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Adaptive cluster synchronization in networks with time-varying and distributed coupling delays

Kezan Li [a,*], Jin Zhou [b], Wenwu Yu [c], Michael Small [d], Xinchu Fu [e]

[a] School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 541004, People's Republic of China
[b] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People's Republic of China
[c] Department of Mathematics, Southeast University, Nanjing 210096, People's Republic of China
[d] School of Mathematics and Statistics, The University of Western Australia, Crawley, WA 6009, Australia
[e] Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China

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ABSTRACT

This paper studies the adaptive cluster synchronization of a generalized linearly coupled network with time-varying delay and distributed delays. This network includes nonidentical nodes displaying different local dynamical behaviors, while for each cluster of that network the internal dynamics is uniform (such as chaotic, periodic, or stable behavior). In particular, the generalized coupling matrix of this network can be asymmetric and weighted. Two different adaptive laws of time-varying coupling strength and a linear feedback control are designed to achieve the cluster synchronization of this network. Some sufficient conditions to ensure the cluster synchronization are obtained by using the invariant principle of functional differential equations and linear matrix inequality (LMI). Numerical simulations verify the efficiency of our proposed adaptive control method.

1. Introduction

In complex network science, synchronization behavior is one of the main research fields. The idea of synchronization is making two dynamical systems (master–slave systems) oscillate in a synchronized manner. With the important discovery of small-world and scale-free networks, synchronization in complex networks has attracted great attention [1–9] in the last decade, due to its important potential applications in real-world dynamical systems [10–16].

Among various synchronization patterns, cluster synchronization has been widely studied. By cluster synchronization in a network, we mean that the dynamical nodes synchronize with each other in each cluster (formed by certain nodes of this network), but with no synchronization appearing between any two nodes from different clusters. Many important results and research methods have been presented in Refs. [17–21] and references cited therein.

Because of the finiteness of signal transmission and switching speed, delay is inevitable in real dynamical systems. Thus the dynamical networks with coupling delays can describe many real complex systems more precisely [22–24]. Based on the LaSalle invariant principle and adaptive control technique, adaptive complete synchronization between two neural networks with time-varying delay and distributed delay was discussed in [22], where all nodes in the whole network must be controlled. By employing the Lyapunov function and matrix inequality technique, reference [23] investigated the global exponential complete synchronization of linearly coupled dynamical networks with distributed coupling delay, where all the nodes are identical (*i.e., the local dynamics of the nodes are all the same*). Cluster synchronization of linearly coupled complex

* Corresponding author. Tel./fax: +86 773 2290700.
E-mail address: lkzzr@guet.edu.cn (K. Li).

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networks without delay but with adaptive control was studied in [24], where all nodes are identical and the adaptive control law needs the information of all synchronous states. These mentioned works have provided important approaches to achieve and control adaptive cluster synchronization on complex networks. However, these results are almost always based on reduced network models with some rigorous assumptions. So some assumptions should be relaxed for real applications. For examples, in most real dynamical networks: the individual nodes are not completely identical [25]; the corresponding coupling matrices may be be asymmetric and weighted; and, the number of controllers should be a fraction of whole nodes for the minimal control cost.

In order to construct more realistic models and design more effective control to achieve cluster synchronization, we incorporate the following four features into our study simultaneously: (i) the nodes in the considered network are not identical (for instances, in metabolic, neural or software community networks, the individual nodes in each community can be viewed as the identical functional units, while any pair of nodes in different communities are essentially different according to their functions [25]); (ii) the network under consideration includes time-varying delay or distributed delay; (iii) a fraction of nodes should be controlled (it seems unrealistic to control all the nodes in a large-scale network to achieve expected state); and, (iv) the coupling matrices can be asymmetric and weighted. The difference between this paper and former work is mainly that we will take into account these four factors simultaneously. By designing control methods, our goal of this paper is that the nodes within each cluster (composed of identical nodes) in a network can be fully synchronized to a dynamic state, while the nodes in separate clusters behave differently.

To achieve this goal, we design two kinds of adaptive control scheme, one of which only needs the state information of individual nodes without using the synchronous states. By using the LaSalle invariant principle and linear matrix inequality, some sufficient conditions to realize the cluster synchronization are obtained. In addition, we use a 3-D neural network and Chua's circuit as the local dynamics of individual nodes to perform the numerical simulations, which shows the validity of our proposed control schemes.

The outline of this paper is as follows. In Section 2, we introduce the network model discussed in this paper and provide some preliminary definitions and assumptions. Section 3 gives stability analysis of cluster synchronization of the network with time-varying delay and distributed delay. In Section 4, some numerical examples are presented to verify our theoretical results, including adaptive cluster synchronization on a regular network and on a small-world network. Finally, Section 5 concludes the paper.

2. Preliminaries

For clarity, we first make some mathematical definitions for a network with cluster structure as follows. Suppose that the network considered in this paper includes N nodes, and is divided into m clusters which depend on the local dynamics of nodes. For $i \in \{1, \dots, m\}, U_i = \{l_{i-1}+1, \dots, l_i\}$ denotes the index set of all nodes in the i–th cluster, where $l_0 = 0, l_m = N$ and $l_{i-1} < l_i, l_i \in \mathbb{N}$. Let $\mathbb{N}_k$ be the first k positive integers, and define a cluster index function $\theta : \mathbb{N}_N \to \mathbb{N}_m$ as $\theta(i) = j$, which means that the node i belongs to the j-th cluster, i.e., $i \in U_j$. The size of cluster U_i is given by $o(i) = l_i - l_{i-1}, i = 1, 2, \dots, m$.

In this paper, we will consider the following linearly coupled network under control

$$\dot{x}_i(t) = f_{\theta(i)}(x_i(t)) + c(t)\sum_{j=1}^{N} a_{ij}\Gamma x_j(t) + S_\tau(t) + u_i(t), \quad i = 1, 2, \dots, N, \tag{1}$$

where $x_i(t) = (x_{1i}, x_{2i}, \dots, x_{ni})^T \in R^n$ is the state variable of node i at time $t, f_{\theta(i)}$ defines the local nonlinear vector function (i.e., the local dynamics) of node i in cluster $U_{\theta(i)}$, and the positive definite matrix $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \dots, \gamma_n) \in R^{n\times n}$ denotes the inner-coupling matrix. The time-varying coupling strength $c(t) > 0$ is governed by an adaptive law, which will be designed later. The term $u_i(t)$ in (1) is the control input and $S_\tau(t)$ has the following form

$$S_\tau(t) = \bar{c}\sum_{j=1}^{N} b_{ij}\Gamma x_j(t - \tau(t)), \tag{2}$$

or

$$S_\tau(t) = \bar{c}\sum_{j=1}^{N} b_{ij}\Gamma \int_0^{\bar{\tau}} \rho(u)x_j(t-u)du, \tag{3}$$

where the parameters $\bar{c}, \bar{\tau}$ are two constants, and the density function $\rho : [0, \bar{\tau}] \to [0, \infty)$ satisfies $\int_0^{\bar{\tau}} \rho(u)du = 1$. Actually, Eq. (2) defines a time-varying coupling term with continuous delay $\tau(t)(> 0)$, and Eq. (3) describes a distributed-delay coupling term with the maximal delay $\bar{\tau} > 0$. The motivation of these coupling terms is to include time-varying delay or distributed delay into our model. In addition, the matrices $A = (a_{ij})_{N\times N}$ and $B = (b_{ij})_{N\times N}$ represent the coupling matrices of non-delayed coupling term and delayed coupling term in (1), respectively.

Let $\mathcal{C}([-r, 0], R)$ be the function space of all continuous functions from $[-r, 0]$ to R, where $r = \sup_{t\in R}\{\tau(t)\}$ under (2) or $r = \bar{\tau}$ under (3). The initial conditions for solution $x_i(t)$ are $\phi_i(t) \in \mathcal{C}([-r, 0], R)$.

Definition 1. Set $\mathcal{M} = \{x = (x_1, x_2, \ldots, x_N) | x_i = x_j \text{ if } \theta(i) = \theta(j); x_i \in \mathcal{C}([-r, 0], R), i, j = 1, 2, \ldots, N\}$ is called the cluster synchronization manifold of the network (1).

Definition 2. The cluster synchronization manifold $\mathcal{M}$ is said to be globally asymptotically stable if for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\|x_i(t) - x_j(t)\| \leqslant \varepsilon, \quad \text{if } \theta(i) = \theta(j), \tag{4}$$

where $\phi_i, \phi_j \in \mathcal{C}([-r, 0], R), t > T, i = 1, 2, \ldots, N$.

Definition 3. We say $A \in \mathbf{M_1}(p)$ with $A = (a_{ij}) \in R^{p\times p}$, if A is irreducible, $a_{ij} = a_{ji} \geqslant 0$, for $i \neq j$ and $a_{ii} = -\sum_{j=1, j\neq i}^{p} a_{ij}$, for $i = 1, 2, \ldots, p$.

Definition 4. We say $A \in \mathbf{M_2}(p)$, if for a p-order matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{12} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}, \tag{5}$$

with $m \leqslant p$, its each block is a zero-row-sum matrix. Further, we say $A \in \mathbf{M_3}(p)$, if $A \in \mathbf{M_2}(p)$ and the diagonal $o(q)$-order matrix $A_{qq} \in \mathbf{M_1}(o(q))$ for $q = 1, 2, \ldots, m$.

One will see clearer in Section 4 that each $o(q)$-order matrix A_{qq} corresponds to coupling relation between nodes in a cluster. The condition $A \in \mathbf{M_2}(p)$ is used to guarantee the existence of cluster synchronization manifold $\mathcal{M}$.

Hypothesis 1 (*H1*). For every $i \in \{1, 2, \ldots, m\}$, there exist positive definite matrix $P_i = \text{diag}(p_i^1, p_i^2, \ldots, p_i^n)$ and matrix $\Delta_i = \text{diag}(\delta_i^1, \delta_i^2, \ldots, \delta_i^n)$ such that

$$(x - y)^T P_i \{f_i(x) - f_i(y) - \Delta_i(x - y)\} \leqslant -w_i (x - y)^T (x - y) \tag{6}$$

for some $w_i > 0$ and all $x, y \in R^n$.

Hypothesis 2 (*H2*). For every $i \in \{1, 2, \ldots, m\}$, $f_i(\cdot)$ is Lipschitz continuous, i.e., there exists a constant $L_i > 0$ such that

$$\|f_i(x) - f_i(y)\| \leqslant L_i \|x - y\| \tag{7}$$

for all $x, y \in R^n$.

Hypothesis 3 (*H3*). The delay function $\tau(t)$ is differentiable and satisfies

$$0 \leqslant \dot{\tau}(t) \leqslant h < 1,$$

where h is a positive constant.

Hypothesis 4 (*H4*). For matrices $\overline{P}_k = \text{diag}(p^k_{\theta(1)}, p^k_{\theta(2)}, \ldots, p^k_{\theta(N)})$ and $A \in \mathbf{M_3}(N)$, there exists a diagonal matrix $D = \text{diag}(D_1, D_2, \ldots, D_m)$ such that

$$\overline{P}_k A + A^T \overline{P}_k - 2\overline{P}_k D < 0, \tag{8}$$

where $D_i = \text{diag}(0, 0, \ldots, d_i)_{o(i)\times o(i)}, d_i > 0$, $i = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, n$.

We quote below some lemmas, which will be needed to prove our main results in the next section.

Lemma 1 (*see [8]*). *The LMI*

$$\begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix} < 0, \tag{9}$$

is equivalent to one of the following conditions

(i) $M_1 < 0, M_3 - M_2^T M_1^{-1} M_2 < 0,$

(ii) $M_3 < 0, M_1 - M_2 M_3^{-1} M_2^T < 0,$

where $M_1 = M_1^T, M_3 = M_3^T$.

Lemma 2 (*see [26,27]*). *Suppose that* $D = \mathrm{diag}(0,\ldots,0,d)$ *with* $d > 0$ *and* $A \in \mathbf{M_1}(p)$ *are two matrices with the same size p. Then the matrix* $A - D$ *is negative definite.*

Lemma 3 (*see [23]*). *Let Q be a* $n \times n$ *positive-definite matrix. Then*

$$\int_0^{\bar{\tau}} \rho(u)z(t-u)^T Q z(t-u)du \geqslant \left(\int_0^{\bar{\tau}} \rho(u)z(t-u)du\right)^T Q\left(\int_0^{\bar{\tau}} \rho(u)z(t-u)du\right),$$

where $z \in R^n$, $\int_0^{\bar{\tau}} \rho(u)du = 1$.

Lemma 4 (*see [24]*). *For a matrix* $M = (m_{ij})_{p\times q}$, *the inequality*

$$x^T M y \leqslant \pi(M)(x^T x + y^T y), \tag{10}$$

holds for all $x \in R^p, y \in R^q$, *where* $\pi(M) = \frac{1}{2}\max\{p,q\} \times \max_{ij}\{|m_{ij}|\}$.

3. Stability analysis of cluster synchronization

Define the synchronization error variables $e_i(t) = x_i(t) - s_{\theta(i)}(t), i = 1,2,\ldots,N$, where $s_{\theta(i)}(t)$ is a solution of equation $\dot{s}_{\theta(i)}(t) = f_{\theta(i)}(s_{\theta(i)}(t))$. Define a set $U_c = \{l_1, l_2, \ldots, l_m\}$ and construct the linear feedback control as

$$u_i(t) = \begin{cases} -c(t)d_{\theta(i)}e_i(t), & i \in U_c; \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

where $d_i \geqslant 0, i = 1,2,\ldots,m$ denote the control gains. For a large-scale network, if the number of clusters is far less than its scale, i.e., $0 < (m/N) \ll 1$, then this control method can be considered as the pinning control [26,27]. There are some other methods to determine the number m of controlled nodes [28,29].

3.1. Stability of cluster synchronization under the time-varying delay

Under the time-varying coupling term $S_\tau(t)$ specified by (2), the globally dynamical behavior of these error variables is dominated by the following coupled equations

$$\dot{e}_i(t) = f_{\theta(i)}(x_i(t)) - f_{\theta(i)}(s_{\theta(i)}(t)) + c(t)\sum_{j=1}^N a_{ij}\Gamma e_j(t) + \bar{c}\sum_{j=1}^N b_{ij}\Gamma e_j(t-\tau) + c(t)\sum_{j=1}^N a_{ij}\Gamma s_{\theta(j)}(t) + \bar{c}\sum_{j=1}^N b_{ij}\Gamma s_{\theta(j)}(t-\tau) + u_i(t), \tag{12}$$

where $i = 1,2,\ldots,N$. Then, we can have the following results.

Theorem 1. *Suppose that* **H1,H3** *and* **H4** *hold. If* $B \in \mathbf{M_2}(N)$ *and the coupling strength satisfies the adaptive law*

$$\dot{c}(t) = \alpha\sum_{j=1}^N e_i^T(t)P_{\theta(i)}e_i(t), \tag{13}$$

with constant $\alpha > 0$, *then the cluster synchronization manifold* $\mathcal{M}$ *of network* (1) *equipped with* (2) *is globally asymptotically stable.*

Proof. Since $A \in \mathbf{M_3}(N), B \in \mathbf{M_2}(N)$, with the control (11) the Eqs. (12) can be rewritten as

$$\dot{e}_i(t) = f_{\theta(i)}(x_i(t)) - f_{\theta(i)}(s_{\theta(i)}(t)) + c(t)\sum_{j=1}^N a_{ij}\Gamma e_j(t) + \bar{c}\sum_{j=1}^N b_{ij}\Gamma e_j(t-\tau) + u_i(t), \quad i = 1,2,\ldots,N. \tag{14}$$

For this system, we construct the following Lyapunov function

$$V(t) = \frac{1}{2}\sum_{i=1}^N e_i^T(t)P_{\theta(i)}e_i(t) + \frac{\beta}{2\alpha}(c_0 - c(t))^2 + \sum_{i=1}^N \int_{t-\tau(t)}^t e_i^T(u)Q_{\theta(i)}e_i(u)du, \tag{15}$$

where the constants $\beta, c_0 > 0$ and matrix $Q_i = \mathrm{diag}(q_i^1, q_i^2, \ldots, q_i^n) > 0, i = 1,2,\ldots,m$ will be determined later. Calculating the derivative of (15) along the trajectories of the error system (14), we get

$$\frac{dV(t)}{dt} = \sum_{i=1}^{N} e_i^T(t)P_{\theta(i)}(f_{\theta(i)}(x_i(t)) - f_{\theta(i)}(s_{\theta(i)}(t))) + c(t)\sum_{i=1}^{N} e_i^T(t)P_{\theta(i)}\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + \bar{c}\sum_{i=1}^{N} e_i^T(t)P_{\theta(i)}\sum_{j=1}^{N} b_{ij}\Gamma e_j(t-\tau(t))$$
$$- \sum_{i=1}^{N}\beta(c_0 - c(t))e_i^T(t)P_{\theta(i)}e_i(t) + \sum_{i=1}^{N} e_i^T(t)Q_{\theta(i)}e_i(t) - \sum_{i=1}^{N}(1-\dot{\tau}(t))e_i^T(t-\tau(t))Q_{\theta(i)}e_i(t-\tau(t))$$
$$- c(t)\sum_{i\in U_c} d_{\theta(i)}e_i^T(t)P_{\theta(i)}e_i(t). \tag{16}$$

Defining new matrices $W = \mathrm{diag}(w_{\theta(1)}, w_{\theta(2)}, \ldots, w_{\theta(N)})$, $\widetilde{P}_k = \mathrm{diag}(p^k_{\theta(1)}\delta^k_{\theta(1)}, p^k_{\theta(2)}\delta^k_{\theta(2)}, \ldots, p^k_{\theta(N)}\delta^k_{\theta(N)})$, $\overline{Q}_k = \mathrm{diag}(q^k_{\theta(1)}, q^k_{\theta(2)}, \ldots, q^k_{\theta(N)})$, and $\bar{e}_k = (e_{k1}, e_{k2}, \ldots, e_{kN})^T$ for every $k \in \{1, 2, \ldots, n\}$, then from (16) and **H1,H3** we have

$$\frac{dV(t)}{dt} \leqslant -\sum_{k=1}^{n}\bar{e}_k^T(t)W\bar{e}_k(t) + \sum_{k=1}^{n}\bar{e}_k^T(t)\widetilde{P}_k\bar{e}_k(t) + c(t)\sum_{k=1}^{n}\gamma_k\bar{e}_k^T(t)\overline{P}_k(A-D)\bar{e}_k(t) + \bar{c}\sum_{k=1}^{n}\gamma_k\bar{e}_k^T(t)\overline{P}_kB\bar{e}_k(t-\tau(t)) - \sum_{k=1}^{n}\bar{e}_k^T(t)[\beta(c_0$$
$$- c(t))\overline{P}_k - \overline{Q}_k]\bar{e}_k(t) - (1-h)\sum_{k=1}^{n}\bar{e}_k^T(t-\tau(t))\overline{Q}_k\bar{e}_k(t-\tau(t)),$$

where the matrix D is defined by $D = \mathrm{diag}(D_1, D_2, \ldots, D_m)$, $D_i = \mathrm{diag}(0, 0, \ldots, d_i)_{o(i)\times o(i)}, i = 1, 2, \ldots, m$. If set new variables $E_k(t) = (\bar{e}_k(t)^T, \bar{e}_k(t-\tau(t))^T)^T \in R^{2N}$ and let

$$\Lambda_k = \begin{bmatrix} M_k & \frac{1}{2}\bar{c}\gamma_k\overline{P}_kB \\ \frac{1}{2}\bar{c}\gamma_kB^T\overline{P}_k & -(1-h)\overline{Q}_k \end{bmatrix}, \tag{17}$$

with $M_k = -W + \widetilde{P}_k + \overline{Q}_k - \beta c_0\overline{P}_k + c(t)\overline{P}_k(\beta I_N + \gamma_k(A-D))$, then we attain

$$\frac{dV(t)}{dt} \leqslant \sum_{k=1}^{n} E_k^T(t)\Lambda_kE_k(t) = \frac{1}{2}\sum_{k=1}^{n} E_k^T(t)(\Lambda_k + \Lambda_k^T)E_k(t),$$

where I_N denotes the identity matrix with size N.

From Lemma 1, we know that $\Lambda_k + \Lambda_k^T < 0$ if and only if the matrix

$$\Lambda_k^s = -W + \widetilde{P}_k + \overline{Q}_k - \beta c_0\overline{P}_k + c(t)[\beta\overline{P}_k + \frac{1}{2}\gamma_k(\overline{P}_kA + A^T\overline{P}_k - 2\overline{P}_kD)] + \frac{1}{4(1-h)}\bar{c}^2\gamma_k^2(\overline{P}_kB)\overline{Q}_k^{-1}(B^T\overline{P}_k) < 0.$$

According to condition $\overline{P}_kA + A^T\overline{P}_k - 2\overline{P}_kD < 0$ in **H4** and $c(t) > 0$, we can select sufficiently small β such that $c(t)[\beta\overline{P}_k + \frac{1}{2}\gamma_k(\overline{P}_kA + A^T\overline{P}_k - 2\overline{P}_kD)] < 0$. Then, we can further select sufficiently large $c_0 > 0$ such that $\Lambda_k^s < 0$ for all $k \in \{1, 2, \ldots, n\}$. Therefore, we can get $dV(t)/dt \leqslant 0$ under the condition in this Theorem.

From (16), we can see that $dV(t)/dt = 0$ if and only if $(e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T = 0$. Defining $\mathcal{E} = \{(e_1^T, e_2^T, \ldots, e_N^T, c)^T \in R^{nN+1} | (e_1^T, e_2^T, \ldots, e_N^T)^T = 0, c = c_0'\}$, then we can easily verify that $\mathcal{E}$ is the largest invariant set contained in set $\{(e_1^T, e_2^T, \ldots, e_N^T, c)^T \in R^{nN+1} | dV(t)/dt = 0\}$. By using the well-known invariant principle of functional differential equations, we obtain that $e_i(t) \to 0, c(t) \to c_0'$ as $t \to \infty$. This ends the proof. □

Corollary 1. *Suppose that* **H2** *and* **H3** *hold. If* $A = A^T \in \mathbf{M_3}(N), B \in \mathbf{M_2}(N), A - D < 0$, *and the coupling strength satisfies the adaptive law*

$$\dot{c}(t) = \alpha\sum_{i=1}^{N} e_i^T(t)e_i(t), \tag{18}$$

with the constant $\alpha > 0$, *then the cluster synchronization manifold* $\mathcal{M}$ *of network* (1) *with* (2) *is globally asymptotically stable.*

Proof. From **H2**, we have

$$2e_i^T(t)(f_{\theta(i)}(x_i(t)) - f_{\theta(i)}(s_{\theta(i)}(t))) \leqslant \|e_i(t)\| + \|f_{\theta(i)}(x_i(t)) - f_{\theta(i)}(s_{\theta(i)}(t))\| \leqslant \|e_i(t)\| + L_i\|e_i(t)\| \leqslant (1+L_i)e_i^T(t)e_i(t), i = 1, 2, \ldots, N. \tag{19}$$

Construct the positive function for the error system (12) as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{\beta}{2\alpha}(c_0 - c(t))^2 + \sum_{i=1}^{N}\int_{t-\tau(t)}^{t} e_i^T(u)e_i(u)du. \tag{20}$$

By noticing the inequality (19) and following a similar proof in Theorem 1, we can obtain this Corollary. Actually, we just need to substitute the identity matrix for replacing each matrix $P_i, Q_i, i \in \{1, 2, \ldots, m\}$ in Theorem 1. □

For each $i \in 1, 2, \ldots, m$, we define $o(i)$-order matrices O_i, T_i as

$$O_i = \begin{bmatrix} o(i) & 0 & \cdots & 0 \\ 0 & o(i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & o(i) \end{bmatrix}, \quad T_i = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

and $\widehat{P}_k = \overline{P}_k \times \mathrm{diag}(O_1, O_2, \ldots, O_m)$, $\check{P}_k = \overline{P}_k \times \mathrm{diag}(T_1, T_2, \ldots, T_m)$, for $k = 1, 2, \ldots, n$, then we can obtain the following result.

Theorem 2. *Suppose that* **H1** *and* **H3** *hold. If* $A \in \mathbf{M_3}(N), B \in \mathbf{M_2}(N)$, *and there exists matrices* D *and* $\overline{Q}_k, k = 1, 2, \ldots, n$ *such that*

$$\widetilde{P}_k + \overline{Q}_k + \frac{c(0)\gamma_k}{2}(\overline{P}_k A + A^T \overline{P}_k - 2\overline{P}_k D) + \frac{\bar{c}^2 \gamma_k^2}{4(1-h)}(\overline{P}_k B)\overline{Q}_k^{-1}(B^T \overline{P}_k) < 0 \tag{21}$$

and the adaptive law is set as

$$\dot{c}(t) = \alpha \sum_{i=1}^{N} \sum_{j \in U_i} (x_i(t) - x_j(t))^T P_{\theta(i)} (x_i(t) - x_j(t)), \tag{22}$$

with constant $\alpha > 0$, *then the cluster synchronization manifold* $\mathcal{M}$ *of network* (1) *with* (2) *is globally asymptotically stable.*

Proof. From Eq. (22), we can further expand it as

$$\begin{aligned}
\dot{c}(t) &= \alpha \sum_{i=1}^{N} \sum_{j \in U_i} (e_i(t)^T P_{\theta(i)} e_i(t) + e_j(t)^T P_{\theta(i)} e_j(t) - 2e_i(t)^T P_{\theta(i)} e_j(t)) \\
&= 2\alpha \sum_{i=1}^{N} o(\theta(i)) e_i(t)^T P_{\theta(i)} e_i(t) - 2\alpha \sum_{i=1}^{N} \sum_{j \in U_i} e_i(t)^T P_{\theta(i)} e_j(t) = 2\alpha \sum_{k=1}^{n} \bar{e}_k^T(t) \widehat{P}_k \bar{e}_k(t) - 2\alpha \sum_{k=1}^{n} \sum_{l=1}^{m} \sum_{i,j \in U_l} p_l^k e_{ki}(t) e_{kj}(t) \\
&= 2\alpha \sum_{k=1}^{n} \bar{e}_k^T(t)(\widehat{P}_k - \check{P}_k)\bar{e}_k(t),
\end{aligned} \tag{23}$$

where $\widehat{P}_k$ and $\check{P}_k$ are defined as above. For system (12), we still use the positive function

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t) P_{\theta(i)} e_i(t) + \frac{\beta}{2\alpha}(c_0 - c(t))^2 + \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(u) Q_{\theta(i)} e_i(u) du. \tag{24}$$

Under the conditions in this Theorem, the derivative of $V(t)$ along the solution of system (12) and (22) is given by

$$\begin{aligned}
\frac{dV(t)}{dt} &= \sum_{i=1}^{N} e_i^T(t) P_{\theta(i)} (f_{\theta(i)}(x_i(t)) - f_{\theta(i)}(s_{\theta(i)}(t))) + c(t) \sum_{i=1}^{N} e_i^T(t) P_{\theta(i)} \sum_{j=1}^{N} a_{ij} \Gamma e_j(t) + \bar{c} \sum_{i=1}^{N} e_i^T(t) P_{\theta(i)} \sum_{j=1}^{N} b_{ij} \Gamma e_j(t - \tau(t)) \\
&\quad - 2\beta(c_0 - c(t)) \sum_{k=1}^{n} \bar{e}_k^T(t)(\widehat{P}_k - \check{P}_k)\bar{e}_k(t) + \sum_{i=1}^{N} e_i^T(t) Q_{\theta(i)} e_i(t) - \sum_{i=1}^{N} (1 - \dot{\tau}(t)) e_i^T(t - \tau(t)) Q_{\theta(i)} e_i(t - \tau(t)) \\
&\quad - c(t) \sum_{i \in U_c} d_{\theta(i)} e_i^T(t) P_{\theta(i)} e_i(t) \leqslant \sum_{k=1}^{n} E_k^T(t) \Lambda_k E_k(t) = \frac{1}{2}\sum_{k=1}^{n} E_k^T(t)(\Lambda_k + \Lambda_k^T) E_k(t),
\end{aligned}$$

where all notations are as the were when defined previously, apart from the matrix Λ_k which is defined as

$$\Lambda_k = \begin{bmatrix} M_k & \frac{1}{2}\bar{c}\gamma_k \overline{P}_k B \\ \frac{1}{2}\bar{c}\gamma_k B^T \overline{P}_k & -(1-h)\overline{Q}_k \end{bmatrix}, \tag{25}$$

where $M_k = -W + \widetilde{P}_k + \overline{Q}_k - 2\beta c_0(\widehat{P}_k - \check{P}_k) + c(t)[2\beta(\widehat{P}_k - \check{P}_k) + \gamma_k \overline{P}_k(A - D)]$. From Lemma 1, we know that $\Lambda_k + \Lambda_k^T < 0$ if the matrix

$$\begin{aligned}
\Lambda_k^s &= -W + \widetilde{P}_k + \overline{Q}_k - 2\beta c_0(\widehat{P}_k - \check{P}_k) + 2c(t)\beta(\widehat{P}_k - \check{P}_k) + \frac{c(t)\gamma_k}{2}(\overline{P}_k A + A^T \overline{P}_k - 2\overline{P}_k D) + \frac{1}{4(1-h)}\bar{c}^2 \gamma_k^2 (\overline{P}_k B)\overline{Q}_k^{-1}(B^T \overline{P}_k) \\
&< -W + \widetilde{P}_k + \overline{Q}_k - 2\beta c_0(\widehat{P}_k - \check{P}_k) + 2c(0)\beta(\widehat{P}_k - \check{P}_k) + \frac{c(0)\gamma_k}{2}(\overline{P}_k A + A^T \overline{P}_k - 2\overline{P}_k D) + \frac{1}{4(1-h)}\bar{c}^2 \gamma_k^2 (\overline{P}_k B)\overline{Q}_k^{-1}(B^T \overline{P}_k) \\
&< 0,
\end{aligned}$$

which is guaranteed for all $k \in \{1, 2, \ldots, n\}$ by the conditions in this Theorem and sufficiently small β. So, we furthermore get $dV(t)/dt \leqslant 0$. In the same way, by using the invariant principle of functional differential equations, we can obtain this Theorem. □

Remark 1. The comparison between the adaptive laws (13) in Theorem 1 and (22) in Theorem 2 is detailed as follows: On one hand, the inequality (21) is stricter than (8). On the other hand, the adaptive law (22) only needs the state information of individual nodes without using the synchronous states, while they are utilized by adaptive law (13). So, in applications we should choose them purposefully for efficient control.

Remark 2. Actually, the inequality conditions (8) and (21) can be easily achieved under suitable parameters, which is shown in the Appendix A.

3.2. Stability of cluster synchronization under the distributed delay

Under the distributed-delay coupling term $S_\tau(t)$ shown by (3), the cluster error system of (1) can be described by

$$\begin{aligned}\dot{e}_i(t) = {} & f_{\theta(i)}(x_i(t)) - f_{\theta(i)}(s_{\theta(i)}(t)) + c(t)\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + \bar{c}\sum_{j=1}^{N} b_{ij}\Gamma\int_0^{\bar{\tau}} \rho(u)e_j(t-u)du + c(t)\sum_{j=1}^{N} a_{ij}\Gamma s_{\theta(j)}(t) \\ & + \bar{c}\sum_{j=1}^{N} b_{ij}\Gamma\int_0^{\bar{\tau}} \rho(u)s_{\theta(j)}(t-u)du + u_i(t),\end{aligned} \tag{26}$$

where $i = 1, 2, \ldots, N$ and the control $u_i(t)$ is provided by (11).

Theorem 3. *Suppose that* **H1** *and* **H4** *hold. If* $B \in \mathbf{M_2}(N)$ *and the coupling strength satisfies the adaptive law*

$$\dot{c}(t) = \alpha\sum_{j=1}^{N} e_i^T(t)P_{\theta(i)}e_i(t), \tag{27}$$

with a constant $\alpha > 0$*, then the cluster synchronization manifold* $\mathcal{M}$ *of network* (1) *with* (3) *is globally asymptotically stable.*

Proof. Since $A \in \mathbf{M_3}(N)$ and $B \in \mathbf{M_2}(N)$, the cluster error system (26) can be reduced to

$$\dot{e}_i(t) = f_{\theta(i)}(x_i(t)) - f_{\theta(i)}(s_{\theta(i)}(t)) + c(t)\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + \bar{c}\sum_{j=1}^{N} b_{ij}\Gamma\int_0^{\bar{\tau}} \rho(u)e_j(t-u)du + u_i(t), i = 1, 2, \ldots, N. \tag{28}$$

Construct a Lyapunov function for systems (27) and (28) as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)P_{\theta(i)}e_i(t) + \frac{\beta}{2\alpha}(c_0 - c(t))^2 + \sum_{i=1}^{N}\int_0^{\bar{\tau}}\int_{t-u}^{t} \rho(u)e_i^T(v)Q_{\theta(i)}e_i(v)dvdu. \tag{29}$$

Then the derivative of $V(t)$ along with the solution of systems (27) and (28) is calculated as

$$\begin{aligned}\frac{dV(t)}{dt} = {} & \sum_{i=1}^{N} e_i^T(t)P_{\theta(i)}(f_{\theta(i)}(x_i(t)) - f_{\theta(i)}(s_{\theta(i)}(t))) + c(t)\sum_{i=1}^{N} e_i^T(t)P_{\theta(i)}\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + \bar{c}\sum_{i=1}^{N} e_i^T(t)P_{\theta(i)}\sum_{j=1}^{N} b_{ij}\Gamma\int_0^{\bar{\tau}} \rho(u)e_j(t \\ & - u)du - \sum_{i=1}^{N}\beta(c_0 - c(t))e_i^T(t)P_{\theta(i)}e_i(t) + \sum_{i=1}^{N} e_i^T(t)Q_{\theta(i)}e_i(t) - \sum_{i=1}^{N}\int_0^{\bar{\tau}} \rho(u)e_i^T(t-u)Q_{\theta(i)}e_i(t-u)du \\ & - c(t)\sum_{i\in U_c} d_{\theta(i)}e_i^T(t)P_{\theta(i)}e_i(t).\end{aligned} \tag{30}$$

By Lemma 3, we get

$$\int_0^{\bar{\tau}} \rho(u)e_i^T(t-u)Q_{\theta(i)}e_i(t-u)du \geqslant \left(\int_0^{\bar{\tau}} \rho(u)e_i(t-u)du\right)^T Q_{\theta(i)}\left(\int_0^{\bar{\tau}} \rho(u)e_i(t-u)du\right).$$

Define new variables $\bar{e}_k(t) = (e_{k1}(t), e_{k2}(t), \ldots, e_{kN}(t))^T$, $\xi_i(t) = \int_0^{\bar{\tau}} \rho(u)e_i(t-u)du \in R^n$ and $\bar{\xi}_k(t) = (\xi_{k1}(t), \xi_{k2}(t), \ldots, \xi_{kN}(t))^T \in R^N$. Moreover, set $E_k(t) = (\bar{e}_k^T(t), \bar{\xi}_k^T(t))^T$. Then combining (30), **H1** and the above integral inequality, we have $dV(t)/dt \leqslant \sum_{k=1}^{n} E_k^T(t)\Lambda_k E_k(t)$, where

$$\Lambda_k = \begin{bmatrix} M_k & \frac{1}{2}\bar{c}\gamma_k\overline{P}_k B \\ \frac{1}{2}\bar{c}\gamma_k B^T\overline{P}_k & -\overline{Q}_k \end{bmatrix}, \tag{31}$$

with $M_k = -W + \widetilde{P}_k + \overline{Q}_k - \beta c_0\overline{P}_k + c(t)\overline{P}_k(\beta I_N + \gamma_k(A-D))$. By noting **H4** and choosing sufficiently small β and sufficiently large c_0, we can get $\frac{1}{2}(\Lambda_k + \Lambda_k^T) < 0$ resulting in $dV(t)/dt \leqslant 0$. Similarly, by applying the invariant principle of functional differential equations, we can complete this Theorem. □

Corollary 2. *Suppose that* **H2** *holds. If* $A = A^T \in \mathbf{M_3}(N), B \in \mathbf{M_2}(N), A - D < 0$, *and the coupling strength satisfies the adaptive law*

$$\dot{c}(t) = \alpha\sum_{i=1}^{N} e_i^T(t)e_i(t), \tag{32}$$

with a constant $\alpha > 0$, *then the cluster synchronization manifold* $\mathcal{M}$ *of network* (1) *with* (3) *is globally asymptotically stable.*

Proof. By the similar discussions in the proof of Corollary 1, we can also get this Corollary. □

Theorem 4. *Suppose that* **H1** *holds. If* $A \in \mathbf{M_3}(N), B \in \mathbf{M_2}(N)$, *and there exists matrices* D *and* $\overline{Q}_k, k = 1, 2, \ldots, n$ *such that*

$$\widetilde{P}_k + \overline{Q}_k + \frac{c(0)\gamma_k}{2}(\overline{P}_k A + A^T\overline{P}_k - 2\overline{P}_k D) + \frac{\bar{c}^2\gamma_k^2}{4(1-h)}(\overline{P}_k B)\overline{Q}_k^{-1}(B^T\overline{P}_k) < 0 \tag{33}$$

and the adaptive law is set as

$$\dot{c}(t) = \alpha\sum_{i=1}^{N}\sum_{j\in U_i}(x_i(t) - x_j(t))^T P_{\theta(i)}(x_i(t) - x_j(t)), \tag{34}$$

with a constant $\alpha > 0$, *then the cluster synchronization manifold* $\mathcal{M}$ *of network* (1) *with* (3) *is globally asymptotically stable.*

Proof. By adopting the positive function (29) and the integral inequality in Lemma 3, we can obtain this Theorem by a similar way presented in the proof of Theorem 2. □

Remark 3. For the moment, in the above analysis we select the controlled nodes in the index set $U_c = \{l_1, l_2, \ldots, l_m\}$ and obtain the corresponding control matrix D as defined in **H4**. Actually, we can choose other control index set $U_c = \{i_1, i_2, \ldots, i_m\}$ if $l_{k-1} < i_k \leqslant l_k$ for $k = 1, 2, \ldots, m$, as the assumption **H4** can also be satisfied according to the analysis in Appendix A by setting suitable control gains d_i. This means that our control schemes only need one control node in each cluster of the network to achieve the cluster synchronization.

4. Numerical simulations

In this section, we will perform some numerical simulations to verify the theoretical results in Part 3.1 of Section 3, and omit the corresponding similar numerical analysis of Part 3.2 for simplicity.

4.1. Synchronization on a regular network

We consider a regular network with size $N = 6$ and its topological structure is shown by Fig. 1. This network can be divided into two clusters, one of which is marked by the gray nodes. By the previous notations, we can have $U_1 = \{1, 2, 3\}$ (gray nodes), $U_2 = \{4, 5, 6\}$ (white nodes), $o(1) = o(2) = 3$ and the controlled nodes $l_1 = 3$ and $l_2 = 6$ according to (11). Here, we should note that the division of clusters depends on the local dynamics of nodes. For example, if the nodes 1, 2, 4 have a same local dynamics and nodes 3, 5, 6 have another identical local dynamics, then we can set $U_1 = \{1, 2, 4\}$ and $U_2 = \{3, 5, 6\}$. Our goal is to realize complete synchronization of identical nodes, whose idea may come from "*Birds of a feather flock together*".

The local dynamics of nodes in U_1 is governed by a 3-D neural network which is described by

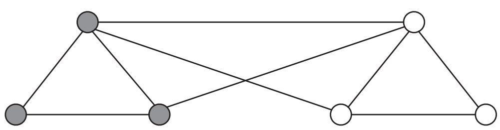

Fig. 1. The network topology with size $N = 6$ and two clusters $U_1 = \{1, 2, 3\}$ (gray nodes), $U_2 = \{4, 5, 6\}$ (white nodes). The division of clusters depends on the local dynamics of nodes.

$$\dot{y}(t) = -y(t) + Hg(y(t)), \tag{35}$$

where $y = (y_1, y_2, y_3) \in R^3$,

$$H = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{pmatrix}$$

and $g(y) = (g(y_1), g(y_2), g(y_3))^T$ with $g(z) = (|z+1| - |z-1|)/2$. This system has a double-scrolling chaotic attractor [24] and (6) is satisfied if we choose $P_1 = I_3$ and $\Delta_1 = 5.5682 I_3$. The local dynamics of nodes in U_2 is characterized by a Chua's circuit which is described by

$$\begin{cases} \dot{y}_1(t) = k[y_2(t) - h(y_1(t))], \\ \dot{y}_2(t) = y_1(t) - y_2(t) + y_3(t), \\ \dot{y}_3(t) = -ly_2(t), \end{cases} \tag{36}$$

where $k = 9, l = 100/7$ and $h(z) = (2/7)z - (3/14)[|z+1| - |z-1|]$. For this system [24], we can choose $P_1 = I_3$ and $\Delta_1 = 10I_3$ to achieve the inequality (6). By the definition in Section 3, we attain that $\widetilde{P}_k = \mathrm{diag}(5.5682, 5.5682, 5.5682, 10, 10, 10)$, for $k = 1, 2, 3$.

For simplicity, the coupling matrices of non-delayed coupling term of network (1) and delayed coupling term (2) are set as

$$A = \begin{pmatrix} -2 & 1 & 1 & 1 & 0 & -1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ -1 & 0 & 1 & 1 & 1 & -2 \end{pmatrix} \tag{37}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0.4 & -1 & 0.6 \\ 0 & 0 & 0 & 0.5 & -1 & 0.5 \\ 0 & 0 & 0 & -1 & 0.8 & 0.2 \\ -1 & 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & -1 & 0.5 & 0 & 0 & 0 \\ 0.2 & -1 & 0.8 & 0 & 0 & 0 \end{pmatrix}, \tag{38}$$

respectively, which are asymmetric and weighted. It is easy to verify that $A \in \mathbf{M_3}(6)$ and $B \in \mathbf{M_2}(6)$. Certainly, one can design other different coupling matrices as long as $A \in \mathbf{M_3}(6)$ and $B \in \mathbf{M_2}(6)$.

Without loss of generality, define the inner-coupling matrix $\Gamma = I_3, \bar{c} = 1$ and time-varying delay $\tau(t) \equiv 1$. The adaptive law of coupling strength is designed by (22) with $\alpha = 1$. Under these settings, we know that the assumptions **H1** and **H3** hold. Now, we should further solve the matrix inequality (21) by select suitable initial coupling strength $c(0)$, matrices D and $\overline{Q}_k, k = 1, 2, 3$, to realize the cluster synchronization based on Theorem 2 in Section 3. By using the MATLAB LMI Toolbox,

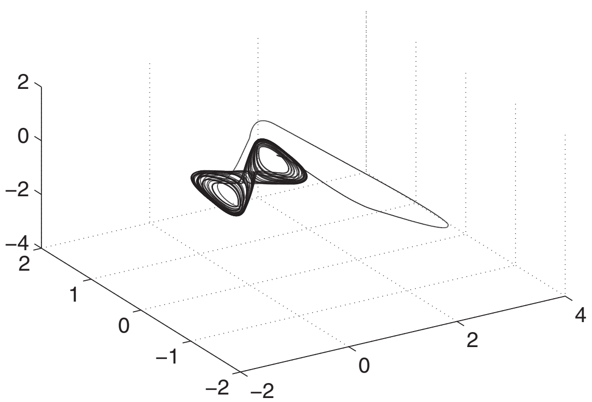

Fig. 2. Phase portrait of node 1 converging to a single 3-D neural network, i.e., s_1.

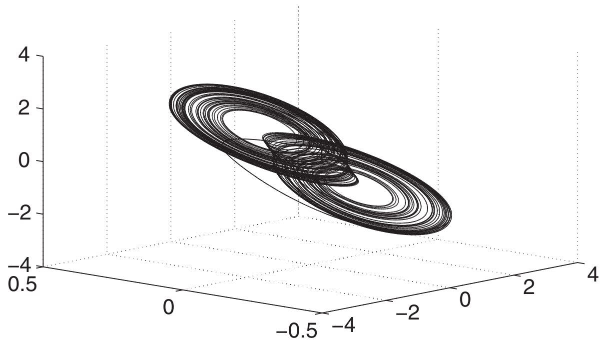

Fig. 3. Phase portrait of node 4 converging to a single Chua's circuit, i.e., s_2.

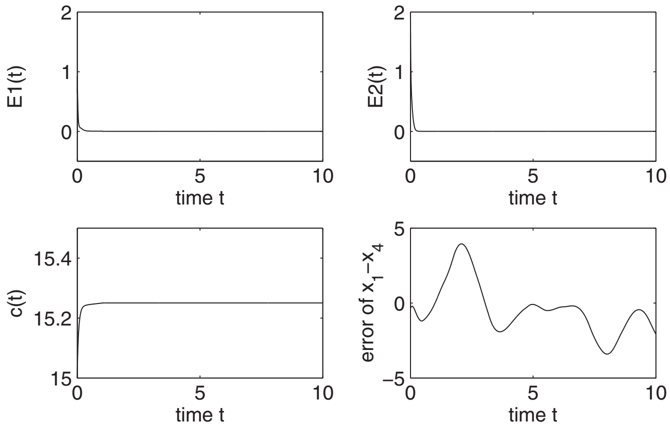

Fig. 4. Cluster synchronization error and adaptive coupling strength $c(t)$ of network (1) under $c(0) = 15$, $d_1 = d_2 = 20$, and time-varying delay $\tau(t) = 1$.

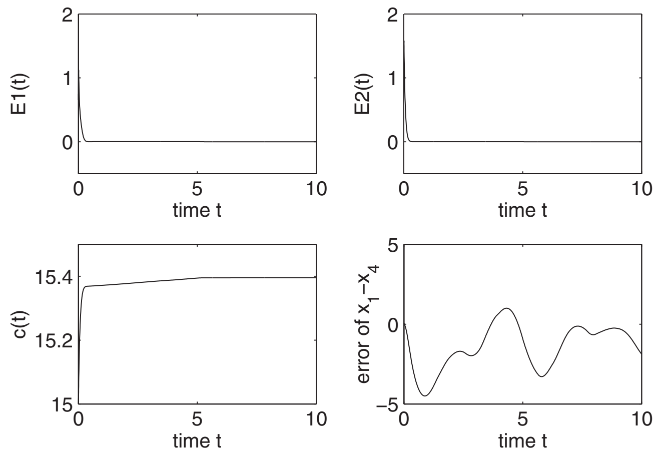

Fig. 5. Cluster synchronization error and adaptive coupling strength $c(t)$ of network (1) under $c(0) = 15$, $d_1 = d_2 = 20$, and time-varying delay $\tau(t) = 5$.

we can obtain the feasible solution of matrix inequality (21) as $c(0) = 15, d_1 = d_2 = 20$ and $\overline{Q}_k = I_6, k = 1, 2, 3$. The initial conditions of network (1) are chosen from interval $[0, 1]$ randomly. Figs. 2 and 3 show the phase portraits of nodes 1 and 4, respectively, and the cluster synchronization error is exhibited by Fig. 4, where

$$E1(t) = \|x_1 - x_2\|_2 + \|x_2 - x_3\|_2 + \|x_1 - x_3\|_2,$$
$$E2(t) = \|x_4 - x_5\|_2 + \|x_5 - x_6\|_2 + \|x_4 - x_6\|_2.$$

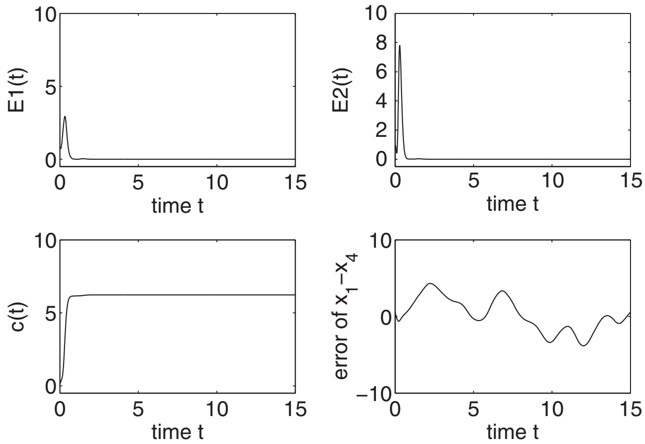

Fig. 6. Cluster synchronization error and adaptive coupling strength $c(t)$ of network (1) under $c(0) = 0.1$, $d_1 = d_2 = 20$, and time-varying delay $\tau(t) = 1$.

In this figure, we can see that the cluster synchronization is realized asymptotically as $E1(t) \to 0, E2(t) \to 0$ and $x_1(t) - x_4(t) \nrightarrow 0$ as $t \to \infty$.

Since our obtained theoretical results are all delay-independent, we can change the delay without destroying the cluster synchronization, which can be seen from Fig. 5 with $\tau(t) = 5$. Fig. 6 gives a simulation with small initial coupling strength $c(0) = 0.1$, which shows that the synchronization is still evident, indicating that the proposed control method is very effective.

4.2. Synchronization on a small-world network

In order to demonstrate the effectiveness of the proposed synchronization control method in Section 3 in a real world context, here we consider synchronization on a random network with small-world structure [25], whose local dynamics is controlled by Chua's circuit (36) with different parameters l. This means that the local behaviors of nodes are governed by the same equation but with different parameters.

This network is generated by the NW small-world algorithm [25] with size $N = 100$ and adding connection probability 0.001, where each node is symmetrically connected with its 2 nearest neighbors in its initial nearest-neighbor network. And the other added long-connections are $(1, 52)$ and $(30, 52)$, where the node pair (i, j) means that node i and j are connected. We will consider two kinds of cluster synchronization in the following simulations.

First, assume that these nodes with number from 1 to 50 are identical and their local dynamics is controlled by Chua's circuit (36) with parameter $l = 103/7$. And these nodes with number from 51 to 100 are also identical and their local dynamics is controlled by Chua's circuit (36) with parameter $l = 100/7$. Based on the division method of clusters in our model, this network can be divided into two clusters $U_1 = \{1, 2, \ldots, 50\}$, $U_2 = \{51, 52, \ldots, 100\}$, and the two controlled nodes can be chosen as $l_1 = 50$ and $l_2 = 100$ according to (11). Without loss of generality, we suppose that the coupling matrix of non-delayed coupling term of network (1) is

$$A = (A(i,j))_{100\times100} = \begin{pmatrix} A_1 & * \\ * & A_1 \end{pmatrix}_{100\times100}, \tag{39}$$

where

$$A_1 = \begin{pmatrix} -1 & 1 & 0 & & \cdots & & 0 \\ 1 & -2 & 1 & & & & \\ 0 & 1 & -2 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & -2 & 1 & 0 \\ & & & & 1 & -2 & 1 \\ 0 & & \cdots & & 0 & 1 & -1 \end{pmatrix}_{50\times50}$$

and the elements in part "$*$" are zeros, except that $A(1,52) = 1, A(1,100) = -1, A(30,52) = -2, A(30,100) = 2, A(52,1) = -3, A(52,30) = 3, A(100,1) = 4, A(100,30) = -4$. The coupling matrix of delayed coupling term (2) is

$$B = (B(i,j))_{100\times 100} = \begin{pmatrix} \mathbf{0} & * \\ * & \mathbf{0} \end{pmatrix}_{100\times 100}, \tag{40}$$

where the elements in part "$*$" are zeros, except that $B(1,52) = 4, B(1,100) = -4, B(30,52) = -3, B(30,100) = 3, B(52,1) = -2, B(52,30) = 2, B(100,1) = 4, B(100,30) = -4$. Obviously, $A \in \mathbf{M_3}(12)$ and $B \in \mathbf{M_2}(12)$.

Define the inner-coupling matrix $\Gamma = I_3, \bar{c} = 1$ and time-varying delay $\tau(t) \equiv 1$. The adaptive law of coupling strength is designed by (22) with $\alpha = 1$. For this two Chua's circuit systems, we can choose $P_1 = P_2 = I_3$ and $\Delta_1 = \Delta_2 = 10I_3$ to achieve the inequality (6). By simple computation, we attain that $\widetilde{P}_k = 10I_6$, for $k = 1,2,3$. The assumptions **H1** and **H3** hold with these settings. By using the MATLAB LMI Toolbox, we can obtain the feasible solution of matrix inequality (21) in Theorem 2 in Section 3 as $c(0) = 20, d_1 = d_2 = 20$ and $\overline{Q}_k = I_6, k = 1,2,3$. The initial conditions of network (1) are chosen from interval $[0,1]$ randomly, i.e., $x_{ij}(0) \in [0,1], i = 1,2,\ldots,100, j = 1,2,3$. In Fig. 7, (a) is the phase portrait of Chua's circuit with $l = 103/7$, i.e., s_1, (b) is the phase portrait of Chua's circuit with $l = 100/7$, i.e., s_2, and sub-figure (c) shows the difference of two synchronous states s_1 and s_2. Fig. 8 gives the trajectories of all state variables $x_{ij}(t), i = 1,2,\ldots,100, j = 1,2,3$ and coupling strength $c(t)$. From this figure, we can see that the adaptive cluster synchronization of dynamical network (1) with delayed coupling term (2) and adaptive coupling strength (22) is realized, as the trajectories of these nodes in U_1 converge to s_1 and the trajectories of these nodes in U_2 converge to s_2, and the coupling strength $c(t)$ converges to a constant.

Then, keeping the small-world network structure fixed, we assume that these nodes with number from 1 to 30 are identical and their local dynamics is controlled by Chua's circuit (36) with parameter $l = 103/7$. And these nodes with number from 31 to 100 are also identical and their local dynamics is controlled by Chua's circuit with parameter $l = 100/7$. This network can be divided into two clusters $U_1 = \{1,2,\ldots,30\}, U_2 = \{31,32,\ldots,100\}$, and the two controlled nodes can be chosen as $l_1 = 30$ and $l_2 = 100$ according to (11). Without loss of generality, we suppose that the coupling matrix of non-delayed coupling term of network (1) is

$$A = (A(i,j))_{100\times 100} = \begin{pmatrix} A_2 & * \\ * & A_3 \end{pmatrix}_{100\times 100}, \tag{41}$$

where

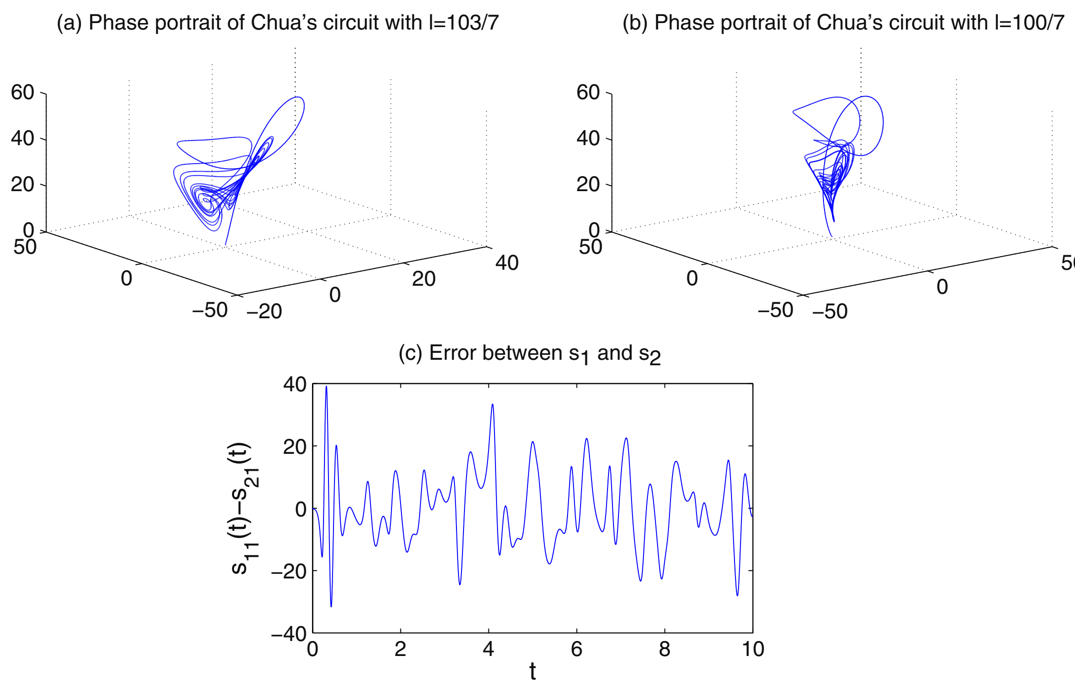

Fig. 7. Different local dynamics. Sub-figure (a): the phase portrait of Chua's circuit with $l = 103/7$, i.e., s_1; (b): the phase portrait of Chua's circuit with $l = 100/7$, i.e., s_2; (c): the difference of two synchronous states s_1 and s_2.

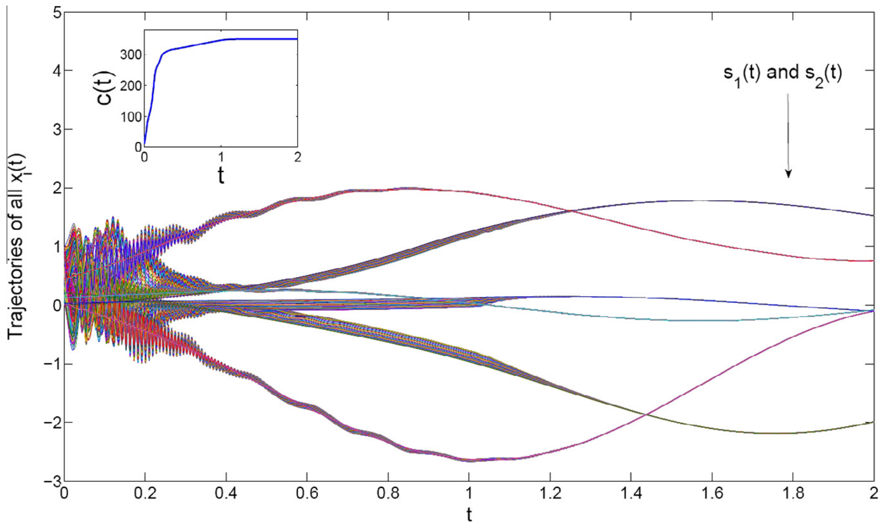

Fig. 8. Dynamics of a small-world network (1) with delayed coupling term (2) and adaptive coupling strength (22). The trajectories of all state variables $x_{ij}(t), i = 1, 2, \ldots, 100, j = 1, 2, 3$ and coupling strength $c(t)$. The trajectories of these nodes in $U_1 = \{1, 2, \ldots, 50\}$ converge to s_1 and the trajectories of these nodes in $U_2 = \{51, 52, \ldots, 100\}$ converge to s_2.

$$A_2 = \begin{pmatrix} -1 & 1 & 0 & & \cdots & & 0 \\ 1 & -2 & 1 & & & & \\ 0 & 1 & -2 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & -2 & 1 & 0 \\ & & & & 1 & -2 & 1 \\ 0 & & \cdots & & 0 & 1 & -1 \end{pmatrix}_{30\times 30},$$

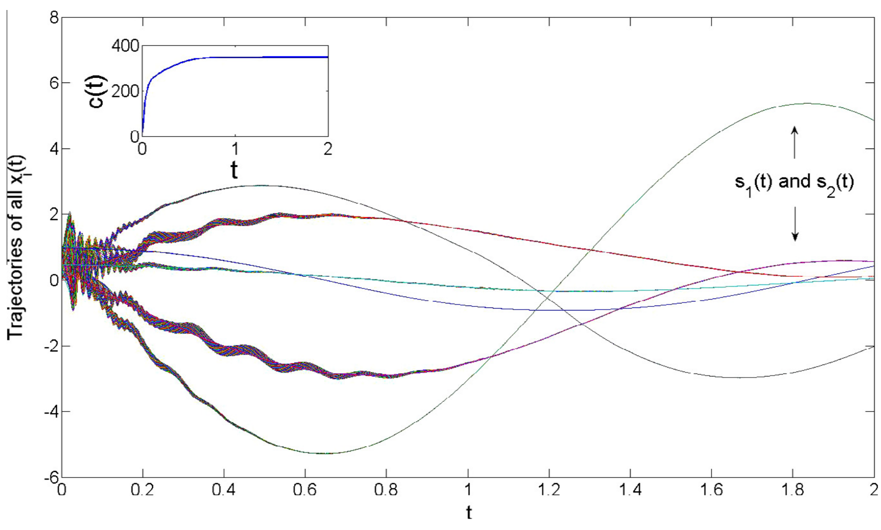

Fig. 9. Dynamics of a small-world network (1) with delayed coupling term (2) and adaptive coupling strength (22). The trajectories of all state variables $x_{ij}(t), i = 1, 2, \ldots, 100, j = 1, 2, 3$ and coupling strength $c(t)$. The trajectories of these nodes in $U_1 = \{1, 2, \ldots, 30\}$ converge to s_1 and the trajectories of these nodes in $U_2 = \{31, 32, \ldots, 100\}$ converge to s_2.

$$A_3 = \begin{pmatrix} -1 & 1 & 0 & & \cdots & & 0 \\ 1 & -2 & 1 & & & & \\ 0 & 1 & -2 & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & -2 & 1 & 0 \\ & & & & 1 & -2 & 1 \\ 0 & & \cdots & & 0 & 1 & -1 \end{pmatrix}_{70\times 70}$$

and the elements in part "$*$" are zeros, except that $A(1,52)=1, A(1,100)=-1, A(30,52)=-2, A(30,100)=2, A(52,1)=-3, A(52,30)=3, A(100,1)=4, A(100,30)=-4$. The coupling matrix of delayed coupling term (2) is

$$B=(B(i,j))_{100\times 100}=\begin{pmatrix} \mathbf{0} & * \\ * & \mathbf{0} \end{pmatrix}_{100\times 100}, \tag{42}$$

where the elements in part "$*$" are zeros, except that $B(1,52)=4, B(1,100)=-4, B(30,52)=-3, B(30,100)=3, B(52,1)=-2, B(52,30)=2, B(100,1)=4, B(100,30)=-4$. The condition in Theorem 2 can be verified similarly. The simulation result is given by Fig. 9. In this case, the cluster synchronization is still realized by the proposed adaptive control method.

5. Conclusion

In this paper, by using the adaptive control method, we have investigated the cluster synchronization on a dynamical network with time-varying delay or distributed delay, where the coupling strength is changed instantaneously according to two kinds of adaptive law. Since the discussed network has nonidentical nodes and the coupling matrices can be both asymmetric and weighted, this research work may be more applicable for general real-world dynamical networks, compared to previous works about cluster synchronization.

Adopting the LaSalle invariant principle and linear matrix inequality, we have obtained some sufficient conditions to ensure the globally asymptotical stability of cluster synchronization. These conditions can be realized easily as verified in the simulation section and Appendix A. Numerical simulations have verified these theoretical results and shown the effectiveness of the proposed control scheme.

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Appendix A. The method to ensure the inequality (8)

We now explain that the assumption **H4** is reasonable, i.e., the inequality (8) can be achieved if we set the parameters suitably. First, we can rewrite the N-order matrix $\overline{P}_kA+A^T\overline{P}_k$ as

$$\overline{P}_kA+A^T\overline{P}_k=\begin{bmatrix} 2p_1^kA_{11} & F_{12}^k & \cdots & F_{1m}^k \\ F_{21}^k & 2p_2^kA_{22} & \cdots & F_{2m}^k \\ \vdots & \vdots & \ddots & \vdots \\ F_{m1}^k & F_{m2}^k & \cdots & 2p_m^kA_{mm} \end{bmatrix},$$

where $F_{ij}^k=p_i^kA_{ij}+p_j^kA_{ji}^T$ for $i\neq j$. For any vector $y=(y_1,y_2,\ldots,y_N)^T\in R^N$, let

$$\bar{y}_i=(y_{l_{i-1}+1},y_{l_{i-1}+2},\ldots,y_{l_i})^T,\quad i=1,2,\ldots,m.$$

Then we update y as $y=(\bar{y}_1^T,\bar{y}_2^T,\ldots,\bar{y}_m^T)^T$, and by Lemma 4 we have

$$y^T(\overline{P}_kA + A^T\overline{P}_k - 2\overline{P}_kD)y = \sum_{i=1}^{m}\sum_{j\neq i}\bar{y}_i^TF_{ij}^k\bar{y}_j + 2\sum_{i=1}^{m}p_i^k\bar{y}_i^T(A_{ii} - D_i)\bar{y}_i \leqslant \sum_{i=1}^{m}\sum_{j\neq i}\pi(F_{ij}^k)(\bar{y}_i^T\bar{y}_i + \bar{y}_j^T\bar{y}_j) + 2\sum_{i=1}^{m}p_i^k\bar{y}_i^T(A_{ii} - D_i)\bar{y}_i$$
$$\leqslant \sum_{v=1}^{m}2(m-1)(\max_{i\neq j}\pi(F_{ij}^k))\bar{y}_v^T\bar{y}_v + 2\sum_{v=1}^{m}p_v^k\bar{y}_v^T(A_{vv} - D_v)\bar{y}_v$$
$$= \sum_{v=1}^{m}\bar{y}_v^T\left[2(m-1)(\max_{i\neq j}\pi(F_{ij}^k))I_{o(v)} + 2p_v^k(A_{vv} - D_v)\right]\bar{y}_v.$$

Since $A_{vv} \in \mathbf{M_1}(o(v))$, from Lemma 2, $A_{vv} - D_v < 0$. Then, under suitable parameters, the matrix $2(m-1)(\max_{i\neq j}\pi(F_{ij}^k))I_{o(v)} + 2p_v^k(A_{vv} - D_v)$ can be negative definite for all v, i.e., the matrix $\overline{P}_kA + A^T\overline{P}_k - 2\overline{P}_kD$ is negative definite.

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