Node Importance in Controlled Complex Networks

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***Abstract*—The problem of node importance in a general controlled complex network is investigated in this brief. An index, ControlRank index, to measure the node importance is presented. The index is relatively intuitive and helps unify various indexes that are effective only in the networks with some particular topology. If all nodes except one are pinned in a network, the importance of the unpinned node is sorted in descending order of degree; even if a node is the most significant node, it does not necessarily belong to the most efficient node set; it is very crucial to have an eye on the information balance as for information dissemination in a network; it is reasonable to rank node importance by sorting the node degrees in a scale-free network. The results and phenomena provide us with a broader and clearer perspective to view node importance in complex networks.**

***Index Terms*—Complex networks, node importance, synchronization, control.**

I. Introduction

DESPITE various studies of dynamics and topology of complex networks [1]–[4], node importance is an increasingly hot topic [5]–[13]. In many social and scientific fields, node importance is called centrality. To study centrality of a network is to determine which nodes have more influence to the whole network than others. Many problems in our society are actually problems of centrality. For examples, why did a software bug affect an estimated 55 million people in U.S. during the Northeast blackout of 2003? How to seed a message in the Internet so that the message spreads more quickly and more widely? What kind of immunization strategy should we take to avoid the outbreak of an epidemic?

When it comes to node importance in a controlled network, there are several effective but not so intuitive indexes. References [14], [15] addressed the problem of centrality by the eigenvectors of the adjacency matrix of a network. Reference [16] used the smallest eigenvalue of the controllability Gramian matrix to evaluate node importance in linear systems by estimating the bound of that eigenvalue. Different from these algorithms, a relatively intuitive ControlRank index is proposed by the stability analysis in this brief.

As for intuitive indexes, node degree is the simplest one. Degree centrality [5], [6] indicates that the number of neighbour nodes highlights the importance of a node. Another approach is the so called k-shell decomposition [9]. This decomposition skins the nodes shell by shell, the remaining nodes in the core are of more importance. Besides, there are many other approaches such as betweenness centrality [8], closeness centrality [11], Page Rank [10], Leader Rank [17] and H-index [18]. All these methods, however, are effective only in the networks with some particular topology. Examples include degree centrality used in the networks with a broad degree distribution; k-shell decomposition for the networks with highly connected but marginal nodes. The ControlRank index to be proposed in this brief aims to unify these different methods in a network.

Through proposing a sufficient and necessary condition (for a static network or a dynamical network), the ControlRank index is presented by sorting the minimum eigenvalues of matrices. It efficiently identifies node importance without isolated node and loop. We show that the major factor for node importance is that the important nodes should be chosen so that each cluster can receive the information efficiently. Another factor is node degree; if a node in the cluster has a relatively large degree, the chance of being controlled increases.

This brief is organized as follows. Some basic preliminaries are introduced in Section II. In Section III, the main results of node importance and helpful remarks are presented, including those in static networks and in dynamical networks. Examples and further illustrations are provided in Section IV. Finally, concluding remarks are given in Section V.

II. Preliminaries

To gain our main results, the following Lemmas [19] on normal matrices are necessary. A matrix in the set $\mathbb{M} \triangleq \{\mathcal{M} \,|\, \mathcal{M}^*\mathcal{M} = \mathcal{M}\mathcal{M}^*\}$ is called a normal matrix, where $\mathcal{M}^*$ is the conjugate transpose of $\mathcal{M}$.

Lemma 1: A matrix $\mathcal{M} \in \mathbb{M}$ if and only if there exists a diagonal matrix $\mathcal{A}$ and a unitary matrix $\mathcal{U}$ such that $\mathcal{U}^*\mathcal{M}\mathcal{U} = \mathcal{A}$.

Lemma 2: A matrix $\mathcal{M}_{N\times N} \in \mathbb{M}$ if and only if there is a real orthogonal matrix $\mathcal{V}$ such that

$$\mathcal{V}^\top \mathcal{M}\mathcal{V} = \begin{pmatrix} \mathcal{M}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathcal{M}_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathcal{M}_M \end{pmatrix},$$

where $\mathcal{M}_0 \triangleq diag\{\hat{\lambda}_1, \ldots, \hat{\lambda}_\psi\}$ is a real diagonal matrix whose elements are $\hat{\lambda}_1$ to $\hat{\lambda}_\psi$, $\mathcal{M}_m$ $(1 \le m \le M)$ is a real matrix of the form $\begin{pmatrix} \alpha_m & \beta_m \\ -\beta_m & \alpha_m \end{pmatrix}$, $\psi + 2M = N$.

It is seen that $\mathcal{M}$'s real eigenvalues are $\hat{\lambda}_1, \ldots, \hat{\lambda}_\psi$ and complex eigenvalues are $\alpha_m \pm i\beta_m$ $(1 \le m \le M)$, where

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$i = \sqrt{-1}$. Denote $\mathcal{M}^s \triangleq \frac{\mathcal{M}^\top + \mathcal{M}}{2}$. Since $\mathcal{V}^\top \mathcal{M}^s \mathcal{V} = \frac{1}{2}\left((\mathcal{V}^\top \mathcal{M} \mathcal{V})^\top + \mathcal{V}^\top \mathcal{M} \mathcal{V}\right) = \begin{pmatrix} \mathcal{M}_0 & \mathbf{0} \\ \mathbf{0} & \mathcal{M}' \end{pmatrix}$, where $\mathcal{M}' \triangleq diag\{\alpha_1, \alpha_1, \ldots, \alpha_m, \alpha_m\}$, one concludes that the real part of $\mathcal{M}$'s eigenvalue is just the eigenvalue of $\mathcal{M}^s$. This conclusion is very helpful to simplify our main results.

III. Main Results

A complex network is called a static network if the individual dynamics is static, namely the dynamics of a node is time invariant in the absence of control and node interactions; otherwise it is called a dynamical network. This brief investigates the node importance in both networks. To begin with, it is reasonable to assume that there is no isolated node and loop in either the static networks or the dynamical networks.

A. Static Network

Generally, a complex static network consisting of N individuals is depicted by $\dot{\mathbf{x}}_j = -c\sum_{k=1}^N l_{jk}\,\mathbf{x}_k$, where $j = 1, 2, \ldots, N$, $\mathbf{x}_j = (x_{j1}, x_{j2}, \ldots, x_{jn})^\top \in \mathbb{R}^n$ denotes the state vector of the j-th node, $c \in \mathbb{R}^+$ is the coupling strength, $\mathbf{L} = (l_{jk})_{N\times N} \in \mathbb{M}$ represents the Laplacian matrix. Introducing controllers, the controlled static network is depicted by $\dot{\mathbf{x}}_j = -c\sum_{k=1}^N l_{jk}\,\mathbf{x}_k + \mathbf{u}_j(\mathbf{x}_1, \ldots, \mathbf{x}_N)$, where $\mathbf{u}_j \in \mathbb{R}^n$ is the control inputs satisfying $\mathbf{u}_j(\mathbf{x}, \ldots, \mathbf{x}) = \mathbf{0}$.

To determine which node is the most important, consider the network with only one controller. Without loss of generality (the reason will be detailed in the proof of Th 3), one can select the first node to be controlled by the simple linear feedback controller $\mathbf{u}_1 = -c\,d_1\,(\mathbf{x}_1 - \mathbf{s})$, where $\mathbf{s} \in \mathbb{R}^n$ is the desired equilibrium. Then the controlled network is recast as

$$\begin{cases} \dot{\mathbf{x}}_1 = -c\sum_{k=1}^N l_{1k}\,\mathbf{x}_k - c\,d_1\,(\mathbf{x}_1 - \mathbf{s}) \\ \dot{\mathbf{x}}_j = -c\sum_{k=1}^N l_{jk}\,\mathbf{x}_k, \end{cases} \tag{1}$$

where $j = 2, \ldots, N$. Introducing the denotations $\mathbf{e}_j \triangleq \mathbf{x}_j - \mathbf{s}$ $(1 \le j \le N)$, $\mathbb{e} \triangleq (\mathbf{e}_1^\top, \mathbf{e}_2^\top, \ldots, \mathbf{e}_N^\top)^\top$ and $\mathbf{D} \triangleq diag\{d_1, 0, \ldots, 0\}$, we have the error system

$$\dot{\mathbb{e}} = -c\,[(\mathbf{L}+\mathbf{D}) \otimes \mathbf{I}_n]\,\mathbb{e}, \tag{2}$$

where $diag\{d_1, 0, \ldots, 0\}$ is a diagonal matrix whose elements are $d_1, 0, \ldots, 0$, $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ is the identity matrix. According to [20] and [21], the zero solution of the linear system (2) is stable if and only if

$$Re[\lambda\,(\mathbf{L}+\mathbf{D})] > \mathbf{0}, \tag{3}$$

where $\lambda(\cdot)$ is any eigenvalue of a matrix, $Re[\cdot]$ denotes the real part of a complex number. Based on Lemma 2, $Re[\lambda\,(\mathbf{L}+\mathbf{D})]$ is just the eigenvalue of the symmetric matrix $\frac{\mathbf{L}^\top + \mathbf{L}}{2} + \mathbf{D} \triangleq \mathbf{L}^s + \mathbf{D}$. Then, Eq. (3) turns into

$$\lambda(\mathbf{L}^s + \mathbf{D}) > 0. \tag{4}$$

Partition $\mathbf{L}^s + \mathbf{D}$ into blocks $l_{11} + d_1 \in \mathbb{R}$, $\acute{\mathbf{L}} \in \mathbb{R}^{1\times(N-1)}$, $\grave{\mathbf{L}} \in \mathbb{R}^{(N-1)\times 1}$ and $\hat{\mathbf{L}} \in \mathbb{R}^{(N-1)\times(N-1)}$. Due to

$$\begin{pmatrix} 1 & -\acute{\mathbf{L}}\,\hat{\mathbf{L}}^{-1} \\ \mathbf{0} & \mathbf{I}_{N-1} \end{pmatrix} \begin{pmatrix} l_{11}+d_1 & \acute{\mathbf{L}} \\ \grave{\mathbf{L}} & \hat{\mathbf{L}} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ -\hat{\mathbf{L}}^{-1}\,\grave{\mathbf{L}} & \mathbf{I}_{N-1} \end{pmatrix} = \begin{pmatrix} l_{11}+d_1 - \acute{\mathbf{L}}\,\hat{\mathbf{L}}^{-1}\,\grave{\mathbf{L}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{L}} \end{pmatrix},$$

one concludes that $\mathbf{L}^s + \mathbf{D} > \mathbf{0}$ equals to $\hat{\mathbf{L}} > \mathbf{0}$ and $l_{11} + d_1 - \acute{\mathbf{L}}\,\hat{\mathbf{L}}^{-1}\,\grave{\mathbf{L}} > 0$ based on the partitioned matrix theory [4], [22]. Let the linear feedback gain $d_1 > \acute{\mathbf{L}}\,\hat{\mathbf{L}}^{-1}\,\grave{\mathbf{L}} - l_{11}$, one has $l_{11} + d_1 - \acute{\mathbf{L}}\,\hat{\mathbf{L}}^{-1}\,\grave{\mathbf{L}} > 0$. Therefore, $\mathbf{L}^s + \mathbf{D} > \mathbf{0}$ is equivalent to $\hat{\mathbf{L}} > 0$. As a result, Eq. (4) holds if and only if

$$\lambda_{\min}(\hat{\mathbf{L}}) > 0. \tag{5}$$

As a result, we have a sufficient and necessary theorem.

Theorem 1: For the static network (1), the zero solution of system (2) is stable if and only if (5) holds, where $\hat{\mathbf{L}}$ is the minor matrix of $\mathbf{L}^s_{N\times N}$ by deleting the 1-st row-column pair.

According to Th 1 and the linear system theory [20], the larger the value of $\lambda_{\min}(\hat{\mathbf{L}})$ is, the faster the convergence rate of the zero solution of system (2) is.

For bidirectionally coupled networks, the lower right block $\hat{\mathbf{L}}$ of $\mathbf{L}$ is a diagonally dominant matrix, and then the minimum eigenvalue of $\hat{\mathbf{L}}$ is positive [3]. For unidirectionally coupled networks, the row-sums of $\mathbf{L}$ are 0s, while those of $\mathbf{L}^s$ may not be. Accordingly the minimum eigenvalue of $\hat{\mathbf{L}}$ may not be positive. If $\lambda_{\min}(\hat{\mathbf{L}}) < 0$, there exists an eigenvalue $\lambda_0(\mathbf{L}^s + \mathbf{D}) < 0$, such that $\lim_{t\to\infty} \|\mathbb{e}\| = \infty$, where $\|\cdot\|$ means any norm of a vector. Even though the control objective can not be achieved, it is obvious that the smaller the value of $\lambda_{\min}(\hat{\mathbf{L}})$, the faster the divergence rate of $\|\mathbb{e}\|$ is. In other words, the larger the value of $\lambda_{\min}(\hat{\mathbf{L}})$, the better. Thus $\lambda_{\min}(\hat{\mathbf{L}})$ is a crucial index for determining node importance in a static network.

B. Dynamical Network

Usually the individuals in a network evolve dynamically, and their evolutions in the absence of couplings are time-varying. Consider a controlled general complex dynamical network consisting of N identical nodes with linearly diffusive couplings, which is described by $\dot{\mathbf{x}}_j = \mathbf{f}(\mathbf{x}_j(t)) - c\sum_{k=1}^N l_{jk}\,\mathbf{x}_k + \mathbf{u}_j(\mathbf{x}_1, \ldots, \mathbf{x}_N)$, where $j = 1, 2, \ldots, N$, $\mathbf{f}: \Omega \times \mathbb{R}^+ \to \mathbb{R}^n$ is a smooth nonlinear vector field, $\Omega \subseteq \mathbb{R}^n$. The other denotations are the same as those in Section III-A. Assume that the desired state $\mathbf{s} = \mathbf{s}(t;\, t_0,\, \mathbf{x}_0) \in \mathbb{R}^n$ is a solution of the individual system satisfying $\dot{\mathbf{s}} = \mathbf{f}(\mathbf{s}(t))$. A single controller is applied to the first node: $\mathbf{u}_1 = -c\,d_1\,(\mathbf{x}_1 - \mathbf{s})$. The controlled dynamical network is transformed into

$$\begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}(\mathbf{x}_1(t)) - c\sum_{k=1}^N l_{1k}\,\mathbf{x}_k - c\,d_1\,(\mathbf{x}_1 - \mathbf{s}), \\ \dot{\mathbf{x}}_j = \mathbf{f}(\mathbf{x}_j(t)) - c\sum_{k=1}^N l_{jk}\,\mathbf{x}_k, \end{cases} \tag{6}$$

where $j = 2, \ldots, N$. Denoting $\mathbf{e} \triangleq (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N)$ and linearizing the error system at the trajectory $\mathbf{s}$, we obtain

$$\dot{\mathbf{e}} = \partial\,\mathbf{f}(\mathbf{s})\,\mathbf{e} - c\,\mathbf{e}\,(\mathbf{L}^\top + \mathbf{D}), \tag{7}$$

where $\partial\,\mathbf{f}(\mathbf{s})$ is the Jacobian matrix of $\mathbf{f}$ evaluated at $\mathbf{s}$.

According to the definition of the normal matrix in Section II, $\mathbf{L}, \mathbf{D} \in \mathbb{M}$ leads to $\mathbf{L}^\top + \mathbf{D} \in \mathbb{M}$. Further, there exists a diagonal matrix Λ and a unitary matrix $\mathbf{P}$ such that $\mathbf{P}^*\,(\mathbf{L}^\top + \mathbf{D})\,\mathbf{P} = \Lambda$ based on Lemma 1. The diagonal entries of Λ are the eigenvalues of $\mathbf{L}^\top + \mathbf{D}$, denoted as $\bar{\lambda}_1, \ldots, \bar{\lambda}_N$. Post-multiply Eq. (7) by $\mathbf{P}$ and let $\xi = (\xi_1, \ldots, \xi_N) \triangleq \mathbf{e}\,\mathbf{P}$. One gets

$$\dot{\xi}_j = (\partial\,\mathbf{f}(\mathbf{s}) - c\,\bar{\lambda}_j\,\mathbf{I}_n)\,\xi_j, \tag{8}$$

where $j = 1, 2, \ldots, N$. Since $\mathbf{P}$ is a unitary matrix, $\lim_{t\to\infty} \mathbf{e} = \mathbf{0}$ is equivalent to $\lim_{t\to\infty} \xi = \mathbf{0}$.

Based on [21], the only equilibrium $\mathbf{0}$ of system (8) is asymptotically stable if and only if

$$\lambda(\mathbf{L}^s + \mathbf{D}) - \frac{1}{c} Re[\lambda\,(\partial\,\mathbf{f}(\mathbf{s}))] > 0. \tag{9}$$

Similar to the deduction of Eq. (5), Eq. (9) equals to

$$\lambda_{\min}(\hat{\mathbf{L}}) > \frac{1}{c} Re[\lambda\,(\partial\,\mathbf{f}(\mathbf{s}))], \tag{10}$$

where $\hat{\mathbf{L}}_{(N-1)\times(N-1)}$ is the lower right block of $\mathbf{L}^s_{N\times N}$. Accordingly a sufficient and necessary condition for a dynamical network is obtained.

Theorem 2: For the dynamical network (6), the zero solution of the error system (7) is asymptotically stable if and only if Eq. (10) holds.

Suppose that the norm of $\partial\,\mathbf{f}(\mathbf{s})$ is upper bounded by $\theta > 0$, namely $\|\,\partial\,\mathbf{f}(\mathbf{s})\,\| \le \theta$, where $\|\cdot\|$ means any induced norm of a matrix. Due to that [23]

$$Re[\lambda\,(\partial\,\mathbf{f}(\mathbf{s}))] \le \rho(\partial\,\mathbf{f}(\mathbf{s})) \le \|\partial\,\mathbf{f}(\mathbf{s})\|,$$

where $\rho(\cdot)$ represents the spectral radius of a matrix, Eq. (10) holds if

$$\lambda_{\min}(\hat{\mathbf{L}}) > \frac{\theta}{c}. \tag{11}$$

Eq. (11) provides a sufficient condition of synchronization, which is consistent with the results in [4].

C. ControlRank Index

To sum up, we have gained from Th 1 and Th 2 that $\lambda_{\min}(\hat{\mathbf{L}})$ is important to node significance in both a static and a dynamical network. To sequence the node significance, we present Th 3.

Theorem 3: Node importance sequencing is determined in descending order of $\lambda_q\,(\mathbf{L}^s)$ $(1 \le q \le N)$, where $\lambda_q\,(\mathbf{L}^s)$ is the minimum eigenvalue of the minor matrix of $\frac{\mathbf{L}^\top + \mathbf{L}}{2}$ by removing the q-th row-column pair.

Proof: It is known that the bigger value of $\lambda_{\min}(\hat{\mathbf{L}})$ is beneficial to control a network with the first node controlled. If the q-th $(1 < q \le N)$ node is controlled, the diagonal matrix $\mathbf{D}$ should be $diag\{0, \ldots, 0, d_q, 0, \ldots, 0\}$. Applying similarity transformation, $\mathbf{L}^s + \mathbf{D}$ is transformed into $\mathbf{L}^s_q + \mathbf{D}_q$, where $\mathbf{L}^s_q$ is obtained by switching the q-th row-column pair with the 1-st row-column pair of $\mathbf{L}^s$, $\mathbf{D}_q = diag\{d_q, 0, \ldots, 0\}$. Actually this situation can be considered as changing the ordinals of the q-th and 1-st node in the network. Being similar matrices, $\mathbf{L}^s_q + \mathbf{D}_q$ and $\mathbf{L}^s + \mathbf{D}$ share the same eigenvalues. Based on the partitioned matrix theory and the deduction from Eq. (4) to Eq. (5), $\lambda_{\min}(\hat{\mathbf{L}}_q) > 0$ is equivalent to $\lambda_{\min}(\mathbf{L}^s_q + \mathbf{D}_q) > 0$, being $\hat{\mathbf{L}}_q$ the minor matrix of $\mathbf{L}^s_q$ by removing the first row-column pair. Since $\hat{\mathbf{L}}_q$ can be gotten by removing the q-th row-column pair of $\mathbf{L}^s$, one has $\lambda_{\min}(\hat{\mathbf{L}}_q) = \lambda_q\,(\mathbf{L}^s)$. And then the node significance gets greater with the value of $\lambda_q\,(\mathbf{L}^s)$. This completes the proof. ■

Th 3 reveals the significance of $\lambda_q\,(\mathbf{L}^s)$ $(1 \le q \le N)$, which is defined as the ControlRank index here. In many realistic networks, the Laplacian and the corresponding modified matrices are large-scale and sparse. If the ControlRank indexes are too complex to be calculated, the standard inverse power method and its various improvements (to accelerate the convergence rate or expand the application of the algorithm) can be applied to simplify the calculation [24].

Sometimes it is required to identify which $l (l \ge 2)$ nodes instead of only one node are the most important in a network. In this case, one should determine which l nodes' corporation will encourage the spread of information in the whole network.

Theorem 4: The importance sequencing of l nodes in a network is determined in descending order of $\lambda_q\,(\mathbf{L}^s)$ $(1 \le q \le C_N^l)$, where $\lambda_q\,(\mathbf{L}^s)$ is the minimum eigenvalue of the minor matrix of $\frac{\mathbf{L}^\top + \mathbf{L}}{2}$ by removing the corresponding l row-column pairs.

Proof: By similar derivation from Eq. (4) to Eq.(5) and the partitioned matrix theory, node set importance gets greater with the value of $\lambda_q\,(\mathbf{L}^s)$, where $\lambda_q\,(\mathbf{L}^s)$ is the minimum eigenvalues of all the possible minor matrices by removing the corresponding l row-column pairs of $\mathbf{L}^s$.

It is seen that the rank of the ControlRank indexes $\lambda_q\,(\mathbf{L}^s)$ $(1 \le q \le C_N^l)$ uncovers the importance of the node set. If $N - 1$ nodes are controlled in a complex network with size N, we have the precedence of uncontrolled node.

Theorem 5: The importance of the uncontrolled node is sorted in descending order of degree.

Proof: Provided that the $N - 1$ nodes are controlled except for node q, one gains that

$$\lambda_q(\mathbf{L}^s) = \mathbf{L}^s_{qq} = \mathbf{L}_{qq} = \Bbbk_q,$$

being $\Bbbk_q$ the degree of node q. Thus the uncontrolled node is sorted in descending order of its degree.

Th 5 tells that instead of being controlled, the node with the maximum degree has the priority of being uncontrolled. Even if the node with the largest degree may be the most important one when controlling one node, it is not necessarily chosen as one element of the important node set when controlling more than one node.

All the results can be extended to weighted complex networks, being the ControlRank index the minimum eigenvalue of a modified matrix of the weight matrix.

IV. Examples

To verify our main results of node significance, we compare our algorithm with the well-known degree centrality and k-shell decomposition. It is known that degree centrality is valid in the networks with a broad degree distribution [5], [6], and k-shell decomposition is efficient in the networks with marginal hubs [9]. The proposed ControlRank index, which is valid for general connected networks, ignores the peculiar topology structure, whether in a bi-star network or in a Barabśi-Albert scale-free network. The simulations show the effectiveness of our index.

A. Bi-Star Network

Consider a network coupled with 11 Chua circuits [25].

$$\begin{cases} \dot{s}_1 = -\gamma_1(s_1 - s_2 + g(s_1)) \\ \dot{s}_2 = s_1 - s_2 + s_3 \\ \dot{s}_3 = -\gamma_2 s_2, \end{cases}$$

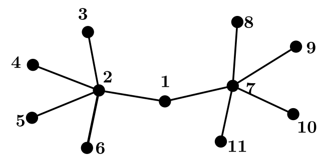

Fig. 1. Topology of the bi-star network.

where $g(s_1) = \gamma_4 s_1 + 0.5(\gamma_3 - \gamma_4)[|s_1 + 1| - |s_1 - 1|]$. The system is chaotic when $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (9, 100/7, -8/7, -5/7)$. The topology of the network is a bi-star that can be depicted by Fig. 1.

To reveal the node importance, we calculate the ControlRank index of each node. Record these indexes, and use a ControlRank index matrix

$$\Upsilon^1 \triangleq (0.17, 0.09, 0.05, 0.05, 0.05, 0.05, 0.09, 0.05, 0.05, 0.05, 0.05),$$

where Υ^1_j $(1 \le j \le 11)$ represents the minimum eigenvalue of the minor matrix by removing the j-th row-column pair of $\mathbf{L}$. Ranking the elements $\Upsilon^1_1 > \Upsilon^1_2 = \Upsilon^1_7 > \Upsilon^1_3 = \ldots = \Upsilon^1_6 = \Upsilon^1_8 = \ldots = \Upsilon^1_{11}$, we get the sequence of node importance according to Th 3. In particular, node 1 is the most efficient for control. While based on the degree centrality, node 2 or node 7 should be the most important, which is irreconcilable with our results. Actually, node 1 can be considered as a "bridge" between the two stars. This bridge plays a major role of information dissemination. The importance of node 1 is succeeded by node 2 and node 7, which share the same importance in the network. The other nodes, sharing the same significance too, follow these two nodes. If each cluster in a network can receive the desired information smoothly, degree is taken into account. It is shown that information balance outweighs information channels.

In reality, determining the node pair importance in a network is sometimes required. If two nodes are the desired information receivers, it is necessary to find out which two whose cooperation are the most efficient for control. By calculation, we develop a ControlRank index matrix

$$\Upsilon^2 \triangleq \begin{pmatrix} 0 & 0.17 & 0.17 & 0.17 & 0.17 & 0.17 & 0.17 & 0.17 & 0.17 & 0.17 & 0.17 \\ & 0 & 0.09 & 0.09 & 0.09 & 0.09 & 1.00 & 0.28 & 0.28 & 0.28 & 0.28 \\ & & 0 & 0.07 & 0.07 & 0.07 & 0.28 & 0.19 & 0.19 & 0.19 & 0.19 \\ & & & 0 & 0.07 & 0.07 & 0.28 & 0.19 & 0.19 & 0.19 & 0.19 \\ & & & & 0 & 0.07 & 0.28 & 0.19 & 0.19 & 0.19 & 0.19 \\ & & & & & 0 & 0.28 & 0.19 & 0.19 & 0.19 & 0.19 \\ & & & & & & 0 & 0.09 & 0.09 & 0.09 & 0.09 \\ & & & & & & & 0 & 0.07 & 0.07 & 0.07 \\ & & & & & & & & 0 & 0.07 & 0.07 \\ & & & & & & & & & 0 & 0.07 \\ & & & & & & & & & & 0 \end{pmatrix},$$

where $\Upsilon^2_{j,k}$ $(j \le k,\ 1 \le j \le 10,\ 2 \le k \le 11)$ represents the minimum eigenvalue by removing the j-th and the k-th row-column pairs of $\mathbf{L}^s$.

Υ^2 reveals many interesting phenomena. First of all, node pair 2-7 has the first ranking for significance, excluding the 1-st node even if it is the most meaningful. That is, the joining of the most efficient individuals does not necessarily lead to the most efficient pair. Second, the same value of the elements in the first row means that any node w $(2 \le w \le 11)$ has the same significance if node 1 has been controlled, regardless of node w's degree. Third, although any node in the set $\{3, \ldots, 6, 8, \ldots, 11\}$ has the same degree, controlling node 2 and the node from $\{8, 9, 10, 11\}$ is better than the node from $\{3, 4, 5, 6\}$. That uncovers the essential of having an eye on the information balance in clusters.

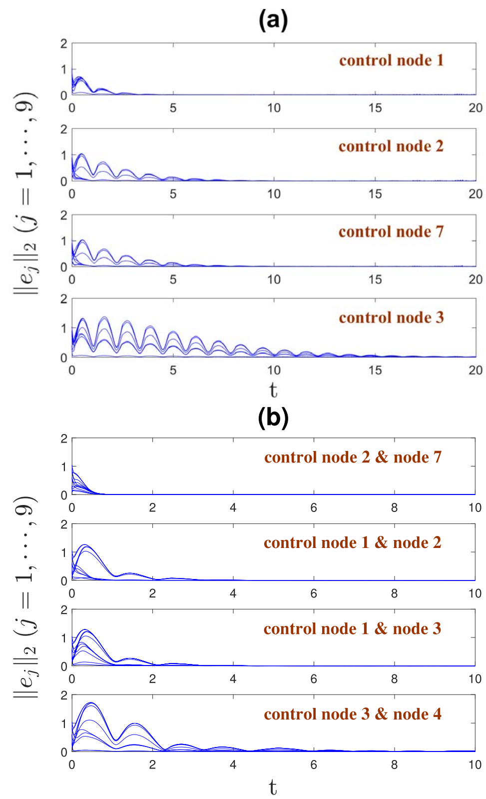

Fig. 2. Norm of error $\|\mathbf{e}_j\|_2$ $(j = 1, 2, \ldots, 11)$ vs. time t. (a) Controlling node j_1 $(j_1 \in \{1, 2, 7, 3\})$. (b) Controlling node j_1 and j_2, where j_1-j_2 $\in$ {2-7, 1-2, 1-3, 3-4}.

To verify our results, we control the 1-st, the 2-nd, the 7-th and the 3rd nodes respectively. It is shown in Fig. 2 (a) that the time sequence of achieving synchronization is in agreement with the sequence of node importance. Similarly, choose the node pairs 2-7, 1-2, 1-3 and 3-4 to be controlled respectively. According to our results, the significance of controlling these node pairs decreases progressively. It is clear in Fig. 2 (b) that the times of achieving network synchronization are consistent with the node pair importance. Besides, controlling two nodes is more beneficial to network synchronization than controlling only one node. That is the reason why the synchronization times in Fig. 2 (b) are shorter than those in Fig. 2 (a).

B. Barabśi-Albert Scale-Free Network

To determine node importance in a Barabśi-Albert (BA) scale-free network [1], we use a network coupled with 200 Chua circuits for example. In generating the BA network, the initial network size ω_0 and the number of links ω added each time are $\omega_0 = \omega = 5$.

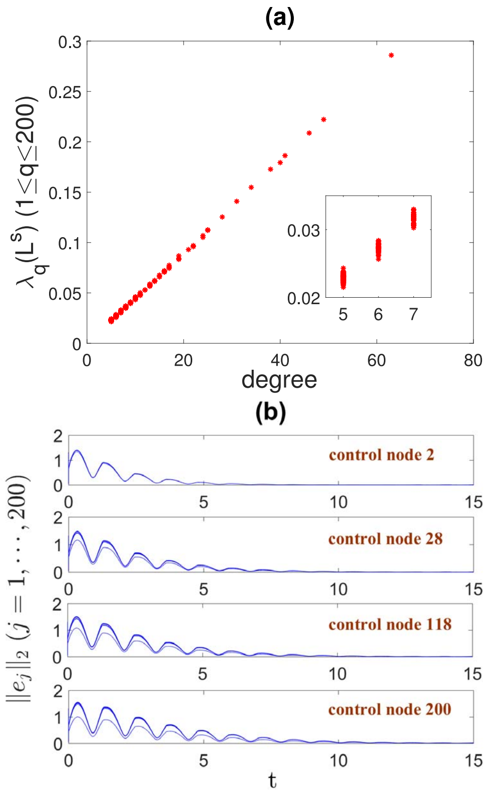

Fig. 3. (a) The ControlRank index $\lambda_q(\mathbf{L}^s)$ vs. the degree of node q. (b) Norm of error $\|\mathbf{e}_j\|_2$ $(j = 1, 2, \ldots, 200)$ vs. time t by controlling node q $(q \in \{2, 28, 118, 200\})$.

Denote the ControlRank index of the q-th node $\lambda_q(\mathbf{L}^s)$ as the minimum eigenvalue of the minor matrix by removing the q-th row-column pair of $\mathbf{L}^s$. We plot in Fig. 3 (a) the relationship between $\lambda_q(\mathbf{L}^s)$ and the degree of node q. It is seen that the node significance scales linearly with the node degree on the whole. This relationship illustrates that it is reasonable to determine the node importance by using its degree in a scale-free network.

Consider the importance of the 2-nd, the 28-th, the 118-th and the 200-th nodes, whose degrees are the 1-st, the 50-th, the 100-th and the 200-th largest respectively. Fig. 3 (b) provides the times of achieving synchronization by controlling the 2-nd, the 28-th, the 118-th and the 200-th node respectively. It is shown that with the decrease in node degree, the synchronization times increase. These phenomena illustrate the effectiveness of our algorithm.

V. Conclusion

In this brief, we have investigated the problem of node importance in a general controlled complex network. We have proposed a ControlRank index to measure the node (node set) importance in a controlled network. Through this algorithm, we have found many interesting phenomena. If $N-1$ nodes are pinned in a network, the importance of the unpinned node is sorted in descending order of degree. Even if a node is the most significant node, it does not necessarily belong to the most efficient node pair. Information balance is more crucial than information channels for spreading information in a network. It is reasonable to rank node importance by sorting the node degrees in a scale-free network. In addition, all the results can be applied to weighted complex networks without isolated node and loop. The results and phenomena in this brief are expected to be applied in controllability and pinning control of complex networks and multilayer networks.

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