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Brief paper

Pinning adaptive synchronization of a general complex dynamical network☆

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Abstract

There are two challenging fundamental questions in pinning control of complex networks: (i) How many nodes should a network with fixed network structure and coupling strength be pinned to reach network synchronization? (ii) How much coupling strength should a network with fixed network structure and pinning nodes be applied to realize network synchronization? To fix these two questions, we propose a general complex dynamical network model and then further investigate its pinning adaptive synchronization. Based on this model, we attain several novel adaptive synchronization criteria which indeed give the positive answers to these two questions. That is, we provide a simply approximate formula for estimating the detailed number of pinning nodes and the magnitude of the coupling strength for a given general complex dynamical network. Here, the coupling-configuration matrix and the inner-coupling matrix are not necessarily symmetric. Moreover, our pinning adaptive controllers are rather simple compared with some traditional controllers. A Barabási–Albert network example is finally given to show the effectiveness of the proposed synchronization criteria.

Keywords: Complex networks; Adaptive synchronization; Pinning control

1. Introduction

Nowadays complex networks lie in everywhere in our daily life, such as the Internet, World Wide Web, communication networks, power grid networks, social networks, genetic regulatory networks, and so on (Chen & Zhou, 2006; Hu, Yang, & Liu, 1998; Lü & Chen, 2005; Lü, Han, Yu, & Chen, 2004; Lü, Yu, & Chen, 2004; Lü, Yu, Chen, & Cheng, 2004; Newman, Barabási, & Watts, 2006; Pandit & Amritkar, 1999; Pavel, 2004; Strogatz, 2001; Timme, Wolf, & Geisel, 2004; Wang & Chen, 2002; Wu, 2006; Zhou, Lu, & Lü, 2006). A complex network is a large set of interconnected nodes, where the nodes and connections can be everything. Over the past one decade, complex networks have been intensively investigated in various disciplines, such as mathematics, physics, biology, engineering, and social sciences (Lü & Chen, 2005; Strogatz, 2001).

Synchronization is a kind of typical collective behaviors and basic motions in nature (Pecora & Carroll, 1990; Wu & Chua, 1995). Recently, one of the interesting and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes in a network. It is well known that there are many useful network synchronization phenomena in our real life, such as the synchronous transfer of digital or analog signals in communication networks. More recently, adaptive synchronization in networks or coupled oscillators has received an increasing attention (Newman et al., 2006; Strogatz, 2001).

As we know now, the real-world complex networks normally have a large number of nodes. Therefore, it is usually difficult to control a complex network by adding the controllers to all nodes. To reduce the number of the controllers, a natural approach is to control a complex network by pinning part of nodes. Grigoriev, Cross, and Schuster (1997) studied the pinning control of spatiotemporal chaos. Parekh, Parthasarathy,

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and Sinha (1998) further investigated the global and local control of spatiotemporal chaos in coupled map lattices. Wang and Chen (2002) proposed an effective measure to pin a scale-free dynamical network to its equilibrium. Sorrentino, Bernardo, Garofalo, and Chen (2007) explored the controllability of complex networks via pinning. However, we do not know how many nodes (at least) a complex network should be pinned to realize network synchronization. In particular, what is the minimum number of nodes which can guarantee network synchronization? Therefore, it is very interesting to ask the following two fundamental questions in pinning control of complex dynamical networks: (i) How many nodes should a network with fixed network structure and coupling strength be pinned to achieve network synchronization? (ii) How much coupling strength should a network with fixed network structure and pinning nodes be applied to realize network synchronization? In the following, based on a given general complex dynamical network, we will give the positive answers to these two basic questions. In brief, we provide a simply approximate formula for estimating the detailed number of pinning nodes and the magnitude of the coupling strength.

Using Lyapunov stability theory, we attain several locally and globally asymptotically stable network synchronization criteria for a general complex dynamical network. In particular, the coupling configuration matrix and the inner-coupling matrix are not necessary symmetric. Compared with some similar designs (Grigoriev et al., 1997; Newman et al., 2006; Parekh et al., 1998; Sorrentino et al., 2007; Strogatz, 2001; Wang & Chen, 2002), our pinning adaptive controllers are very simple. In addition, the pinning nodes can be randomly selected. It indeed provides some new insights for the future practical engineering design.

The left paper is organized as follows. A general complex dynamical network model and several mathematical preliminaries are introduced in Section 2. In Section 3, several locally and globally pinning adaptive synchronization criteria for the general complex dynamical networks are deduced. A Barabási–Albert (BA) network (Barabási & Albert, 1999) example is then given to show the effectiveness of the proposed network synchronization criteria in Section 4. Conclusions are finally drawn in Section 5.

2. Preliminaries

This section proposes a generally controlled complex dynamical network model and gives several necessary mathematical preliminaries.

2.1. A generally controlled complex dynamical network model

Consider a generally controlled complex dynamical network consisting of N identical nodes with linearly diffusive couplings, which is described by

$$\dot{\mathbf{x}}_i = \mathbf{g}(\mathbf{x}_i, t) + \sum_{j=1}^{N} c_{ij}\mathbf{A}\mathbf{x}_j + \mathbf{v}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N), \tag{1}$$

where $1 \leqslant i \leqslant N$, $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})^{\mathrm{T}} \in \mathbf{R}^n$ is the state vector of the ith node, $\mathbf{g} : \Omega \times \mathbf{R}^+ \to \mathbf{R}^n$ is a smooth nonlinear vector field, node dynamics is $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, t)$, $\mathbf{v}_i \in \mathbf{R}^n$ are the control inputs satisfying $\mathbf{v}_i(\mathbf{x}, \ldots, \mathbf{x}) = \mathbf{0}$. Here, $\mathbf{A} \in \mathbf{R}^{n\times n}$ is the inner-coupling matrix and $\mathbf{C} = (c_{ij})_{N\times N} \in \mathbf{R}^{N\times N}$ is the coupling configuration matrix. If there is a link from node i to node $j\,(j \neq i)$, then $c_{ij} > 0$ and c_{ij} is the coupling strength; otherwise, $c_{ij} = 0$. Assume that $\mathbf{C}$ is a diffusive matrix satisfying

$$\sum_{j=1}^{N} c_{ij} = 0.$$

Suppose that the coupling matrix $\mathbf{C}$ is irreducible. Note that here the coupling matrix $\mathbf{C}$ and inner coupling matrix $\mathbf{A}$ do not need to be symmetric. Hereafter, let $\mathbf{x} = \mathbf{s}(t; t_0, \mathbf{x}_0) \in \mathbf{R}^n$ with $\mathbf{x}_0 \in \mathbf{R}^n$, denoted as $\mathbf{s}(t)$, be a solution of the node system $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, t)$. Then $\mathbf{S}(t) = (\mathbf{s}^{\mathrm{T}}(t), \mathbf{s}^{\mathrm{T}}(t), \ldots, \mathbf{s}^{\mathrm{T}}(t))^{\mathrm{T}} \in \mathbf{R}^{n\times N}$ is a synchronous solution of the controlled complex dynamical network (1) because it is a diffusive coupling network. Note that $\mathbf{s}(t)$ can be an equilibrium point, a periodic orbit, an aperiodic orbit, even a chaotic orbit in the phase space.

2.2. Mathematical preliminaries

To begin with, one presents a rigorous mathematical definition for network synchronization.

Definition 1. Let $\mathbf{x}_i(t; t_0, \mathbf{X}_0)(1 \leqslant i \leqslant N)$ be a solution of the controlled network (1), where $\mathbf{X}_0 = (\mathbf{x}_1^0, \mathbf{x}_2^0, \ldots, \mathbf{x}_N^0) \in \mathbf{R}^{n\times N}$. Assume that $\mathbf{g} : \Omega \times \mathbf{R}^+ \to \mathbf{R}^n$ and $\mathbf{v}_i : \Omega \times \cdots \times \Omega \to \mathbf{R}^n (1 \leqslant i \leqslant N)$ are continuously differentiable, $\Omega \subseteq \mathbf{R}^n$. If there is a nonempty subset $\Gamma \subseteq \Omega$, with $\mathbf{x}_i^0 \in \Gamma (1 \leqslant i \leqslant N)$, such that $\mathbf{x}_i(t; t_0, \mathbf{X}_0) \in \Omega$ for all $t \geqslant t_0$, $1 \leqslant i \leqslant N$, and

$$\lim_{t\to\infty} \|\mathbf{x}_i(t; t_0, \mathbf{X}_0) - \mathbf{s}(t; t_0, \mathbf{x}_0)\|_2 = 0, \quad 1 \leqslant i \leqslant N, \tag{2}$$

where $\mathbf{x}_0 \in \Omega$, then the controlled network (1) is said to achieve network synchronization and $\Gamma \times \cdots \times \Gamma$ is called the region of synchrony for the dynamical network (1).

Define error vectors as

$$\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t), \tag{3}$$

where $1 \leqslant i \leqslant N$. According to the controlled network (1), the error system is then described by

$$\dot{\mathbf{e}}_i = \mathbf{g}(\mathbf{x}_i, t) - \mathbf{g}(\mathbf{s}, t) + \sum_{j=1}^{N} c_{ij}\mathbf{A}\mathbf{e}_j + \mathbf{v}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N), \tag{4}$$

where $1 \leqslant i \leqslant N$.

To realize network synchronization, the controllers $\mathbf{v}_i$ should guide the error vectors (3) to approach zero as t goes to infinity. That is,

$$\lim_{t\to+\infty} \|\mathbf{e}_i(t)\|_2 = 0, \tag{5}$$

where $1 \leqslant i \leqslant N$.

In the following, a useful hypothesis and a lemma are introduced.

Hypothesis 1. (**H1**). Suppose that $\|\mathbf{Dg}(\mathbf{s})\|_2$ is bounded, where $\mathbf{Dg}(\mathbf{s})$ is the Jacobian of $\mathbf{g}$ evaluated at $\mathbf{x}=\mathbf{s}$. That is, there exists a nonnegative constant α satisfying $\|\mathbf{Dg}(\mathbf{s})\|_2 \leqslant \alpha$.

Hereafter, assume that $\mathbf{A} \neq 0$ and $\|\mathbf{A}\|_2 = \gamma > 0$. Denote $\rho_{\min}$ as the minimum eigenvalue of the matrix $(\mathbf{A}+\mathbf{A}^{\mathrm{T}})/2$.

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ be the eigenvalues of the matrix $(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\mathrm{T}})/2$, where $\hat{\mathbf{C}}$ is a modifying matrix of $\mathbf{C}$ via replacing the diagonal elements c_{ii} by $(\rho_{\min}/\gamma)c_{ii}$. Note that matrix $\hat{\mathbf{C}}$ does not possess the property of zero row sums. Moreover, there does not exist a definite relationship between the eigenvalues of $\mathbf{C}$ and those of $\hat{\mathbf{C}}$ for the general matrix $\mathbf{C}$.

Lemma 1. (Wilkinson, 1965; Zheng, Chen, Mo, & Huang, 2002). *Assume that* $\mathbf{E}, \mathbf{F}$ *are the* $N \times N$ *Hermitian matrices. Suppose that* $\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_N$, $\zeta_1 \geqslant \zeta_2 \geqslant \cdots \geqslant \zeta_N$, *and* $\varsigma_1 \geqslant \varsigma_2 \geqslant \cdots \geqslant \varsigma_N$ *are the eigenvalues of* $\mathbf{E}, \mathbf{F}, \mathbf{E}+\mathbf{F}$, *respectively. Then one has*

$$\xi_i + \zeta_N \leqslant \varsigma_i \leqslant \xi_i + \zeta_1, \quad 1 \leqslant i \leqslant N.$$

3. Pinning adaptive synchronization of a controlled complex dynamical network

This section further investigates the local and global pinning adaptive synchronization of a controlled complex dynamical network. Several network synchronization criteria are attained by using Lyapunov stability theory.

3.1. Local pinning adaptive synchronization

Without loss of generality, assume that the first l nodes $1 \leqslant i \leqslant l$ are selected and pinned with the adaptive controllers, which are described by

$$\begin{cases} \mathbf{v}_i = -p_i \mathbf{e}_i, \quad \dot{p}_i = q_i \|\mathbf{e}_i\|_2^2, \quad 1 \leqslant i \leqslant l, \\ \mathbf{v}_i = 0, \quad \text{otherwise}, \end{cases} \tag{6}$$

where $q_i (i = 1, \ldots, l)$ are any positive constants. Thus the controlled network (1) can be rewritten as follows:

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{g}(\mathbf{x}_i, t) + \sum_{j=1}^{N} c_{ij} \mathbf{A} \mathbf{x}_j - p_i \mathbf{e}_i, \quad 1 \leqslant i \leqslant l, \\ \dot{p}_i = q_i \|\mathbf{e}_i\|_2^2, \quad 1 \leqslant i \leqslant l, \\ \dot{\mathbf{x}}_i = \mathbf{g}(\mathbf{x}_i, t) + \sum_{j=1}^{N} c_{ij} \mathbf{A} \mathbf{x}_j \quad \text{otherwise}. \end{cases} \tag{7}$$

According to (4) and (6), linearizing system (7) at zero yields

$$\begin{cases} \dot{\mathbf{e}}_i = \mathbf{Dg}(\mathbf{s}) \mathbf{e}_i + \sum_{j=1}^{N} c_{ij} \mathbf{A} \mathbf{e}_j - p_i \mathbf{e}_i, \quad 1 \leqslant i \leqslant l, \\ \dot{p}_i = q_i \|\mathbf{e}_i\|_2^2, \quad 1 \leqslant i \leqslant l, \\ \dot{\mathbf{e}}_i = \mathbf{Dg}(\mathbf{s}) \mathbf{e}_i + \sum_{j=1}^{N} c_{ij} \mathbf{A} \mathbf{x}_j, \quad (l+1) \leqslant i \leqslant N, \end{cases} \tag{8}$$

where $\mathbf{Dg}(\mathbf{s})$ is the Jacobian of $\mathbf{g}$ evaluated at $\mathbf{x}=\mathbf{s}$.

Theorem 1. *Suppose that* (**H1**) *holds. If there exists a natural number* $1 \leqslant l < N$ *satisfying* $\lambda_{l+1} < -\frac{\alpha}{\gamma}$, *then the synchronous solution* $\mathbf{S}(t)$ *of controlled network* (1) *is locally asymptotically stable under the pinning adaptive controllers*

$$\begin{cases} \mathbf{v}_i = -p_i \mathbf{e}_i, \; \dot{p}_i = q_i \|\mathbf{e}_i\|_2^2, \quad 1 \leqslant i \leqslant l, \\ \mathbf{v}_i = 0, \quad (l+1) \leqslant i \leqslant N, \end{cases} \tag{9}$$

where q_i *are positive constants for* $1 \leqslant i \leqslant l$.

Proof. Construct a Lyapunov candidate as follows:

$$V = \frac{1}{2} \sum_{i=1}^{N} \mathbf{e}_i^{\mathrm{T}} \mathbf{e}_i + \frac{1}{2} \sum_{i=1}^{l} \frac{(p_i - p)^2}{q_i}, \tag{10}$$

where p are positive constants satisfying $p > \alpha + \gamma \lambda_1$.

Thus the differential coefficient of V is described by

$$\begin{aligned} \dot{V} &= \frac{1}{2} \sum_{i=1}^{N} (\dot{\mathbf{e}}_i^{\mathrm{T}} \mathbf{e}_i + \mathbf{e}_i^{\mathrm{T}} \dot{\mathbf{e}}_i) + \sum_{i=1}^{l} \frac{(p_i - p)\dot{p}_i}{q_i} \\ &= \sum_{i=1}^{N} \mathbf{e}_i^{\mathrm{T}} \left(\frac{\mathbf{Dg}(\mathbf{s}) + \mathbf{Dg}^{\mathrm{T}}(\mathbf{s})}{2} \right) \mathbf{e}_i \\ &\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} \mathbf{e}_i^{\mathrm{T}} \mathbf{A} \mathbf{e}_j - \sum_{i=1}^{l} p \mathbf{e}_i^{\mathrm{T}} \mathbf{e}_i \\ &= \sum_{i=1}^{N} \mathbf{e}_i^{\mathrm{T}} \left(\frac{\mathbf{Dg}(\mathbf{s}) + \mathbf{Dg}^{\mathrm{T}}(\mathbf{s})}{2} \right) \mathbf{e}_i + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij} \mathbf{e}_i^{\mathrm{T}} \mathbf{A} \mathbf{e}_j \\ &\quad + \sum_{i=1}^{N} c_{ii} \mathbf{e}_i^{\mathrm{T}} \left(\frac{\mathbf{A} + \mathbf{A}^{\mathrm{T}}}{2} \right) \mathbf{e}_i - \sum_{i=1}^{l} p \mathbf{e}_i^{\mathrm{T}} \mathbf{e}_i \\ &\leqslant \sum_{i=1}^{N} \alpha \mathbf{e}_i^{\mathrm{T}} \mathbf{e}_i + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \gamma c_{ij} \|\mathbf{e}_i\|_2 \|\mathbf{e}_j\|_2 \\ &\quad + \sum_{i=1}^{N} c_{ii} \rho_{\min} \mathbf{e}_i^{\mathrm{T}} \mathbf{e}_i - \sum_{i=1}^{l} p \mathbf{e}_i^{\mathrm{T}} \mathbf{e}_i \\ &= \mathbf{e}^{\mathrm{T}} (\alpha \mathbf{I}_N + \gamma \hat{\mathbf{C}} - \mathbf{D}) \mathbf{e} \\ &= \mathbf{e}^{\mathrm{T}} \left(\alpha \mathbf{I}_N + \gamma \frac{\hat{\mathbf{C}} + \hat{\mathbf{C}}^{\mathrm{T}}}{2} - \mathbf{D}\right) \mathbf{e}, \end{aligned}$$

where $\mathbf{D} = \mathrm{diag}\{\underbrace{p, \ldots, p}_{l}, \underbrace{0, \ldots, 0}_{N-l}\}$, $\mathbf{e} = (\|\mathbf{e}_1\|_2, \|\mathbf{e}_2\|_2, \ldots, \|\mathbf{e}_N\|_2)^{\mathrm{T}}$.

Assume that $\hat{\lambda}_1 \geqslant \hat{\lambda}_2 \geqslant \cdots \geqslant \hat{\lambda}_N$ are the eigenvalues of the matrix $\left(\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\mathrm{T}}}{2} - \frac{\mathbf{D}}{\gamma}\right)$. According to Lemma 1, since $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ are the eigenvalues of the matrix $\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\mathrm{T}}}{2}$, one has

$$\begin{cases} -\frac{p}{\gamma} + \lambda_N \leqslant \hat{\lambda}_i \leqslant -\frac{p}{\gamma} + \lambda_1, \quad 1 \leqslant i \leqslant l, \\ \lambda_N \leqslant \hat{\lambda}_i \leqslant \lambda_1, \quad l+1 \leqslant i \leqslant N \end{cases}$$

and

$$-\frac{p}{\gamma}+\lambda_i \leqslant \hat{\lambda}_i \leqslant \lambda_i, \quad 1 \leqslant i \leqslant N.$$

Since $\left(\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\mathrm{T}}}{2}-\frac{\mathbf{D}}{\gamma}\right)$ is a real symmetric matrix, then there exists an orthogonal matrix $\mathbf{P}$ satisfying $\left(\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\mathrm{T}}}{2}-\frac{\mathbf{D}}{\gamma}\right)=\mathbf{P}^{\mathrm{T}}\mathrm{diag}\{\hat{\lambda}_1,\ldots,\hat{\lambda}_N\}\mathbf{P}$.

Therefore, one gets

$$\begin{aligned}\dot{V} &\leqslant \mathbf{e}^{\mathrm{T}}\left(\alpha\mathbf{I}_N+\gamma\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\mathrm{T}}}{2}-\mathbf{D}\right)\mathbf{e} \\ &= (\mathbf{Pe})^{\mathrm{T}}\mathrm{diag}\{(\alpha+\gamma\hat{\lambda}_1),\ldots,(\alpha+\gamma\hat{\lambda}_l), \\ &\quad (\alpha+\gamma\hat{\lambda}_{l+1}),\ldots,(\alpha+\gamma\hat{\lambda}_N)\}(\mathbf{Pe}) \\ &\leqslant (\mathbf{Pe})^{\mathrm{T}}\mathrm{diag}\{(\alpha-p+\gamma\lambda_1),\ldots,(\alpha-p+\gamma\lambda_1), \\ &\quad (\alpha+\gamma\lambda_{l+1}),\ldots,(\alpha+\gamma\lambda_N)\}(\mathbf{Pe}) \\ &= (\mathbf{Pe})^{\mathrm{T}}\mathbf{Q}(\mathbf{Pe}),\end{aligned}$$

where $\mathbf{Q}=\mathrm{diag}\{(\alpha-p+\gamma\lambda_1),\ldots,(\alpha-p+\gamma\lambda_1),(\alpha+\gamma\lambda_{l+1}),\ldots,(\alpha+\gamma\lambda_N)\}$.

According to the assumption of Theorem 1, $p>\alpha+\gamma\lambda_1$, one has $\alpha+\gamma\lambda_1-p<0$. Then $\alpha+\gamma\lambda_i<0$ for $(l+1)\leqslant i\leqslant N$.

Therefore, $\mathbf{Q}$ is a negative definite matrix. It follows that $\mathbf{Pe}\to 0$ as $t\to+\infty$. Since $\mathbf{P}$ is an orthogonal matrix, the error vector $\eta=(\mathbf{e}_1^{\mathrm{T}},\mathbf{e}_2^{\mathrm{T}},\ldots,\mathbf{e}_N^{\mathrm{T}})^{\mathrm{T}}\to\mathbf{0}$ as $t\to+\infty$. That is, the synchronous solution $\mathbf{S}(t)$ of controlled network (1) is locally asymptotically stable under the pinning adaptive controllers (9).

Thus the proof is completed. □

Theorem 1 indicates that the network synchronization depends on three basic elements: node dynamics (α), network structure (λ_{l+1}), and inner coupling means (γ, $\rho_{\min}$). In detail, the inequality $\lambda_{l+1}<-\frac{\alpha}{\gamma}$ indeed gives a sufficient condition of these three basic elements as above for the network synchronization.

Corollary 1. *Suppose that* **H1** *holds and* $\mathbf{C}=c\overline{\mathbf{C}}$, where $\overline{\mathbf{C}}$ *is a diffusive coupling matrix with* $\bar{c}_{ij}=0$ or $1(j\neq i)$. *For a given* c *(*l*), if* $\bar{\lambda}_{l+1}<-\alpha/(c\gamma)$ *(*$\bar{\lambda}_{l+1}<0$ *and* $c>-\alpha/(\gamma\bar{\lambda}_{l+1})$*), then the synchronous solution* $\mathbf{S}(t)$ *of controlled network* (1) *is locally asymptotically stable under the pinning adaptive controllers* (9)*, where* $\bar{\lambda}_1\geqslant\bar{\lambda}_2\geqslant\cdots\geqslant\bar{\lambda}_N$ *are the eigenvalues of the matrix* $(\hat{\overline{\mathbf{C}}}+\hat{\overline{\mathbf{C}}}^{\mathrm{T}})/2$ *and* $\hat{\overline{\mathbf{C}}}$ *is a modifying matrix of* $\overline{\mathbf{C}}$ *via replacing the diagonal elements* $\bar{c}_{ii}$ *by* $(\rho_{\min}/\gamma)\bar{c}_{ii}$.

The conclusion is obvious and the proof is omitted here.

Remark 1.

(1) From Theorem 1, for the controlled complex dynamical network (1) with fixed network structure and coupling strength, one can approximately estimate the minimum number of the pinning nodes which can achieve network synchronization. Moreover, one can also randomly select l ($\lambda_{l+1}<-\frac{\alpha}{\gamma}$) pinning nodes to add the controllers since Theorem 1 does not limit the detailed positions of the pinning controllers. Furthermore, for the controlled complex dynamical network (1) with fixed network structure and pinning nodes, one can approximately estimate the magnitude of the coupling strength which can reach network synchronization. In addition, all constant gains $q_i\,(1\leqslant i\leqslant l)$ are any positive numbers.

(2) Assume that $\mathbf{C}=c\overline{\mathbf{C}}$ is symmetric and $\rho_{\min}=\gamma$. Thus the eigenvalues satisfy the condition: $0=\bar{\lambda}_1>\bar{\lambda}_2\geqslant\cdots\geqslant\bar{\lambda}_N$. If $c>-\frac{\alpha}{\gamma\bar{\lambda}_2}$, then network (1) can achieve network synchronization under the adaptive controller (9) with $l=1$.That is, a single controller is enough to pin control a complex dynamical network provided that the coupling strength is larger than a given upper critical value $-\alpha/(\gamma\bar{\lambda}_2)$.

3.2. Global pinning adaptive synchronization

This subsection studies the globally pinning adaptive synchronization of the controlled complex dynamical network (1). Several network synchronization criteria are drawn.

Rewrite the node system $\dot{\mathbf{x}}=\mathbf{g}(\mathbf{x},t)$ as follows:

$$\dot{\mathbf{x}}=\mathbf{G}\mathbf{x}+\mathbf{h}(\mathbf{x},t),$$

where $\mathbf{Gx}$ is the linear part of the node dynamics with $\mathbf{G}\in\mathbf{R}^{n\times n}$ and $\mathbf{h}:\Omega\times\mathbf{R}^{+}\to\mathbf{R}^{n}$ is a continuously differential nonlinear function. Thus the controlled network (1) is recasted as follows:

$$\dot{\mathbf{x}}_i=\mathbf{G}\mathbf{x}_i+\mathbf{h}(\mathbf{x}_i,t)+\sum_{j=1}^{N}c_{ij}\mathbf{A}\mathbf{x}_j+\mathbf{v}_i(\mathbf{x}_1,\ldots,\mathbf{x}_N), \tag{11}$$

where $1\leqslant i\leqslant N$.

Since $\mathbf{G}$ is a given constant matrix, then there exists a non-negative constant β satisfying $\|\mathbf{G}\|_2\leqslant\beta$.

Hypothesis 2. (**H2**). Suppose that $\mathbf{h}(\mathbf{x},t)$ is Lipschitz continuous. That is, there exists a Lipschitz constant μ satisfying $\|\mathbf{h}(\mathbf{x}_i,t)-\mathbf{h}(\mathbf{s},t)\|_2\leqslant\mu\|\mathbf{e}_i\|_2$ for $1\leqslant i\leqslant N$.

Similarly, one can select the first l nodes as the pinning nodes. According to (6) and (11), the error system is then described by

$$\begin{cases}\dot{\mathbf{e}}_i=\mathbf{G}\mathbf{e}_i+\mathbf{h}(\mathbf{x}_i,t)-\mathbf{h}(\mathbf{s},t)+\sum_{j=1}^{N}c_{ij}\mathbf{A}\mathbf{e}_j-p_i\mathbf{e}_i, & 1\leqslant i\leqslant l,\\ \dot{p}_i=q_i\|\mathbf{e}_i\|_2^2, & 1\leqslant i\leqslant l,\\ \dot{\mathbf{e}}_i=\mathbf{G}\mathbf{e}_i+\mathbf{h}(\mathbf{x}_i,t)-\mathbf{h}(\mathbf{s},t)+\sum_{j=1}^{N}c_{ij}\mathbf{A}\mathbf{x}_j, & (l+1)\leqslant i\leqslant N.\end{cases} \tag{12}$$

Thus one can obtain the following globally pinning adaptive synchronization criterion.

Theorem 2. *Suppose that* (H2) *holds. If there exists a natural number* $1\leqslant l<N$ *satisfying* $\lambda_{l+1}<-(\beta+\mu)/\gamma$*, then the* *synchronous solution* $\mathbf{S}(t)$ *of controlled network* (11) *is globally asymptotically stable under the pinning adaptive controllers*

$$\begin{cases} \mathbf{v}_i = -p_i \mathbf{e}_i,\ \dot{p}_i = q_i \|\mathbf{e}_i\|_2^2, & 1 \leqslant i \leqslant l, \\ \mathbf{v}_i = 0, & (l+1) \leqslant i \leqslant N, \end{cases} \tag{13}$$

where q_i *are positive constants for* $1 \leqslant i \leqslant l$.

Proof. Similarly, construct Lyapunov function (10) with $p > \beta + \mu + \gamma\lambda_1$, one gets

$$\begin{aligned} \dot{V} &= \frac{1}{2}\sum_{i=1}^{N}(\dot{\mathbf{e}}_i^{\mathrm{T}}\mathbf{e}_i + \mathbf{e}_i^{\mathrm{T}}\dot{\mathbf{e}}_i) + \sum_{i=1}^{l}\frac{(p_i - p)\dot{p}_i}{q_i} \\ &= \sum_{i=1}^{N}\mathbf{e}_i^{\mathrm{T}}\left(\frac{\mathbf{G}+\mathbf{G}^{\mathrm{T}}}{2}\mathbf{e}_i + \mathbf{h}(\mathbf{x}_i, t) - \mathbf{h}(\mathbf{s}, t)\right) \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\mathbf{e}_i^{\mathrm{T}}\mathbf{A}\mathbf{e}_j - \sum_{i=1}^{l} p\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_i \\ &\leqslant \sum_{i=1}^{N}(\beta+\mu)\,\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_i + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N}\gamma c_{ij}\|\mathbf{e}_i\|_2\|\mathbf{e}_j\|_2 \\ &\quad + \sum_{i=1}^{N} c_{ii}\rho_{\min}\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_i - \sum_{i=1}^{l} p\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_i \\ &= \mathbf{e}^{\mathrm{T}}[(\beta+\mu)\,\mathbf{I}_N + \gamma\hat{\mathbf{C}} - \mathbf{D}]\mathbf{e} \\ &= \mathbf{e}^{\mathrm{T}}\left[(\beta+\mu)\,\mathbf{I}_N + \gamma\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\mathrm{T}}}{2} - \mathbf{D}\right]\mathbf{e} \\ &\leqslant (\mathbf{P}\mathbf{e})^{\mathrm{T}}\bar{\mathbf{Q}}(\mathbf{P}\mathbf{e}), \end{aligned}$$

where $\mathbf{D} = \mathrm{diag}\{\underbrace{p,\ldots,p}_{l},\underbrace{0,\ldots,0}_{N-l}\}$, $\mathbf{e} = (\|\mathbf{e}_1\|_2, \|\mathbf{e}_2\|_2, \ldots, \|\mathbf{e}_N\|_2)^{\mathrm{T}}$, and $\bar{\mathbf{Q}} = \mathrm{diag}\{(\beta+\mu+\gamma\lambda_1 - p), \ldots, (\beta+\mu+\gamma\lambda_1 - p), (\beta+\mu+\gamma\lambda_{l+1}), \ldots, (\beta+\mu+\gamma\lambda_N)\}$.

According to the assumption of Theorem 2, $p > \beta+\mu+\gamma\lambda_1$, one has $\beta+\mu+\gamma\lambda_1 - p < 0$. Then $\beta+\mu+\gamma\lambda_i < 0$ for $(l+1) \leqslant i \leqslant N$.

Therefore, $\bar{\mathbf{Q}}$ is a negative definite matrix. It follows that $\mathbf{P}\mathbf{e} \to 0$ as $t \to +\infty$. Since $\mathbf{P}$ is an orthogonal matrix, the error vector $\eta = (\mathbf{e}_1^{\mathrm{T}}, \mathbf{e}_2^{\mathrm{T}}, \ldots, \mathbf{e}_N^{\mathrm{T}})^{\mathrm{T}} \to \mathbf{0}$ as $t \to +\infty$. That is, the synchronous solution $\mathbf{S}(t)$ of controlled network (11) is globally asymptotically stable under the pinning adaptive controllers (13).

The proof is thus completed. □

Corollary 2. *Suppose that* (H2) *holds and* $\mathbf{C} = c\overline{\mathbf{C}}$, *where* $\overline{\mathbf{C}}$ *is a diffusive coupling matrix with* $\bar{c}_{ij} = 0$ *or* $1 (j \neq i)$. *For a given* c (l), *if* $\bar{\lambda}_{l+1} < -(\beta+\mu)/(c\gamma)$ $\left(\bar{\lambda}_{l+1} < 0 \text{ and } c > -(\beta+\mu)/(\gamma\bar{\lambda}_{l+1})\right)$, *then the synchronous solution* $\mathbf{S}(t)$ *of controlled network* (11) *is globally asymptotically stable under the pinning adaptive controllers* (13), *where* $\bar{\lambda}_1 \geqslant \bar{\lambda}_2 \geqslant \cdots \geqslant \bar{\lambda}_N$ *are the eigenvalues of the matrix* $(\hat{\overline{\mathbf{C}}} + \hat{\overline{\mathbf{C}}}^{\mathrm{T}})/2$ *and* $\hat{\overline{\mathbf{C}}}$ *is a modifying matrix of* $\overline{\mathbf{C}}$ *via replacing the diagonal elements* $\bar{c}_{ii}$ *by* $(\rho_{\min}/\gamma)\bar{c}_{ii}$.

This conclusion can be easily deduced from Theorem 2 and thus the proof is omitted here.

4. Numerical simulation

In this section, a simple example is used to explain the effectiveness of the proposed network synchronization criteria.

Lorenz, Chen, and Lü systems are the typical benchmark three dimensional chaotic systems (Lü & Chen, 2002; Lü, Chen, Cheng, & Celikovsky, 2002). Here we assume that the controlled network (1) consists of 500 identical Lü systems (Lü & Chen, 2002), which is described by

$$\dot{\mathbf{x}}_i = \mathbf{G}\mathbf{x}_i + \mathbf{h}(\mathbf{x}_i) + c\sum_{j=1}^{500}\bar{c}_{ij}\mathbf{A}\mathbf{x}_j + \mathbf{v}_i, \tag{14}$$

where $1 \leqslant i \leqslant 500$, $\mathbf{A} = \mathrm{diag}\{1, 1.2, 1\}$, $\overline{\mathbf{C}} = (\bar{c}_{ij})_{500\times 500}$ is a symmetrically diffusive coupling matrix with $\bar{c}_{ij} = 0$ or $1 (j \neq i)$. Here the coupling coefficient $c = 40$. The node dynamics is then given by

$$\begin{aligned} \dot{\mathbf{x}}_i &= \begin{pmatrix} -r_1 & r_1 & 0 \\ 0 & r_3 & 0 \\ 0 & 0 & -r_2 \end{pmatrix}\begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} \\ &= \mathbf{G}\mathbf{x}_i + \mathbf{h}(\mathbf{x}_i), \end{aligned}$$

where $1 \leqslant i \leqslant 500$ and $r_1 = 36, r_2 = 3, r_3 = 20$.

Obviously, $\gamma = \|\mathbf{A}\|_2 = 1.2$, $\beta = \|\mathbf{G}\|_2 \approx 52.9843$, and

$$\mathbf{h}(\mathbf{x}_i) - \mathbf{h}(\mathbf{s}) = \begin{pmatrix} 0 \\ -x_{i1}x_{i3} + s_1 s_3 \\ x_{i1}x_{i2} - s_1 s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_{i3}e_{i1} - s_1 e_{i3} \\ x_{i2}e_{i1} + s_1 e_{i2} \end{pmatrix},$$

where $1 \leqslant i \leqslant 500$.

It is well known that Lü attractor is bounded. Here we suppose that all nodes are running in the given bounded region. Our theoretical and numerical analyses show that there exist constants M_1=25, M_2=30 and M_3=45 satisfying $\|x_{ij}\|, \|s_j\| \leqslant M_j$ for $1 \leqslant i \leqslant 500$ and $1 \leqslant j \leqslant 3$ (Li, Lu, Wu, & Chen, 2006). Thus one gets

$$\begin{aligned} &\|\mathbf{h}(\mathbf{x}_i) - \mathbf{h}(\mathbf{s})\|_2 \\ &\quad\leqslant \sqrt{(-x_{i3}e_{i1} - s_1 e_{i3})^2 + (x_{i2}e_{i1} + s_1 e_{i2})^2} \\ &\quad\leqslant \sqrt{2M_1^2 + M_2^2 + M_3^2}\|\mathbf{e}_i\|_2 \\ &\quad\approx 64.6142\|\mathbf{e}_i\|_2. \end{aligned}$$

Let $\mu = 64.6142$. Then one has

$$-\frac{\beta+\mu}{c\gamma} = -\frac{52.9843 + 64.6142}{40 \times 1.2} = -2.4500.$$

Here we assume that the network structure of (14) obeys the scale-free distribution of the BA model (Barabási & Albert, 1999). The parameters of BA model are given by $m_0 = m = 5$, $N = 500$. Without loss of generality, one randomly chooses the pinning nodes. Since $\bar{\lambda}_{26} = -2.4436$ and $\bar{\lambda}_{27} = -2.4611$,

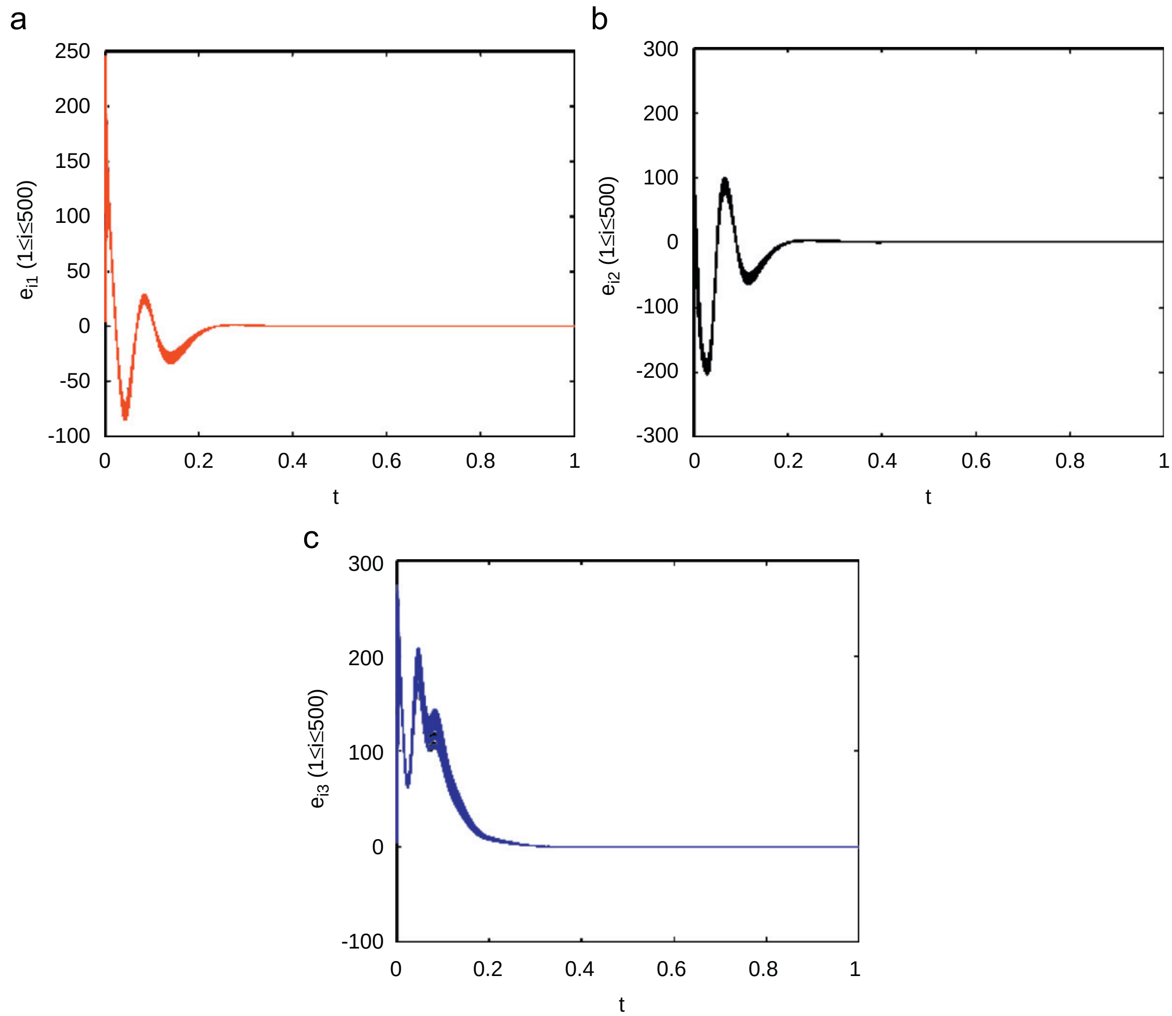

Fig. 1. Synchronization errors $e_{ij}(1 \leqslant i \leqslant 500, 1 \leqslant j \leqslant 3, 0 \leqslant t \leqslant 1)$ of the controlled network (14). (a) e_{i1} $(1 \leqslant i \leqslant 500)$ for $0 \leqslant t \leqslant 1$. (b) e_{i1} $(1 \leqslant i \leqslant 500)$ for $0 \leqslant t \leqslant 1$. (c) e_{i2} $(1 \leqslant i \leqslant 500)$ for $0 \leqslant t \leqslant 1$.

then there exists a natural number $l = 26$ satisfying $\bar{\lambda}_{26+1} = -2.4611 < -2.4500$. That is, one only needs to pin 26 nodes of 500 nodes for realizing network synchronization. From Corollary 2, the synchronous solution $\mathbf{S}(t)$ of the controlled network (14) is globally asymptotically stable under the pinning adaptive controllers (13).

In all numerical simulations, the parameters are given as follows: $l = 26$, $q_i = 0.1$ and $p_i(0) = 1$ for $1 \leqslant i \leqslant 26$, $q_i = p_i(0) = 0$ for $27 \leqslant i \leqslant 500$, $x_i(0) = (4 + 0.5i, 5 + 0.5i, 6 + 0.5i)^{\mathrm{T}}$, $s(0) = (4, 5, 6)^{\mathrm{T}}$, where $1 \leqslant i \leqslant 500$. The synchronous errors $e_i(1 \leqslant i \leqslant 500)$ are shown in Fig. 1. Clearly, the controlled network (14) is globally asymptotically stable at zero under the pinning controllers (13) with $l = 26$.

Our numerical simulations show that the convergence speed of the intentional pinning scheme is much faster than that of the random pinning scheme. However, the random pinning scheme has more wide application than the intentional pinning scheme since the random pinning scheme does not need to know the exact network degree distribution. It should be especially pointed out that our network synchronization criteria are also valid for the above two pinning schemes. That is, if we know the exact network degree distribution information, then we can use the intentional pinning scheme; however, if we do not know the exact network degree distribution information, then we can use the random pinning scheme.

5. Conclusions

We have further explored the following two fundamental questions in complex dynamical networks: (i) How many nodes should a network with fixed network structure and coupling strength be pinned to achieve network synchronization? (ii) How much coupling strength should a network with fixed network structure and pinning nodes be applied to realize network synchronization? Based on a given network model, several adaptive synchronization criteria are attained by using Lyapunov stability theory and the pinning control method. In particular, these synchronization criteria indeed give the positive answers to the proposed two questions. That is, we provide a simply approximate formula for estimating the detailed number of pinning nodes and the magnitude of the coupling strength for a given complex dynamical network. In addition, the coupling configuration matrix and the inner-coupling matrix are not necessarily symmetric. Moreover, our pinning adaptive controllers are rather simple compared with some traditional controllers. Finally, these adaptive synchronization criteria also provide some new insights for the network synchronization and the possible applications in the real-world engineering systems.

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