Generalized outer synchronization between complex dynamical networks

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In this paper, the problem of generalized outer synchronization between two completely different complex dynamical networks is investigated. With a nonlinear control scheme, a sufficient criterion for this generalized outer synchronization is derived based on Barbalat's lemma. Two corollaries are also obtained, which contains the situations studied in two lately published papers as special cases. Numerical simulations further demonstrate the feasibility and effectiveness of the theoretical results. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3072787]

Synchronization of complex networks have been extensively investigated in many research and application fields. Most of this research has been focused upon a coherent behavior within a network, where each node of the network arrives at the same steady state. This kind of synchronization, which was called "inner synchronization" in Ref. 20 has attracted broad attention. As a matter of fact, in real-world situations, there also exist other kinds of synchronization for complex networks, such as "outer synchronization" between two networks as considered in Refs. 20 and 23 below, where under the assumption that all individuals in two networks have completely identical behaviors a "complete outer synchronization" was studied. However, this kind of assumption may not seem practical. Take the predator-prey interactions in ecological communities as an example, where predators and preys influence one another's evolution. Without preys there would not be predators, while too many predators would bring the preys into extinction. The communities of predators and preys will finally reach harmonious coexistence without man made sabotage. It is worth noting that inside the networks of predators or preys, one individual always behaves differently from another. Thus it is more practical to assume that each node has different dynamics. Furthermore, the interactions of predators themselves usually differ from that of preys, that is, the topological structure of the predators community is different from that of the preys community. Therefore, synchronization between two different complex networks, where the difference results from node diverseness as well as topological difference, is a more practical and significant problem worth investigating.

I. INTRODUCTION

Complex networks have received rapidly increasing attention from different fields in recent years. From the internet to the world wide web, from communication networks to social organizations, from food webs to ecological communities, etc., complex networks widely exist in our life and are presently prominent candidates to describe sophisticated collaborative dynamics in many sciences.[1–6]

So far, the dynamics of complex networks has been extensively investigated, in which synchronization is a typical topic that has attracted lots of interests. Synchronization is a fundamental phenomenon that enables coherent behavior in networks as a result of interactions. Pecora *et al.* used the master stability function approach to determine the stability of the synchronous state in coupled systems.[7,8] Chen *et al.* imposed constraints on the coupling strengths to ensure stability of the synchronized states in arbitrarily coupled dynamical systems based on the master stability function together with Gershörin disk theory.[9] Wu *et al.* investigated synchronization in linearly coupled identical dynamical systems by the Lyapunov direct method and proved that strong enough mutual diffusive coupling will synchronize an array of identical cells.[10] The Lyapunov function method was also employed in some of Wu's later works on synchronization of coupled systems.[11–14] Wang and Chen studied synchronization in two specific kinds of networks: Scale-free networks and small-world networks.[15,16] Lü and Chen introduced a time-varying dynamical network and further investigated its synchronization criteria.[17] Zhou *et al.* and Lu considered synchronization in networks by integrating network models and an adaptive technique.[18,19]

Researches on synchronization of networks mentioned above focused on the phenomenon that all nodes in a network achieve a coherent behavior, which was called inner synchronization,[20] as it is a collective behavior within a network. In reality, there exist other kinds of network synchronization, for example, synchronization between two or more networks, which was termed outer synchronization in Ref. 20. A representative illustration is predator-prey interactions in ecological communities,[21] where predators and preys can influence one another's evolution. For example, plant-eating animals, such as mice, rats, and rabbits, would soon strip the land bare without the controlling effect of predators. Areas where large predators have been reduced through trapping, shooting, and other predator-control methods often develop

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large populations of mice, rats, and rabbits that can destroy the plants needed by other wildlife species for both food and shelter. If the predators had been allowed to remain, the prey species probably would have been kept under control. In a word, the relationship between the network of predators and that of preys is important in maintaining balance among different animal species. Mankind has been trying every means to maintain this balance. Another example is the balance of intestinal microflora for human beings.[22] A vast amount of good (or beneficial) bacteria living inside our digestive system which perform very important functions exist alongside with bad (or pathogenic) bacteria which produce harmful substance and create serious problems. The bacteria are essentially competing with one other for space and nutrients. A good balance of the two communities of good and bad bacteria provides protection against a broad range of pathogens while discomforts and symptoms of disease can result when factors like antibiotics, poor diet, and stress cause this balance to be disrupted. There are a great many examples about relationships between different networks, which also indicates that it is necessary and significant to investigate the dynamics between different networks.

Recently, Li *et al.*[20] pioneered in studying outer synchronization between two unidirectionally coupled networks and derived a criterion for the synchronization between two networks with identical topological structures. Shortly after, Tang *et al.* analyzed outer synchronization between two complex networks with nonidentical topologies using adaptive controllers.[23] In these two papers, it is assumed that each node in both networks has identical dynamics, and the corresponding nodes in two networks manifest completely the same dynamics, so strictly speaking, it is complete outer synchronization between two networks. However, nodes in different networks usually have different dynamics (parameter mismatch or structural discrepancy), while the two networks may still behave in a synchronous way. This kind of synchronization is called generalized synchronization,[24–27] which represents another degree of coherence. For instance, in the aforementioned predator and prey networks, predators and preys may finally reach a synchronous state even though they have entirely different behaviors (even individuals inside a network may behave in quite diverse ways).

Motivated by the above discussions, in this paper, we introduce the concept of generalized outer synchronization between two complex dynamical networks, where nodes in one network synchronize with their counterparts in the other network through some smooth functions. A criterion on generalized outer synchronization is derived based on Barbalat's lemma, and then two corollaries are drawn. In our study, each network can be undirected or directed, connected or disconnected, and nodes in either network may have identical or different dynamics. As an extension of complete synchronization, the generalized outer synchronization studied here has a much wider application range than complete outer synchronization.

The rest of this paper is organized as follows: Network models and some preliminaries are introduced in Sec. II. In Sec. III, using nonlinear control, we present a criterion for generalized outer synchronization between two networks with arbitrary node dynamics and topological structures. Some numerical simulations are provided to illustrate the feasibility and effectiveness of the proposed approach in Sec. IV. Finally, concluding remarks are given in Sec. V.

II. PROBLEM DESCRIPTION

A. Network models

Some necessary notations that will be used throughout this paper are first introduced. $\top$ denotes the transpose of a matrix or a vector. $\|\xi\|$ indicates the 2-norm of a vector ξ, i.e., $\|\xi\|=\sqrt{\xi^\top\xi}$. $I_i\in\mathbb{R}^{i\times i}$ represents the identity matrix with dimension i. $\otimes$ denotes the Kronecker product of two matrices.[28] $\lambda_m(A)$ represents the maximum eigenvalue of a square matrix A.

Consider a weighted general complex dynamical network consisting of N dynamical nodes with linear couplings, which is characterized by

$$\dot{\mathbf{x}}_i(t)=\mathbf{f}_i(\mathbf{x}_i(t))+\sum_{j=1}^{N}c_{ij}P\mathbf{x}_j(t),\quad i=1,2,\ldots,N. \tag{1}$$

Here $\mathbf{x}_i(t)=(x_{i1}(t),x_{i2}(t),\ldots,x_{in}(t))^\top\in\mathbb{R}^n$ is the state vector of the ith node, and $\mathbf{f}_i:\mathbb{R}^n\to\mathbb{R}^n$ is a smooth nonlinear vector-valued function governing the evolution of $\mathbf{x}_i(t)$ in the absence of interactions with other nodes. $P\in\mathbb{R}^{n\times n}$ is an inner-coupling matrix determining the interaction of variables. $C=(c_{ij})_{N\times N}\in\mathbb{R}^{N\times N}$ is the coupling configuration matrix representing the coupling strength and the topological structure of the network, in which c_{ij} is defined as follows: if there is a connection from node i to node $j(j\neq i)$, $c_{ij}\neq 0$; otherwise, $c_{ij}=0$. The diagonal elements of matrix C are defined as

$$c_{ii}=-\sum_{j=1,j\neq i}^{N}c_{ij},\quad i=1,2,\ldots,N.$$

Consider another complex dynamical network containing N dynamical nodes as follows:

$$\dot{\mathbf{y}}_i(t)=\mathbf{G}_i(\mathbf{y}_i(t))+\sum_{j=1}^{N}d_{ij}Q\mathbf{y}_j(t),\quad i=1,2,\ldots,N, \tag{2}$$

where $\mathbf{y}_i(t)=(y_{i1}(t),y_{i2}(t),\ldots,y_{im}(t))^\top\in\mathbb{R}^m$ is the state vector of the ith node, and $\mathbf{G}_i:\mathbb{R}^m\to\mathbb{R}^m$ is a smooth nonlinear vector-valued function governing the evolution of the ith isolated node $\mathbf{y}_i(t)$. $Q\in\mathbb{R}^{m\times m}$ is the inner-coupling matrix, and $D=(d_{ij})_{N\times N}\in\mathbb{R}^{N\times N}$ is the coupling configuration matrix, which has the same meaning as that of matrix C. In the following, we will take Eq. (1) as the drive network and Eq. (2) as the response network.

It is well-known that many systems, such as the Lorenz system, Chen system, Lü system, Rössler system, Chua's circuit, hyperchaotic Rössler system, hyperchaotic Chen and Lü system, can be written in the following form:

$$\dot{\mathbf{y}}=A\mathbf{y}+\mathbf{g}(\mathbf{y}),$$

where $A\in\mathbb{R}^{m\times m}$ is the Jacobian matrix of the system at the origin, and $\mathbf{g}(\mathbf{y})$ is the nonlinear part. Therefore, without loss of generality, we can describe the response network (2) with control as

$$\dot{\mathbf{y}}_i(t) = A_i\mathbf{y}_i(t) + \mathbf{g}_i(\mathbf{y}_i(t)) + \sum_{j=1}^{N} d_{ij}Q\mathbf{y}_j(t) + \mathbf{u}_i, \quad i = 1,2,\ldots,N, \tag{3}$$

where $A_i\mathbf{y}_i$ and $\mathbf{g}_i$ are, respectively, the linear and nonlinear part of the ith node, and $\mathbf{u}_i$ is the ith controller to be designed according to the specific node dynamics and topological structures of the drive and response networks.

B. Preliminaries

In this subsection, we will first give a definition of generalized outer synchronization between two networks, followed by an assumption and a lemma which will be needed in the subsequent study.

Definition 1: Let $\phi_i:\mathbb{R}^n \to \mathbb{R}^m (i=1,2,\ldots,N)$ be continuously differentiable vector maps. Network (1) is said to achieve generalized outer synchronization with network (3) if

$$\lim_{t\to\infty} \sum_{i=1}^{N} \|\mathbf{y}_i(t) - \phi_i(\mathbf{x}_i(t))\| = 0.$$

Usually, function $\mathbf{g}_i(\cdot)$ is globally Lipschitz continuous, i.e., the following assumption is satisfied.

Assumption 1: For function $\mathbf{g}_i(z)(i=1,2,\ldots,N)$, there exists a positive constant $h_i(i=1,2,\ldots,N)$ such that

$$\|\mathbf{g}_i(\mathbf{z}_1) - \mathbf{g}_i(\mathbf{z}_2)\| \leqslant h_i\|\mathbf{z}_1 - \mathbf{z}_2\|$$

holds for any $\mathbf{z}_1$ and $\mathbf{z}_2$.

Lemma 1: (Barbalat's lemma[29]) If $f(t)$ is non-negative, integrable, and uniformly continuous on $[a,+\infty)$, then $f(t) \to 0$ as $t\to\infty$.

III. GENERALIZED OUTER SYNCHRONIZATION CRITERIA

With the network models and the definition given previously, we arrive at the following main theorem.

Theorem 1: Suppose that Assumption 1 holds. The drive network (1) can achieve generalized outer synchronization with the response network (3) under the following control law:

$$\mathbf{u}_i = \mathcal{D}\phi_i(\mathbf{x}_i)\cdot\mathbf{f}_i(\mathbf{x}_i) - A_i\phi_i(\mathbf{x}_i) - \mathbf{g}_i(\phi_i(\mathbf{x}_i)) - k\mathbf{e}_i - \sum_{j=1}^{N} d_{ij}Q\phi_j(\mathbf{x}_j) + \mathcal{D}\phi_i(\mathbf{x}_i)\sum_{j=1}^{N} c_{ij}P\mathbf{x}_j, \quad i=1,2,\ldots,N, \tag{4}$$

where $\mathcal{D}\phi_i(\mathbf{x}_i)$ is the Jacobian matrix of the map $\phi_i(\mathbf{x}_i)$, $\mathbf{e}_i = \mathbf{y}_i - \phi_i(\mathbf{x}_i)$, and k is a sufficiently large positive constant.

Proof: Since $\mathbf{e}_i = \mathbf{y}_i - \phi_i(\mathbf{x}_i)$, from networks (1) and (3), together with the control scheme (4), we obtain the error dynamical network described by

$$\begin{aligned}\dot{\mathbf{e}}_i &= \dot{\mathbf{y}}_i - \mathcal{D}\phi_i(\mathbf{x}_i)\cdot\dot{\mathbf{x}}_i\\ &= A_i\mathbf{e}_i - k\mathbf{e}_i + \mathbf{g}_i(\mathbf{y}_i) - \mathbf{g}_i(\phi_i(\mathbf{x}_i)) + \sum_{j=1}^{N} d_{ij}Q\mathbf{e}_j,\\ & i=1,2,\ldots,N.\end{aligned} \tag{5}$$

Let $\mathbf{e} = (\mathbf{e}_1^\top(t), \mathbf{e}_2^\top(t), \ldots, \mathbf{e}_N^\top(t))^\top \in \mathbb{R}^{mN}$, and consider the following Lyapunov candidate function:

$$V(t) = \frac{1}{2}\mathbf{e}^\top\mathbf{e} = \frac{1}{2}\sum_{i=1}^{N}\mathbf{e}_i^\top(t)\mathbf{e}_i(t). \tag{6}$$

Calculating its derivative along the trajectories of Eq. (5), under Assumption 1, we obtain

$$\begin{aligned}\dot{V}(t)|_{(5)} &= \sum_{i=1}^{N}\mathbf{e}_i^\top\dot{\mathbf{e}}_i\\ &= \sum_{i=1}^{N}\mathbf{e}_i^\top A_i\mathbf{e}_i - k\sum_{i=1}^{N}\mathbf{e}_i^\top\mathbf{e}_i + \sum_{i=1}^{N}\mathbf{e}_i^\top[\mathbf{g}_i(\mathbf{y}_i) - \mathbf{g}_i(\phi_i(\mathbf{x}_i))]\\ &\quad + \sum_{i=1}^{N}\mathbf{e}_i^\top\sum_{j=1}^{N} d_{ij}Q\mathbf{e}_j\\ &\leqslant \sum_{i=1}^{N}\mathbf{e}_i^\top A_i\mathbf{e}_i - k\sum_{i=1}^{N}\mathbf{e}_i^\top\mathbf{e}_i + \sum_{i=1}^{N} h_i\mathbf{e}_i^\top\mathbf{e}_i\\ &\quad + \sum_{i=1}^{N}\mathbf{e}_i^\top\sum_{j=1}^{N} d_{ij}Q\mathbf{e}_j\\ &= \mathbf{e}^\top\mathcal{A}\mathbf{e} - k\mathbf{e}^\top\mathbf{e} + \sum_{i=1}^{N} h_i\mathbf{e}_i^\top\mathbf{e}_i + \mathbf{e}^\top\mathcal{Q}\mathbf{e}\\ &\leqslant \left(\lambda_m\left(\frac{\mathcal{A}+\mathcal{A}^\top}{2}\right) - k + \max_i\{h_i\} + \lambda_m\left(\frac{\mathcal{Q}+\mathcal{Q}^\top}{2}\right)\right)\mathbf{e}^\top\mathbf{e},\end{aligned}$$

where $\mathcal{A} = \mathrm{diag}(A_1, A_2, \ldots, A_N) \in \mathbb{R}^{mN\times mN}$ (i.e., the ith diagonal square block of $\mathcal{A}$ is A_i), and $\mathcal{Q} = D\otimes Q$. Taking $k \geqslant k^* = \max_i\{h_i\} + \lambda_m(\mathcal{A}+\mathcal{A}^\top/2) + \lambda_m(\mathcal{Q}+\mathcal{Q}^\top/2) + 1$, we get

$$\dot{V}(t)|_{(5)} \leqslant -\mathbf{e}^T\mathbf{e}. \tag{7}$$

Obviously, $\dot{V}(t) \leqslant 0$, so $V(t)$ is uniformly continuous. Furthermore, we have

$$V(t) \leqslant V(0)\exp(-2t), \tag{8}$$

thus $\lim_{t\to\infty}\int_0^t V(\tau)d\tau$ exists, namely, $V(t)$ is integrable on $[0,+\infty)$. According to Barbalat's lemma, we obtain $\lim_{t\to+\infty} V(t) = 0$, which implies $\lim_{t\to+\infty}\mathbf{e}_i(t) = 0$ for $i = 1,2,\ldots,N$. Therefore, networks (1) and (3) asymptotically achieve generalized outer synchronization. This completes the proof. □

Remark 1: In the theorem, the configuration matrices C and D need not be symmetric or irreducible, which means that networks (1) and (3) can be undirected or directed networks, and they may also contain isolated nodes and clusters. In addition, there is not any constraint imposed on the inner-coupling matrices P and Q. Moreover, each node may have different node dynamics. Therefore, our method is applicable to a large variety of complex dynamical networks.

Remark 2: The feedback gain k can be chosen properly to adjust the synchronization speed. Theoretically, a larger k may lead to faster synchronization. However, it is noted that $k \geq k^*$ is only a sufficient condition but not a necessary one. Later simulations will show that a small value of k can also lead to generalized outer synchronization quickly.

Based on Theorem 1, we can easily derive the following corollaries:

Corollary 1: Suppose that Assumption 1 holds. If networks (1) and (3) have the same topological structures and uniform inner-coupling matrices, then the two networks can achieve complete outer synchronization under the following control scheme:

$$\mathbf{u}_i = \mathbf{f}_i(\mathbf{x}_i) - A_i\mathbf{x}_i - \mathbf{g}_i(\mathbf{x}_i) - k\mathbf{e}_i, \quad i = 1,2,\ldots,N, \tag{9}$$

where $k \geq k^* = \max_i\{h_i\} + \lambda_m(\mathcal{A}+\mathcal{A}^\top/2) + \lambda_m(\mathcal{Q}+\mathcal{Q}^\top/2) + 1$.

Corollary 2: Suppose that Assumption 1 holds. If the ith nodes in networks (1) and (3) have identical dynamics, namely, $\mathbf{f}_i = \mathbf{G}_i (i=1,2,\ldots,N)$, then the two networks can achieve complete outer synchronization under the following control scheme:

$$\mathbf{u}_i = -k\mathbf{e}_i + \sum_{j=1}^{N} (c_{ij}P - d_{ij}Q)\mathbf{x}_j, \quad i = 1,2,\ldots,N, \tag{10}$$

where $k \geq k^* = \max_i\{h_i\} + \lambda_m(\mathcal{A}+\mathcal{A}^\top/2) + \lambda_m(\mathcal{Q}+\mathcal{Q}^\top/2) + 1$.

Remark 3: This corollary presents a control scheme on complete outer synchronization for the case as studied in Ref. 23. In that paper, an adaptive technique was employed, and for two networks of size N, N^2+N additional adaptive controllers have to be utilized, which immensely increases the control cost. However, according to our Corollary 2, only N simple linear controllers are needed, which are much easier to implement.

In particular, if the two networks (1) and (3) have identical topological structures and inner-coupling matrices, then the controllers (10) are further simplified into

$$\mathbf{u}_i = -k\mathbf{e}_i \quad (i = 1,2,\ldots,N)$$

for complete outer synchronization. This is an extension of the case as studied in Ref. 20. Note that in Ref. 20 under the additional assumption that $\mathbf{f}_i = \mathbf{f} = \mathbf{G}_i$ for $i=1,2,\ldots,N$, a criterion for local synchronization was derived. In contrast, the synchronization obtained herein is global synchronization.

IV. NUMERICAL SIMULATIONS

In this section, illustrative examples will be provided to verify the effectiveness of the control scheme obtained in the preceding section. For this purpose, we consider several benchmark chaotic systems, such as Lorenz system, Chen system, and Lü systems.

Lorenz system is known to be a simplified model of several physical systems. It was originally derived from a model of the Earth's atmospheric convection flow heated from below and cooled from above.[30] Furthermore, it has been reported that Lorenz equations may describe such different systems as laser devices, disk dynamos, and several problems related to convection.[31] Later on, the Lorenz attractor was mathematically confirmed to exist.[32] The Lorenz system is represented by

$$\dot{\mathbf{x}} = \begin{pmatrix} a(x_2 - x_1) \\ cx_1 - 0x_1x_3 - x_2 \\ x_1x_2 - bx_3 \end{pmatrix}, \tag{11}$$

which has a chaotic attractor when $a=10$, $b=8/3$, $c=28$.

Chen system is a typical chaos anticontrol model, which has a more complicated topological structure than the Lorenz attractor.[33] It has been implemented by circuitry,[34] and has wide application potential in secure communications. The nonlinear differential equations that describe the Chen system are

$$\dot{\mathbf{x}} = \begin{pmatrix} a(x_2 - x_1) \\ (c-a)x_1 - x_1x_3 + cx_2 \\ x_1x_2 - bx_3 \end{pmatrix}, \tag{12}$$

which has a chaotic attractor when $a=35$, $b=3$, $c=28$.

Lü system is a typical transition system, which connects the Lorenz and Chen attractors and represents the transition from one to the other.[35] Lü system is described by

$$\dot{\mathbf{x}} = \begin{pmatrix} a(x_2 - x_1) \\ -x_1x_3 + cx_2 \\ x_1x_2 - bx_3 \end{pmatrix}, \tag{13}$$

which has a chaotic attractor when $a=36$, $b=3$, $c=20$. Later on Lü *et al.* proposed a unified chaotic system,[36] which contains Lorenz system and Chen system as two extremes and Lü system as a special case. The unified chaotic system is described by

$$\dot{\mathbf{x}} = \begin{pmatrix} (25\theta + 10)(x_2 - x_1) \\ (28 - 35\theta)x_1 - x_1x_3 + (29\theta - 1)x_2 \\ x_1x_2 - \dfrac{\theta + 8}{3}x_3 \end{pmatrix}, \tag{14}$$

where $\theta \in [0,1]$. Obviously, system (14) is the original Lorenz system for $\theta=0$ while it reduces to the original Chen system for $\theta=1$. When $\theta=0.8$, system (14) is just the critical system, Lü system. In fact, system (14) bridges the gap between Lorenz system and Chen system. Especially, system (14) is always chaotic over the whole interval $\theta \in [0,1]$.

As is known, there are many hyperchaotic systems discovered in the high-dimensional social and economical systems. Typical examples are the four-dimensional (4D) hyperchaotic Rössler system,[37] hyperchaotic Lorenz–Haken system,[38] hyperchaotic Chua's circuit,[39] and hyperchaotic Chen system.[40] Since hyperchaotic systems have the characteristics of high capacity, high security, and high efficiency, they have broad application potential in secure communications, nonlinear circuits, biological systems, neural networks, etc. In Ref. 41, Chen *et al.* presented the hyperchaotic Lü system,[41] which is described by

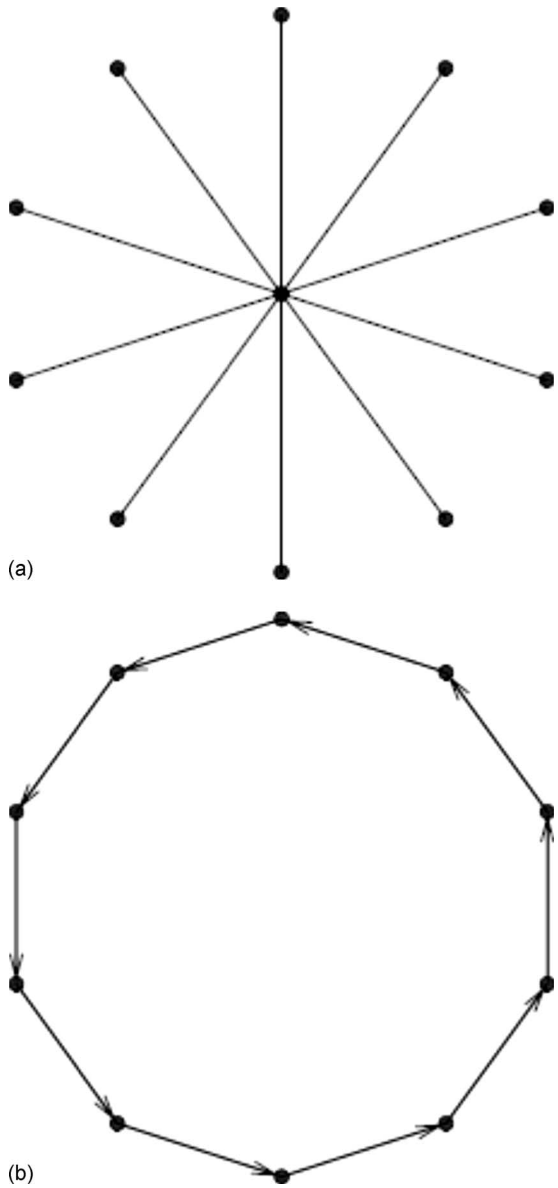

FIG. 1. (Top) a star coupled network; (bottom) a directed ring network.

$$\dot{\mathbf{x}} = \begin{pmatrix} a(x_2 - x_1) + x_4 \\ -x_1x_3 + cx_2 \\ x_1x_2 - bx_3 \\ x_1x_3 + dx_4 \end{pmatrix}. \tag{15}$$

When $a=36$, $b=3$, $c=20$, system (15) has a periodic orbit for $-1.03 \leqslant d \leqslant -0.46$, a chaotic attractor for $-0.46 < d \leqslant -0.35$, and a hyperchaotic attractor for $-0.35 < d \leqslant 1.30$.

In what follows, we will take the unified chaotic system and the hyperchaotic Lü system as node dynamics to illustrate our proposed method for generalized outer synchronization. For brevity, we always take P and Q as identity matrices with proper dimensions. In all the following simulations, we assume the drive network is a star network, while the response network is a directed ring network, as shown in Fig. 1. All the coupling strength is set to be 1 and the network size N is taken as 50.

A. Complete synchronization between two networks

Consider the unified chaotic system (14) as isolated node dynamics. Let θ for the ith node be $|\sin i|$ in the drive network and $|\sin i^2|$ in the response network, namely, each node in the two networks are different but all chaotic. Thus we have

$$\mathbf{f}_i(\mathbf{x}_i) = \begin{pmatrix} (25|\sin i| + 10)(x_{i2} - x_{i1}) \\ (28 - 35|\sin i|)x_{i1} - x_{i1}x_{i3} + (29|\sin i| - 1)x_{i2} \\ x_{i1}x_{i2} - \dfrac{|\sin i| + 8}{3}x_{i3} \end{pmatrix}, \tag{16}$$

$$A_i = \begin{pmatrix} -(25|\sin i^2| + 10) & 25|\sin i^2| + 10 & 0 \\ 28 - 35|\sin i^2| & 29|\sin i^2| - 1 & 0 \\ 0 & 0 & \dfrac{|\sin i^2| + 8}{3} \end{pmatrix}, \tag{17}$$

and

$$\mathbf{g}_i(\mathbf{y}) = \mathbf{g}(\mathbf{y}) = \begin{pmatrix} 0 \\ -y_1y_3 \\ y_1y_2 \end{pmatrix}. \tag{18}$$

For any vectors $\mathbf{y}$ and $\mathbf{z}$ of the unified chaotic system (14), there exists a positive constant $M=57$ such that $\|y_p\| \leqslant M$, $\|z_p\| \leqslant M$ for $1 \leqslant p \leqslant 3$ since the unified chaotic system is bounded in a certain region.[42] Therefore, one has

$$\begin{aligned} &\|\mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{z})\| \\ &= \sqrt{(y_1(y_3 - z_3) + z_3(y_1 - z_1))^2 + (y_1(y_2 - z_2) + z_2(y_1 - z_1))^2} \\ &\leqslant \sqrt{2}M\|\mathbf{y} - \mathbf{z}\|, \end{aligned}$$

that is, Assumption 1 is satisfied. So we may take $h_i = \sqrt{2}M$ for $i=1,2,\ldots,N$.

For complete synchronization, the map ϕ_i is defined as

$$\mathbf{y}_i = \phi_i(\mathbf{x}_i) = \phi(\mathbf{x}_i) = \mathbf{x}_i,$$

then

$$\mathcal{D}\phi_{\mathbf{i}}(\mathbf{x}_{\mathbf{i}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, $\lambda_m(\mathcal{A} + \mathcal{A}^\top/2) = 31$, $\lambda_m(\mathcal{Q} + \mathcal{Q}^\top/2) = 0$, which gives $k^* = 112.6$.

With the parameters specified above, the controllers are designed according to the control law (4). The initial values for the ith node in the drive and response networks are set to be $x_i(0) = (0.1i, -0.2i, 0.3i)^\top$ and $y_i(0) = (0.2i, -0.3i, 0.4i)^\top$, respectively. Let $E(t) = \Sigma_{i=1}^N \|y_i(t) - \phi_i(x_i(t))\|$ denote the synchronization error between the two networks. Figure 2 shows the phase diagrams of node 5 in the two networks without control. Figure 3(a) displays the evolution of $E(t)$ along time t without control. When the control law is imposed, the synchronization error quickly tends to zero, as displayed in

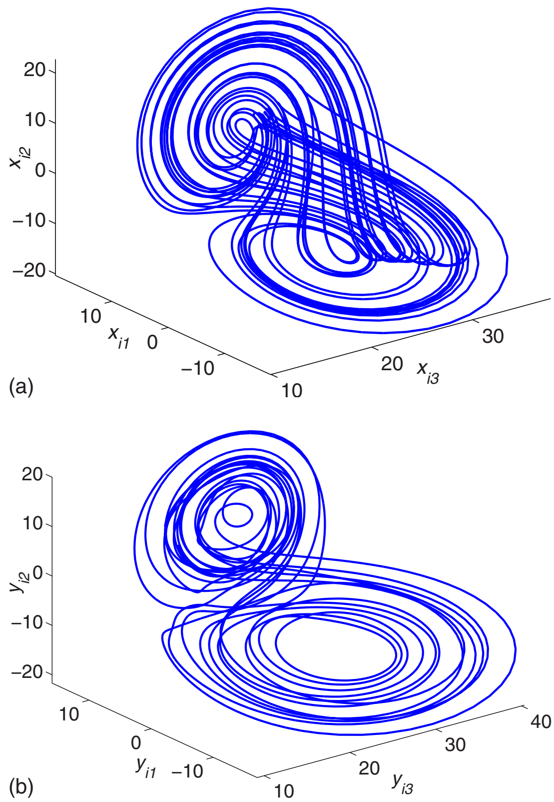

FIG. 2. (Color online) Phase diagrams of node 5 without control. (a) Node 5 in the drive network with $\theta = |\sin 5| = 0.9589$; (b) Node 5 in the response network with $\theta = |\sin 5^2| = 0.1324$.

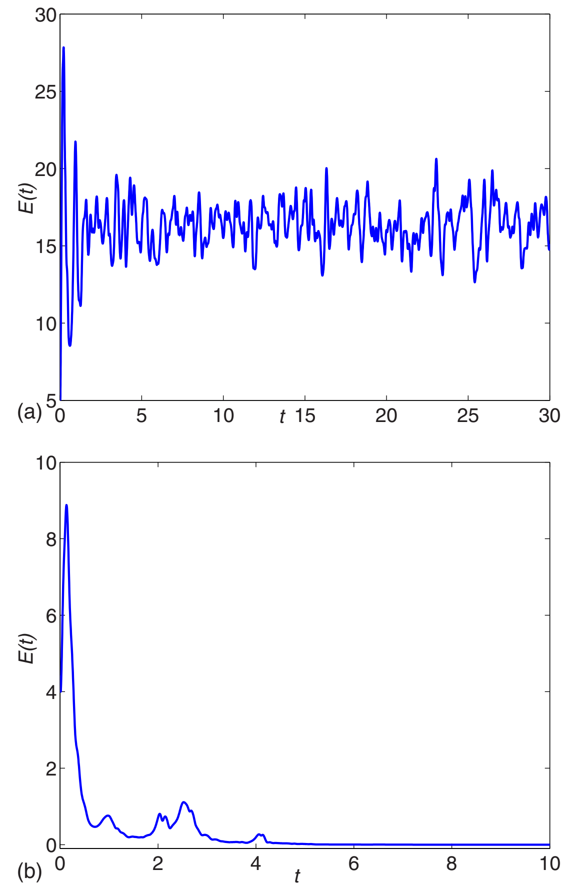

FIG. 3. (Color online) Complete synchronization errors of the drive and response networks, where the ith node in the drive and response networks is a unified chaotic system (14) with $\theta = |\sin i|$ and $\theta = |\sin i^2|$, respectively. (a) without control; (b) control imposed with $k=5$.

Fig. 3(b) for $k=5$. This further indicates that $k \geqslant k^*$ is only a sufficient condition, and a small value of k can also render a fast synchronization.

B. Generalized synchronization with uniform node dimension in the drive and response networks

Still take the unified chaotic system as node dynamics, with θ for the ith node being $|\sin i|$ and $|\sin i^2|$ in the drive and response networks, respectively.

Define two different types of maps ϕ_i as

$$\mathbf{y}_i = \phi_i(\mathbf{x}_i) = \phi(\mathbf{x}_i) = (2x_{i1}, x_{i2} + 1, x_{i3}^2)^{\top}, \quad i = 1, 2, \ldots, \frac{N}{2}$$

and

$$\mathbf{y}_i = \phi_i(\mathbf{x}_i) = \phi(\mathbf{x}_i) = (x_{i2}, x_{i1}x_{i2}, x_{i1} + 2x_{i3})^{\top},$$
$$i = \frac{N}{2} + 1, \frac{N}{2} + 2, \ldots, N.$$

Then

$$\mathcal{D}\phi_i(\mathbf{x}_i) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2x_{i3} \end{pmatrix}, \quad i = 1, 2, \ldots, \frac{N}{2};$$

and

$$\mathcal{D}\phi_i(\mathbf{x}_i) = \begin{pmatrix} 0 & 1 & 0 \\ x_{i2} & x_{i1} & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad i = \frac{N}{2} + 1, \frac{N}{2} + 2, \ldots, N.$$

With the expressions above, we design controllers u_i ($i = 1, 2, \ldots, N$) according to the control scheme (4). The initial conditions are set as the same as those in the previous subsection. The synchronization error with $k=3$ is shown in Fig. 4. It is seen from the figure that the generalized outer

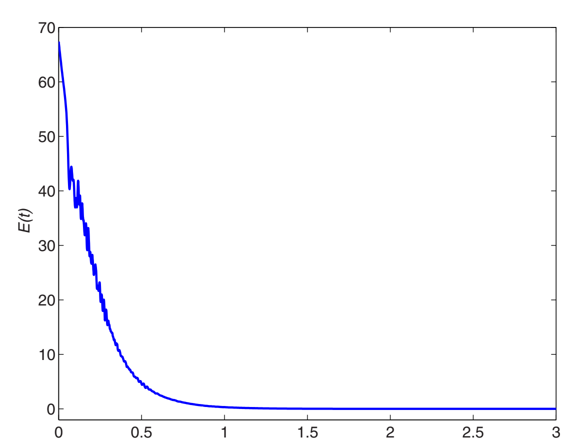

FIG. 4. (Color online) Generalized synchronization error between the drive and response networks with $k=3$.

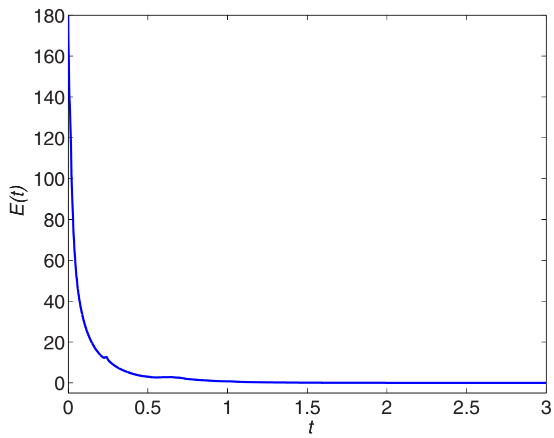

FIG. 5. (Color online) Generalized synchronization error between the drive and response networks with $k=15$, where each node in the drive network displays a hyperchaotic Lü attractor with $a=36$, $b=3$, $c=20$, $d=-0.4$.

synchronization is attained after a quite short transient period.

Remark 4: In this simulation, though there are two different forms of the maps $\phi_i(\mathbf{x}_i)$ which vary for nodes, the generalized outer synchronization is achieved under our proposed scheme. For more different types of the maps $\phi_i(\mathbf{x}_i)$, similar work can be generalized easily.

C. Generalized synchronization with different node dimensions in the drive and response networks

In this subsection, we consider the hyperchaotic Lü system as node dynamics in the drive network, and the unified chaotic system as node dynamics in the response network, where θ for the ith node is still taken as $|\sin i^2|$. The maps ϕ_i are defined variously for different nodes, with

$$\mathbf{y}_i = \phi_i(\mathbf{x}_i) = (2ix_{i1}, x_{i2} + 0.5x_{i4}, x_{i3} - i)^{\top}, \quad i = 1,2,\ldots,N.$$

Therefore,

$$\mathcal{D}\phi_i(\mathbf{x}_i) = \begin{pmatrix} 2i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The controllers $\mathbf{u}_i$ $(i=1,2,\ldots,N)$ are then designed according to the control law (4). Select the initial values as $\mathbf{x}_i(0)=(-0.1i,-0.2i,0.3i,0.4i)^{\top}$ in the drive network and $\mathbf{y}_i(0)=(0.2i,-0.3i,0.4i)^{\top}$ in the response network. To begin with, let all the nodes in the drive network be the identical hyperchaotic Lü system with $a=36$, $b=3$, $c=20$, $d=-0.4$, where each node displays a chaotic attractor. Figure 5 plots the generalized outer synchronization error $E(t)$ along time t, with $k=15$. Figure 6 shows the dynamics of node 5 in the drive and response networks, where projections on different phase space are displayed. Next, we assume that each node in the drive network is distinct. Assume the ith node is a hyperchaotic Lü system with $a=36$, $b=3$, $c=20$ but $d=-1+(2.3i/N)$. So d varies from about -1 to 1.3, and the nodes transit from a periodic orbit to a hyperchaotic attractor, as discussed previously. Figure 7 displays the generalized outer synchronization error $E(t)$ with $k=15$, which tends to zero

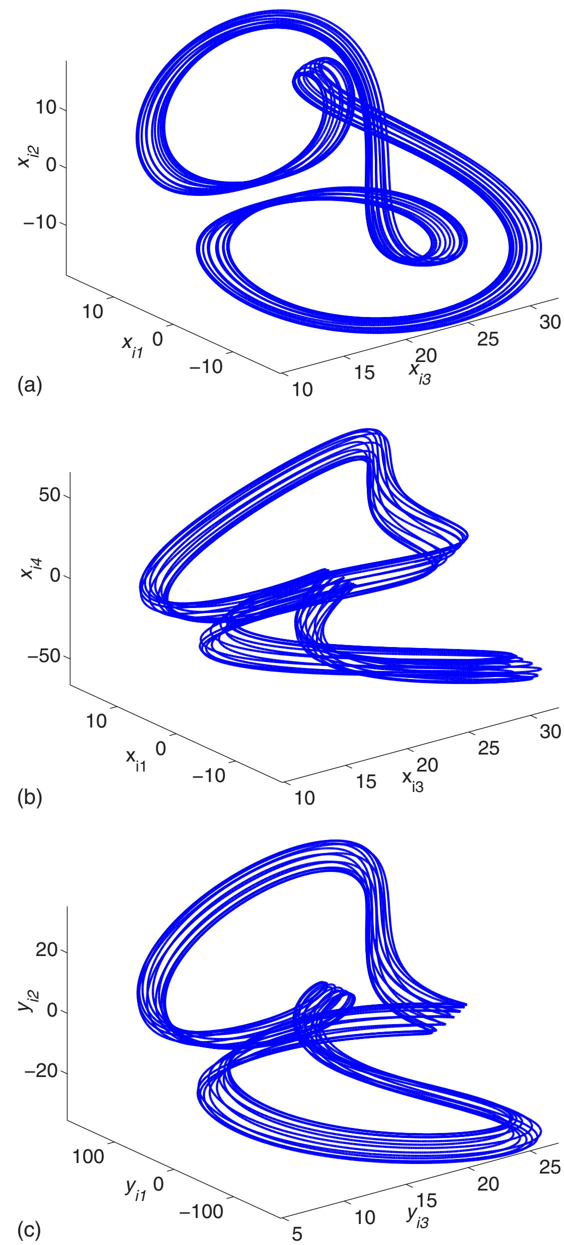

FIG. 6. (Color online) Phase diagrams for node 5, where $(y_{i1},y_{i2},y_{i3})=(2ix_{i1},x_{i2}+0.5x_{i4},x_{i3}-i)$ with $i=5$. (a) projection in the (x_{i3},x_{i1},x_{i2})-phase space of node 5 in the drive network; (b) projection in the (x_{i3},x_{i1},x_{i4})-phase space of node 5 in the drive network; (c) controlled node 5 in the response network with $\theta=|\sin 5^2|=0.1324$.

after a short transient period. To take a clearer view of the relationships between dynamics of nodes in two networks, we also depict some corresponding subvariables of node 25 in the xy plane, as shown in Fig. 8, where transients are discarded.

V. CONCLUSIONS

Synchronization within complex networks has been extensively studied in the past decade. However, investigation on synchronization between two networks (called outer synchronization) is still at the initial stage. To the best of our knowledge, there have been only a few papers in the literature that focus on complete outer synchronization between

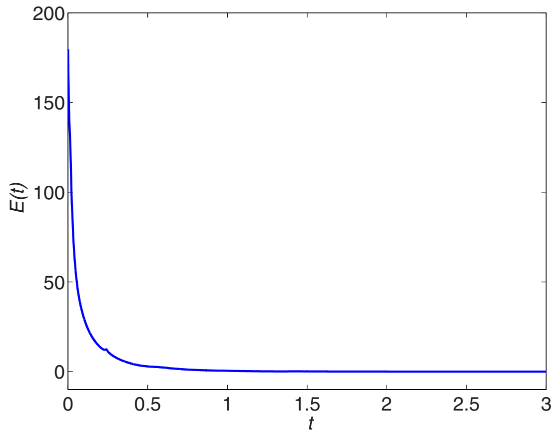

FIG. 7. (Color online) Generalized synchronization error between the drive and response networks with $k=15$, where the ith node in the drive network displays a hyperchaotic Lü attractor with $a=36$, $b=3$, $c=20$, and $d=-1+(2.3i/N)$.

two networks, where it is required that all the nodes should have the same dynamical behaviors. In this paper, we have investigated generalized outer synchronization between two complex dynamical networks with different topologies and diverse node dynamics. We have proposed a nonlinear control scheme which is guaranteed to achieve this generalized outer synchronization. When two complex networks have the same topological structures or identical dynamics, the proposed control scheme for achieving generalized outer synchronization reduces to a simpler form. The applicability of the theoretical findings has been validated by the computer simulations.

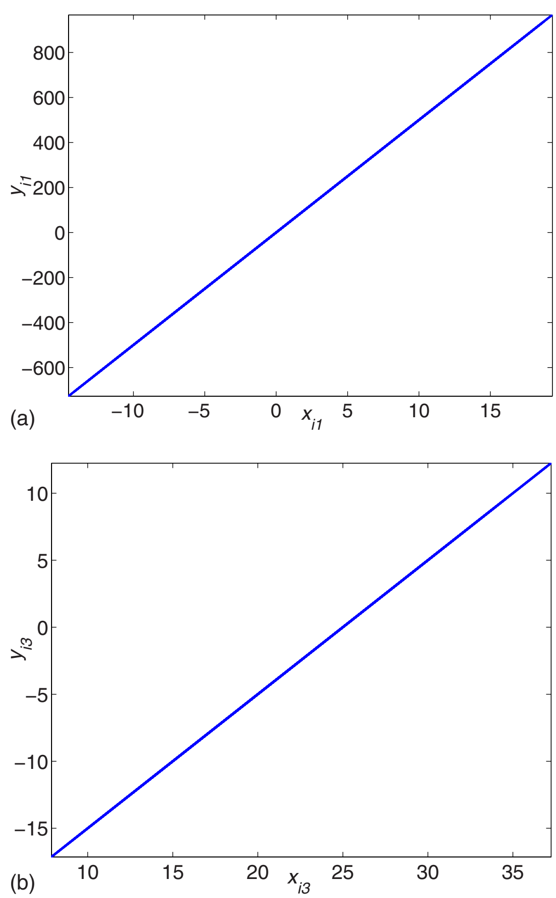

FIG. 8. (Color online) Relationships between the subvariables for node 25 in the drive and response networks. (a) $y_{i1}=2ix_{i1}$ for $i=25$; (b) $y_{i3}=x_{i3}-i$ for $i=25$.

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