A second multisine signal was used as a validation signal. This has $N = 4096, V = 1$ and has odd harmonics uniform and even harmonics suppressed. Using this signal, the even-order components contribute to 1.67% of the total output power, and the normalized mean squared error (NMSE) is 5.32×10^{-3}. The performance of the model is very good, as can be seen from Fig. 5. This is in fact better than the model obtained using theoretical analysis in Section II, which gave $\mathrm{NMSE} = 1.23 \times 10^{-1}$. The reason for this is due to the restriction on the model orders for L_1 and L_2 in the theoretical analysis.

With α_0 and β_0 unchanged, the proposed modeling approach was tested on first-order systems with ρ_0 varying from -0.01 to -0.1, representing an increasing amount of nonlinearity. In all cases, LIFRED and ELiS could be applied to obtain the transfer functions L_1 and L_2. For example, for $\rho_0 = -0.1, L_1(z^{-1}) = 0.168z^{2.28}/(1 - 0.832z^{-1})$ and $L_2(z^{-1}) = (-0.956 + 0.681z^{-1})z^{-2.73}/(1 - 0.150z^{-1} - 0.575z^{-2})$. Note that for all these cases, a first-order model was able to capture the dynamics of L_1 in the frequency range of interest, but a second-order model was required for L_2.

When positive values of ρ_0 were used, the same estimates were obtained for the frequency responses of L_1 and L_2 as those for the corresponding negative values of ρ_0, but with an additional phase shift of 180° for L_2. This is consistent with theoretical results for which a change in the sign of ρ_0 causes a change in the sign of the contribution from even-order terms.

Next, a second-order bilinear system with $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = 0.2$ and $\rho_0 = \rho_1 = -0.05$ was perturbed using the training signal at $T = 1$ s. The Volterra kernel is plotted in Fig. 6. The application of LIFRED resulted in smooth estimates of the frequency responses of L_1 and L_2. Subsequent application of ELiS gave

$$L_1(z^{-1}) = \frac{(0.254 - 0.428z^{-1} + 0.219z^{-2})z^{0.29}}{1 - 2.257z^{-1} + 1.779z^{-2} - 0.476z^{-3}}$$

$$\text{and} \quad L_2(z^{-1}) = \frac{(-0.524 + 0.336z^{-1})z^{-0.09}}{1 - 1.412z^{-1} + 0.601z^{-2}}.$$

Note that while L_1 is lowpass, L_2 is a bandpass system. Using the validation signal, $\mathrm{NMSE} = 0.589$, with $k = 0.337$. In this case, the even-order terms contribute to 3.86% of the total power at the bilinear system output.

One possible reason for the relatively high NMSE is due to the effects of higher order nonlinearity on the measurement of the second-order kernel. The contours in Fig. 6 can be observed to be less smooth compared to those in Fig. 3. The kernel estimation could be improved by using a longer and sparser NID multisine that eliminates the effects of fourth-order nonlinearity [9]. Alternatively, if a Wiener–Hammerstein model with a higher order of nonlinearity is required, the LIFRED technique can be extended to cater for this [11], with the multisine again designed using the algorithm in [9].

IV. Conclusion

The modeling of nonlinear effects in bilinear systems using a Wiener–Hammerstein structure has been considered. Theoretical analysis through output matching is possible only in the simplest cases, and under certain constraints on the system parameters. In order to reduce the complexity of the theoretical approximation, the model parameters were obtained using LIFRED. This method allows higher model orders to be used for the linear subsystems. The higher flexibility in the model orders leads to an improvement in the quality of the approximation, with an additional benefit of reducing the constraints imposed when applying the theoretical approximation.

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Adaptive Synchronization of an Uncertain Complex Dynamical Network

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***Abstract*—This note further investigates the locally and globally adaptive synchronization of an uncertain complex dynamical network. Several network synchronization criteria are deduced. Especially, our hypotheses and designed adaptive controllers for network synchronization are rather simple in form. It is very useful for future practical engineering design. Moreover, numerical simulations are also given to show the effectiveness of our synchronization approaches.**

***Index Terms*—Adaptive synchronization, complex networks, uncertain systems.**

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I. Introduction

Over the past decade, complex networks have been intensively studied in various disciplines, such as social, biological, mathematical, and engineering sciences [1]–[8]. A complex network is a large set of interconnected nodes, where the nodes and connections can be anything. Detailed examples are the World Wide Web, Internet, communication networks, metabolic systems, food webs, electrical power grids, and so on.

Recently, one of the interesting and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes in a network. In fact, synchronization is a kind of typical collective behaviors and basic motions in nature. For example, the synchronization of coupled oscillators can explain well many natural phenomena. Furthermore, some synchronization phenomena are very useful in our daily life, such as the synchronous transfer of digital or analog signals in communication networks. Specifically, synchronization in networks of coupled chaotic systems has received a great deal of attention. Some synchronization criteria of two or three Lorenz systems have been obtained in the literature. However, it is often difficult to get the exact estimation of the coupling coefficients since we do not know the exact boundary for most chaotic systems. Up to now, we can only estimate the boundary of very few chaotic systems [9]–[13], such as the Lorenz, Chen, and Lü systems [14]. Moreover, we often know very little information on the network structure, which makes network design very difficult. To overcome these difficulties, an effectively adaptive synchronization approach is proposed based on an uncertain complex dynamical network model in this note.

Slotine *et al.* [16], [17] further discussed the synchronization of nonlinearly coupled continuous and hybrid oscillators networks by using the contraction analysis approach [18]. Bohacek and Jonckheere [19], [20] proposed the so-called linear dynamically varying method based on discrete time dynamical systems. In the following, by using Lyapunov stability theory, several novel locally and globally asymptotically stable network synchronization criteria are deduced for an uncertain complex dynamical network. Compared with some similar results [3], [5], [15], our sufficient conditions for network synchronization are rather broad and the controllers are very simple. It is very useful for future practical engineering design. Moreover, our analysis method and network model are very different from those of the above referenced literature [16]–[20]. However, for some complex systems (e.g., biological systems) with unknown couplings, our conditions are hard to be verified. In fact, it is impossible to propose a universal synchronization criterion for various complex networks since there are many uncertain factors, such as network structures and coupling mechanisms.

This note is organized as follows. An uncertain complex dynamical network model and several necessary hypotheses are given in Section II. In Section III, locally and globally adaptive synchronization criteria for uncertain complex dynamical networks are proposed. In Section IV, a simple example is provided to verify the effectiveness of the proposed method. Finally, conclusions are given in Section V.

II. Preliminaries

This section introduces an uncertain complex dynamical network model and gives some preliminary definitions and hypotheses.

A. An Uncertain Complex Dynamical Network Model

Consider an uncertain complex dynamical network consisting of N identical nonlinear oscillators with uncertain nonlinear diffusive couplings, which is described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i,\, t) + \mathbf{h}_i(\mathbf{x}_1,\, \mathbf{x}_2,\, \ldots,\, \mathbf{x}_N) + \mathbf{u}_i \tag{1}$$

where $1 \le i \le N$, $\mathbf{x}_i = (x_{i1},\, x_{i2},\, \ldots,\, x_{in})^T \in \mathbf{R}^n$ is the state vector of the ith node, $\mathbf{f} : \Omega \times \mathbf{R}^+ \to \mathbf{R}^n$ is a smooth nonlinear vector field, node dynamics is $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\, t)$, $\mathbf{h}_i : \Omega \times \cdots \times \Omega \to \mathbf{R}^n$ are unknown nonlinear smooth diffusive coupling functions, $\mathbf{u}_i \in \mathbf{R}^n$ are the control inputs, and the coupling-control terms satisfy $\mathbf{h}_i(\mathbf{s},\, \mathbf{s},\, \ldots,\, \mathbf{s}) + \mathbf{u}_i = \mathbf{0}$, where $\mathbf{s}$ is a synchronous solution of the node system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\, t)$.

B. Preliminaries

Network synchronization is a typical collective behavior. In the following, a rigorous mathematical definition is introduced for the concept of network synchronization.

Definition 1: Let $\mathbf{x}_i(t;\, t_0,\, \mathbf{X}_0)$ $(1 \le i \le N)$ be a solution of the dynamical network (1), where $\mathbf{X}_0 = (\mathbf{x}_1^0,\, \mathbf{x}_2^0,\, \ldots,\, \mathbf{x}_N^0)$, $\mathbf{f} : \Omega \times \mathbf{R}^+ \to \mathbf{R}^n$, and $\mathbf{h}_i : \Omega \times \cdots \times \Omega \to \mathbf{R}^n$ $(1 \le i \le N)$ are continuously differentiable, $\Omega \subseteq \mathbf{R}^n$. If there is a nonempty subset $\Lambda \subseteq \Omega$, with $\mathbf{x}_i^0 \in \Lambda$ $(1 \le i \le N)$, such that $\mathbf{x}_i(t;\, t_0,\, \mathbf{X}_0) \in \Omega$ for all $t \ge t_0$, $1 \le i \le N$, and

$$\lim_{t\to\infty} \|\mathbf{x}_i(t;\, t_0,\, \mathbf{X}_0) - \mathbf{s}(t;\, t_0,\, \mathbf{x}_0)\|_2 = \mathbf{0}, \qquad 1 \le i \le N \tag{2}$$

where $\mathbf{s}(t;\, t_0,\, \mathbf{x}_0)$ is a solution of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\, t)$ with $\mathbf{x}_0 \in \Omega$, then the dynamical network (1) is said to realize *synchronization* and $\Lambda \times \cdots \times \Lambda$ is called the *region of synchrony* for the dynamical network (1).

Hereafter, denote $\mathbf{s}(t;\, t_0,\, \mathbf{x}_0)$ as $\mathbf{s}(t)$. Then $\mathbf{S}(t) = (\mathbf{s}^T(t),\, \mathbf{s}^T(t),\, \ldots,\, \mathbf{s}^T(t))^T$ is a synchronous solution of uncertain dynamical network (1) since it is a diffusive coupling network. Here, $\mathbf{s}(t)$ can be an equilibrium point, a periodic orbit, an aperiodic orbit, or a chaotic orbit in the phase space.

Define error vector

$$\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t), \qquad 1 \le i \le N\,. \tag{3}$$

Then the objective of controller $\mathbf{u}_i$ is to guide the dynamical network (1) to synchronize. That is

$$\lim_{t\to+\infty} \|\mathbf{e}_i(t)\|_2 = 0\,, \qquad 1 \le i \le N\,. \tag{4}$$

Since $\dot{\mathbf{s}} = \mathbf{f}(\mathbf{s},\, t)$, from network (1), we have

$$\dot{\mathbf{e}}_i = \bar{\mathbf{f}}(\mathbf{x}_i,\, \mathbf{s},\, t) + \bar{\mathbf{h}}_i(\mathbf{x}_1,\, \mathbf{x}_2,\, \ldots,\, \mathbf{x}_N,\, \mathbf{s}) + \mathbf{u}_i \tag{5}$$

where $1 \le i \le N$, $\bar{\mathbf{f}}(\mathbf{x}_i,\, \mathbf{s},\, t) = \mathbf{f}(\mathbf{x}_i,\, t) - \mathbf{f}(\mathbf{s},\, t)$, and

$$\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s}) = \mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) - \mathbf{h}_i(\mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}).$$

In the following, we give several useful hypotheses.

Hypothesis 1: H1) Assume that there exists a nonnegative constant α satisfying $\|\mathrm{D}\mathbf{f}(\mathbf{s},\, t)\|_2 = \|\mathbf{A}(t)\|_2 \le \alpha$, where $\mathbf{A}(t)$ is the Jacobian of $\mathbf{f}(\mathbf{s},\, t)$.

Hypothesis 2: H2) Suppose that there exist nonnegative constants γ_{ij} $(1 \le i,\, j \le N)$ satisfying $\|\bar{\mathbf{h}}_i(\mathbf{x}_1,\, \mathbf{x}_2,\, \ldots,\, \mathbf{x}_N,\, \mathbf{s})\|_2 \le \sum_{j=1}^{N} \gamma_{ij} \|\mathbf{e}_j\|_2$ for $1 \le i \le N$.

Remark 1: If H1) holds, then we get $\left\|(\mathbf{A}(t) + \mathbf{A}^T(t))/2\right\|_2 \le \alpha$.

III. Adaptive Synchronization of an Uncertain Complex Dynamical Network

This section discusses the local synchronization and global synchronization of the uncertain complex dynamical network (1). Several network synchronization criteria are given.

A. Local Synchronization

Linearizing error system (5) around zero gives

$$\dot{\mathbf{e}}_i = \mathbf{A}(t)\mathbf{e}_i(t) + \bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s}) + \mathbf{u}_i \qquad (6)$$

where $1 \le i \le N$ and recall that $\mathbf{A}(t) = \mathrm{D}\mathbf{f}(\mathbf{s}, t)$ is the Jacobian of $\mathbf{f}$ evaluated at $\mathbf{x} = \mathbf{s}(t)$.

Based on H1) and H2), a network synchronization criterion is deduced as follows.

Theorem 1: Suppose that H1 and H2 hold. Then, the synchronous solution $\mathbf{S}(t)$ of uncertain dynamical network (1) is locally asymptotically stable under the adaptive controllers

$$\mathbf{u}_i = -d_i\mathbf{e}_i, \qquad 1 \le i \le N \qquad (7)$$

and updating laws

$$\dot{d}_i = k_i\,\mathbf{e}_i^T\mathbf{e}_i = k_i\|\mathbf{e}_i\|_2^2, \qquad 1 \le i \le N \qquad (8)$$

where k_i ($1 \le i \le N$) are positive constants.

Proof: Define a Lyapunov candidate as follows:

$$V = \frac{1}{2}\sum_{i=1}^{N}\mathbf{e}_i^T\mathbf{e}_i + \frac{1}{2}\sum_{i=1}^{N}\frac{(d_i - \hat{d}_i)^2}{k_i} \qquad (9)$$

where $\hat{d}_i$ ($1 \le i \le N$) are positive constants to be determined. Thus, one gets

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\sum_{i=1}^{N}(\dot{\mathbf{e}}_i^T\mathbf{e}_i + \mathbf{e}_i^T\dot{\mathbf{e}}_i) + \frac{1}{2}\sum_{i=1}^{N}\frac{(d_i - \hat{d}_i)\dot{d}_i}{k_i}\\
&= \sum_{i=1}^{N}\mathbf{e}_i^T\left(\frac{\mathbf{A}(t) + \mathbf{A}^T(t)}{2} - d_i\mathbf{I}_n\right)\mathbf{e}_i\\
&\quad + \sum_{i=1}^{N}\mathbf{e}_i^T\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s}) + \sum_{i=1}^{N}(d_i - \hat{d}_i)\mathbf{e}_i^T\mathbf{e}_i\\
&\le \sum_{i=1}^{N}\mathbf{e}_i^T\left(\frac{\mathbf{A}(t) + \mathbf{A}^T(t)}{2} - \hat{d}_i\mathbf{I}_n\right)\mathbf{e}_i\\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}\|\mathbf{e}_i\|_2\|\mathbf{e}_j\|_2\\
&\le \sum_{i=1}^{N}(\alpha - \hat{d}_i)\|\mathbf{e}_i\|_2^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}\|\mathbf{e}_i\|_2\|\mathbf{e}_j\|_2\\
&= \mathbf{e}^T(\boldsymbol{\Gamma} + \mathrm{diag}\{\boldsymbol{\alpha} - \hat{\mathrm{d}}_\mathbf{1}, \boldsymbol{\alpha} - \hat{\mathrm{d}}_\mathbf{2}, \ldots, \boldsymbol{\alpha} - \hat{\mathrm{d}}_\mathbf{N}\})\mathbf{e}
\end{aligned}$$

where $\mathbf{e} = (\|\mathbf{e}_1\|_2, \|\mathbf{e}_2\|_2, \ldots, \|\mathbf{e}_N\|_2)^T$ and $\boldsymbol{\Gamma} = (\boldsymbol{\gamma}_{\mathbf{ij}})_{\mathbf{N}\times\mathbf{N}}$.

Since α and γ_{ij} ($1 \le i, j \le N$) are nonnegative constants, one can select suitable constants $\hat{d}_i$ ($1 \le i \le N$) to make $\boldsymbol{\Gamma} + \mathrm{diag}\{\boldsymbol{\alpha} - \hat{\mathrm{d}}_\mathbf{1}, \boldsymbol{\alpha} - \hat{\mathrm{d}}_\mathbf{2}, \ldots, \boldsymbol{\alpha} - \hat{\mathrm{d}}_\mathbf{N}\}$ a negative definite matrix. Thus it follows that the error vector $\boldsymbol{\eta} = (\mathbf{e}_\mathbf{1}^\mathbf{T}, \mathbf{e}_\mathbf{2}^\mathbf{T}, \ldots, \mathbf{e}_\mathbf{N}^\mathbf{T})^\mathbf{T} \to \mathbf{0}$ as $t \to +\infty$. That is, the synchronous solution $\mathbf{S}(t)$ of uncertain dynamical network (1) is locally asymptotically stable under the adaptive controllers (7) and updating laws (8).

The proof is thus completed.

Assume that the coupling of network (1) is linear satisfying $\mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = \sum_{j=1}^{N} b_{ij}\mathbf{x}_j$ for $1 \le i \le N$, where b_{ij} ($1 \le i, j \le N$) are constants. Then, the uncertain network (1) is recasted as follows:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j=1}^{N} b_{ij}\mathbf{x}_j + \mathbf{u}_i, \qquad 1 \le i \le N. \qquad (10)$$

For linear coupling, H2) is naturally satisfied. Thus, one gets the following corollaries.

Corollary 1: Suppose that H1) holds. Then, the synchronous solution $\mathbf{S}(t)$ of the uncertain dynamical network (10) is locally asymptotically stable under the adaptive controllers (7) and updating laws (8).

Moreover, for the coupling scheme $\mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = \sum_{j=1}^{N} b_{ij}\mathbf{p}(\mathbf{x}_j)$ with $1 \le i \le N$, where b_{ij} ($1 \le i, j \le N$) are constants satisfying $\sum_{j=1}^{N} b_{ij} = 0$ for $1 \le i \le N$ and $\|\mathrm{D}\mathbf{p}(\boldsymbol{\xi})\|_2 \le \delta$ for $\boldsymbol{\xi} \in \Omega$, the network (1) is rewritten as follows:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j=1}^{N} b_{ij}\mathbf{p}(\mathbf{x}_j) + \mathbf{u}_i, \qquad 1 \le i \le N. \qquad (11)$$

If H1) holds, then one has

$$\begin{aligned}
\|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s})\|_2 &= \sum_{j=1}^{N}|b_{ij}|\,\|\mathbf{p}(\mathbf{x}_j) - \mathbf{p}(\mathbf{s})\|_2\\
&\le \sum_{j=1}^{N}\delta\,|b_{ij}|\|\mathbf{e}_j\|_2
\end{aligned}$$

for $1 \le i \le N$. That is, H2) holds and one gets the following corollary.

Corollary 2: Assume that H1) holds. Then, under the adaptive controllers (7) and updating laws (8), the synchronous solution $\mathbf{S}(t)$ of the uncertain dynamical network (11) is locally asymptotically stable.

In the following subsection, we discuss the global synchronization case.

B. Global Synchronization

This section presents two global network synchronization criteria.

Rewrite node dynamics $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t)$ as $\dot{\mathbf{x}}_i = \mathbf{B}\mathbf{x}_i(t) + \mathbf{g}(\mathbf{x}_i, t)$, where $\mathbf{B} \in \mathbf{R}^{n\times n}$ is a constant matrix and $\mathbf{g}: \Omega \times \mathbf{R}^+ \to \mathbf{R}^n$ is a smooth nonlinear function. Thus, network (1) is described by

$$\dot{\mathbf{x}}_i = \mathbf{B}\mathbf{x}_i(t) + \mathbf{g}(\mathbf{x}_i, t) + \mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) + \mathbf{u}_i \qquad (12)$$

where $1 \le i \le N$. Similarly, one can get the error system

$$\dot{\mathbf{e}}_i = \mathbf{B}\mathbf{e}_i(t) + \bar{\mathbf{g}}(\mathbf{x}_i, \mathbf{s}, t) + \bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s}) + \mathbf{u}_i \qquad (13)$$

where $1 \le i \le N$ and $\bar{\mathbf{g}}(\mathbf{x}_i, \mathbf{s}, t) = \mathbf{g}(\mathbf{x}_i, t) - \mathbf{g}(\mathbf{s}, t)$.

Hypothesis 3: H3) Suppose that there exists a nonnegative constant μ satisfying $\|\bar{\mathbf{g}}(\mathbf{x}_i, \mathbf{s}, t)\|_2 \le \mu\,\|\mathbf{e}_i\|_2$.

Then one can get the following global network synchronization criterion.

Theorem 2: Suppose that H2) and H3) hold. Then, the synchronous solution $\mathbf{S}(t)$ of uncertain dynamical network (1) is globally asymptotically stable under the adaptive controllers

$$\mathbf{u}_i = -d_i\mathbf{e}_i, \qquad 1 \le i \le N \qquad (14)$$

and updating laws

$$\dot{d}_i = k_i\,\mathbf{e}_i^T\mathbf{e}_i = k_i\|\mathbf{e}_i\|_2^2, \qquad 1 \le i \le N \qquad (15)$$

where k_i $(1 \le i \le N)$ are positive constants.

Proof: Since $\mathbf{B}$ is a given constant matrix, there exists a nonnegative constant β such that $\|\mathbf{B}\|_2 \le \beta$. It follows that $\left\|(\mathbf{B}+\mathbf{B}^T)/2\right\|_2 \le \beta$.

Similarly, construct Lyapunov function (9), then one has

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N} \mathbf{e}_i^T \left(\frac{\mathbf{B}+\mathbf{B}^T}{2} - \hat{d}_i \mathbf{I}_n\right)\mathbf{e}_i + \sum_{i=1}^{N} \mathbf{e}_i^T \bar{\mathbf{g}}_i(\mathbf{x}_i, \mathbf{s}, t) \\ &\quad + \sum_{i=1}^{N} \mathbf{e}_i^T \bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s}) \\ &\le \sum_{i=1}^{N} (\beta + \mu - \hat{d}_i)\|\mathbf{e}_i\|_2^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} \gamma_{ij}\|\mathbf{e}_i\|_2\|\mathbf{e}_j\|_2 \\ &= \mathbf{e}^T(\mathbf{\Gamma} + \mathrm{diag}\{\boldsymbol{\beta} + \boldsymbol{\mu} - \hat{\mathrm{d}}_1, \ldots, \boldsymbol{\beta} + \boldsymbol{\mu} - \hat{\mathrm{d}}_\mathbf{N}\})\mathbf{e}\end{aligned}$$

where $\mathbf{e} = (\|\mathbf{e}_1\|_2, \|\mathbf{e}_2\|_2, \ldots, \|\mathbf{e}_N\|_2)^T$ and $\mathbf{\Gamma} = (\boldsymbol{\gamma}_{\mathbf{ij}})_{\mathbf{N}\times\mathbf{N}}$.

Since β, μ and γ_{ij} $(1 \le i, j \le N)$ are nonnegative constants, one can select suitable constants $\hat{d}_i$ $(1 \le i \le N)$ to make $\mathbf{\Gamma} + \mathrm{diag}\{\boldsymbol{\beta} + \boldsymbol{\mu} - \hat{\mathrm{d}}_1, \boldsymbol{\beta} + \boldsymbol{\mu} - \hat{\mathrm{d}}_2, \ldots, \boldsymbol{\beta} + \boldsymbol{\mu} - \hat{\mathrm{d}}_\mathbf{N}\}$ a negative definite matrix. Then the error vector $\boldsymbol{\eta} = (\mathbf{e}_\mathbf{1}^\mathbf{T}, \mathbf{e}_\mathbf{2}^\mathbf{T}, \ldots, \mathbf{e}_\mathbf{N}^\mathbf{T})^\mathbf{T} \to \mathbf{0}$ as $t \to +\infty$. That is, the synchronous solution $\mathbf{S}(t)$ of uncertain dynamical network (1) is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

This completes the proof.

Similarly, one gets the following two corollaries of global network synchronization.

Corollary 3: Suppose that H3) holds. Then the synchronous solution $\mathbf{S}(t)$ of uncertain linearly coupled dynamical network (10) is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

Corollary 4: Suppose that H3) holds. Then, the synchronous solution $\mathbf{S}(t)$ of uncertain dynamical network (11) is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

Proof: According to (11), one has

$$\begin{aligned}\|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s})\|_2 &= \left\|\sum_{j=1}^{N} b_{ij}(\mathbf{B}\mathbf{e}_j + \bar{\mathbf{g}}_i(\mathbf{x}_j, \mathbf{s}, t))\right\|_2 \\ &\le \sum_{j=1}^{N} b_{ij}(\|\mathbf{B}\|_2 + \mu)\|\mathbf{e}_j\|_2\end{aligned}$$

thus H2) holds. Therefore, from Theorem 2, the synchronous solution $\mathbf{S}(t)$ of network (11) is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

The proof is thus completed.

Hypothesis 4: H4) Assume that $\mathbf{g}(\mathbf{x}, t)$ satisfies the Lipschitz condition. That is, there exists a positive constant κ satisfying $\|\mathbf{g}(\mathbf{x}, t) - \mathbf{g}(\mathbf{y}, t)\| \le \kappa\|x - y\|$, where κ is the Lipschitz constant.

Obviously, H4) implies H3). Now, one has the following synchronization criterion.

Theorem 3: Suppose that H2) and H4) hold. Then the synchronous solution $\mathbf{S}(t)$ of uncertain dynamical network (1) is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

IV. Example

This section presents an example to show the effectiveness of the above synchronization criteria.

Consider a dynamical network consisting of 50 identical Lorenz systems. Here, node dynamics is described by

$$\begin{pmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{pmatrix} = \mathbf{A}\begin{pmatrix}x_{i1}\\ x_{i2}\\ x_{i3}\end{pmatrix} + \begin{pmatrix}0\\ -x_{i1}x_{i3}\\ x_{i1}x_{i2}\end{pmatrix}$$

where

$$\mathbf{A} = \begin{pmatrix}-a & a & 0\\ c & -1 & 0\\ 0 & 0 & -b\end{pmatrix}$$

$a = 10, b = 8/3, c = 28$, and $1 \le i \le 50$. The networked system is defined as follows:

$$\begin{aligned}\begin{pmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{pmatrix} &= \mathbf{A}\begin{pmatrix}x_{i1}\\ x_{i2}\\ x_{i3}\end{pmatrix} + \begin{pmatrix}0\\ -x_{i1}x_{i3}\\ x_{i1}x_{i2}\end{pmatrix} \\ &\quad + \begin{pmatrix}f_1(\mathbf{x}_i) - 2f_1(\mathbf{x}_{i+1}) + f_1(\mathbf{x}_{i+2})\\ 0\\ f_2(\mathbf{x}_i) - 2f_2(\mathbf{x}_{i+1}) + f_2(\mathbf{x}_{i+2})\end{pmatrix} + d_i\mathbf{e}_i \qquad (16)\end{aligned}$$

and

$$\dot{d}_i = k_i\,\|\mathbf{e}_i\|_2^2 \qquad (17)$$

$f_1(\mathbf{x}_i) = a(x_{i2} - x_{i1}), f_2(\mathbf{x}_i) = x_{i1}x_{i2} - bx_{i3}, x_{51} \equiv x_1, x_{52} \equiv x_2$, and $1 \le i \le 50$.

Obviously, one gets

$$\bar{\mathbf{g}}(\mathbf{x}_i, s, t) = \begin{pmatrix}0\\ -x_{i1}x_{i3} + s_1s_3\\ x_{i1}x_{i2} - s_1s_2\end{pmatrix} = \begin{pmatrix}0\\ -x_{i3}e_{i1} - s_1e_{i3}\\ x_{i2}e_{i1} + s_1e_{i2}\end{pmatrix}$$

where $1 \le i \le 50$.

Since Lorenz attractor is confined to a bounded region $\Phi \subset \mathbf{R}^3$ [9]–[13], there exists a constant M satisfying $|x_{ij}|, |s_j| \le M$ for $1 \le i \le 50$ and $j = 1, 2, 3$. Therefore

$$\begin{aligned}\|\bar{\mathbf{g}}(\mathbf{x}_i, s, t)\|_2 &= \sqrt{(x_{i3}e_{i1} + s_1e_{i3})^2 + (x_{i2}e_{i1} + s_1e_{i2})^2} \\ &\le 2M\|\mathbf{e}_i\|_2.\end{aligned}$$

Similarly, one has

$$\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s}) = \begin{pmatrix}f_1(\mathbf{e}_i) - 2f_1(\mathbf{e}_{i+1}) + f_1(\mathbf{e}_{i+2})\\ 0\\ f_3(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{s})\end{pmatrix}$$

where

$$\begin{aligned}f_3(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{s}) &= -be_{i3} + 2be_{i+1,3} - be_{i+2,3} + x_{i2}e_{i1} \\ &\quad + s_1e_{i2} - 2(x_{i+1,2}e_{i+1,1} + s_1e_{i+1,2}) \\ &\quad + x_{i+2,2}e_{i+2,1} + s_1e_{i+2,2}\end{aligned}$$

and $1 \le i \le 50$.

Since

$$\begin{aligned}&\|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s})\|_2^2 \\ &= (f_1(\mathbf{e}_i) - 2f_1(\mathbf{e}_{i+1}) + f_1(\mathbf{e}_{i+2}))^2 \\ &\quad + (f_3(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{s}))^2 \\ &\le (a\|\mathbf{e}_i\|_1 + 2a\|\mathbf{e}_{i+1}\|_1 + a\|\mathbf{e}_{i+2}\|_1)^2 \\ &\quad + (M\|\mathbf{e}_i\|_1 + 2M\|\mathbf{e}_{i+1}\|_1 + M\|\mathbf{e}_{i+2}\|_1)^2 \\ &\le 6(a^2 + M^2)(\|\mathbf{e}_i\|_1^2 + \|\mathbf{e}_{i+1}\|_1^2 + \|\mathbf{e}_{i+2}\|_1^2) \\ &\le 18(a^2 + M^2)(\|\mathbf{e}_i\|_2^2 + \|\mathbf{e}_{i+1}\|_2^2 + \|\mathbf{e}_{i+2}\|_2^2) \\ &\le 18(a^2 + M^2)(\|\mathbf{e}_i\|_2 + \|\mathbf{e}_{i+1}\|_2 + \|\mathbf{e}_{i+2}\|_2)^2\end{aligned}$$

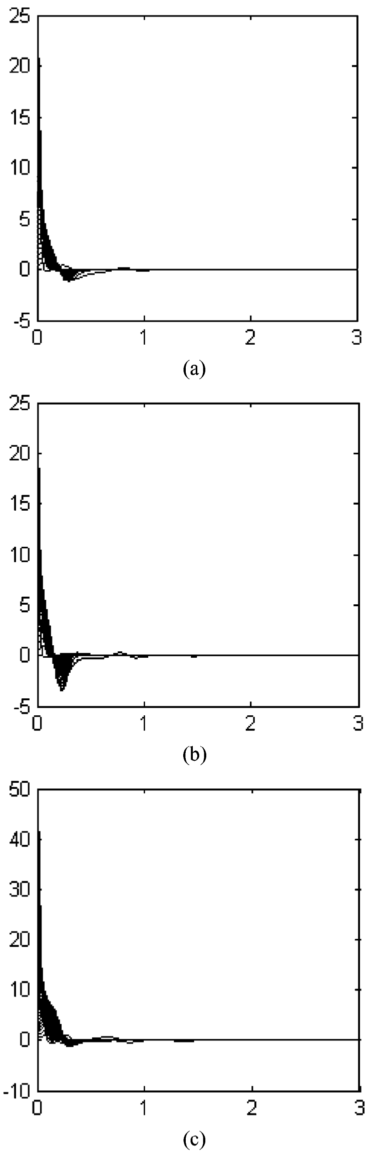

Fig. 1. Synchronization errors of network (16)–(17). (a) $\mathbf{e}_{i1}$ $(1 \leq i \leq 50)$. (b) $\mathbf{e}_{i2}$ $(1 \leq i \leq 50)$. (c) $\mathbf{e}_{i3}$ $(1 \leq i \leq 50)$.

where $1 \leq i \leq 50$, one gets

$$\|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{s})\|_2 \leq 3\sqrt{2(a^2 + M^2)}(\|\mathbf{e}_i\|_2 + \|\mathbf{e}_{i+1}\|_2 + \|\mathbf{e}_{i+2}\|_2).$$

Thus, H2) and H3) hold. According to Theorem 2, the synchronous solution $\mathbf{S}(t)$ of dynamical network (16)–(17) is globally asymptotically stable.

Assume that $k_i = 1, d_i(0) = 1, \mathbf{x}_i(0) = (4 + 0.5i,\, 5 + 0.5i,\, 6 + 0.5i)$ for $1 \leq i \leq 50$ and $\mathbf{s}(0) = (4,\, 5,\, 6)$. The synchronous error $\mathbf{e}_i$ is shown in Fig. 1. Obviously, the zero error is globally asymptotically stable for dynamical network (16)–(17).

Remark 2: It is well known that the nearest-neighbor coupled ring lattices are very hard to synchronize. This is because the coupling coefficient c satisfies $c = O(N^2)$. However, the above example shows that the synchronization of nearest-neighbor coupled ring lattice will be relatively easy by adding a simple adaptive controller.

V. Conclusion

We have further studied the locally and globally adaptive synchronization of an uncertain complex dynamical network. Several novel network synchronization criteria have been proved by using Lyapunov stability theory. Compared with some similar results, our assumptions and adaptive controllers are very simple. Furthermore, the effectiveness of these synchronization criteria have been demonstrated by numerical simulations.

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