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Synchronizability of two-layer networks

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Abstract. In this paper, we investigate the synchronizability of two-layer networks according to the master stability method. We define three particular couplings: positively correlated, randomly correlated and negatively correlated couplings. When the inter-layer coupling strength is fixed, negatively correlated coupling leads to the best synchronizability of a two-layer network, and synchronizability of networks with randomly and positively correlated couplings follow consecutively. For varying inter-layer coupling strength, the trend of network synchronizability with an unbounded synchronous region differs from that with a bounded one. If the synchronous region is unbounded, synchronizability of the two-layer network keeps enhancing, but it has a threshold. If the synchronous region is bounded, the synchronizability of the two-layer network keeps improving until the inter-layer coupling strength reaches a certain value, and then the synchronizability gets weakened with ever-increasing inter-layer coupling strength. To summarise, there exists an optimal value of the inter-layer coupling strength for maximising synchronizability of two-layer networks, regardless of the synchronous region types and coupling patterns. The findings provided in this paper shed new light on understanding synchronizability of multilayer networks, and may find potential applications in designing optimal inter-layer couplings for synchronization of two-layer networks.

1 Introduction

As is well known, complex networks in reality are usually interconnected. For example, in a social relationship network [1], a person connects a family network with a friend network, according to his different roles in the two networks. To describe interconnected networks, a new kind of network named multiplex networks was proposed by Mucha et al. [2] in 2010. Since that, multilayer networks have attracted more and more attention. Various aspects regarding multilayer networks have been studied, such as network topologies and dynamic properties [3,4], diffusion dynamics [5,6], the spectrum [7], game theory [8,9], synchronization [10–13], asynchronization [13–15], among many others [1,16,17]. There are significant differences between the properties of multilayer networks and those of traditional single-layer complex networks.

As for traditional complex networks, many realistic networks are scale-free, namely only a few nodes within a network have many connections and most of the remainders have much fewer links. For instance, the degrees of nodes obey a power-law distribution in the Internet, the World Wide Web, social networks, and even metabolic networks, which exhibit scale-free properties. The BA scale-free network [18], proposed by Barabási and Albert, is a frequently discussed model because of its strong robustness and extensive existence. Due to the ubiquity of scale-free networks in reality, the investigation of multilayer networks composed of BA scale-free networks is of great significance. Even though there have been many studies on single-layer BA networks, such as stopping hacker attack or preventing epidemics from spreading [19], multilayer networks coupled with two (identical or distinct) BA networks have not yet been studied.

For the numerous dynamical phenomena of multilayer networks, synchronization is a typical one [19]. Though there are some papers [10–15] focusing on synchronization or asynchronization of multilayer networks, the research is still in its initial stages. In these papers, different methods are applied such as the Master Stability Method [10], the Mean-Field Approximation [11], the Lyapunov Method [12] and the Semi-Tensor Product Approach [13–15]. In this paper, we will illustrate the synchronizability of two-layer networks formed by two identical BA networks using the Master Stability Method [20–22]. Similar to reference [5], we assume that the nodes in the two layers are identical, the intra-layer edges in one layer are independent of those in the other one, and that each node in one layer is connected to a counterpart in the other layer (see Fig. 1).

Three inter-layer coupling patterns are considered: positively correlated, negatively correlated and randomly correlated couplings. Specifically, let the nodes within each layer be ordered according to ascending degrees, that is, $\mathbf{k}^1 = (k_1^1, k_2^1, \ldots, k_N^1)$ and $\mathbf{k}^2 = (k_1^2, k_2^2, \ldots, k_N^2)$, where

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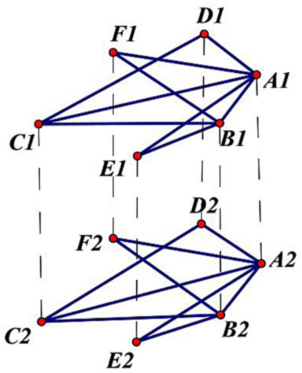

Fig. 1. A two-layer network composed of two identical single-layer networks. The nodes in the two layers carry the same dynamics; the intra-layer edges in one layer are independent to that of the other one, and each node in one layer is connected to a counterpart in the other layer. Specifically, PC means $A1 \sim A2, \ldots, F1 \sim F2$ and NC represents $A1 \sim F2, \ldots, F1 \sim A2$.

$k_1^\tau \leq k_2^\tau \leq \ldots \leq k_N^\tau$ $(\tau = 1, 2)$. Positively correlated coupling, simplified as PC, means that the correlation coefficient of $\mathbf{k}^1$ and $\mathbf{k}^2$ is maximised. For example, a person with many links in the friendship network layer is very likely to have many links in another social network layer, too. A special case is the assortative mixing (nodes with large degrees in one layer are likely to have more links in the other layer as well), namely, $k_1^1 \sim k_1^2, k_2^1 \sim k_2^2, \ldots, k_N^1 \sim k_N^2$. The second coupling strategy is randomly correlated coupling, abbreviated as RC, which represents a random matching between a node in one layer with a node in the other layer. Likewise, in the negatively correlated coupling (NC), nodes' degrees in different layers are maximally anti-correlated in their degree order. A special case of the NC coupling is disassortative mixing, such as $k_1^1 \sim k_N^2, k_2^1 \sim k_{N-1}^2, \ldots, k_N^1 \sim k_1^2$.

For the three coupling patterns, we will discuss the synchronizability of two-layer networks with bounded or unbounded synchronous regions. We observe that, when the inter-layer coupling strength is invariant, negatively correlated coupling leads to the best synchronizability of a two-layer network regardless of types of the synchronous regions. Those of randomly and positively correlated couplings follow. The synchronizability of two-layer networks in the three coupling patterns progressively approach each other as their single-layer networks become denser, and is worse than the synchronizability of single layers. When the inter-layer coupling strength is increasing, the synchronizability of the two-layer networks with an unbounded region keeps enhancing and finally reaches a threshold. If the synchronous region is bounded, the synchronizability of the two-layer network is kept enhanced with increasing inter-layer coupling until the coupling strength reaches a certain value, then the synchronizability gets weakened with ever-increasing coupling strength. The findings shed new light on understanding synchronizability of multilayer networks, and may find potential applications in designing optimal inter-layer couplings for synchronization of two-layer networks.

The rest of the paper is organised as follows. Some preliminaries are introduced in Section 2. The synchronizability of the two-layer BA networks with unbounded and bounded synchronous regions are illustrated in details in Sections 3 and 4, respectively. Finally, some conclusions are drawn in Section 5.

2 Preliminaries

Consider a multilayer network composed of M layers, with each layer consisting of N nodes. The following equation describes the dynamics of the ith node in the Kth layer:

$$\dot{x}_i^K = f\left(x_i^K\right) + d_K \sum_{j=1}^{N} w_{ij}^K \Gamma\left(x_j^K\right) + \sum_{L=1}^{M} d_i^{KL} \Gamma\left(x_i^L\right), \quad (1)$$

where $1 \leq i \leq N$, $1 \leq K \leq M$, $x_i^K \in \mathscr{R}^n$ is the state of the ith node in the Kth layer, $f : \mathscr{R}^n \to \mathscr{R}^n$ is a smooth nonlinear vector-valued function governing the dynamics of isolated node $x_i^K (1 \leq i \leq N,\ 1 \leq K \leq M)$. The continuous function $\Gamma : \mathscr{R}^n \to \mathscr{R}^n$ represents both the intra-layer and inter-layer coupling functions. d_K represents the intra-layer coupling strength of the Kth layer. Here, $W^K = (w_{ij}^K) \in \mathscr{R}^{N\times N}$ is the coupling weight configuration matrix of the Kth layer. If there is a link from node j to node i $(i \neq j)$, $w_{ij}^K = 1$, otherwise $w_{ij}^K = 0$. It is clear that W^K is diffusive by taking

$$w_{ii}^K = -\sum_{j=1, j\neq i}^{N} w_{ij}^K,$$

thus $L^K = -d_K W^K$ is a Laplacian matrix. Denotation d_i^{KL} is the inter-layer coupling strength between the ith nodes in the Kth and Lth layers, satisfying

$$d_i^{KK} = -\sum_{L=1, L\neq K}^{M} d_i^{KL}.$$

It is obvious that $D_i = (d_i^{KL}) \in \mathscr{R}^{M\times M}$ is also a negative Laplacian matrix.

Similar to reference [5], let $\mathcal{L}$ be the Supra-Laplacian matrix of equation (1), $\mathcal{L}_I$ be the Supra-Laplacian matrix representing the inter-layer topology, and $\mathcal{L}_L$ be the Supra-Laplacian matrix describing the intra-layer topology. Then $\mathcal{L}$ can be written as:

$$\mathcal{L} = \mathcal{L}_I + \mathcal{L}_L. \quad (2)$$

Taking L_I to be the Laplacian matrix of the inter-layer networks, we have

$$\mathcal{L}_I = L_I \otimes I_N, \quad (3)$$

where $\otimes$ is the Kronecker product, I_N is the $N\times N$ identity matrix. As for $\mathcal{L}_L$, it can be represented by the direct sum of the Laplacian matrix L^K within each layer, namely,

$$\mathcal{L}_L = \begin{pmatrix} L^1 & 0 & \dots & 0 \\ 0 & L^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L^M \end{pmatrix} = \bigoplus_{K=1}^{M} L^K. \quad (4)$$

Then for a two-layer network, we can safely conclude that:

(1) If $d_i^{KL} = d$, one gets

$$L_I = \begin{pmatrix} d & -d \\ -d & d \end{pmatrix}$$

and $\Lambda(L_I) = \{0, 2d\}$, where $\Lambda(L_I)$ is the set of eigenvalues of matrix L_I. It is obvious that $\Lambda(L_I) \subset \Lambda(\mathcal{L})$ [5].

(2) The set of eigenvalues of $\mathcal{L}_L$ is the union of that of L^1 and L^2, namely $\Lambda(\mathcal{L}_L) = \Lambda(L^1) \cup \Lambda(L^2)$. Let $\Lambda(\mathcal{L}_L) = \{0 = \bar{\lambda}_1 = \bar{\lambda}_2 \le \bar{\lambda}_3 \ldots \le \bar{\lambda}_{2N}\}$, where $\bar{\lambda}_i$ $(1 \le i \le 2N)$ are the eigenvalues of $\mathcal{L}_L$.

To get the main results, the following three Lemmas are needed.

Lemma 1 [23]**.** Assuming that U and V are $n \times n$ Hermitian matrices, the eigenvalues of U, V and $U + V$ are $\xi_1 \ge \xi_2 \ge \ldots \ge \xi_n$, $\zeta_1 \ge \zeta_2 \ge \ldots \ge \zeta_n$, and $\varsigma_1 \ge \varsigma_2 \ge \ldots \ge \varsigma_n$ respectively, then one has

$$\xi_i + \zeta_n \le \varsigma_i \le \xi_i + \zeta_1, 1 \le i \le n. \quad (5)$$

According to Lemma 1, one obtains:

$$\begin{aligned} 0 = \lambda_1 &= \bar{\lambda}_1, \\ 0 \le \lambda_2 &\le \bar{\lambda}_2 + 2d = 2d, \quad (6) \\ \bar{\lambda}_3 \le \lambda_3 &\le \bar{\lambda}_3 + 2d, \\ &\ldots \\ \bar{\lambda}_{2N} \le \lambda_{2N} &\le \bar{\lambda}_{2N} + 2d, \quad (7) \end{aligned}$$

where $0 = \lambda_1 < \lambda_2 \le \lambda_3 \ldots \le \lambda_{2N}$ are eigenvalues of the Supra-Laplacian matrix of a two-layer network.

Lemma 2 [24]**.** The relationship between the eigenvalues of the two Laplacian matrices L^1, L^2 and that of the sum matrix $L^s = (L^1 + L^2)/2$ is $\lambda_s \ge (\lambda_2^1 + \lambda_2^2)/2 \ge \min(\lambda_2^1, \lambda_2^2)$, where λ_s is the nonzero minimum eigenvalue of L^s, λ_2^1 and λ_2^2 are the nonzero minimum eigenvalues of L^1 and L^2, respectively.

Lemma 3 [25,26]**.** Provided that $m = m_0$ in a BA scale-free network, the estimation of the nonzero minimum eigenvalue λ_2 of the Laplacian matrix is

$$\hat{\lambda}_2 = \begin{cases} 0.000696, & m = 1 \\ 0.7744 * m - 1.049, & 2 \leqslant m \leqslant 5 \\ 0.9493 * m - 2.161, & 6 \leqslant m \leqslant 30 \end{cases} \quad (8)$$

when the network size N is sufficiently large, where m_0 is the initial network size and m is the number of existing nodes that are connected to a newly introduced node.

This lemma implies that when $N \to \infty$ in a BA scale-free network, λ_2 tends to a constant $\hat{\lambda}_2$ which is determined by m.

The Master Stability Function [20–22] tells the criteria for determining synchronizability of a network. For a network whose synchronous region is unbounded, the nonzero minimum eigenvalue λ_2 of the Laplacian matrix determines the synchronizability. The larger the λ_2 is, the better synchronizability the network has. For a network with a bounded synchronous region, the synchronizability is decided by the eigenratio of the maximum eigenvalue and the nonzero minimum eigenvalue λ_{max}/λ_2 of the network's Laplacian matrix. If λ_{max}/λ_2 is small enough (1 is the best), the network has strong synchronizability. The synchronous region of a complex network is mainly determined by $\mathbf{f}$ and Γ.

3 Two-layer networks with unbounded synchronous regions

In this section, we consider networks with unbounded synchronous regions. Specifically, the two-layer BA-BA network is composed of two interconnected identical BA networks of size $N = 500$ and the intra-layer coupling strength is $d_1 = d_2 = 1$.

3.1 Invariant inter-layer coupling strength

To begin with, we study synchronizability of two-layer networks with invariant inter-layer coupling strength, that is, $d_i^{KL} = d = 1\,(K, L = 1, 2, K \ne L, 1 \le i \le N)$. Corresponding to the three different coupling patterns– PC, NC and RC, the nonzero minimum eigenvalues of the Supra-Laplacian matrices are denoted by λ_2^p, λ_2^n and λ_2^r, respectively. Similarly, the eigenratios are denoted by $\lambda_{2N}^p/\lambda_2^p$, $\lambda_{2N}^n/\lambda_2^n$ and $\lambda_{2N}^r/\lambda_2^r$ for the respective coupling patterns.

According to (8), it is concluded that $\lambda_2^1 = \lambda_2^2 < 2$ if $m \le 3$ and $\lambda_2^1 = \lambda_2^2 > 2$ otherwise. To obtain some representative results, take $m = 3$ and $m = 6$ as examples.

When $m = 3$, it is observed that $\lambda_2^1 = \lambda_2^2 = \lambda_2^p \le \lambda_2^r \le \lambda_2^n$ (Fig. 2a) from numerical simulations. It reveals that the single-layer BA network has the same synchronizability as that of the PC two-layer network, which is weaker than that of the RC or NC two-layer networks. The synchronizability of the NC two-layer network is the best. The reason is probably due to the fact that the homogeneous degree distribution is beneficial to synchronization of a network. The coupling tendencies of NC and RC make the sum-degrees (the sum of degrees of a node in the two layers) more homogeneous in the networks than that in the PC networks. Thus the synchronizability of PC two-layer network is weaker than that of NC and RC networks. Meanwhile, the reason why the NC network has slightly better synchronizability than that of the RC network is

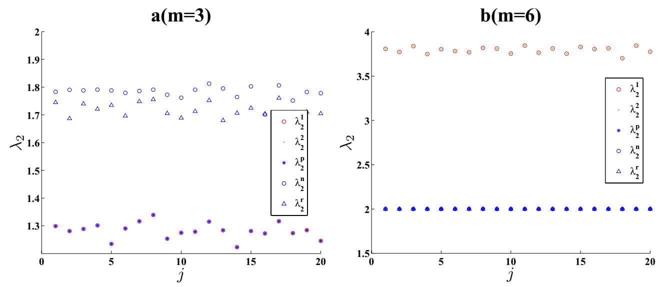

Fig. 2. The values of λ_2 by 20 trials for $m = 3$ (a) and $m = 6$ (b). The red $\circ$ represents λ_2^1; the green $\cdot$ represents λ_2^2; the blue $*$ is λ_2^p; the blue $\circ$ is λ_2^n; the blue $\triangle$ is λ_2^r.

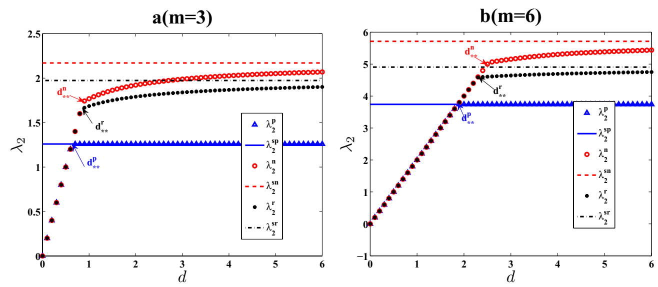

Fig. 3. The values of λ_2 versus d for $m = 3$ (a) and $m = 6$ (b). The black $\cdot$ is λ_2^p, the red $\circ$ is λ_2^n, and the blue $\triangle$ is λ_2^r. The transverse lines are λ_2^{sn}, λ_2^{sr} and λ_2^{sp}, from top to bottom.

that the former one has more uniform sum-degrees than the latter. Besides, it is clear that all these values are less than 2. This can be explained by (6) and (8), because the nonzero minimum eigenvalues of the coupled two-layer networks satisfy $0 \le \lambda_2^p, \lambda_2^r, \lambda_2^n \le 2$ according to (6), and the nonzero minimum eigenvalues of the Laplacian matrices satisfy $\lambda_2^1 = \lambda_2^2 < 2$ according to (8).

Figure 2b shows the nonzero minimum eigenvalues of the Laplacian matrices for $m = 6$ by 20 trials. On the one hand, it can be seen from the panel that $\lambda_2^1 = \lambda_2^2 > 2$ for $m = 6$ and is larger than that for $m = 3$, which is coincident with (8). It means that the synchronizability of single-layer BA networks for $m = 6$ is better than that for $m = 3$. It is obvious since the former network has a larger average degree than the latter one, and the eigenvalues increase on the whole. On the other hand, the panel reveals $\lambda_2^p = \lambda_2^n = \lambda_2^r = 2$, which is smaller than $\lambda_2^1 = \lambda_2^2$. This implies that the synchronizability of the coupled two-layer networks is weaker than that of the single-layer ones, regardless of the coupling patterns. Besides, due to the larger average degree of the single layers for $m = 6$ than that for $m = 3$, the connectivity in each layer is denser and more robust, thus the two-layer networks have reduced synchronizabilities compared with the corresponding single layers [27]. Since $0 \le \lambda_2 \le \bar{\lambda}_2 + 2 = 2$ can be obtained from (6), synchronizability of the current two-layer network is maximised for all the coupling patterns.

3.2 Varying inter-layer coupling strength

To analyse the synchronizability of two-layer networks varying with inter-layer coupling strength, we introduce the notations λ_2^{sp}, λ_2^{sn} and λ_2^{sr}. They represent the nonzero minimum eigenvalues of the average Laplacian matrices $L_{sp} = (L_1^p + L_2^p)/2$, $L_{sn} = (L_1^n + L_2^n)/2$ and $L_{sr} = (L_1^r + L_2^r)/2$, respectively.

It can be seen from Figure 3 that there are similar tendencies of λ_2 of the Supra-Laplacian matrices for $m = 3$ and $m = 6$. Generally, λ_2 increases as the inter-layer coupling strength d increases. In detail, λ_2 increases sharply at the beginning (nearly linearly), and then experiences much slower increase, and finally almost levels off at some upper bounded values. Coincidentally, when d increases to a certain value d_{**}, λ_2^p, λ_2^n, λ_2^r begins to increase slowly and finally arrives at some upper bounds. It is observed that λ_2^{sp}, λ_2^{sn} and λ_2^{sr} are basically the corresponding upper bounds. It reveals that synchronizability of the two-layer networks is enhanced almost linearly with increasing d and then grows much more slowly to be upper bounded. In applications, synchronizability of a two-layer network can be enhanced by means of raising d, but it is impractical for too large d. That is to say, the inter-layer coupling strength d attains its optimal value d_{**} for maximizing synchronizability of network with minimum cost. In addition, the figure shows $\lambda_2^{sp} \le \lambda_2^{sr} \le \lambda_2^{sn}$. It means that as far as the synchronizability is concerned, the RC pattern is better than the PC pattern, while the NC pattern is the best method for interconnection. This is because the coupling methods of NC, RC and PC give two-layer networks an increasing uniformity of sum-degrees. The larger values of d_{**} and the higher upper bounds of λ_2 in the network for $m = 6$ than that for $m = 3$ illustrates that the synchronizability of the former is better than that of the latter.

4 Two-layer networks with bounded synchronous regions

According to the Master Stability Function framework, synchronous regions of a network can be divided into four types [20–22]. Besides the unbounded synchronous region, there exists a bounded one, an empty one, and a union of several unbounded or bounded regions. The unbounded and bounded synchronous regions are more usually taken into consideration since they are more commonly seen in real-world networks than the other two types. In this section, we consider networks with bounded synchronous regions.

4.1 Invariant inter-layer coupling strength

Let the inter-layer coupling strength between the two layers be $d_i^{KL} = d = 1$. Networks in each layer are still generated according to the BA algorithm with $m = 3$ or $m = 6$. Figure 4 displays the values of λ_{max}/λ_2. Similar to the result of $\lambda_2^p \le \lambda_2^r \le \lambda_2^n$ for networks with unbounded synchronous regions, it can be seen from

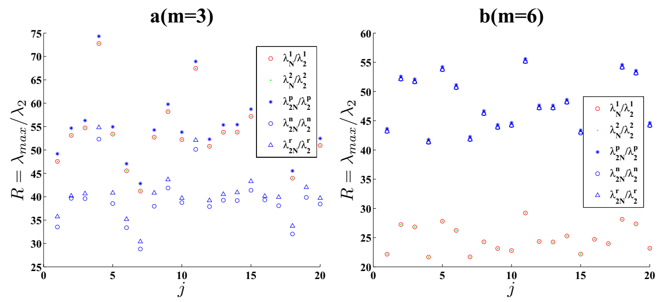

Fig. 4. The values of λ_{max}/λ_2 for $m = 3$ (a) and $m = 6$ (b) by 20 trials. The red ∘ represents λ_N^1/λ_2^1; the green · represents λ_N^2/λ_2^2; the blue $*$ is λ_N^p/λ_2^p; the blue ∘ is λ_N^n/λ_2^n; the blue △ is λ_N^r/λ_2^r.

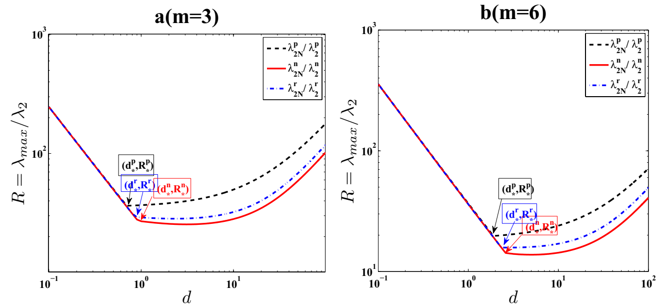

Fig. 5. A log-log plot of λ_{max}/λ_2 versus d for $m = 3$ (a) and $m = 6$ (b). The black dash represents $\lambda_{2N}^p/\lambda_2^p$, the red line is $\lambda_{2N}^n/\lambda_2^n$, and the blue dash dot is $\lambda_{2N}^r/\lambda_2^r$.

Figure 4 that $\lambda_{2N}^n/\lambda_2^n \leq \lambda_{2N}^r/\lambda_2^r \leq \lambda_{2N}^p/\lambda_2^p$. The inequalities can be reasoned as follows. According to (7), one gains $\bar{\lambda}_{2N} \leq \lambda_{2N} \leq \bar{\lambda}_{2N} + 2$, where λ_{2N} is the largest eigenvalue of the Supra-Laplacian matrix of the coupled two-layer network, while $\bar{\lambda}_{2N}$ is the largest eigenvalue of the Supra-Laplacian matrix of the intra-layer network. In other words, λ_{2N} can be regarded as a small perturbation of $\bar{\lambda}_{2N}$ when N is sufficiently large, and then the values of $\lambda_{2N}^n, \lambda_{2N}^r$ and λ_{2N}^p are very close to each other. Therefore, the relationship among $\lambda_{2N}^n/\lambda_2^n, \lambda_{2N}^r/\lambda_2^r$ and $\lambda_{2N}^p/\lambda_2^p$ is similar to that among $1/\lambda_2^n, 1/\lambda_2^r$ and $1/\lambda_2^p$. For the same reason of homogeneity, the synchronizability of the RC two-layer networks is weaker than that of the NC networks, and better than that of the PC networks.

It can be seen that $1 < \lambda_{2N}^n/\lambda_2^n \leq \lambda_{2N}^r/\lambda_2^r \leq \lambda_N^1/\lambda_2^1 = \lambda_N^2/\lambda_2^2 \leq \lambda_{2N}^p/\lambda_2^p$ for $m = 3$ in Figure 4a, which reveals that the single-layer BA network has better synchronizability than that of the PC two-layer network, and weaker than that of the RC and NC two-layer networks. When $m = 6$, $1 < \lambda_N^1/\lambda_2^1 = \lambda_N^2/\lambda_2^2 \leq \lambda_{2N}^n/\lambda_2^n \leq \lambda_{2N}^r/\lambda_2^r \leq \lambda_{2N}^p/\lambda_2^p$ is observed in Figure 4b. This suggests that the single-layer BA network has better synchronizability than interconnected two-layer networks. Because the connections in the two-layer networks with $m = 6$ are denser than that with $m = 3$, the networks are more robust. This leads to a quicker drop of the eigenratios after two networks are coupled into two-layer networks for $m = 6$ than for $m = 3$.

If m keeps increasing, the single-layer BA network becomes more and more robust, and the synchronizability of the two-layer networks becomes weaker and weaker compared with the single-layer networks. When m turns into a large value, we will have approximately $\lambda_{2N}^n/\lambda_2^n = \lambda_{2N}^r/\lambda_2^r = \lambda_{2N}^p/\lambda_2^p$. Thus two-layer networks with the three different interconnection patterns have roughly identical synchronizability.

4.2 Varying inter-layer coupling strength

In this subsection, we provide some insight into the tendencies of eigenratios with varying d. The eigenratios of the Supra-Laplacian matrices for $m = 3$ and $m = 6$ are displayed in Figures 5a and 5b, respectively. It is found that λ_{2N}/λ_2 decreases as a power-law function of d at the beginning, and then increases after d attains a certain value d_*, irrespective of the coupling pattern. That is, the synchronizability of the two-layer networks is getting better at first with increasing d and begins to get weaker after d attains a certain value. From Figure 5, it can be seen that d_* is varied for different coupling patterns and $d_*^p \leq d_*^r \leq d_*^n$. With d increasing to a certain value d_*, λ_{2N}/λ_2 reaches its minimum R_*. Therefore, d_* is an optimal inter-layer coupling strength as far as enhancing synchronizability of two-layer networks is concerned. Furthermore, when d exceeds the critical value, it is the NC networks that have the best synchronizability for a fixed inter-layer coupling strength.

5 Conclusion

We have investigated the synchronizability of positively correlated, randomly correlated and negatively correlated two-layer networks, which are formed by two interconnected identical BA scale-free networks. We have concluded that when the inter-layer coupling strength is fixed to be 1, negatively correlated coupling leads to the best synchronizability of a two-layer network, and randomly and positively correlated couplings follow. Synchronizability of two-layer networks with three different connection patterns progressively approach each other as m increases, and they become worse and worse compared with those of the isolated single-layers. When the inter-layer coupling strength d grows, network synchronizability trend with an unbounded synchronous region is different from that with a bounded one. With an unbounded synchronous region, the synchronizability of a two-layer network continues to be enhanced, but it has a threshold. Among the thresholds of λ_2, that of the NC network is the largest, the next is the RC network, and the smallest is for the PC network. If the synchronous region is bounded, the synchronizability of a two-layer network is enhanced with initially increasing d, and then weakened after d gets larger than a certain value d_*. In particular, the synchronizability index λ_{max}/λ_2 attains its minimum R_* at a critical value d_*, and we have $R_*^n \leq R_*^r \leq R_*^p$. Therefore, there exists an optimal value of the inter-layer coupling strength for maximiszing synchronizability of two-layer networks, regardless of the types of synchronous regions and the coupling patterns. Even though these results are obtained by using two-layer networks with identical structures in the two layers, they will provide insight into understanding synchronizability of multilayer networks, and may potentially be applicable in the design of optimal inter-layer couplings for synchronizing two-layer networks, such as power grids [28] and ad hoc mobile networks [29].

Author contribution statement

Mingming Xu, Jin Zhou, Jun-an Lu carried out the analysis. Mingming Xu, Jin Zhou implemented numerical simulations. Mingming Xu, Jin Zhou, Xiaoqun Wu wrote the main text of the manuscript.

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References

1. G. D'Agostino, A. Scala, *Networks of Networks: the last Frontier of Complexity* (Springer, Berlin, 2014)
2. P.J. Mucha, T. Richardson, K. Macon, M.A. Porter, J.P. Onnela, Science **328**, 5980 (2010)
3. R. Gutiérrez, I. Sendiñna-Nadal, M. Zanin, D. Papo, S. Boccaletti, Sci. Rep. **2**, 396 (2012)
4. G. Bianconi, Phys. Rev. E **87**, 6 (2013)
5. S. Gómez, A. Díaz-Guilera, J. Gómez-Gardeñes, C.J. Pérez-Vicente, Y. Moreno, A. Arenas, Phys. Rev. Lett. **110**, 2 (2013)
6. C. Granell, S. Gomez, A. Arenas, Phys. Rev. Lett. **111**, 12 (2013)
7. A. Sole-Ribalta, M. De Domenico, N.E. Kouvaris, A. Díaz-Guilera, S. Gómez, A. Arenas, Phys. Rev. E **88**, 3 (2013)
8. J. Gómez-Gardeñes, I. Reinares, A. Arenas, L.M. Floría, Sci. Rep. **2**, 620 (2012)
9. J. Gómez-Gardeñes, C. Gracia-Lázaro, L.M. Floría, Y. Moreno, Phys. Rev. E **86**, 5 (2012)
10. J. Aguirre, R. Sevilla-Escoboza, R. Gutiérrez, D. Papo, J.M. Buldú, Phys. Rev. Lett. **112**, 24 (2014)
11. J. Um, P. Minnhagen, B.J. Kim, Chaos **21**, 2 (2011)
12. R. Lu, W. Yu, J. Lü, A. Xue, IEEE T. Neur. Net. Lear. **25**, 11 (2014)
13. H. Zhang, X. Wang, X. Lin, IEEE/ACM T. Comput. Bi. **11**, 5 (2014)
14. C. Luo, X. Wang, H. Liu, Sci. Rep. **4**, 7522 (2014)
15. C. Luo, X. Wang, H. Liu, Chaos **24**, 3 (2014)
16. M. Kivelä, A. Arenas, M. Barthelemy, J.P. Gleeson, Y. Moreno, M.A. Porter, J. Com. Net. **2**, 3 (2013)
17. S. Boccaletti, G. Bianconi, R. Criado, C.I. del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendiña-Nadal, Z. Wang, M. Zanin, Phys. Rep. **544**, 1 (2014)
18. A.L. Barabási, R. Albert, Science **286**, 5439 (1999)
19. X.F. Wang, X. Li, G. Chen, *Complex Networks Theory and its Application* (Tsinghua University Press, Beijing, 2006)
20. L.M. Pecora, T.L. Carroll, Phys. Rev. Lett. **80**, 10 (1998)
21. L. Tang, J.-A. Lu, J. Lü, X. Yu, Int. J. Bifurc. Chaos **22**, 11 (2012)
22. L. Tang, J.-A. Lu, J. Lü, X. Wu, Int. J. Bifurc. Chaos **24**, 1 (2014)
23. B. Mohar, Y. Alavi, G. Chartrand, O. Oellermann, Graph Theory, Combinatorics, and Applications **18**, 7 (1991)
24. G. Chen, *The Theory and Application of Matrix* (Science Press, Beijing, 2007)
25. C.W. Wu, Phys. Lett. A **319**, 5 (2003)
26. Y.Y. Gao, J.-A. Lu, Complex Systems and Complexity Science **9**, 3 (2012)
27. S.J. Wang, X.J. Xu, Z.X. Wu, Y.H. Wang, Phys. Rev. E. **74**, 4 (2006)
28. A.E. Motter, S.A. Myers, M. Anghel, T. Nishikawa, Nat. Phys. **9**, 3 (2013)
29. A. Díaz-Guilera, J. Gómez-Gardeñes, Y. Moreno, M. Nekovee, Int. J. Bifurc. Chaos **19**, 2 (2009)