Flocking of multi-agent dynamical systems based on pseudo-leader mechanism[☆]

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ABSTRACT

Flocking behavior of multiple agents can be widely observed in nature such as schooling fish and flocking birds. Recent literature has proposed the possibility that flocking can be achieved even with only a small fraction of informed agents with the desired position and velocity. However, it is still a challenging problem to determine which agents should be informed or have the ability to detect the desired information. This paper aims to address this problem. By combining the ideas of virtual force and pseudo-leader mechanism, where a pseudo-leader represents an informed agent who can detect the desired information, we propose a scheme for choosing pseudo-leaders in a multi-agent group, which can be applied to an unconnected or switching neighbor graph. Numerical examples are given to show the effectiveness of the methods presented in this paper.

1. Introduction

Flocking is a collective behavior of a large group of mobile agents. Typical flocking phenomena include flocks of birds, schools of fish, herds of animals, colonies of bacteria, etc. In nature, animals achieve flocking for various reasons. For example, in order to keep safety in numbers and also to confuse predators, they will form flocks for protection. An agent is more likely to be attacked if it strays away from the flocking group.

As a common demonstration of emergent behavior, flocking was first simulated on a computer by Craig Reynolds [1]. In 1987, he started with a boid model to build a simulated flock and introduced three rules for flocking:

- Collision Avoidance: steer to avoid collision with nearby flock-mates (short range repulsion).
- Velocity Matching: steer to match velocity with nearby flock-mates.
- Flock Centering: steer to stay close to nearby flockmates (long range attraction).

The mechanism, known as "separation", "alignment" and "cohesion", results in all agents moving in a formation with the same heading and a fixed network structure. From then on, these three rules have been widely used to study flocking behavior.

As a special case of Reynolds' model, in [2], Vicsek et al. presented a simulation model based on nearest neighborhood law, in which each agent's heading is updated by the average of the headings of its nearest neighbors and itself. It was shown that the headings of all the group agents converge to a common value.

Many works have been devoted to study Reynolds' and Vicsek's models in recent years [3–13]. To thoroughly and systematically investigate flocking behavior, artificial potential functions (APFs) have been widely used. In [4], Tanner et al. presented an APF in a network with fixed topology which is a differential, nonnegative and radially unbounded function of the distance between two agents. Then, he modified the APF to a nonsmooth one in a network with switching topology which captures the fact that there is no interaction between two agents beyond a proper distance [5]. Later, Olfati–Saber modified Tanner's APF and defined another APF which is a bounded and smooth one for switching topology [7].

In view of the pitfall of regular fragmentation [7], which is a phenomenon of flocking failure most likely occurring for a generic set of initial states and a large number of agents, Olfati–Saber also introduced a flocking mechanism based on a virtual leader [7]. Even though the initial state is selected randomly, the mechanism can guarantee a flocking behavior. However, this method assumes that all the agents know information about the virtual leader. In consideration of practical use, Shi et al. and Su et al. showed that flocking appears even when only some agents are informed [8–10], which are called pseudo-leaders in this paper.

Though Shi et al. and Su et al. showed that flocking may appear when only a fraction of agents are informed in a group, precisely which agents should be informed has not yet been considered. In this paper, we focus on investigating which agents should be selected as pseudo-leaders for flocking in a weighted network. Another difference from previous work is that, in the absence of control, the acceleration of each agent is dynamic. This can be seen in our system model in Section 2. That is, if there are no attractive/repulsive controls, no information exchange with others and no information received from the virtual leader for an agent, its acceleration is dynamic instead of constant, as is the case in previous work. By using the Lyapunov stability theory, a simple criterion for choosing pseudo-leaders is proposed.

The reminder of this paper is organized as follows. In Section 2, model depiction, preliminaries about graph theory, and mathematical analysis are briefly introduced. In Section 3, the main results are proposed. We investigate how to select pseudo-leaders for flocking in a multi-agent dynamical group with fixed topology. Section 4 gives some extensions and discussion of the main results so that one can gain a useful insight into the problem of choosing pseudo-leaders. Two numerical examples in small-sized and large-scale groups are simulated to illustrate effectiveness of the proposed approach in Section 5. Finally, we summarize the main ideas and conclusions in Section 6.

2. Preliminaries

2.1. Graph theory

To make this paper self-contained, some basic concepts and results of graph theory are briefly introduced [12–14].

A *graph* $\bar{\mathcal{G}}$ is a pair of sets $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite non-empty set of elements called *vertices*, and $\mathcal{E}$ is a set of unordered pairs of distinct vertices called *edges*. The set $\mathcal{V}$ and $\mathcal{E}$ are the vertex set and edge set of $\bar{\mathcal{G}}$, and are often denoted by $\mathcal{V}(\bar{\mathcal{G}})$ and $\mathcal{E}(\bar{\mathcal{G}})$, respectively. If $i, j \in \mathcal{V}$ and $(i, j) \in \mathcal{E}$, then i and j are *adjacent vertices*, or *neighbors*. The set of neighbors of a vertex i is denoted by $\mathcal{N}_i$. A *walk* in a graph is a sequence of vertices and edges $j_0, e_1, j_1, \ldots, e_k, j_k$, in which each edge is $e_r = (j_{r-1}, j_r)$. A *path* is a walk in which no vertex is repeated. If there is a path between any two vertices of a graph $\bar{\mathcal{G}}$, then $\bar{\mathcal{G}}$ is said to be *connected*. If $\bar{\mathcal{G}}$ and $\bar{\mathcal{G}}_1$ are graphs with $\mathcal{V}(\bar{\mathcal{G}}_1) \subseteq \mathcal{V}(\bar{\mathcal{G}})$ and $\mathcal{E}(\bar{\mathcal{G}}_1) \subseteq \mathcal{E}(\bar{\mathcal{G}})$, then $\bar{\mathcal{G}}_1$ is a *subgraph* of $\bar{\mathcal{G}}$.

The neighbor graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ is a weighted graph consisting of a set of indexed vertices and a set of ordered edges, where $\mathbf{W} = (w_{ij}) \in \mathbb{R}^{N \times N}$, $N \in \mathbb{R}^+$ is the weight matrix which represents the weighted coupling coefficients of interaction between the agents. If there is a link from vertex i to vertex j ($j \neq i$), then $w_{ij} = w_{ji} > 0$ and w_{ij} is the weight; otherwise, $w_{ij} = 0$. Throughout the paper, assume that $\mathbf{W}$ is a symmetric matrix satisfying *diffusive condition*

$$\sum_{j=1}^{N} w_{ij} = 0.$$

2.2. Model depiction

Consider a multi-agent system consisting of N agents. Here, the dynamics of each agent in the group is given by

$$\dot{\mathbf{p}}_i = \mathbf{v}_i,$$
$$\dot{\mathbf{v}}_i = \mathbf{f}(\mathbf{v}_i) + \mathbf{u}_i, \tag{1}$$

where $1 \leq i \leq N$, $\mathbf{p}_i \in \mathbb{R}^n$, $\mathbf{v}_i \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{v}_i) \in \mathbb{R}^n$ denote the position, velocity and the acceleration dynamics (without control input) of the i-th agent respectively, and $\mathbf{u}_i \in \mathbb{R}^n$ is the control input of agent i.

The motion model for the virtual leader is

$$\dot{\mathbf{p}}_l = \mathbf{v}_l,$$
$$\dot{\mathbf{v}}_l = \mathbf{f}(\mathbf{v}_l), \tag{2}$$

where $\mathbf{p}_l \in \mathbb{R}^n$, $\mathbf{v}_l \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{v}_l) \in \mathbb{R}^n$ represent the position, velocity and the acceleration dynamics of the virtual leader, respectively.

The flocking task is to design an appropriate control input $\mathbf{u}_i$ based on Reynolds' rules.

2.3. Mathematical preliminaries

In this subsection, some useful mathematical definitions, lemmas and assumptions are outlined.

Definition 1. A matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{N \times N}$ is called *reducible* if the indices $1, 2, \ldots, N$ can be divided into two disjoint nonempty sets $i_1, i_2, \ldots, i_{\beta_1}$ and $j_1, j_2, \ldots, j_{\beta_2}$ (with $\beta_1 + \beta_2 = N$) such that

$$a_{i_{\kappa_1} j_{\kappa_2}} = 0$$

for all $1 \leq \kappa_1 \leq \beta_1$ and $1 \leq \kappa_2 \leq \beta_2$. A matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{N \times N}$ is *irreducible* if and only if it is not reducible.

A matrix is reducible if and only if it can be placed into block upper-triangular form by simultaneous row/column permutations. In addition, a matrix is reducible if and only if its associated graph is not connected [15].

Definition 2 ([16]). A matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{N \times N}$ is called *diagonally dominant* if $|a_{ii}| \geq \sum_{j=1, j \neq i}^{N} |a_{ij}|$ for $1 \leq i \leq N$. $\mathbf{A}$ is called *strictly diagonally dominant* if $|a_{ii}| > \sum_{j=1, j \neq i}^{N} |a_{ij}|$ for $1 \leq i \leq N$.

The Gershgorin circle theorem [16] results in many interesting conclusions. A strictly diagonally dominant matrix is nonsingular. A symmetric diagonally dominant real matrix with nonnegative diagonal entries is positive semi-definite. If a symmetric matrix is strictly diagonally dominant and all its diagonal elements are positive, then its eigenvalues are positive; if all its diagonal elements are negative, then its eigenvalues are negative. Thus is obvious that the real part of eigenvalues of the weight matrix $\mathbf{W}$ which are all negative except an eigenvalue 0 with multiplicity one. Throughout the paper, we denote a positive definite (positive semi-definite, negative definite, negative semi-definite) matrix $\mathbf{A}$ as $\mathbf{A} > \mathbf{0}$ ($\geq \mathbf{0}$, $< \mathbf{0}$, $\leq \mathbf{0}$).

Definition 3 ([17]). A set Ω is said to be a *positively invariant set* with respect to an equation if a solution $x(t)$ of this equation satisfies

$$x(0) \in \Omega \Rightarrow x(t) \in \Omega \quad \forall t \geq 0.$$

Definition 4 ([17]). A continuous function $g : [0, \bar{\rho}) \rightarrow [0, \infty)$ is said to belong to *class* $\mathcal{K}$ if it is strictly increasing and $g(0) = 0$. It is said to belong to *class* $\mathcal{K}_\infty$ if $\bar{\rho} = \infty$ and $g(\rho) \rightarrow \infty$ as $\rho \rightarrow \infty$.

Definition 5. The *2-norm*, also called *Euclidean norm*, of a vector ξ is defined as

$$\|\xi\|_2 = \sqrt{\xi^\top \xi},$$

where $\top$ denotes the transpose of a vector or a matrix.

Definition 6 ([7]). σ-*norm* of a vector ξ is defined as

$$\|\xi\|_\sigma = \frac{1}{\sigma} \left[\sqrt{1 + \sigma \|\xi\|_2^2} - 1 \right],$$

where σ is a positive constant.

Due to the fact that $\|\xi\|_\sigma = 0$ as $\|\xi\|_2 = 0$, $\|\xi\|_\sigma \rightarrow \infty$ as $\|\xi\|_2 \rightarrow \infty$, and that $\frac{d\|\xi\|_\sigma}{d\|\xi\|_2} = \frac{\|\xi\|_2}{\sqrt{1 + \sigma \|\xi\|_2^2}} > 0$ when $\|\xi\|_2 \neq 0$, $\|\xi\|_\sigma$ is a class $\mathcal{K}_\infty$ function of $\|\xi\|_2$. On the other hand, note that even though $\|\xi\|_\sigma$ is not a norm in the sense of algebra, it is differentiable everywhere while $\|\xi\|_2$ is not. Thus $\|\xi\|_\sigma$ instead of $\|\xi\|_2$ is used to construct APF in this paper.
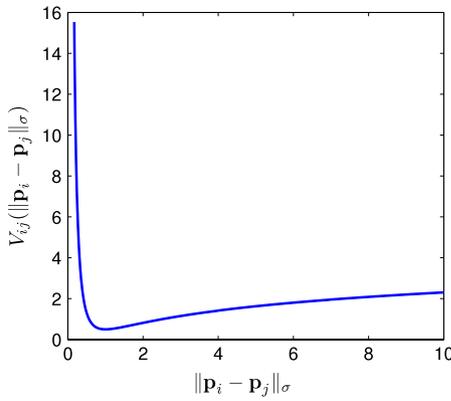
Fig. 1. The APF $V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) = \frac{1}{2} \ln \|\mathbf{p}_i - \mathbf{p}_j\|_\sigma^2 + \frac{1}{2\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma^2}$ versus $\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma$.

Definition 7. *Artificial Potential Function (APF)* V_{ij} is a differentiable, nonnegative, radially unbounded [17] function of $\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma$, which is the σ-norm of the position error between the *i*-th and the *j*-th agents. $V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma)$ has the following properties:

(1) $V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) \to \infty$ as $\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma \to 0$,
(2) $V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma)$ attains its unique minimum when the *i*-th agent and the *j*-th agent are located at a desired distance.

An example of APF is

$$V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) = c_1 \quad \ln \|\mathbf{p}_i - \mathbf{p}_j\|_\sigma^2 + \frac{c_2}{\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma^2}, \tag{3}$$

where $c_1 > 0$, $c_2 > 0$ are two constants. The potential function approaches infinity as $\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma$ tends to 0, and attains its unique minimum when $\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma = \sqrt{\frac{c_2}{c_1}}$. The APF $V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma)$ with $c_1 = c_2 = \frac{1}{2}$ is depicted in Fig. 1.

In order to derive the main results, the following lemmas and assumption are needed.

Lemma 1 (*Schur Complement [18,19]*). *The following linear matrix inequality (LMI)*

$$\begin{pmatrix} \mathcal{A}(x) & \mathcal{B}(x) \\ \mathcal{B}(x)^\top & \mathcal{C}(x) \end{pmatrix} < \mathbf{0},$$

where $\mathcal{A}(x)^\top = \mathcal{A}(x)$, $\mathcal{C}(x)^\top = \mathcal{C}(x)$, *is equivalent to either of the following conditions:*

(a) $\mathcal{A}(x) < \mathbf{0}$ and $\mathcal{C}(x) - \mathcal{B}(x)^\top \mathcal{A}(x)^{-1} \mathcal{B}(x) < \mathbf{0}$
(b) $\mathcal{C}(x) < \mathbf{0}$ and $\mathcal{A}(x) - \mathcal{B}(x) \mathcal{C}(x)^{-1} \mathcal{B}(x)^\top < \mathbf{0}$.

Lemma 2 (*[20]*). *If a symmetric matrix* $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{N \times N} (a_{ij} > 0, \ i \neq j)$ *is irreducible and satisfies diffusive condition, then* $\mathbf{A} - \mathbf{D}_i < \mathbf{0}$ *holds for a diagonal matrix* $\mathbf{D}_i = \mathrm{diag}\{0, \ldots, 0, \delta, 0, \ldots, 0\} \in \mathbb{R}^{N \times N}$, *where the i-th* $(1 \leq i \leq N)$ *element* δ *is any positive constant and the others are 0.*

Assumption 1 (*A1*). Suppose that there exists a positive constant α satisfying $(\xi_2 - \xi_1)^\top (\mathbf{f}(\xi_2) - \mathbf{f}(\xi_1)) \leq \alpha \|\xi_2 - \xi_1\|_2^2$ for any two vectors $\xi_1, \xi_2 \in \mathbb{R}^n$.

This assumption is achievable for a large class of systems, such as linear systems, piecewise linear systems, and a numerous of chaotic systems (for example, Chua's circuit, Lorenz family, etc.).

3. Main results

In this section, a group of mobile agents with fixed topology is considered. Assume that the position and velocity neighbor graph

$\mathscr{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ is connected. That is, the weight matrix $\mathbf{W}$ is irreducible. For the case that $\mathscr{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ is unconnected, discussion will be found in Section 4.3.

Generally, there should be attractive and repulsive mechanisms in the control input, where the attraction indicates that any two agents intend to move closer to each other and repulsion provides the fact that each agent does not want to be too close to nearby flockmates. These two mechanisms can be jointly embodied in APF (actually there are many ways to achieve attraction and repulsion).

Besides "separation" and "cohesion", "alignment" is an important rule in flocking. We use the approach that group members receive moving information from a virtual leader to realize alignment. Based on this approach, recent literature has shown the phenomenon that flocking will appear even if not all the agents are informed in the group. Here, we investigate which agents should be selected as informed ones for flocking in a weighted network. An agent, which utilizes the motion information of the virtual leader as a reference in its controller, is a pseudo-leader of the group. Suppose that the i_1-th, i_2-th, $\ldots$, i_m-th agents are chosen as the pseudo-leaders.

The control law for the *i*-th agent is given by

$$\mathbf{u}_i = \mathbf{u}_i^1 + \mathbf{u}_i^2 + \mathbf{u}_i^3,$$

$$\mathbf{u}_i^1 = -\sum_{j \in \mathcal{N}_i} w_{ij}(\mathbf{v}_i - \mathbf{v}_j) = \sum_{j=1}^N w_{ij}\mathbf{v}_j,$$

$$\mathbf{u}_i^2 = -\sum_{j \in \mathcal{N}_i} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma)$$

$$= -\sum_{j=1, j\neq i}^N \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma), \tag{4}$$

$$\mathbf{u}_i^3 = \begin{cases} -h_{i_r}(\mathbf{v}_{i_r} - \mathbf{v}_l), \quad \dot{h}_{i_r} = k_{i_r}(\|\mathbf{v}_{i_r} - \mathbf{v}_l\|_2^2) \\ \qquad i \in \{i_r | 1 \leq r \leq m, \ 1 \leq i_r \leq N\}, \\ \mathbf{0} \qquad \qquad \qquad \qquad \qquad \text{otherwise,} \end{cases}$$

where ∇ represents the gradient of a function, h_{i_r} $(1 \leq r \leq m, 1 \leq i_r \leq N)$ is the adaptive feedback gain, $k_{i_r} > 0$ is a constant. For agent *i*, $\mathbf{u}_i^1$ is the velocity coupling of interactions between agents, and $\mathbf{u}_i^2$ stands for the coupling gradient of APF with respect to its position. The term $\mathbf{u}_i^3$, which will disappear if agent *i* is not a pseudo-leader, controls the received information from the virtual leader. It provides an adaptive feedback adjusting mechanism and will be more practical in engineering than those with linear feedback ones.

Define the position and velocity error between agent *i* and the virtual leader as $\mathbf{e}_i^\mathbf{p} \triangleq \mathbf{p}_i - \mathbf{p}_l$ and $\mathbf{e}_i^\mathbf{v} \triangleq \mathbf{v}_i - \mathbf{v}_l$, then we have

$$\dot{\mathbf{e}}_i^\mathbf{p} = \mathbf{e}_i^\mathbf{v},$$

$$\dot{\mathbf{e}}_i^\mathbf{v} = \mathbf{f}(\mathbf{v}_i) - \mathbf{f}(\mathbf{v}_l) + \mathbf{u}_i.$$

Since A1 holds, we obtain

$$\mathbf{e}_i^{\mathbf{v}\top}(\mathbf{f}(\mathbf{v}_i) - \mathbf{f}(\mathbf{v}_l)) \leq \alpha \|\mathbf{e}_i^\mathbf{v}\|_2^2$$

for $1 \leq i \leq N$. In addition, it is easy to get

$$\nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) = \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{e}_i^\mathbf{p} - \mathbf{e}_j^\mathbf{p}\|_\sigma)$$

$$= \nabla_{\mathbf{e}_i^\mathbf{p}} V_{ij}(\|\mathbf{e}_i^\mathbf{p} - \mathbf{e}_j^\mathbf{p}\|_\sigma)$$

and

$$\nabla_{\mathbf{e}_i^\mathbf{p}} V_{ij}(\|\mathbf{e}_i^\mathbf{p} - \mathbf{e}_j^\mathbf{p}\|_\sigma) = -\nabla_{\mathbf{e}_j^\mathbf{p}} V_{ij}(\|\mathbf{e}_i^\mathbf{p} - \mathbf{e}_j^\mathbf{p}\|_\sigma).$$

Then it follows

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) = \dot{\mathbf{e}}_i^{\mathbf{p}\top} \nabla_{\mathbf{e}_i^\mathbf{p}} V_{ij}(\|\mathbf{e}_i^\mathbf{p} - \mathbf{e}_j^\mathbf{p}\|_\sigma)$$

$$+ \dot{\mathbf{e}}_j^{\mathbf{p}\top} \nabla_{\mathbf{e}_j^\mathbf{p}} V_{ij}(\|\mathbf{e}_i^\mathbf{p} - \mathbf{e}_j^\mathbf{p}\|_\sigma)$$

$$= \mathbf{e}_i^{\mathbf{v}\top} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma)$$

$$+ \mathbf{e}_j^{\mathbf{v}\top} \nabla_{\mathbf{p}_j} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma).$$

Theorem 1. *Suppose that* A1 *holds. If* $\mathbf{W}_{N-m} + \alpha \mathbf{I}_{N-m} < \mathbf{0}$, *where* $\mathbf{W}_{N-m}$ *is the minor matrix of the weight matrix* $\mathbf{W}$ *by removing all the* i_r-th $(1 \leq r \leq m, 1 \leq i_r \leq N)$ *row–column pairs for a natural number* m $(1 \leq m < N)$, *flocking behavior appears in system* (1) *by the control strategy* (2) *and* (4). *The velocities of all agents approach the desired velocity* $(\mathbf{v}_i \to \mathbf{v}_l$ *for all* $1 \leq i \leq N)$ *asymptotically and collisions between agents are avoided* $(\mathbf{p}_i \nrightarrow \mathbf{p}_j$ *for any* $1 \leq i \leq j \leq N)$. *Furthermore, the global potentials of all agents in the group are minimized with the final configuration.*

Proof. Consider a positive semi-definite function as

$$L = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma)$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \mathbf{e}_i^{\mathbf{v}\top} \mathbf{e}_i^{\mathbf{v}} + \frac{1}{2} \sum_{r=1}^{m} \frac{(h_{i_r} - h)^2}{k_{i_r}},$$

where $h > 0$ is a constant to be determined.

We now prove that $\Omega_c = \{(\mathbf{p}_i - \mathbf{p}_j, \mathbf{e}_i^{\mathbf{v}}) \mid L \leq c, \ c > 0\}$, the sub-level set of L, is compact. Firstly, from $L \leq c$ one gets $\|\mathbf{e}_i^{\mathbf{v}}\|_2^2 = \mathbf{e}_i^{\mathbf{v}\top} \mathbf{e}_i^{\mathbf{v}} \leq 2\,c$ and $w_{ij} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) \leq 2\,c$ for $1 \leq i, j \leq N$. Due to the fact that the continuous function V_{ij} is radially unbounded, $V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) \to \infty$ as $\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma \to 0$, and that $\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma$ is a class $\mathcal{K}_\infty$ function with respect to $\|\mathbf{p}_i - \mathbf{p}_j\|_2$, there exists a positive constant c' such that $\|\mathbf{p}_i - \mathbf{p}_j\|_2 \leq c'$. Thus Ω_c is a bounded set. Secondly, because $\Omega_c' = \{L \mid L \leq c, \ c > 0\}$ is closed, Ω_c is a closed set for the continuity of function L. Then according to the Heine–Borel Theorem [21], Ω_c is compact.

Regarding L as a Lyapunov candidate, its derivative along the trajectories of (1), (2) and (4) is

$$\dot{L} = \sum_{i=1}^{N} \mathbf{e}_i^{\mathbf{v}\top} \left(\mathbf{f}(\mathbf{v}_i) - \mathbf{f}(\mathbf{v}_l) + \sum_{j=1}^{N} w_{ij} \mathbf{e}_j^{\mathbf{v}} \right.$$

$$\left. - \sum_{j=1, j\neq i}^{N} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) \right)$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \left(\mathbf{e}_i^{\mathbf{v}\top} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) \right.$$

$$\left. + \mathbf{e}_j^{\mathbf{v}\top} \nabla_{\mathbf{p}_j} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) \right)$$

$$- \sum_{r=1}^{m} h_{i_r} \mathbf{e}_{i_r}^{\mathbf{v}\top} \mathbf{e}_{i_r}^{\mathbf{v}} + \sum_{r=1}^{m} (h_{i_r} - h) \mathbf{e}_{i_r}^{\mathbf{v}\top} \mathbf{e}_{i_r}^{\mathbf{v}}$$

$$= \sum_{i=1}^{N} \mathbf{e}_i^{\mathbf{v}\top} \left(\mathbf{f}(\mathbf{v}_i) - \mathbf{f}(\mathbf{v}_l) + \sum_{j=1}^{N} w_{ij} \mathbf{e}_j^{\mathbf{v}} \right.$$

$$\left. - \sum_{j=1, j\neq i}^{N} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) \right)$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \mathbf{e}_i^{\mathbf{v}\top} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma)$$

$$+ \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1, i\neq j}^{N} \mathbf{e}_i^{\mathbf{v}\top} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) - \sum_{r=1}^{m} h \mathbf{e}_{i_r}^{\mathbf{v}\top} \mathbf{e}_{i_r}^{\mathbf{v}}$$

$$= \sum_{i=1}^{N} \mathbf{e}_i^{\mathbf{v}\top} \left(\mathbf{f}(\mathbf{v}_i) - \mathbf{f}(\mathbf{v}_l) + \sum_{j=1}^{N} w_{ij} \mathbf{e}_j^{\mathbf{v}} \right.$$

$$\left. - \sum_{j=1, j\neq i}^{N} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) \right)$$

$$+ \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \mathbf{e}_i^{\mathbf{v}\top} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) - \sum_{r=1}^{m} h \mathbf{e}_{i_r}^{\mathbf{v}\top} \mathbf{e}_{i_r}^{\mathbf{v}}$$

$$\leq \sum_{i=1}^{N} \alpha \mathbf{e}_i^{\mathbf{v}\top} \mathbf{e}_i^{\mathbf{v}} + \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{e}_i^{\mathbf{v}\top} \mathbf{e}_j^{\mathbf{v}} - \sum_{r=1}^{m} h \mathbf{e}_{i_r}^{\mathbf{v}\top} \mathbf{e}_{i_r}^{\mathbf{v}}$$

$$= \sum_{i=1}^{N} \alpha \|\mathbf{e}_i^{\mathbf{v}}\|_2^2 + \sum_{i=1}^{N} w_{ii} \|\mathbf{e}_i^{\mathbf{v}}\|_2^2$$

$$+ \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} w_{ij} \mathbf{e}_i^{\mathbf{v}\top} \mathbf{e}_j^{\mathbf{v}} - \sum_{r=1}^{m} h \|\mathbf{e}_{i_r}^{\mathbf{v}}\|_2^2$$

$$\leq \sum_{i=1}^{N} \alpha \|\mathbf{e}_i^{\mathbf{v}}\|_2^2 + \sum_{i=1}^{N} w_{ii} \|\mathbf{e}_i^{\mathbf{v}}\|_2^2$$

$$+ \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} w_{ij} \|\mathbf{e}_i^{\mathbf{v}}\|_2 \|\mathbf{e}_j^{\mathbf{v}}\|_2 - \sum_{r=1}^{m} h \|\mathbf{e}_{i_r}^{\mathbf{v}}\|_2^2$$

$$\triangleq \mathbf{e}^{\mathbf{v}\top} \mathbf{Q} \mathbf{e}^{\mathbf{v}},$$

where $\mathbf{e}^{\mathbf{v}} = (\|\mathbf{e}_1^{\mathbf{v}}\|_2, \|\mathbf{e}_2^{\mathbf{v}}\|_2, \ldots, \|\mathbf{e}_N^{\mathbf{v}}\|_2)^\top$, $\mathbf{Q} = \alpha \mathbf{I}_N + \mathbf{W} - \mathbf{H}$, and $\mathbf{H}$ is a diagonal matrix whose i_r-th $(1 \leq r \leq m)$ elements are h and the others are 0.

After applying row–column permutation, $\mathbf{Q}$ can be changed into

$$\begin{pmatrix} \mathbf{W}^* + \alpha \mathbf{I}_m - h \mathbf{I}_m & \mathbf{W}^{**} \\ \mathbf{W}^{**\top} & \mathbf{W}_{N-m} + \alpha \mathbf{I}_{N-m} \end{pmatrix},$$

where $\mathbf{W}^*, \mathbf{W}^{**}$ are the corresponding matrices with compatible dimensions. Since $\mathbf{W}_{N-m} + \alpha \mathbf{I}_{N-m} < \mathbf{0}$, $\mathbf{W}_{N-m} + \alpha \mathbf{I}_{N-m}$ is invertible. According to Lemma 1, choosing h be a positive constant satisfying $\mathbf{W}^* + \alpha \mathbf{I}_m - h \mathbf{I}_m - \mathbf{W}^{**} (\mathbf{W}_{N-m} + \alpha \mathbf{I}_{N-m})^{-1} \mathbf{W}^{**\top} < \mathbf{0}$, we have $\mathbf{Q} < \mathbf{0}$. Furthermore, $\dot{L} \leq 0$, which indicates that $L(t)$ is a non-increasing function of t. Thus any solution of (1), (2) and (4) starting in Ω_c will stay in it. Namely, Ω_c is a positively invariant set.

Because Ω_c is compact and positively invariant, every solution of the system converges to the largest invariant set Ω^* of the set $\{(\mathbf{p}_i - \mathbf{p}_j, \mathbf{e}_i^{\mathbf{v}}) \mid \dot{L} = 0\}$ on the basis of LaSalle's invariance principle [17]. In Ω^*, $\dot{\mathbf{e}}_i^{\mathbf{p}} = \mathbf{e}_i^{\mathbf{v}} = \mathbf{0}$, which means that all agents' final velocities are equal and their position differences remain unchanged in the steady state.

Furthermore, $\mathbf{e}_i^{\mathbf{v}} = \mathbf{0}$ leads to $\dot{\mathbf{e}}_i^{\mathbf{v}} = \mathbf{0}$ in Ω^*. Combining with Eqs. (1), (2) and (4), we have $\sum_{j=1, j\neq i}^{N} \nabla_{\mathbf{p}_i} V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) = 0$ for $1 \leq i \leq N$. This indicates that the group final configuration is a minima of global potential function of agent i.

Collision avoidance can be proved by contradiction. Assume that there exists a time $t_1 > 0$ so that the position difference between two distinct agents i^* and i^{**} satisfies $\|\mathbf{p}_{i^*} - \mathbf{p}_{i^{**}}\|_2 \to 0$ as $t \to t_1$. Then $\|\mathbf{p}_{i^*} - \mathbf{p}_{i^{**}}\|_\sigma \to 0$ since $\|.\|_\sigma$ is a class $\mathcal{K}_\infty$ function with respect to $\|.\|_2$. According to the definition of APF, $V_{i^* i^{**}}(\|\mathbf{p}_{i^*} - \mathbf{p}_{i^{**}}\|_\sigma) \to \infty$. This is in contradiction with the fact that Ω_c is a positively invariant set. Therefore, no two agents collide at any time $t \geq 0$.

Thus the proof is complete. □

From this theorem, if $\lambda_{\max}(\mathbf{W}_{N-m}) < -\alpha$, where $\lambda_{\max}(.)$ represents the maximum eigenvalue of a symmetric matrix, $\bar{\mathcal{V}} = \{i_1, i_2, \ldots, i_m\}$ can be chosen as the pseudo-leader set to guarantee flocking in system (1), (2) and (4).

4. Discussion and extensions

In this section, some remarks and extensions are discussed.

4.1. Other flocking models

Previous papers [4–9] studied a multi-agent dynamical model for flocking behavior with fixed velocity. The motion models of the

i-th agent and the virtual leader respectively are

$$\dot{\mathbf{p}}_i = \mathbf{v}_i, \tag{5}$$
$$\dot{\mathbf{v}}_i = \mathbf{u}_i,$$

and

$$\dot{\mathbf{p}}_l = \mathbf{v}_l, \tag{6}$$
$$\dot{\mathbf{v}}_l = \mathbf{0},$$

where $\mathbf{p}_i, \mathbf{v}_i$ ($i = 1, 2, \ldots, N, l$) and $\mathbf{u}_i$ ($i = 1, 2, \ldots, N$) have the same meaning with Eqs. (1) and (2). For the virtual leader, $\mathbf{f}(\mathbf{v}_l) = 0$ implies that it moves along a straight line with a desired velocity $\mathbf{v}_l(0)$. Letting $\alpha = 0$, we have the following theorem for choosing pseudo-leaders in the classical model.

Theorem 2. *If* $\mathbf{W}_{N-m} < \mathbf{0}$ *holds, flocking behavior will appear in system* (5) *by the control strategy* (4) *and* (6). *The velocities of all agents approach the desired velocity asymptotically and collisions between agents are avoided. In addition, the global potentials of all these agents are minimized with the group final configuration.*

For other flocking models such as taking velocity damping into consideration [10] *(which is frequently unavoidable when objects move with high speeds or in a viscous environment), the schemes for determining pseudo-leaders can be attained by similar analysis.*

4.2. A single pseudo-leader is enough for flocking

Consider the classical model at first. Denote the minor matrix of $\mathbf{W}$ by deleting any row–column pair as $\mathbf{W}_{N-1}$. It can be rewritten as $\mathbf{W}_{N-1} = \hat{\mathbf{W}}_{N-1} + \sum_i \mathbf{D}_i$, where $1 \le i \le \chi, 1 \le \chi \le N - 1$, $\hat{\mathbf{W}}_{N-1}$ is the corresponding symmetric and diffusive matrix, and $\mathbf{D}_i$ is the diagonal matrix where the i-th element is negative and the others are 0. According to Lemma 2, $\mathbf{W}_{N-1}$ is negative definite. Therefore, flocking will occur with just one single pseudo-leader (any agent is available as an option) based on Theorem 2.

Below we will discuss the model presented in Section 2.2. Suppose that the weight matrix $\mathbf{W} = w \mathbf{B}$, where $w > 0$ is the common weight coupling, $\mathbf{B} = (b_{ij}) \in \mathbb{R}^{N \times N}$ is the adjacent matrix with $b_{ij} = b_{ji} = 1$ if there is a link from agent i to agent j ($j \ne i$) and $b_{ij} = 0$ otherwise. Moreover, $\mathbf{B}$ satisfies diffusive condition $\sum_{j=1}^{N} b_{ij} = 0$. For an agent i_r ($1 \le r \le m$, $1 \le i_r \le N$), if the minor matrix $\mathbf{B}_{N-1}$ that is obtained by deleting the i_r-th row–column pair of $\mathbf{B}$ satisfies $\lambda_{\max}(\mathbf{B}_{N-1}) < -\frac{\alpha}{w}$, it can be picked out as a pseudo-leader for flocking according to Theorem 1. Since $\mathbf{B}_{N-1}$ is negative definite by Gerschgorin theorem, it is concluded that flocking will be achieved with just one single pseudo-leader, which can be selected randomly from the vertex set $\mathcal{V}$, provided that the common coupling weight w is large enough.

4.3. The position and velocity neighbor graph is unconnected

In the main results, we assume that the position and velocity neighbor graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ is connected. In reality, however, it is not always the case. Suppose that the graph consists of several connected subgraphs $\mathcal{G}^j = (\mathcal{V}^j, \mathcal{E}^j, \mathbf{W}^j)$ ($j \in \{1, 2, \ldots\}$), where $\bigcup_j \mathcal{V}^j = \mathcal{V}$, $\bigcup_j \mathcal{E}^j = \mathcal{E}$, $\mathbf{W}^j \in R^{N_j \times N_j}$ is the weight matrix of subgraph $\mathcal{G}^j$ which is symmetric, irreducible and diffusive, and $\sum_j N_j = N$. In subgraph $\mathcal{G}^j$, pseudo-leader set $\bar{\mathcal{V}}^j$ can be picked out according to Theorems 1 or 2. Putting the pseudo-leaders in all the subgraphs together, the scheme for choosing pseudo-leader set $\bar{\mathcal{V}} = \bigcup_j \bar{\mathcal{V}}^j$ of the whole unconnected group is obtained.

As discussed in the previous subsection, one single pseudo-leader in each connected subgroup is enough for flocking. If, however, there exists a connected subgroup in which no agent is informed, it is impossible for this subgroup to detect the desired moving information. Thus flocking failure, such as regular fragmentation, may occur under this circumstance.

4.4. The position and velocity neighbor graph is switching

The aforementioned results are on the basis of a multi-agent group whose topology is fixed. We can also consider a switching position and velocity neighbor graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathbf{W}(t))$, where

$$\mathbf{W}(t) = \mathbf{W}(\tau_k) \quad t \in [\tau_k, \tau_{k+1}), \ k = 0, 1, \ldots$$

with $\tau_0 = 0$ being a switching matrix. It is easy to see that the graph switches at some instant time. Since $\mathbf{W}(\tau_k)$ is invariant in $[\tau_k, \tau_{k+1})$, a temporal pseudo-leader set $\bar{\mathcal{V}}(\tau_k)$ in this time interval can be determined based on Theorems 1 or 2. Thus for a multi-agent group whose topology is switching, a switching scheme can be employed to choose pseudo-leaders for flocking.

For those switching models which utilize local information (nearest neighbor law) to establish the group topology [7–9], this technique will be more efficient than using a fixed pseudo-leader set.

4.5. Center of the group members

Define the position and velocity of the center of all group members as

$$\bar{\mathbf{p}} \triangleq \frac{1}{N} \sum_{i=1}^{N} \mathbf{p}_i,$$

$$\bar{\mathbf{v}} \triangleq \frac{1}{N} \sum_{i=1}^{N} \mathbf{v}_i.$$

Then we have the error system

$$\bar{\mathbf{e}}^{\mathbf{p}} \triangleq \bar{\mathbf{p}} - \mathbf{p}_l = \frac{1}{N} \sum_{i=1}^{N} \mathbf{e}_i^{\mathbf{p}},$$

$$\bar{\mathbf{e}}^{\mathbf{v}} \triangleq \bar{\mathbf{v}} - \mathbf{v}_l = \frac{1}{N} \sum_{i=1}^{N} \mathbf{e}_i^{\mathbf{v}}.$$

According to the proof of Theorem 1, every solution of the system converges to the largest invariant set Ω^* of the set $\{(\mathbf{p}_i - \mathbf{p}_j, \mathbf{e}_i^{\mathbf{v}}) \mid \dot{L} = 0\}$, in which $\dot{\mathbf{e}}_i^{\mathbf{p}} = \mathbf{e}_i^{\mathbf{v}} = \mathbf{0}$ for $1 \le i \le N$. This result implies that $\dot{\bar{\mathbf{e}}}^{\mathbf{p}} = \bar{\mathbf{e}}^{\mathbf{v}} = \mathbf{0}$ in Ω^*. That is, in the steady state, the center of all agent velocities is equal to the desired velocity, and the position difference between the center and the virtual leader remains unchanged.

5. Numerical simulations

5.1. Flocking in a small-sized multi-agent group

In the following, Theorem 1 is illustrated by using the Lü system [22] as the dynamical acceleration in system (1) and (2). As a typical benchmark chaotic system, Lü system is given by

$$\dot{\mathbf{x}} = \begin{pmatrix} -a & a & 0 \\ 0 & c & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}$$
$$\triangleq \mathbf{R}\mathbf{x} + \mathbf{T}(\mathbf{x}),$$

which has a chaotic attractor when $a = 36, b = 3, c = 20$ as displayed in Fig. 2. For any two state vectors $\mathbf{y}$ and $\mathbf{z}$ of Lü system, there exist constants M_s such that $\|y_s\|, \|z_s\| \le M_s$ for $1 \le s \le 3$ since the Lü attractor is bounded within a certain region. From simple calculation, $M_1 = 25, M_2 = 30$ and $M_3 = 45$ are obtained.
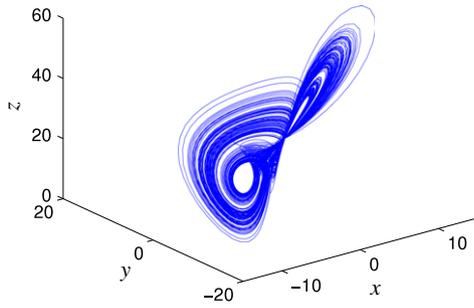
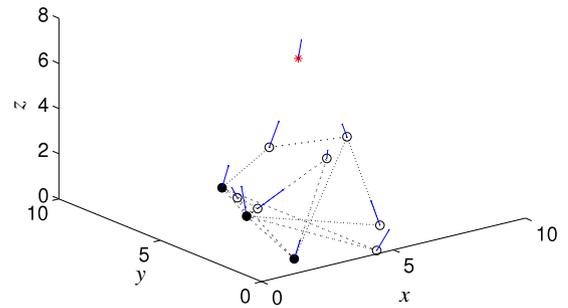**Fig. 2.** The states of the Lü attractor.

Fig. 3. Initial moving state of the group when $N = 10$ and $m = 3$.

Therefore, one has

$$(\mathbf{y} - \mathbf{z})^\top (\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{z}))$$
$$= (\mathbf{y} - \mathbf{z})^\top \mathbf{R} (\mathbf{y} - \mathbf{z}) + (\mathbf{y} - \mathbf{z})^\top (\mathbf{T}(\mathbf{y}) - \mathbf{T}(\mathbf{z}))$$
$$= (\mathbf{y} - \mathbf{z})^\top \mathbf{R} (\mathbf{y} - \mathbf{z}) + (y_1 - z_1)(z_2 y_3 - y_2 z_3)$$
$$= (\mathbf{y} - \mathbf{z})^\top \left(\begin{pmatrix} -a & \frac{a}{2} & 0 \\ \frac{a}{2} & c & 0 \\ 0 & 0 & -b \end{pmatrix} + \begin{pmatrix} 0 & -\frac{y_3}{2} & \frac{y_2}{2} \\ -\frac{y_3}{2} & 0 & 0 \\ \frac{y_2}{2} & 0 & 0 \end{pmatrix} \right)$$
$$\times (\mathbf{y} - \mathbf{z})$$
$$\leq \lambda_{\max} \begin{pmatrix} -a & \frac{a}{2} - \frac{y_3}{2} & \frac{y_2}{2} \\ \frac{a}{2} - \frac{y_3}{2} & c & 0 \\ \frac{y_2}{2} & 0 & -b \end{pmatrix} \|(\mathbf{y} - \mathbf{z})\|^2$$
$$\approx 60.3402 \, \|(\mathbf{y} - \mathbf{z})\|^2.$$

Thus the Lü system satisfies Assumption A1 with $\alpha = 60.3402$.

For simplicity, consider the group with ten nodes whose adjacent matrix is

$$\mathbf{B} = \begin{pmatrix} -7 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & -4 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

The common weight coupling is $w = 70$. By deleting the first three row–column pairs of $\mathbf{B}$, the maximum eigenvalue of the minor matrix is $\lambda_{\max}(\mathbf{B}_7) = -1$. It is easy to see that

$$\lambda_{\max}(\mathbf{B}_7) < -\frac{\alpha}{w}.$$

Thus the first three agents can be picked out as pseudo-leaders in the group according to Theorem 1.

Simulation results are shown in Figs. 3–6. In the simulation, the initial position and velocity states of the ten agents in the group are distributed randomly from the cube $[0, 5]^3$ and $[0, 2]^3$ respectively. The initial position and velocity of the virtual leader, which is marked with a red star in Figs. 3 and 4, are set as $\mathbf{p}_l(0) = (6, 6, 6)^\top$ and $\mathbf{v}_l(0) = (1, 1, 1)^\top$ respectively. Other parameters are chosen as $h_{i_r}(0) = 1$ and $k_{i_r} = 0.01$ $(1 \leq r \leq m, \ 1 \leq i_r \leq N)$. Letting $\sigma = 0.1$ in σ-norm and $c_1 = c_2 = \frac{1}{2}$, we use the following APF

$$V_{ij}(\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma) = c_1 \ \ln \|\mathbf{p}_i - \mathbf{p}_j\|_\sigma^2 + \frac{c_2}{\|\mathbf{p}_i - \mathbf{p}_j\|_\sigma^2}.$$
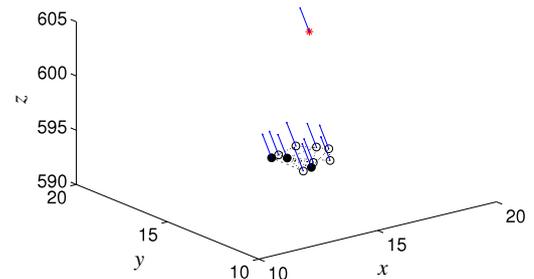

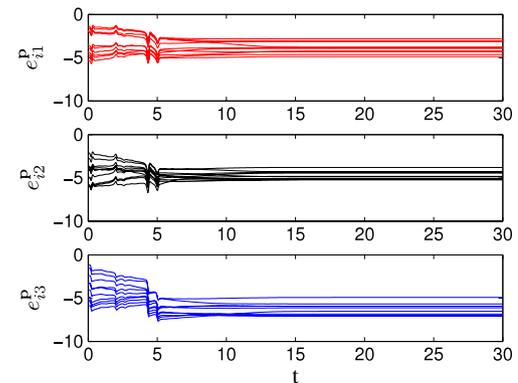**Fig. 4.** Group's moving state when $N = 10$ and $m = 3$ at $t = 30$.

Fig. 5. The position differences $e_{is}^{\mathbf{p}}$ ($1 \leq i \leq 10$, $1 \leq s \leq 3$) between the agents and the virtual leader when $N = 10$ and $m = 3$.

The group's moving states at $t = 0$ and $t = 30$ are illustrated in Figs. 3 and 4. Here, the solid circles represent the pseudo-leaders that receive moving information of the virtual leader, and the hollow ones denote the followers in the group. The arrows display velocity vectors of all the agents. The dashed lines depict the connections between agents. From these two figures, it is easy to see that in spite of the initial disordered state, all the agents flock at $t = 30$.

The state of position and velocity differences between the agents and the virtual leader are illustrated in Figs. 5 and 6 respectively. It is shown that the velocities of all the group members converge to the desired velocity. Moreover, the position errors between agents remain fixed after a period of time.

5.2. Flocking in a 200-sized multi-agent group

Large-scale groups with complex topology are common in nature. It has been demonstrated that the topological information of most large-scale systems display scale-free features, among
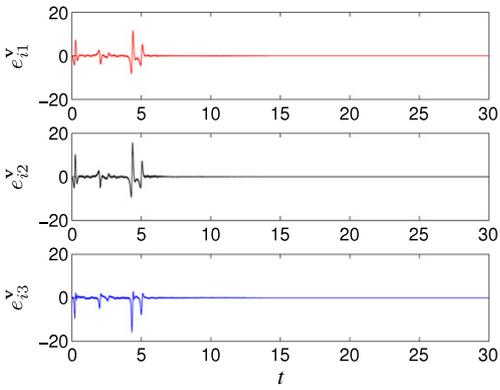
Fig. 6. The velocity differences $e_{is}^{\mathbf{v}}(1 \le i \le 10,\ 1 \le s \le 3)$ between the agents and the virtual leader when $N = 10$ and $m = 3$.

Fig. 7. Initial moving state of the group when $N = 200$ and $m = 6$.

which the Barabási and Albert (BA) model [23] of preferential attachment has become the standard mechanism to explain the emergence of scale-free networks. Nodes are added to the network with a preferential bias toward attachment to nodes with already high degree. This naturally gives rise to hubs with a degree distribution following a power-law.

In this subsection, a BA scale-free network consisting of 200 agents with $q_0 = 5$ and $q = 5$ are considered, where q_0 is the size of the initial network, and q is the number of edges added in each step. Similar to the previous simulation, we take the Lü system as the dynamical acceleration in system (1) and (2). The initial positions and the velocities of the 200 agents are selected randomly from the cube $[0, 50]^3$ and $[0, 2]^3$ respectively, and that of the virtual leader are selected as $\mathbf{p}_l(0) = (60, 60, 60)^\top$ and $\mathbf{v}_l(0) = (1, 1, 1)^\top$ respectively. Assume that the other parameters in the simulation are the same as those in the previous subsection.

Denote $\mathbf{B}_{194}^{BA}$ as the minor matrix of the adjacent matrix $\mathbf{B}^{BA}$ by removing 6 row–column pairs which correspond to the 6 agents with the largest degree in the whole group. Since the maximum eigenvalue of $\mathbf{B}_{194}^{BA}$ satisfies

$$\lambda_{\max}(\mathbf{B}_{194}^{BA}) = -0.9284 < -0.8620 = -\frac{\alpha}{w},$$

the 6 agents with the largest degrees can be picked out as pseudo-leaders of the group. After using the control mechanism presented in Theorem 1, flocking appears by letting just 3% agents be informed. The moving states of all the group members at $t = 0$ and $t = 30$ are exhibited in Figs. 7 and 8. In order to display distinctly, the connections between agents are not depicted in these two figures for their abundance. Since there are too many lines to illustrate the position differences between all the agents and the virtual leader, the position differences $\bar{\mathbf{e}}^\mathbf{p}$ between the center of all agents and the virtual leader, and the velocity differences $\mathbf{e}_i^\mathbf{v}(1 \le i \le 200)$ between all the agents and the virtual leader, are depicted in Figs. 9 and 10 instead.

Clearly, the approaches presented in this paper are of high accuracy with good performance not only for small-sized multi-agent groups but also for larger-scale multi-agent systems.

6. Conclusions

In this paper, we have presented a criterion for choosing pseudo-leaders in a multi-agent dynamical group. Particularly, the weight configuration of the position and velocity neighbor graph is not necessarily irreducible or time invariant. By combining the ideas of virtual force and pseudo-leader mechanism, mathematical analysis has been deduced to illustrate how to determine the pseudo-leader set in a group. The proposed schemes have been proved rigorously by using the Schur complement and Lyapunov
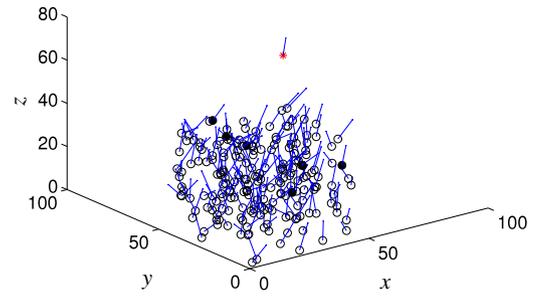

Fig. 8. Group's moving state when $N = 200$ and $m = 6$ at $t = 30$.
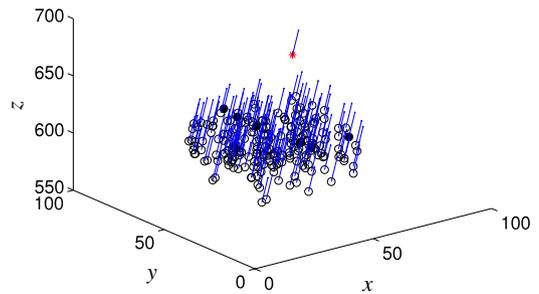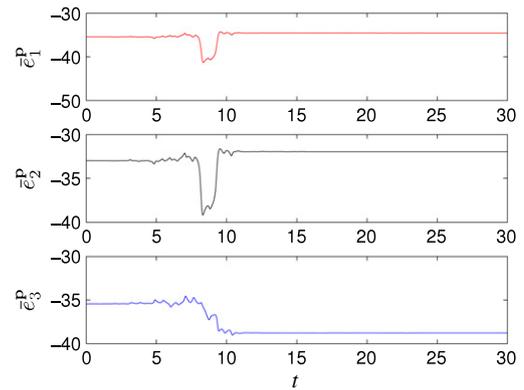

Fig. 9. The position differences $\bar{e}_s^\mathbf{p}(1 \le s \le 3)$ between the center of all agents and the virtual leader when $N = 200$ and $m = 6$.
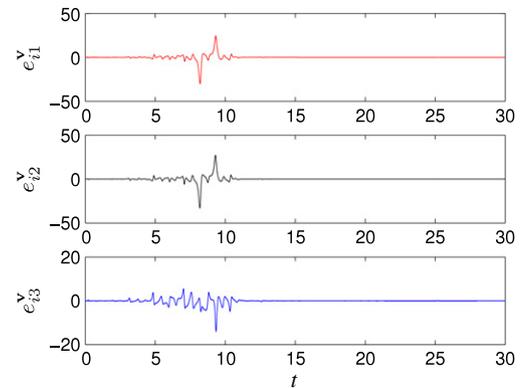

Fig. 10. The velocity differences $e_{is}^{\mathbf{v}}(1 \le i \le 200,\ 1 \le s \le 3)$ between the agents and the virtual leader when $N = 200$ and $m = 6$.

stability theory. Finally, two computational examples including small-sized and larger-scale multi-agent groups have been shown to illustrate the effectiveness of the proposed approach.

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