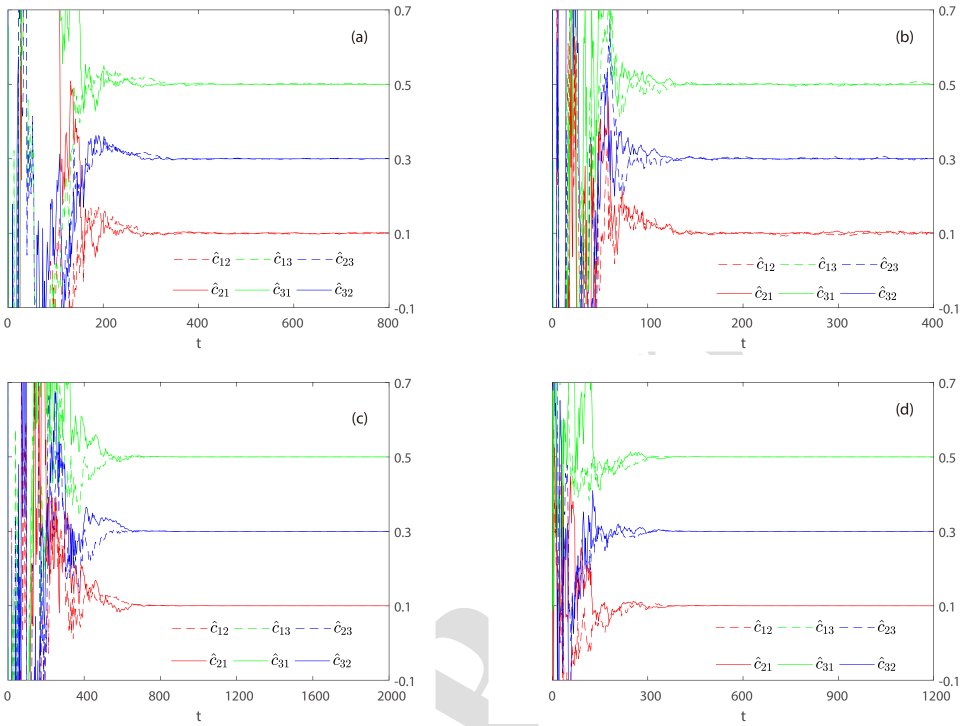


Figure 3: Identification of parameters $\{\hat{c}_{12}, \hat{c}_{13}, \hat{c}_{23}, \hat{c}_{21}, \hat{c}_{31}, \hat{c}_{32}\}$ with $\sigma = 0.1$. (a) $\tau = 0, \mu = 0.2$; (b) $\tau = 0, \mu = 0.2$; (c) $\tau = 0.1, \mu = 0$. (d) $\tau = 0.1, \mu = 0$.

Next, we further explore the influence of the changes of the two parameters of the network: the noise intensity μ and the coupling delay τ on the average identification time with coupling strength $\sigma = 0.1$. Here, the average time is the least time required for $|\hat{c}_{ij} - c_{ij}| < 10^{-3}$ with 100 iterations, which is displayed by the error bar. Rank the data from smallest to largest. Arrange the data from smallest to largest. The upper line length of the error bar is the maximum minus the average, and the lower line length is the average minus the minimum.

It is seen from Fig. 4(a) that as μ increases, the average identification time gradually decreases, and there is no obvious drop tendency when μ varies from 0.75 to 0.95. This is because the appearance of noise makes the motion trajectories of nodes more diverse, which hinders the generation of synchronization and makes the trajectories of nodes different. The feedback information is more abundant, which speeds up the time of topology identification. Stronger noise intensity does not further affect the identification since the nodes are already distinguished from each other. In Fig. 4(b), the average identification time decreases as μ increases, but there is a slight increase when $\tau = 0.9$. It has been explained in Ref.[23] that the coupling delay narrows the synchronization regions and further affects the identification performance.

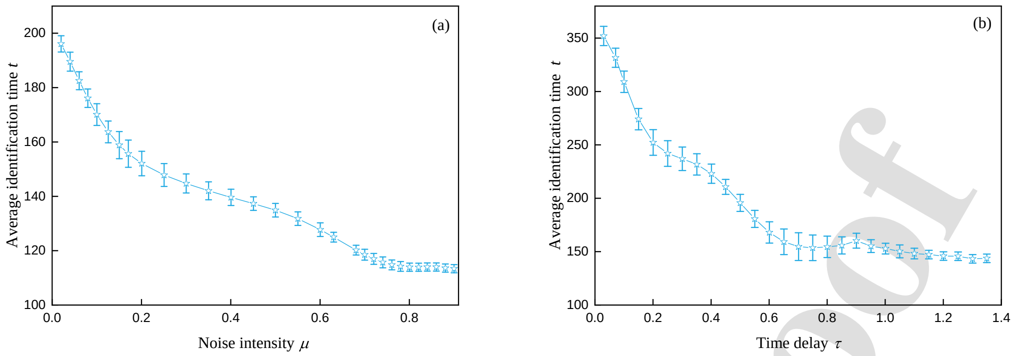


Figure 4: The error bar with fill area of the average identification time versus noise intensity μ (a) and time delay τ (b) for the network (11) with $\sigma = 0.1$.

4.2. Large-scale complex dynamical networks

To verify the effectiveness of our method, large-scale complex networks generated as the above are employed. Fig. 5 (a) and (b) display the identification performance when network size $N = 100$ and 1000, respectively. Here, we set coupling strength $\sigma = 0.1$, coupling delay $\tau = 0.1$ and noise intensity $\mu = 0.2$. In traditional methods [9, 14], the auxiliary network is composed of the same size of nodes as the original network. In that case, for a 100-node network, one needs to construct 100 controllers and 100×100 observers in the auxiliary network, which is too costly to be applied. However, by using our proposed method, only 3 controllers and 3×100 observers are needed. Similarly, for a 1000-node network, the dimension of controllers and updating laws in the auxiliary network is 3003 rather than 1001000. Therefore, for dealing with the same large-scale networks, our method greatly reduces the calculation and computing memory space compared with the traditional methods. The larger the network size, the more obvious the advantages of our method.

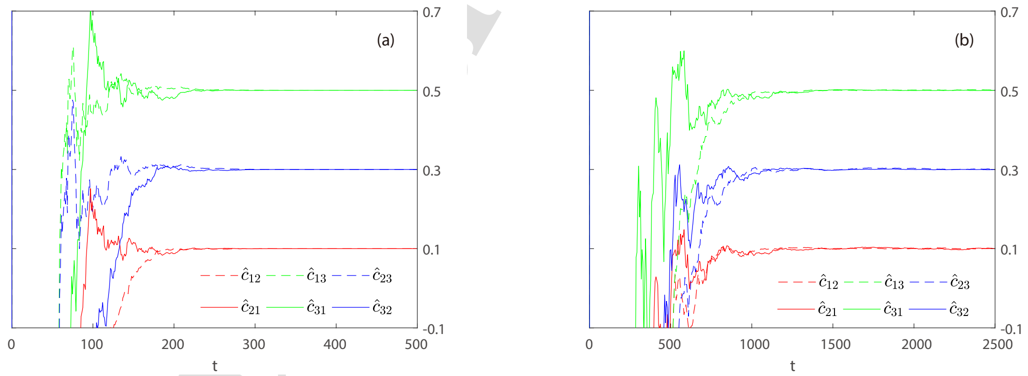


Figure 5: Identifying partial topology with network size (a) $N = 100$ and (b) $N = 1000$: the evolution of parameters $\{\hat{c}_{12}, \hat{c}_{13}, \hat{c}_{23}, \hat{c}_{21}, \hat{c}_{31}, \hat{c}_{32}\}$.

5. Conclusion

In this paper, the partial identification topology has been investigated in the noised-perturbed complex networks affected by coupling delay. The results have leveraged the adaptive feedback control strategy and the pinning mechanism to infer the partial original network. Compared with the existing methods, our method is able to deal with large-scale networks when we are only interested in a small part of the whole topology. In numerical simulations, we have found that the impact of the changes of the noise intensity and the coupling delay on the identification effectiveness has been explored. It has been found that the weak noise intensity facilitated the partial topology identification. The coupling delay sometimes promotes the identification, but sometimes hinders it. For the same large-scale complex dynamical networks, the proposed approach reduces the calculation and the computing memory space compared with the traditional methods. The results are expected to be applied to finite-time identification in further study.

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CRediT authorship contribution statement

Chunyan Chen: Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing – original draft, Funding acquisition. **Jin Zhou**: Methodology, Formal analysis, Writing – original draft, Funding acquisition. **Fenglin Qu**: Software, Formal analysis, Data curation, Writing – original draft. **Changjiang Song**: Software, Conceptualization, Writing – original draft. **Shuaibing Zhu**: Methodology, Software, Data curation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.