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Superdiffusion criteria on duplex networks

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ABSTRACT

Diffusion processes widely exist in nature. Some recent papers concerning diffusion processes focus their attention on multiplex networks. Superdiffusion, a phenomenon by which diffusion processes converge to equilibrium faster on multiplex networks than on single networks in isolation, may emerge because diffusion can occur both within and across layers. Some studies have shown that the emergence of superdiffusion depends on the topology of multiplex networks if the interlayer diffusion coefficient is large enough. This paper proposes some superdiffusion criteria relating to the Laplacian matrices of the two layers and provides a construction mechanism for generating a superdiffusible two-layered network. The method we proposed can be used to guide the discovery and construction of superdiffusible multiplex networks without calculating the second smallest Laplacian eigenvalues.

Compared with single-layer networks, what will happen to the diffusion efficiency of multiplex networks due to the interaction between layers? Gómez *et al.*[1] proved that the diffusion will be faster in a duplex network than in its slowest layer when the interlayer diffusion coefficient is much larger than the intralayer diffusion coefficients. They further found that it is possible that the diffusion of a multiplex network is faster than that of both layers acting separately, which indicates that the multiplex network has superdiffusive behavior. The emergence of superdiffusion depends on the specific structures coupled together. If a network can have superdiffusive behavior when the interlayer and intralayer diffusion coefficients take certain values, we call it superdiffusible. In the following, we will provide a mechanism for constructing superdiffusible networks.

I. INTRODUCTION

Complex networks have become an important tool for researching the dynamics of complex systems, such as power systems, ecosystems, and transportation systems.[2–7] One of the main focuses of the study of complex networks is to investigate the relationship between the network topology and the dynamic processes that occur on them (e.g., diffusion processes and synchronization processes).[8,9]

With the study of complex systems deepening, it is now recognized that real complex systems are rarely formed by single, isolated networks with links of equivalent meanings and connotations.[10–14] A natural extension then is to describe a complex system as a set of networks coupled across levels (multiplex network). For instance, in social multiplex networks, we communicate information through different types of social relationships (friend, colleague, acquaintance, family member, etc.); in transportation multiplex networks, diseases can spread over different layers of transportation (bus, metro, train, etc.). Currently, multiplex networks have been investigated in many fields, including molecular biology, epidemic spreading process, ecology, transportation networks, and infrastructures.[1,15–20] Different from the dynamical progresses on single networks, those on multiplex networks may show new phenomena. For example, in the study of synchronization on multiplex networks, intralayer and interlayer synchronization have been proposed and investigated.[21,22] Another widely studied network model is the higher-order network, which captures the many-body interactions of complex systems. Higher-order networks are usually represented by simplicial complexes which, differently from graphs, are not only formed by nodes and links, but also include triangles, tetrahedra, and so on.[23–26] Several studies on diffusion, synchronization, and epidemic spreading have shown that taking into account the higher-order organization of networks can lead to emergent behavior remarkably different from that of graphs, where interactions are limited to pairs of two nodes only.[27–29]

Some recent studies analyzed diffusion dynamics taking place on multiplex networks and showed that in some cases, multiplex networks can have superdiffusive behavior, that is, that diffusive processes in multiplex networks are faster than in any single networks that form it separately.[1,30–34] The emergence of superdiffusion depends on the topology of multiplex networks and the intralayer and interlayer diffusion coefficients. It has been shown in the literature[1,33] that for an undirected duplex network, the larger the ratio of the interlayer diffusion coefficient to the intralayer diffusion coefficient, the greater the diffusion rate, which means that if the interlayer diffusion coefficient is much larger than the intralayer diffusion coefficient, the emergence of superdiffusion only depends on the topology of duplex networks. The relationship between the network topology and the superdiffusion phenomenon has been studied,[31] and the numerical simulation results show that the emergence of superdiffusion is independent from the presence of an edge overlap between layers. In addition, Tejedor *et al.*[33,35] studied the diffusion dynamics on directed multiplex networks and revealed a new and unexpected signature; that is, differently from their undirected counterparts, they can exhibit a nonmonotonic rate of convergence to a steady state as a function of the degree of coupling, wherein the diffusion is even faster than that of the theoretical limit of undirected systems, i.e., the diffusion in the integrated network obtained from the aggregation of all layers. These and other characteristics of superdiffusible networks remain to be further studied. This work attempts to find out a class of superdiffusible duplex networks.

In this paper, we first find types of easily identifiable networks with algebraic connectivity less than 0.5. Then, we prove that these types of networks can be used to construct a superdiffusible duplex network.

The rest of the paper is structured as follows: the model network and some basic preliminaries are introduced in Sec. II; the main results are presented in Sec. III; Sec. IV shows two examples to illustrate our results; and finally, the conclusions and possible further extensions of this work are given in Sec. V.

II. NETWORK MODEL AND PRELIMINARIES

A. Graph theory preliminaries

To begin with, some necessary concepts in graph theory are introduced below.

A graph can be described by $G = (V, E)$, where V represents the set of vertices and E represents the set of edges. We denote edge joining vertices u and v of a graph by uv. $G_1 (= (V_1, E_1)) \subseteq G_2$ $(= (V_2, E_2))$ means $V_1 = V_2$ and $E_1 \subseteq E_2$.

The adjacency matrix of an undirected graph $G = (V, E)$, whose vertices are explicitly ordered $v_1, v_2, \ldots, v_N$, is a $N \times N$ matrix $A(G) = (a_{ij})$ such that

$$a_{ij} = a_{ji} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where $v_i, v_j \in V$. The Laplacian matrix of an undirected graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix composed of the degrees of the vertices. The Laplacian eigenvalues of G are the eigenvalues of its Laplacian matrix $L(G)$. The second smallest Laplacian eigenvalue, $\lambda_2(G)$, is called the algebraic connectivity of G.[36] Denoting the vector $(1, 1, \ldots, 1)^T$ by $\boldsymbol{e}$ and the set of all column vector $\boldsymbol{z}$ by $\boldsymbol{W}$ such that $\boldsymbol{z}^T\boldsymbol{z} = 1$, $\boldsymbol{z}^T\boldsymbol{e} = 0$, then we have $\lambda_2(G) = \min_{z\in W} \boldsymbol{z}^T L(G)\boldsymbol{z}$ by Courant's theorem.

A walk in a graph G is an alternating sequence of vertices and edges, $W = \hat{v}_0, e_1, \hat{v}_1, e_2, \hat{v}_2, \ldots, e_n, \hat{v}_n$ such that for $j = 1, \ldots, n$, the vertices $\hat{v}_{j-1}$ and $\hat{v}_j$ are the end points of the edge e_j. A graph is connected if there exists at least a walk between every pair of vertices. A vertex of a graph is called as a cut-vertex if its removal increases the number of connected components.[37]

A graph is bipartite if its vertices can be partitioned into two sets in such a way that no edge joins two vertices in the same set.[37] The symbol $K_{p,q}$ denotes a bipartite graph where the two partite sets have cardinalities p and q, and each vertex in one partite set is adjacent to all the vertices in the other partite set. A tree is a connected graph with no cycles. In particular, the tree $K_{1,N}$ is also called a star graph.

B. Network model

Consider a two-layered network with N nodes in each layer. Assume that every single layer is connected, unweighted, and undirected. The diffusion dynamics on the duplex network can be described by

$$\begin{cases} \dfrac{d\boldsymbol{x}_i^{1\,T}}{dt} = C_{intra} \displaystyle\sum_{j=1}^{N} a_{ij}^1 \left(\boldsymbol{x}_j^{1\,T} - \boldsymbol{x}_i^{1\,T}\right) + C_{inter}\left(\boldsymbol{x}_i^{2\,T} - \boldsymbol{x}_i^{1\,T}\right), \\ \dfrac{d\boldsymbol{x}_i^{2\,T}}{dt} = C_{intra} \displaystyle\sum_{j=1}^{N} a_{ij}^2 \left(\boldsymbol{x}_j^{2\,T} - \boldsymbol{x}_i^{2\,T}\right) + C_{inter}\left(\boldsymbol{x}_i^{1\,T} - \boldsymbol{x}_i^{2\,T}\right), \end{cases} \tag{2}$$

where $\boldsymbol{x}_i^k(t) = (x_{i,1}^k, x_{i,2}^k, \ldots, x_{i,M}^k)^T$ represents the state of the ith $(i = 1, \ldots, N)$ node in layer k $(k = 1, 2)$, $A_k = (a_{ij}^k)$ is the adjacency matrix of layer k, C_{intra} denotes the intralayer diffusion coefficient, and C_{inter} stands for the interlayer diffusion coefficient. Let $\boldsymbol{x} = (\boldsymbol{x}^{1\,T}, \boldsymbol{x}^{2\,T})^T = (\boldsymbol{x}_1^{1\,T}, \ldots, \boldsymbol{x}_N^{1\,T}, \boldsymbol{x}_1^{2\,T}, \ldots, \boldsymbol{x}_N^{2\,T})^T$. Then, Eq. (2) can be rewritten as

$$\dot{\boldsymbol{x}} = -(\mathscr{L} \otimes I_M)\boldsymbol{x}, \tag{3}$$

where

$$\mathscr{L} = \begin{pmatrix} C_{intra}L_1 + C_{inter}I_N & -C_{inter}I_N \\ -C_{inter}I_N & C_{intra}L_2 + C_{inter}I_N \end{pmatrix},$$

$\otimes$ denotes the Kronecker product, $L_k = D_k - A_k$, D_k is the diagonal matrix of the vertex degrees of layer k, and I_N and I_M are the identity matrices with dimension N and M, respectively.

The eigenvalues of the supra-Laplacian matrix $\mathscr{L}$ are $0 = \Lambda_1 < \Lambda_2 \leqslant \cdots \leqslant \Lambda_{2N}$. Because $\mathscr{L} \otimes I_M$ is symmetric, the solution of Eq. (2) is $\varphi_j(t) = \varphi_j(0)\, e^{-\hat{\Lambda}_j t}$ $(j = 1, 2, \ldots, 2MN)$, where $\hat{\Lambda}_j$ are the eigenvalues of $\mathscr{L} \otimes I_M$ and $0 = \Lambda_1 = \hat{\Lambda}_1 = \cdots = \hat{\Lambda}_M < \Lambda_2 = \hat{\Lambda}_{M+1} = \cdots = \hat{\Lambda}_{2M} \leqslant \cdots \hat{\Lambda}_{2MN} = \Lambda_{2N}$. Since the general solution is a linear combination of the normal solutions, the time scale of the duplex to reach equilibrium is thus controlled by Λ_2. In other words, the diffusion time scale of the duplex is proportional to $\frac{1}{\Lambda_2}$. Similarly, the diffusion time scale of layer k

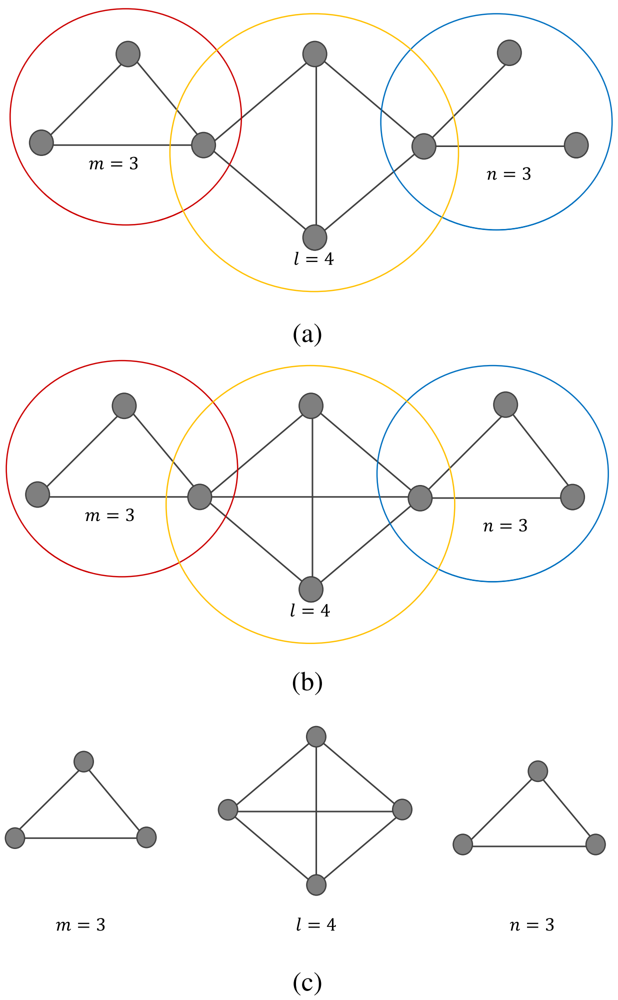

FIG. 1. (a) A network in $\mathbb{G}_{3,3,4}$. Note that the array (m, n, l) may not be unique for some networks. (b) Network $G_{3,3,4}$. (c) Three fully connected parts.

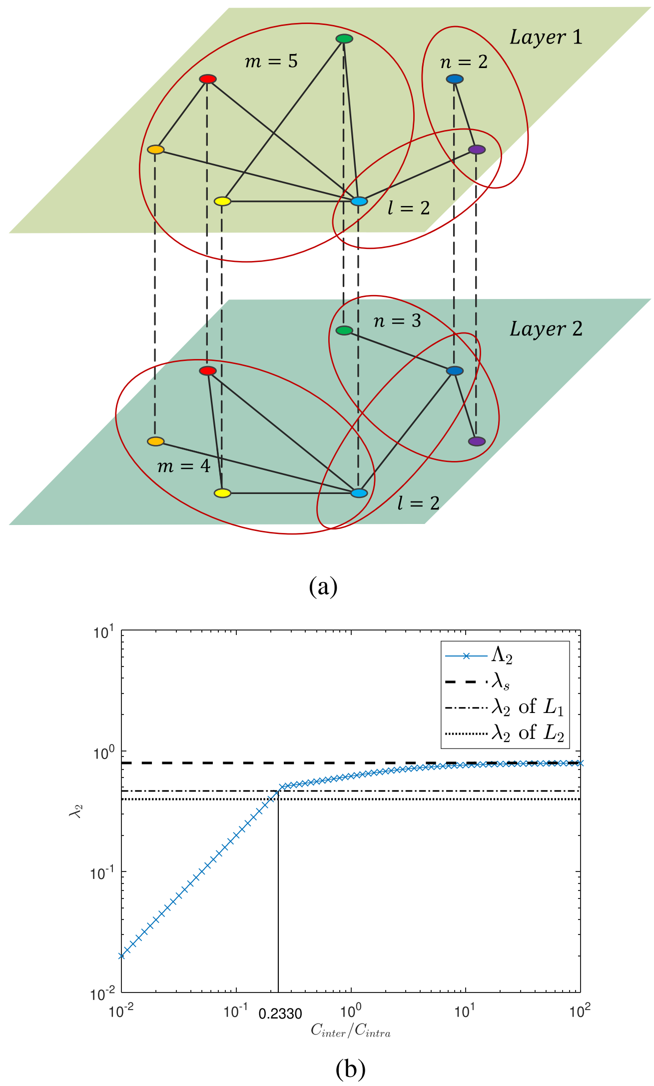

FIG. 2. (a) A duplex network consisting of $G_1 \in \mathbb{G}_{5,2,2}$ and $G_2 \in \mathbb{G}_{4,3,2}$. (b) Comparison between the second smallest eigenvalues λ_2 of the different Laplacian matrices.

is proportional to $\frac{1}{\lambda_2^k}$, where λ_2^k represents the smallest nonzero eigenvalue of L_k.

Definition 1. *Superdiffusion emerges in a duplex network when* $\Lambda_2 > \max\{\lambda_2^1, \lambda_2^2\}$.

Denote the smallest nonzero eigenvalue of $(L_1 + L_2)/2$ as λ_s. Gómez *et al.*[1] have shown that Λ_2 approaches λ_s as $C_{inter} \gg C_{intra}$. Moreover, Tejedor *et al.*[33] have proved that Λ_2 increases when C_{inter}/C_{intra} increases. Thus, it is concluded that λ_s is the supremum of Λ_2, which means that superdiffusion cannot emerge in duplex networks if $\lambda_s \leqslant \max\{\lambda_2^1, \lambda_2^2\}$. Therefore, from the above literature, we have the following definition:

Definition 2. *A duplex network is said to be superdiffusible if* $\lambda_s > \max\{\lambda_2^1, \lambda_2^2\}$.

A duplex network being superdiffusible does not mean that superdiffusion can emerge in a duplex network for any C_{inter} and C_{intra}. For example, when C_{inter} is much less than C_{intra}, superdiffusion cannot emerge in a duplex network.

III. A CLASS OF SUPERDIFFUSIBLE DUPLEX NETWORKS

Consider a network with $n + m + l - 2$ vertices, which contains two cut-vertices. The network is divided into three parts by the two cut-vertices, which contain m, n, and l vertices, respectively [see Fig. 1(a)]. Denote the set of such networks as $\mathbb{G}_{m,n,l}$. Furthermore, if all three parts are fully connected for $G \in \mathbb{G}_{m,n,l}$, we denote it as

$G_{m,n,l}$ [see Fig. 1(b)]. Obviously, $G_{m,n,l}$ can be obtained by adding the necessary edge to a graph $G \in \mathbb{G}_{m,n,l}$.

Lemma 1 (Ref. 36). *$\lambda_2(G_1) \leqslant \lambda_2(G_2)$ if $G_1 \subseteq G_2$.*

According to Lemma 1, for any $G \in \mathbb{G}_{m,n,l}$, one has $\lambda_2(G) \leqslant \lambda_2(G_{m,n,l})$. Thus, we have the following lemma.

Lemma 2. *For a graph $G \in \mathbb{G}_{m,n,l}$, if $2 \leqslant l < \left(3 - m - n + \sqrt{m^2 + 14mn - 16m + n^2 - 16n + 16}\right)/2$, then $\lambda_2(G) < 0.5$.*

Proof. Let

$$H_1 = \begin{pmatrix} m-1 & -1 & \cdots & -1 & -1 \\ -1 & m-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & m-1 & -1 \\ -1 & -1 & \cdots & -1 & m+l-2 \end{pmatrix}_{m\times m},$$

$$H_2 = \begin{pmatrix} n+l-2 & -1 & \cdots & -1 & -1 \\ -1 & n-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & n-1 & -1 \\ -1 & -1 & \cdots & -1 & n-1 \end{pmatrix}_{n\times n},$$

$$H_3 = H_4^T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \end{pmatrix}_{m\times n},$$

$$H_5 = H_6^T = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ -1 & \cdots & -1 \end{pmatrix}_{m\times (l-2)},$$

$$H_7 = H_8^T = \begin{pmatrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n\times (l-2)},$$

$$H_9 = \begin{pmatrix} l-1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & l-1 \end{pmatrix}_{(l-2)\times (l-2)}.$$

For $l = 2$,

$$L(G_{m,n,2}) = \begin{pmatrix} H_1 & H_3 \\ H_4 & H_2 \end{pmatrix}. \tag{4}$$

Let

$$H_1^1 = \begin{pmatrix} m-1-x & -1 & \cdots & -1 & -1 \\ -1 & m-1-x & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & m-1-x & -1 \\ -1 & -1 & \cdots & -1 & m+l-2-x \end{pmatrix}_{m\times m},$$

$$H_2^1 = \begin{pmatrix} n+l-2-x & -1 & \cdots & -1 & -1 \\ -1 & n-1-x & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & n-1-x & -1 \\ -1 & -1 & \cdots & -1 & n-1-x \end{pmatrix}_{n\times n},$$

$$H_1^2 = \begin{pmatrix} 1 & 1 & \cdots & 1 & -\frac{m-1}{1-x} \\ 0 & m-x & \cdots & 0 & -1-\frac{m-1}{1-x} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & m-x & -1-\frac{m-1}{1-x} \\ 0 & 0 & \cdots & 0 & m+l-2-x-\frac{m-1}{1-x} \end{pmatrix}_{m\times m},$$

$$H_2^2 = \begin{pmatrix} n+l-2-x-\frac{n-1}{1-x} & 0 & \cdots & 0 & 0 \\ -1-\frac{n-1}{1-x} & n-x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1-\frac{n-1}{1-x} & 0 & \cdots & n-x & 0 \\ -\frac{n-1}{1-x} & 1 & \cdots & 1 & 1 \end{pmatrix}_{n\times n},$$

one has

$$\begin{aligned} \left|L(G_{m,n,2})-xI\right| &= \begin{vmatrix} H_1^1 & H_3 \\ H_4 & H_2^1 \end{vmatrix} \\ &= (1-x)^2 \begin{vmatrix} H_1^2 & H_3 \\ H_4 & H_2^2 \end{vmatrix} \\ &= f_1(x)(m-x)^{m-2}(n-x)^{n-2}x, \end{aligned} \tag{5}$$

where

$$f_1(x) = x^3 - x^2(m+n+2) + x(mn+m+n+2) - m - n. \tag{6}$$

For $l > 2$,

$$L(G_{m,n,l}) = \begin{pmatrix} H_1 & H_5 & H_3 \\ H_6 & H_9 & H_8 \\ H_4 & H_7 & H_2 \end{pmatrix}. \tag{7}$$

Let

$$H_9^1 = \begin{pmatrix} l-1-x & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & l-1-x \end{pmatrix}_{(l-2)\times(l-2)},$$

one has

$$\begin{aligned} \left|L(G_{m,n,l})-xI\right| &= \begin{vmatrix} H_1^1 & H_5 & H_3 \\ H_6 & H_9^1 & H_8 \\ H_4 & H_7 & H_2^1 \end{vmatrix} \\ &= (1-x)^2 \begin{vmatrix} H_1^2 & H_5 & H_3 \\ H_6 & H_9^1 & H_8 \\ H_4 & H_7 & H_2^2 \end{vmatrix} \\ &= (1-x)^2(m-x)^{m-2}(n-x)^{n-2} \begin{vmatrix} m+l-2-x-\frac{m-1}{1-x} & -1 & \cdots & -1 & -1 \\ -1 & l-1-x & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & l-1-x & -1 \\ -1 & -1 & \cdots & -1 & n+l-2-x-\frac{n-1}{1-x} \end{vmatrix} \\ &= (1-x)^2(m-x)^{m-2}(n-x)^{n-2} \begin{vmatrix} m+l-1-x-\frac{m-1}{1-x} & 0 & \cdots & 0 & 0 \\ -1 & l-1-x & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & l-1-x & -1 \\ -1 & -1 & \cdots & -1 & n+l-2-x-\frac{n-1}{1-x} \end{vmatrix} \end{aligned}$$

$$+\,(1-x)^2(m-x)^{m-2}(n-x)^{n-2}\begin{vmatrix} -1 & -1 & \cdots & -1 & -1 \\ -1 & l-1-x & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & l-1-x & -1 \\ -1 & -1 & \cdots & -1 & n+l-2-x-\frac{n-1}{1-x} \end{vmatrix}$$
$$= f_2(x)(l-x)^{l-3}(m-x)^{m-2}(n-x)^{n-2}x, \tag{8}$$

where

$$f_2(x) = (1-m)(1-n) - (1-x)^2(m+l-x-1)(n+l-x-1). \tag{9}$$

For $l = 2$, the spectrum of the Laplacian matrix $L(G_{m,n,2})$ of $G_{m,n,2}$ is $\underbrace{m,\ldots,m}_{m-2}, \underbrace{n,\ldots,n}_{n-2}, 0, x_1^1, x_2^1, x_3^1$, where x_1^1, x_2^1, x_3^1 are the three roots of $f_1(x) = 0$ and $x_1^1 \leqslant x_2^1 \leqslant x_3^1$. For $l > 2$, the spectrum of the Laplacian matrix $L(G_{m,n,l})$ is $\underbrace{m,\ldots,m}_{m-2}, \underbrace{n,\ldots,n}_{n-2}, \underbrace{l,\ldots,l}_{l-3}, 0, x_1^2, x_2^2, x_3^2, x_4^2$, where $x_1^2, x_2^2, x_3^2, x_4^2$ are the four roots of $f_2(x) = 0$ and $x_1^2 \leqslant x_2^2 \leqslant x_3^2 \leqslant x_4^2$. Because $f_1(1) > 0 (f_2(1) > 0)$ and $f_1(x) \to -\infty (f_2(x) \to -\infty)$ as $x \to -\infty$, one has $x_1^1 < 1 (x_1^2 < 1)$. In other words, $\lambda_2(G_{m,n,l})$ is x_1^1 or x_1^2. We discuss two cases of $l = 2$ and $l > 2$ separately.

1. Case $l = 2$: Because $f_1(0) < 0$, $f_1(1) > 0$, $f_1(\max(m,n)) < 0$, and $f_1(m+n) > 0$, one has $0 < x_1^1 < 1 < x_2^1 < \max(m,n) < x_3^1 < (m+n)$. Thus, $\lambda_2(G_{m,n,2}) < 0.5$ if and only if $f_1(0.5) > 0$; i.e., $m, n \geqslant 2$ and $m + n \geqslant 6$.
2. Case $l > 2$: Because the function $f_2(x)$ increases monotonically on the interval $(-\infty, 1)$, $f_2(x) \to -\infty$ as $x \to -\infty$ and $f_2(1) = (m-1)(n-1) > 0$, one has $\lambda_2(G_{m,n,l}) < 0.5$ if and only if $f_2(0.5) > 0$, i.e., $2 < l < (3 - m - n + \sqrt{m^2 + 14mn - 16m + n^2 - 16n + 16})/2$.

In summary, one has $\lambda_2(G_{m,n,l}) < 0.5$ if $2 \leqslant l < \left(3 - m - n + \sqrt{m^2 + 14mn - 16m + n^2 - 16n + 16}\right)/2$. In addition, since $\lambda_2(G_{m,n,s})$ is an upper bound of $\lambda_2(G)$, the lemma is proved. □

Lemma 3 (Ref. 36). $\lambda_2(K_{p,q}) = \min(p,q)$ *if* $p + q > 2$.

Lemma 2 reveals a class of networks on which diffusion dynamics has a limited diffusion rate. If it can be ensured that the eigenvalue λ_s of a duplex network with a limited diffusion rate on each layer is comparatively large, the duplex network may be superdiffusible. Therefore, according to Lemmas 2 and 3, we give the following theorem.

Theorem 1. *Let $G_1 = (V, E_1), G_2 = (V, E_2)$ be two networks with the same N vertices, and we assume that G_1 and G_2 are connected, unweighted, and undirected. If the following conditions hold:*

1. $G_1 \in \mathbb{G}_{m_1,n_1,l_1}, G_2 \in \mathbb{G}_{m_2,n_2,l_2}$;
2. $2 \leqslant l_i < \left(\sqrt{m_i^2 + 14m_in_i - 16m_i + n_i^2 - 16n_i + 16} - m_i - n_i + 3\right)/2$, *$i = 1,2$; and*
3. $K_{1,N-1} \subseteq G_1 \cup G_2$,

where $G_1 \cup G_2 = (V, E_1 \cup E_2)$, then a duplex network consisting of G_1 and G_2 is superdiffusible.

Proof. Denoting the Laplacian matrix of $G_1 \cup G_2$ by $\widetilde{L}$, one has

$$\begin{aligned}\lambda_s &= \frac{1}{2}\min_{z\in W} \boldsymbol{z}^T(L_1 + L_2)\boldsymbol{z} \\ &\geqslant \frac{1}{2}(\min_{z\in W} \boldsymbol{z}^T\widetilde{L}\boldsymbol{z} + \min_{z\in W} \boldsymbol{z}^T(L_1 + L_2 - \widetilde{L})\boldsymbol{z}) \\ &\geqslant \frac{1}{2}\min_{z\in W} \boldsymbol{z}^T\widetilde{L}\boldsymbol{z} \\ &= \frac{1}{2}\lambda_2(G_1 \cup G_2).\end{aligned} \tag{10}$$

Based on Lemma 1 and condition 3, one has

$$\lambda_2(G_1 \cup G_2) \geqslant \lambda_2(K_{1,N-1}) = 1. \tag{11}$$

Thus, according to Lemma 2 and condition 2, one obtains

$$\lambda_s \geqslant \frac{1}{2}\lambda_2(G_1 \cup G_2) \geqslant 0.5 > \max\{\lambda_2^1, \lambda_2^2\}, \tag{12}$$

which means that superdiffusion could emerge in the duplex network when $C_{inter} \gg C_{intra}$. □

Remark 1. *Denote $\left(\sqrt{m^2 + 14mn - 16m + n^2 - 16n + 16} - m - n + 3\right)/2$ in Theorem 1 by $h(m,n)$. For $m, n \geqslant 1$, we have*

$$\begin{aligned}\frac{\partial h}{\partial m} &= \frac{1}{2}\left(\frac{(m+7n-8) - \sqrt{m^2 + 2m(7n-8) + n^2 - 16n + 16}}{\sqrt{m^2 + 2m(7n-8) + n^2 - 16n + 16}}\right) \\ &= \frac{24(n-1)^2}{\left((m+7n-8) + \sqrt{m^2 + 2m(7n-8) + n^2 - 16n + 16}\right)\sqrt{m^2 + 2m(7n-8) + n^2 - 16n + 16}} \\ &\geqslant 0.\end{aligned} \tag{13}$$

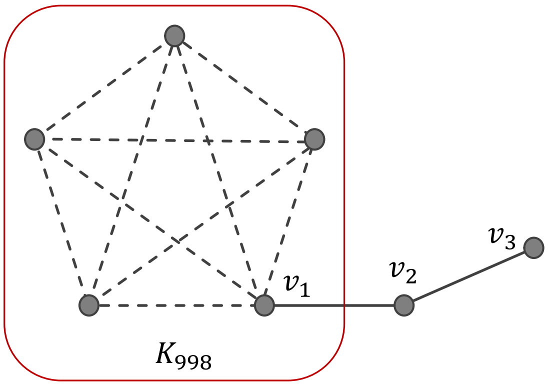

FIG. 3. Network $G_{998,2,2}$, where K_{998} represents the fully connected network with 998 nodes.

Similarly, we have $\partial h/\partial n \geqslant 0$. Thus, $h(m,n)$ is a monotone increasing function with respect to m and n. Therefore, the graph $G_{\hat{m},\hat{n},l}$ satisfies condition 2 if $G_{m,n,l}$ satisfies condition 2 ($\hat{m} \geqslant m$, $\hat{n} \geqslant n$).

Remark 2. *According to Lemma 2, if $T \neq K_{1,N-1}$ is a tree on $N \geqslant 6$ vertices, then $\lambda_2(T) < 0.5$. Thus, the graph $K_{1,N-1}$ is the tree with the largest algebraic connectivity of trees with $N \geqslant 6$ vertices. Any connected graph contains at least one tree. Therefore, we choose $K_{1,N-1}$ as the network contained in $G_1 \cup G_2$ in Theorem 1.*

IV. EXAMPLE

Theorem 1 can be used to discover the superdiffusible duplex network without calculating λ_2. The following example is presented to demonstrate the effectiveness of Theorem 1.

Example 1. *Consider a duplex network shown in Fig. 2(a). The network consists of $G_1 \in \mathbb{G}_{5,2,2}$ and $G_2 \in \mathbb{G}_{4,3,2}$. Meanwhile, $l_1 = 2 < \left(\sqrt{73} - 4\right)/2 = h(5,2)$, $l_2 = 2 < \left(\sqrt{97} - 4\right)/2 = h(4,3)$. Besides, in $G_1 \cup G_2$, the degree of the light blue vertex is 6; i.e., $G_1 \cup G_2$ contains $K_{1,N-1}$. According to Theorem 1, the duplex network is thus superdiffusible. In Fig. 2(b), we have represented, as indicated in the legend, the eigenvalues of each layer λ_2^k, the eigenvalue of the superposition of both layers λ_s, as well as the eigenvalue of the supra-Laplacian matrix Λ_2 as a function of C_{inter}/C_{intra}, respectively. Figure 2(b) shows that Λ_2 increases as C_{inter}/C_{intra} increases and approaches λ_s for large C_{inter}/C_{intra}. When $C_{inter}/C_{intra} > 0.2330$, we see that $\Lambda_2 > \{\max\{\lambda_2^1, \lambda_2^2\}$; that is, superdiffusion occurs in the duplex network, which means that the diffusion time scale associated with the whole duplex network is smaller than that of layer 1 and layer 2 if they were considered independently.*

For a duplex network that satisfies conditions 1 and 2 of Theorem 1, we add edges to a certain layer of the network so that the aggregated network of the two layers contains a star graph. According to Theorem 1, if the duplex network after adding edges still satisfies conditions 1 and 2 of Theorem 1, then we obtain a superdiffusible duplex network.

Example 2. *Consider a duplex network consisting of $G_{998,2,2}$ (see Fig. 3) and a network $\widehat{G}$ consisting of 1000 nodes with two adjacent cut-vertices. Because $\widehat{G}$ contains two adjacent cut-vertices, $\widehat{G} \in G_{m,n,2}$, where $m \geqslant 4$ and $n \geqslant 2$. According to Remark 1, since $\lambda_2(G_{4,2,2}) < 0.5$, we have $\lambda_2(\widehat{G}) \leqslant \lambda_2(G_{m,n,2}) < 0.5$. Similarly, we have $\lambda_2(G_{998,2,2}) < 0.5$. Denote the nodes connected to v_1, v_2, v_3 in $\widehat{G}$ by $\widehat{v}_1$, $\widehat{v}_2$, $\widehat{v}_3$, respectively. If $\widehat{v}_1$ and $\widehat{v}_3$ are directly connected or there is a node in $\widehat{G}$ that is connected to $\widehat{v}_2$ and $\widehat{v}_3$, then $G_{998,2,2} \cup \widehat{G}$ contains $K_{1,999}$ and the duplex consisting of $G_{998,2,2}$ and $\widehat{G}$ is superdiffusible.*

V. CONCLUSION

In this paper, the relationship between the diffusion rate in a network and its topology has been investigated, and novel superdiffusion criteria relating to the Laplacian matrices of the two layers have been proposed. Our result can be used to guide the discovery and construction of superdiffusible duplex networks. Furthermore, our result is practical because any network satisfying the conditions in Lemma 2 and $K_{1,N-1}$ is both easy to identify and construct. Moreover, the method we proposed can also be extended to multiplex networks with more than two layers.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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