Pinning synchronization of delayed neural networks

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This paper investigates adaptive pinning synchronization of a general weighted neural network with coupling delay. Unlike recent works on pinning synchronization which proposed the possibility that synchronization can be reached by controlling only a small fraction of neurons, this paper aims to answer the following question: Which neurons should be controlled to synchronize a neural network? By using Schur complement and Lyapunov function methods, it is proved that under a mild topology-based condition, some simple adaptive feedback controllers are sufficient to globally synchronize a general delayed neural network. Moreover, for a concrete neurobiological network consisting of identical Hindmarsh–Rose neurons, a specific pinning control technique is introduced and some numerical examples are presented to verify our theoretical results. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2995852]

It is well known that there are many useful network synchronization phenomena in our daily life, such as, the synchronous transfer of digital or analog signals in communication networks. In particular, adaptive synchronization in networks or coupled oscillators has received increasing attention. In this paper, we study adaptive synchronization of a general delayed neural network with pinning mechanism, which has been introduced recently to avoid the impracticality of controlling all the vertices in a large-scale network. Unlike recent works on pinning synchronization which proposed the possibility that synchronization can be reached by controlling a small fraction of neurons, this paper aims to answer the following question: Which neurons should be controlled to synchronize a neural network? We present a novel synchronization algorithm based on Schur complement and Lyapunov stability theory, and also prove that under a mild topology-based condition, an adaptive feedback control is sufficient to globally synchronize a general delayed neural network. In particular, for a concrete neurobiological network consisting of identical Hindmarsh–Rose neurons, we introduce a specific pinning control technique. We exhibit numerical results verifying the validity of our pinning control method.

I. INTRODUCTION AND MODEL DEPICTION

The most important component of the central nervous system is a complicated interconnected network of neurons that are responsible for enabling every function of our body. Neurobiological networks have been extensively studied since the 1980s for their potential applications in modelling complex dynamic systems, and various neural networks have been successfully applied to signal processing, linear and nonlinear programming, image processing, pattern recognition,[1,2] and so on.

As an important phenomenon in neural networks, usually the dynamics of each neuron is derived not only by its own dynamical property, but also by the evolution of its neighbors. A widely studied neural network model is formulated as

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t),t) + c\sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{g}(\mathbf{x}_j(t)),$$

where $1\leqslant i\leqslant N$, $\mathcal{N}_i$ represents the neighborhood of neuron i, the state vector of the ith neuron $\mathbf{x}_i(t)=(x_{i1}(t),x_{i2}(t),\ldots,x_{\text{in}}(t))^{\mathsf{T}}\in\mathbb{R}^n$ is a continuous function, $\mathbf{f}:\mathbb{R}^n\times\mathbb{R}^+\to\mathbb{R}^n$ is a smooth nonlinear vector function, individual neuron dynamics is $\dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t),t)$, $\mathbf{g}:\mathbb{R}^n\to\mathbb{R}^n$ is the inner-coupling vector function. The outer-coupling weight configuration matrix $\mathbf{A}=(a_{ij})\in\mathbb{R}^{N\times N}(a_{ij}\in\{0,1\})$ is symmetric and diffusive satisfying $\Sigma_{j=1}^N a_{ij}=0$.

This formulation assumes a common outer-coupling strength for all connections and instantaneous information from their neighborhood for all neurons. In a real neural network, however, this is not always the case. Couplings between neurons are not the same in most circumstances even if the diffusive condition is still satisfied. Apart from instantaneous information, neurons will also usually receive delayed information from their neighbors.[3–7] One method to solve these problems is to add weights and delayed couplings to this model. Thus the following neural network model is introduced and will be considered throughout the paper:

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t),t) + \sum_{j=1}^N a_{ij}\mathbf{g}(\mathbf{x}_j(t)) + \sum_{j=1}^N b_{ij}\mathbf{h}(\mathbf{x}_j(t-\tau(t))), \tag{1}$$

where $1\leqslant i\leqslant N$, $\mathbf{x}_i(t)=(x_{i1}(t),x_{i2}(t),\ldots,x_{in}(t))^{\mathsf{T}}$ is the state vector of the ith node, $\mathbf{g}:\mathbb{R}^n\to\mathbb{R}^n$ and $\mathbf{h}:\mathbb{R}^n\to\mathbb{R}^n$ are the inner-coupling and delayed inner-coupling vector functions, respectively. The initial conditions of Eq. (1) are given by $\mathbf{x}(t)=\Upsilon(t)\in C([-\bar{\tau},0],\mathbb{R}^n)$, where $\bar{\tau}=\sup_{t\in\mathbb{R}^+}\{\tau(t)\}$, $\tau(t)$ is

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the time delay in the couplings, $C([-\bar{\tau},0],\mathbb{R}^n)$ represents the set of all continuous functions from $[-\bar{\tau},0]$ to $\mathbb{R}^n$. The outer-coupling and delayed outer-coupling weight configuration matrices $\mathbf{A}=(a_{ij})\in\mathbb{R}^{N\times N}$ and $\mathbf{B}=(b_{ij})\in\mathbb{R}^{N\times N}$ are diffusive and symmetric. Take $\mathbf{A}$, for example, if there is a link between node i and node $j(j\neq i)$, then $a_{ij}=a_{ji}>0$ and a_{ij} is the coupling weight; otherwise, $a_{ij}=0$. In addition, $a_{ii}=-\sum_{j=1,j\neq i}a_{ij}$.

For a neurobiological network, control and synchronization have been one of the focal points in many research and application fields.[5–12] In particular, adaptive techniques, have emerged as an exciting research topic in nonlinear system control, and have been demonstrated to be an effective way to synchronize a complex network.[8–10] However, it is assumed that all the nodes need to be controlled. In view of impracticality of controlling all the nodes in a large-scale network, pinning control has been introduced in recent years.[13–15] Many existing works on pinning control presented the possibility of controlling a small fraction of nodes in a network to reach synchronization. However, how to carry out pinning and which nodes should be pinned are still challenging problems.

In this paper, a novel criterion for synchronizing a general delayed neural network (1) by pinning control is proposed. In view of the Schur complement and Lyapunov function methods, rigorous theoretical analysis with topology-based conditions is obtained. For a concrete neural network with coupled Hindmarsh–Rose (HR) neurons, a specific pinning control technique is presented by only controlling the membrane potential of each neuron.

The paper is organized as follows: In Sec. II, preliminaries are given. The mechanism of globally adaptive pinning synchronization of a general neural network (1) is discussed in Sec. III. A specific pinning control technique for a neural network consisting of HR neurons is then presented in Sec. IV. In Sec. V, numerical examples are provided to verify the theoretical results. Finally, the conclusion is drawn in Sec. VI.

II. PRELIMINARIES

A. Notations

To begin with, some necessary notations are introduced which will be used throughout the paper. The matrix $\mathbf{I}_{ii}\in\mathbb{R}^{i\times i}$ is the identity matrix with dimension i. T denotes the transpose of a matrix or a vector. $\|\xi\|$ represents the 2-norm of a vector which is defined as $\|\xi\|=\sqrt{\xi^{\mathsf{T}}\xi}$. $\otimes$ denotes the Kronecker product of two matrices.[16] $\lambda_m(\cdot)$ represents the maximum eigenvalue of a square matrix. The symmetric matrix $\mathcal{G}<0$ means that $\mathcal{G}$ is negative definite. The matrix $\mathcal{G}_l$ denotes the minor matrix of a matrix $\mathcal{G}$ by removing all the i_kth $(1\leq i_k\leq N, 1\leq k\leq l, 1\leq l\leq N)$ row-column pairs of $\mathcal{G}$.

B. Lemmas

In order to derive the main results, the following two lemmas are needed.

Lemma 1 (Schur complement[17,18]): The linear matrix inequality (LMI)

$$\begin{pmatrix}\mathcal{A}(x) & \mathcal{B}(x)\\ \mathcal{B}(x)^{\mathsf{T}} & \mathcal{C}(x)\end{pmatrix}<0,$$

where $\mathcal{A}(x)^{\mathsf{T}}=\mathcal{A}(x)$, $\mathcal{C}(x)^{\mathsf{T}}=\mathcal{C}(x)$, is equivalent to either of the following conditions:

(a) $\mathcal{A}(x)<0$ and $\mathcal{C}(x)-\mathcal{B}(x)^{\mathsf{T}}\mathcal{A}(x)^{-1}\mathcal{B}(x)<0$;

(b) $\mathcal{C}(x)<0$ and $\mathcal{A}(x)-\mathcal{B}(x)\mathcal{C}(x)^{-1}\mathcal{B}(x)^{\mathsf{T}}<0$.

Lemma 2: Assume that $\mathcal{M}$ is a diagonal matrix whose i_kth $(1\leq i_k\leq N, 1\leq k\leq l, 1\leq l\leq N)$ diagonal elements are m and the others are 0, where $m>0$ is a constant. Then for a symmetric matrix $\mathcal{G}$ which has the same dimension with $\mathcal{M}$, $\mathcal{G}-\mathcal{M}<0$ is equivalent to $\mathcal{G}_l<0$ when m is large enough.

Proof: After an elementary matrix transformation, $\mathcal{G}-\mathcal{M}$ can be changed into

$$\begin{pmatrix}\mathcal{G}_1-m\mathbf{I}_{ll} & \mathcal{G}^*\\ \mathcal{G}^{*\mathsf{T}} & \mathcal{G}_l\end{pmatrix},$$

where $\mathcal{G}_1$, $\mathcal{G}^*$ are the corresponding matrices with compatible dimension. If $\mathcal{G}-\mathcal{M}<0$, i.e.,

$$\begin{pmatrix}\mathcal{G}_1-m\mathbf{I}_{ll} & \mathcal{G}^*\\ \mathcal{G}^{*\mathsf{T}} & \mathcal{G}_l\end{pmatrix}<0,$$

one has $\mathcal{G}_l<0$ according to Lemma 1. On the other side, since $\mathcal{G}_l<0$, $\mathcal{G}_l$ is invertible, and then $\mathcal{G}_1-m\mathbf{I}_{ll}-\mathcal{G}^*\mathcal{G}_l^{-1}\mathcal{G}^{*\mathsf{T}}<0$ when m is large enough. According to Lemma 1, $\mathcal{G}_l<0$ leads to $\mathcal{G}-\mathcal{M}<0$. The proof is thus completed.

C. Hypotheses

For the purpose of obtaining adaptive pinning synchronization criterion for the neural network (1), the hypotheses which will be used in the main results are outlined below.

Hypothesis 1 (H1): Suppose that the delay function $\tau(t)$ is differentiable and satisfies $-\delta\leq\dot{\tau}(t)\leq\delta$, where $0\leq\delta<1$ is a constant.

This hypothesis is practical in real experiments and engineering since $\tau(t)$ change slowly.

Hypothesis 2 (H2): Assume that there exists a positive constant μ satisfying $(\xi_2-\xi_1)^{\mathsf{T}}(\mathbf{f}(\xi_2,t)-\mathbf{f}(\xi_1,t))\leq\mu\|\xi_2-\xi_1\|^2$ for any two vectors $\xi_1,\xi_2\in\mathbb{R}^n$.

Hypothesis 3 (H3): Suppose that the inner-coupling function $\mathbf{g}(\xi)$, where $\xi\in\mathbb{R}^n$, satisfies $d_1\|\xi_2-\xi_1\|^2\leq(\xi_2-\xi_1)^{\mathsf{T}}(\mathbf{g}(\xi_2)-\mathbf{g}(\xi_1))\leq\|\xi_2-\xi_1\|\|\mathbf{g}(\xi_2)-\mathbf{g}(\xi_1)\|\leq d_2\|\xi_2-\xi_1\|^2$ for any two vectors $\xi_1,\xi_2\in\mathbb{R}^n$, where d_1, d_2 are two positive constants.

Hypothesis 4 (H4): Assume that the delayed inner-coupling function $\mathbf{h}(\xi)$, where $\xi\in\mathbb{R}^n$, satisfies $e_1\|\xi_2-\xi_1\|^2\leq(\xi_2-\xi_1)^{\mathsf{T}}(\mathbf{h}(\xi_2)-\mathbf{h}(\xi_1))\leq\|\xi_2-\xi_1\|\|\mathbf{h}(\xi_2)-\mathbf{h}(\xi_1)\|\leq e_2\|\xi_2-\xi_1\|^2$ for any two vectors $\xi_1,\xi_2\in\mathbb{R}^n$, where e_1, e_2 are two positive constants.

III. PINNING ADAPTIVE SYNCHRONIZATION

In this section, a novel criterion for globally synchronizing the delayed neural network (1) with adaptive controllers is proposed.

Suppose that the synchronous state $\mathbf{x}_0(t)$ is a trajectory of the uncoupled system, i.e., $\dot{\mathbf{x}}_0(t)=\mathbf{f}(\mathbf{x}_0(t),t)$, which can be an equilibrium point, a periodic orbit, an aperiodic orbit, or even a chaotic orbit in the phase space. The objective of synchronization is to control neural network (1) to the given trajectory $\mathbf{x}_0(t)$.

By pinning a small fraction of neurons, we apply some adaptive controllers to the network. Assuming that the i_1th, i_2th, $\ldots, i_l$th neurons are controlled, the following controlled network is considered:

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t),t) + \sum_{j=1}^{N} a_{ij}\mathbf{g}(\mathbf{x}_j(t)) + \sum_{j=1}^{N} b_{ij}\mathbf{h}(\mathbf{x}_j(t-\tau(t))) + \mathbf{u}_i(t), \tag{2}$$

where $1 \leqslant i \leqslant N$, $\mathbf{u}_i$ are the adaptive controllers designed by

$$\begin{aligned} &\mathbf{u}_{i_k}(t) = -\alpha_{i_k}(t)(\mathbf{x}_{i_k}(t) - \mathbf{x}_0(t)) && 1 \leqslant k \leqslant l \\ &\dot{\alpha}_{i_k}(t) = \beta_{i_k}\|\mathbf{x}_{i_k}(t) - \mathbf{x}_0(t)\|^2 && 1 \leqslant k \leqslant l \\ &\mathbf{u}_{i_k}(t) = 0 && \text{otherwise}. \end{aligned} \tag{3}$$

Using the above controllers, a criterion for globally synchronizing a general weighted neural network with time-varying coupling delay is derived according to Lyapunov stability theory.[19] Let $\bar{\mathbf{A}}$ be a modified matrix of $\mathbf{A}$ in which the diagonal elements a_{ii} are replaced by $d_1 a_{ii}$ and other a_{ij} are replaced by $d_2 a_{ij}$, and $\bar{\mathbf{B}}$ be a changed matrix of $\mathbf{B}$ whose diagonal elements and others are $e_1 b_{ii}$ and $e_2 b_{ij}$, respectively.

Theorem 1: Suppose that H1–H4 hold. Then the synchronous solution of the controlled neural network (2) is globally asymptotically stable with adaptive pinning controllers (3), provided that $\lambda_m((\bar{\mathbf{A}} + 1/4\gamma(1-\delta)\bar{\mathbf{B}}^2)_l) < -(\mu+\gamma)$, where γ can be any positive constant.

Proof: Consider a Lyapunov candidate as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}\|\mathbf{x}_i(t) - \mathbf{x}_0(t)\|^2 + \gamma\sum_{i=1}^{N}\int_{t-\tau(t)}^{t}\|\mathbf{x}_i(\sigma) - \mathbf{x}_0(\sigma)\|^2 d\sigma + \sum_{k=1}^{l}\frac{1}{2\beta_{i_k}}(\alpha_{i_k}(t) - \alpha)^2,$$

where α is a sufficiently large positive constant to be determined. Taking the derivative of $V(t)$ along the trajectories of Eqs. (2) and (3), one obtains

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N}(\mathbf{x}_i(t) - \mathbf{x}_0(t))^\top\left[\mathbf{f}(\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{x}_0(t)) + \sum_{j=1}^{N}a_{ij}\mathbf{g}(\mathbf{x}_j(t)) + \sum_{j=1}^{N}b_{ij}\mathbf{h}(\mathbf{x}_j(t-\tau(t)))\right] - \sum_{k=1}^{l}\alpha_{i_k}(t)\|\mathbf{x}_{i_k}(t) - \mathbf{x}_0(t)\|^2 \\
&\quad + \gamma\sum_{i=1}^{N}[\|\mathbf{x}_i(t) - \mathbf{x}_0(t)\|^2 - (1-\dot{\tau}(t))\|\mathbf{x}_i(t-\tau(t)) - \mathbf{x}_0(t-\tau(t))\|^2] + \sum_{k=1}^{l}(\alpha_{i_k}(t) - \alpha)\|\mathbf{x}_{i_k}(t) - \mathbf{x}_0(t)\|^2 \\
&= \sum_{i=1}^{N}(\mathbf{x}_i(t) - \mathbf{x}_0(t))^\top(\mathbf{f}(\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{x}_0(t))) + \sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(\mathbf{x}_i(t) - \mathbf{x}_0(t))^\top(\mathbf{g}(\mathbf{x}_j(t)) - \mathbf{g}(\mathbf{x}_0(t))) + \sum_{i=1}^{N}\sum_{j=1}^{N}b_{ij}(\mathbf{x}_i(t) \\
&\quad - \mathbf{x}_0(t))^\top(\mathbf{h}(\mathbf{x}_j(t-\tau(t))) - \mathbf{h}(\mathbf{x}_0(t-\tau(t)))) + \gamma\sum_{i=1}^{N}[\|\mathbf{x}_i(t) - \mathbf{x}_0(t)\|^2 - (1-\dot{\tau}(t))\|\mathbf{x}_i(t-\tau(t)) - \mathbf{x}_0(t-\tau(t))\|^2] \\
&\quad - \sum_{k=1}^{l}\alpha\|\mathbf{x}_{i_k}(t) - \mathbf{x}_0(t)\|^2 \\
&\leqslant \sum_{i=1}^{N}\mu\|\mathbf{x}_i(t) - \mathbf{x}_0(t)\|^2 + \sum_{i=1}^{N}a_{ii}d_1\|\mathbf{x}_i(t) - \mathbf{x}_0(t)\|^2 + \sum_{i=1}^{N}b_{ii}e_1\|\mathbf{x}_i(t-\tau(t)) - \mathbf{x}_0(t-\tau(t))\|^2 + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}d_2\|\mathbf{x}_i(t) - \mathbf{x}_0(t)\| \\
&\quad \times\|\mathbf{x}_j(t) - \mathbf{x}_0(t)\| + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}b_{ij}e_2\|\mathbf{x}_i(t) - \mathbf{x}_0(t)\|\|\mathbf{x}_j(t-\tau(t)) - \mathbf{x}_0(t-\tau(t))\| + \gamma\sum_{i=1}^{N}[\|\mathbf{x}_i(t) - \mathbf{x}_0(t)\|^2 \\
&\quad - (1-\dot{\tau}(t))\|\mathbf{x}_i(t-\tau(t)) - \mathbf{x}_0(t-\tau(t))\|^2] - \sum_{k=1}^{l}\alpha\|\mathbf{x}_{i_k}(t) - \mathbf{x}_0(t)\|^2 \\
&\leqslant \mu\boldsymbol{\Delta}^\top(t)\boldsymbol{\Delta}(t) + \boldsymbol{\Delta}^\top(t)\bar{\mathbf{A}}\boldsymbol{\Delta}(t) + \boldsymbol{\Delta}^\top(t)\bar{\mathbf{B}}\boldsymbol{\Delta}(t-\tau(t)) + \gamma\boldsymbol{\Delta}^\top(t)\boldsymbol{\Delta}(t) - \gamma(1-\delta)\boldsymbol{\Delta}^\top(t-\tau(t))\boldsymbol{\Delta}(t-\tau(t)) - \boldsymbol{\Delta}^\top(t)\boldsymbol{\Lambda}\boldsymbol{\Delta}(t) \\
&= \begin{pmatrix}\boldsymbol{\Delta}(t) \\ \boldsymbol{\Delta}(t-\tau)\end{pmatrix}^\top\begin{pmatrix}(\mu+\gamma)\mathbf{I}_{NN} + \bar{\mathbf{A}} - \boldsymbol{\Lambda} & \frac{1}{2}\bar{\mathbf{B}} \\ \frac{1}{2}\bar{\mathbf{B}} & -\gamma(1-\delta)\mathbf{I}_{NN}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Delta}(t) \\ \boldsymbol{\Delta}(t-\tau)\end{pmatrix},
\end{aligned}$$

where $\boldsymbol{\Delta}(t)=(\|\mathbf{x}_1(t)-\mathbf{x}_0(t)\|,\|\mathbf{x}_2(t)-\mathbf{x}_0(t)\|,\ldots,\|\mathbf{x}_N(t)-\mathbf{x}_0(t)\|)^\top$, and $\boldsymbol{\Lambda}\in\mathbb{R}^{N\times N}$ is a diagonal matrix whose i_kth $(1\leq k\leq l)$ elements are α and the others are 0.

Due to the fact that $\lambda_m((\bar{\mathbf{A}}+1/4\gamma(1-\delta)\bar{\mathbf{B}}^2)_l)<-(\mu+\gamma)$, one has $(\mu+\gamma)\mathbf{I}_{NN}+\bar{\mathbf{A}}+1/4\gamma(1-\delta)\bar{\mathbf{B}}^2-\boldsymbol{\Lambda}<0$ when α is large enough according to Lemma 2. Furthermore,

$$\begin{pmatrix}(\mu+\gamma)\mathbf{I}_{NN}+\bar{\mathbf{A}}-\boldsymbol{\Lambda} & 1/2\bar{\mathbf{B}}\\ 1/2\bar{\mathbf{B}} & -\gamma(1-\delta)\mathbf{I}_{NN}\end{pmatrix}<0$$

in view of Lemma 1. Based on LaSalle's invariance principle,[19] every solution of the system converges to the largest invariant set $\Gamma=\{\dot{V}=0\}$, in which $\boldsymbol{\Delta}(t)=\mathbf{0}$. Thus $\lim_{t\to\infty}\boldsymbol{\Delta}(t)=\mathbf{0}$, in other words, $\lim_{t\to\infty}x_{ir}(t)-x_{0r}(t)=0(1\leq i\leq N,1\leq r\leq n)$. As a conclusion, the synchronous solution of the controlled weighted delayed neural network (2) is globally asymptotically stable with the adaptive pinning controllers (3). This completes the proof.

Remark 1: From this theorem, we find that for an appropriate positive constant γ, if the topology-based condition $\lambda_m((\bar{\mathbf{A}}+1/4\gamma(1-\delta)\bar{\mathbf{B}}^2)_l)<-(\mu+\gamma)$ is satisfied, the i_1th, i_2th, $\ldots, i_l$th neurons can be controlled to reach globally asymptotic synchronization for this type of neurobiological network.

Remark 2: It should be noted that since the condition in the theorem is just sufficient, it does not mean that synchronization cannot be reached if the condition is not satisfied.

Remark 3: At the first stage of studying pinning synchronization on delayed neural networks, constant and delayed couplings are both considered. The constant μ in Hypothesis 2 can be considered as the passivity degree,[22] where $\mu>0$ means that each node needs energy from outside to stabilize the network, while $\mu<0$ means that each node itself is already stable. The derived conditions in the theorems in this paper imply that $\mu<0$ when $\mathbf{A}=0$. Here, only $\mu>0$ is considered throughout the paper for simplicity; otherwise the network can achieve self-synchronization even without control. There are some results about synchronization in networks with solely delayed couplings where $\mathbf{A}=0$ and $\mu>0$ in the literature, where the problem can be investigated by a different scenario and will be our future works.

IV. A SPECIFIC NEUROBIOLOGICAL NETWORK AND THE CORRESPONDING CONTROL

The motivation for the Hindmarsh–Rose (HR) model was to isolate the essentially mathematical properties of excitation and propagation from the electrochemical properties of sodium and potassium ion flow. The HR neuron model is described by[2]

$$\dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t)),$$

where $\mathbf{x}(t)=(x_1(t),x_2(t),x_3(t))^\top$,

$$\mathbf{f}(\mathbf{x}(t))=\begin{pmatrix}f_1(\mathbf{x}(t))\\ f_2(\mathbf{x}(t))\\ f_3(\mathbf{x}(t))\end{pmatrix}=\begin{pmatrix}\chi_1x_1^2(t)-x_1^3(t)-x_2(t)-x_3(t)\\ (\chi_1+\kappa_1)x_1^2(t)-x_2(t)\\ \kappa_2(\chi_2x_1(t)-x_3(t)+\chi_3)\end{pmatrix}.$$

Here, $x_1(t)$ is the membrane potential, $x_2(t)$ and $x_3(t)$ are the recovery variables, and $\chi_1,\chi_2,\chi_3,\kappa_1,\kappa_2$ are dimensionless parameters.

For a HR model, the following inequality can be attained (a proof is given in the Appendix):

$$\sum_{i=1}^{N}(\mathbf{x}_i-\mathbf{x}_0)^\top(\mathbf{f}(\mathbf{x}_i)-\mathbf{f}(\mathbf{x}_0))\leq\sum_{i=1}^{N}\begin{pmatrix}|x_{i1}-x_{01}|\\ |x_{i2}-x_{02}|\\ |x_{i3}-x_{03}|\end{pmatrix}^\top\begin{pmatrix}2\chi_1M & (\chi_1+\kappa_1)M+\frac{1}{2} & \frac{\kappa_2\chi_2+1}{2}\\ (\chi_1+\kappa_1)M+\frac{1}{2} & -1 & 0\\ \frac{\kappa_2\chi_2+1}{2} & 0 & -\kappa_2\end{pmatrix}\begin{pmatrix}|x_{i1}-x_{01}|\\ |x_{i2}-x_{02}|\\ |x_{i3}-x_{03}|\end{pmatrix}.$$

Usually, in the realistic neurobiological network, neurons are coupled with the first variable, namely, membrane potential. Moreover, only the membrane potentials can be observed by electrodes in practical experiments. Taking that into consideration, we describe this kind of neurobiological network model by

$$\dot{\mathbf{x}}_i(t)=\mathbf{f}(\mathbf{x}_i(t),t)+\sum_{j=1}^{N}a_{ij}\mathbf{g}(\mathbf{x}_j(t))+\sum_{j=1}^{N}b_{ij}\mathbf{h}(\mathbf{x}_j(t-\tau(t))),\tag{4}$$

where $\mathbf{g}(\mathbf{x}_j(t))$ and $\mathbf{h}(\mathbf{x}_j(t-\tau(t)))$ are specified as

$$\mathbf{g}(\mathbf{x}_j(t))=\begin{pmatrix}g_1(x_{j1}(t))\\ 0\\ 0\end{pmatrix}\quad\text{and}$$

$$\mathbf{h}(\mathbf{x}_j(t-\tau(t)))=\begin{pmatrix}h_1(x_{j1}(t-\tau))\\ 0\\ 0\end{pmatrix}.$$

In what follows, we introduce two useful Hypotheses to reach our main results in this section.

Hypothesis 3′ (***H***3′). Suppose that $g_1(\zeta)$, where $\zeta\in\mathbb{R}^n$, satisfies $d_1'\leq(g_1(\zeta_2)-g_1(\zeta_1))/(\zeta_2-\zeta_1)\leq d_2'$ for any two scalar variables $\zeta_1,\zeta_2\in\mathbb{R}$, where d_1', d_2' are two positive constants.

Hypothesis 4′ (***H***4′). Assume that $h_1(\zeta)$, where $\zeta\in\mathbb{R}^n$, satisfies $e_1'\leqslant(h_1(\zeta_2)-h_1(\zeta_1))/(\zeta_2-\zeta_1)\leqslant e_2'$ for any two scalar variables $\zeta_1,\zeta_2\in\mathbb{R}$, where e_1', e_2' are two positive constants.

To achieve synchronization in this kind of neural network consisting of HR neurons, we can design a pinning controlled network as follows:

$$\dot{\mathbf{x}}_i(t)=\mathbf{f}(\mathbf{x}_i(t),t)+\sum_{j=1}^{N}a_{ij}\mathbf{g}(\mathbf{x}_j(t))+\sum_{j=1}^{N}b_{ij}\mathbf{h}(\mathbf{x}_j(t-\tau(t)))+\mathbf{u}_i(t),\quad(5)$$

where the controllers are

$$\begin{aligned}&\mathbf{u}_{i_k}(t)=\begin{pmatrix}-\alpha_{i_k}(t)(x_{i1}(t)-x_{01}(t))\\0\\0\end{pmatrix} && 1\leqslant k\leqslant l\\ &\dot{\alpha}_{i_k}(t)=\beta_{i_k}(x_{i1}(t)-x_{01}(t))^2 && 1\leqslant k\leqslant l\\ &\mathbf{u}_{i_k}(t)=\mathbf{0} && \text{otherwise.}\end{aligned}\quad(6)$$

Denote $\bar{\mathbf{A}}'$ as a modified matrix of $\mathbf{A}$ in which the diagonal elements a_{ii} are replaced by $d_1'a_{ii}$ and other a_{ij} are replaced by $d_2'a_{ij}$, and $\bar{\mathbf{B}}'$ as a similarly modified matrix of $\mathbf{B}$. Then the globally adaptive pinning synchronization theorem for neurobiological network (4) is attained.

Theorem 2: Suppose that H1, H3′, H4′ hold. If $\lambda_m(\bar{\mathbf{A}}'+1/4\gamma(1-\delta)\bar{\mathbf{B}}'^2)_l<-(((\chi_1+\kappa_1)M+\frac{1}{2})^2+1/\kappa_2(\kappa_2\chi_2+1/2)^2+2\chi_1M+\gamma)$, the synchronous solution of the controlled network (5) will be globally asymptotically stable using the adaptive pinning controllers (6).

Proof: Consider a Lyapunov candidate as

$$V(t)=\frac{1}{2}\sum_{i=1}^{N}\|\mathbf{x}_i(t)-\mathbf{x}_0(t)\|^2+\gamma\sum_{i=1}^{N}\int_{t-\tau(t)}^{t}(x_{i1}(\sigma)-x_{01}(\sigma))^2d\sigma+\sum_{k=1}^{l}\frac{1}{2\beta_{i_k}}(\alpha_{i_k}(t)-\alpha)^2,$$

where γ is a positive constant, α is a sufficiently large positive constant to be determined. The derivative of $V(t)$ along the trajectory (5) and (6) is

$$\begin{aligned}\dot{V}(t)&=\sum_{i=1}^{N}(\mathbf{x}_i(t)-\mathbf{x}_0(t))^{\top}\left[\mathbf{f}(\mathbf{x}_i(t))-\mathbf{f}(\mathbf{x}_0(t))+\sum_{j=1}^{N}a_{ij}\mathbf{g}(\mathbf{x}_j(t))+\sum_{j=1}^{N}b_{ij}\mathbf{h}(\mathbf{x}_j(t-\tau(t)))\right]-\sum_{i=1}^{l}\alpha_i(t)(x_{i1}(t)-x_{01}(t))^2\\&\quad+\gamma\sum_{i=1}^{N}[(x_{i1}(t)-x_{01}(t))^2-(1-\dot{\tau}(t))(x_{i1}(t-\tau(t))-x_{01}(t-\tau(t)))^2]+\sum_{k=1}^{l}(\alpha_{i_k}(t)-\alpha)(x_{i_k1}(t)-x_{01}(t))^2\\&\leqslant\sum_{i=1}^{N}(\mathbf{x}_i(t)-\mathbf{x}_0(t))^{\top}(\mathbf{f}(\mathbf{x}_i(t))-\mathbf{f}(\mathbf{x}_0(t)))+\sum_{i=1}^{N}a_{ii}d_1(x_{i1}(t)-x_{01}(t))^2+\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}d_2(x_{i1}(t)-x_{01}(t))(x_{j1}(t)-x_{01}(t))\\&\quad+\sum_{i=1}^{N}b_{ii}e_1(x_{i1}(t-\tau(t))-x_{01}(t-\tau(t)))^2+\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}b_{ij}e_2(x_{i1}(t)-x_{01}(t))(x_{j1}(t-\tau(t))-x_{01}(t-\tau(t)))\\&\quad+\gamma\sum_{i=1}^{N}[(x_{i1}(t)-x_{01}(t))^2-(1-\delta)(x_{i1}(t-\tau(t))-x_{01}(t-\tau(t)))^2]-\sum_{k=1}^{l}\alpha(x_{i_k1}(t)-x_{01}(t))^2\leqslant\eta^{\top}(t)\mathbf{Q}\eta(t),\end{aligned}$$

where $\eta(t)=(|x_{11}(t)-x_{01}(t)|,\ldots,|x_{N1}(t)-x_{01}(t)|,|x_{11}(t-\tau)-x_{01}(t-\tau)|,\ldots,|x_{N1}(t-\tau)-x_{01}(t-\tau)|,|x_{12}(t)-x_{02}(t)|,\ldots,|x_{N2}(t)-x_{02}(t)|,|x_{13}(t)-x_{03}(t)|,\ldots,|x_{N3}(t)-x_{03}(t)|)^{\top}$,

$$\mathbf{Q}=\begin{pmatrix}\mathbf{Q}_1 & \frac{1}{2}\bar{\mathbf{B}}' & \left((\chi_1+\kappa_1)M+\frac{1}{2}\right)\mathbf{I}_{NN} & \frac{\kappa_2\chi_2+1}{2}\mathbf{I}_{NN}\\ \frac{1}{2}\bar{\mathbf{B}}' & -\gamma(1-\delta)\mathbf{I}_{NN} & \mathbf{0} & \mathbf{0}\\ \left((\chi_1+\kappa_1)M+\frac{1}{2}\right)\mathbf{I}_{NN} & \mathbf{0} & -\mathbf{I}_{NN} & \mathbf{0}\\ \frac{\kappa_2\chi_2+1}{2}\mathbf{I}_{NN} & \mathbf{0} & \mathbf{0} & -\kappa_2\mathbf{I}_{NN}\end{pmatrix},$$

$\mathbf{Q}_1=\bar{\mathbf{A}}'+(2\chi_1M+\gamma)\mathbf{I}_{NN}-\mathbf{\Lambda}$, $\mathbf{\Lambda}$ is a diagonal matrix whose i_kth elements are α and the others are 0.

Provided with $\lambda_m(\bar{\mathbf{A}}'+1/4\gamma(1-\delta)\bar{\mathbf{B}}'^2)_l<-(((\chi_1+\kappa_1)M+\frac{1}{2})^2+1/\kappa_2(\kappa_2\chi_2+1/2)^2+2\chi_1M+\gamma)$ in Theorem 2, one has

$$\mathbf{Q}_1 + \frac{1}{4\gamma(1-\delta)}\overline{\mathbf{B}}'^2 + \left(\left((\chi_1+\kappa_1)M + \frac{1}{2}\right)^2 + \frac{1}{\kappa_2}\left(\frac{\kappa_2\chi_2+1}{2}\right)^2\right)\mathbf{I}_{NN} < \mathbf{0}$$

when α is large enough according to Lemma 2. It follows that

$$\begin{pmatrix} \mathbf{Q}_1 - \left(\left((\chi_1+\kappa_1)M + \frac{1}{2}\right)^2 + \frac{1}{\kappa_2}\left(\frac{\kappa_2\chi_2+1}{2}\right)^2\right)\mathbf{I}_{NN} & \frac{1}{2}\overline{\mathbf{B}}' \\ \frac{1}{2}\overline{\mathbf{B}}' & -\gamma(1-\delta)\mathbf{I}_{NN} \end{pmatrix} < \mathbf{0},$$

namely,

$$\begin{pmatrix} \mathbf{Q}_1 & \frac{1}{2}\overline{\mathbf{B}}' \\ \frac{1}{2}\overline{\mathbf{B}}' & -\gamma(1-\delta)\mathbf{I}_{NN} \end{pmatrix} - \mathbf{Q}_2^{\top}\begin{pmatrix} -\mathbf{I}_{NN} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{\kappa_2}\mathbf{I}_{NN} \end{pmatrix}\mathbf{Q}_2 < \mathbf{0},$$

where

$$\mathbf{Q}_2 = \begin{pmatrix} \left((\chi_1+\kappa_1)M + \frac{1}{2}\right)\mathbf{I}_{NN} & \mathbf{0} \\ \frac{\kappa_2\chi_2+1}{2}\mathbf{I}_{NN} & \mathbf{0} \end{pmatrix}.$$

As a result, one gets $\mathbf{Q}<\mathbf{0}$ according to Lemma 1.

Then we have $\lim_{t\to\infty}\eta(t)=\mathbf{0}$, and further, $\lim_{t\to\infty}x_{ir}(t)-x_{0r}(t)=0(1\leqslant i\leqslant N, 1\leqslant r\leqslant n)$. Therefore, the synchronous solution of the controlled weighted delayed neural network (5) is globally asymptotically stable with adaptive pinning controllers (6). Thus the proof is completed.

Remark 1: It is revealed that if $\lambda_m(\overline{\mathbf{A}}'+1/4\gamma(1-\delta)\overline{\mathbf{B}}'^2)_l < -(((\chi_1+\kappa_1)M+\frac{1}{2})^2+1/\kappa_2(\kappa_2\chi_2+1/2)^2+2\chi_1 M+\gamma)$ for an appropriate $\gamma>0$, globally asymptotic synchronization of this type of neural network can be reached by controlling only the membrane potential of the i_1th, i_2th, …, i_lth neurons.

Remark 2: Similarly, it should be noted that the condition in the theorem is just sufficient.

V. NUMERICAL SIMULATION

A. A detailed example with network size 3

In this subsection, some numerical examples are simulated to show the effectiveness of the adaptive pinning control presented in Sec. IV.

For simplicity, three neurons are considered to form a neurobiological network (4) with coupling delay $\tau(t)=1$, and the dynamics of each neuron is a HR equation with parameters $\chi_1=0.2$, $\chi_2=1.5$, $\chi_3=0.1$, $\kappa_1=1.9$, and $\kappa_2=0.1$. In this case, the bound M of the first variable in the HR equation is 0.8.

Suppose that the instantaneous and delayed outer-coupling matrices are

$$\mathbf{A} = \begin{pmatrix} -5 & 0 & 5 \\ 0 & -6 & 6 \\ 5 & 6 & -11 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

respectively. Moreover, $g_1(x_{j1})=h_1(x_{j1})=x_{j1}(1\leqslant j\leqslant N)$ are assumed to be the instantaneous and delayed inner-coupling function. It is easy to check that H1, H3′, and H4′ naturally hold where $d_1'=d_2'=e_1'=e_2'=1$ and $\delta=0$.

Selecting $\gamma=1/2(1-\delta)$, we get $-(((\chi_1+\kappa_1)M+\frac{1}{2})^2+1/\kappa_2(\kappa_2 b+1/2)^2+2\chi_1 M+1/2(1-\delta))=-9.5187$ by simple computation. Since $\lambda_m(\overline{\mathbf{A}}+\frac{1}{2}\overline{\mathbf{B}}^2)_2=-10<-9.5187$ when $i_1=1$, $i_2=2$, the first two neurons can be pinned to reach global synchronization with adaptive controllers (6) according to Theorem 2.

Choose initial values as $\mathbf{x}_i(0)=(0.1+0.1i, 0.2+0.1i, 0.3+0.1i)^{\top}$ for $0\leqslant i\leqslant N$ and $\alpha_k(0)=\beta_k=1$ for $1\leqslant k\leqslant l$ with $N=3$, $l=2$ in the numerical simulation. The state time evolutions of the neurons $x_{ir}(t)(1\leqslant i\leqslant 3, 1\leqslant r\leqslant 3)$ and of the desired synchronous state $x_{0r}(t)$ without control are depicted in Fig. 1 and those with adaptive pinning control are exhibited in Fig. 2. Synchronization errors $x_{ir}-x_{0r}$ without control and with adaptive pinning control are displayed in Fig. 3 and Fig. 4, respectively. The figures reveal that these neurons cannot synchronize with the desired state when no controllers are

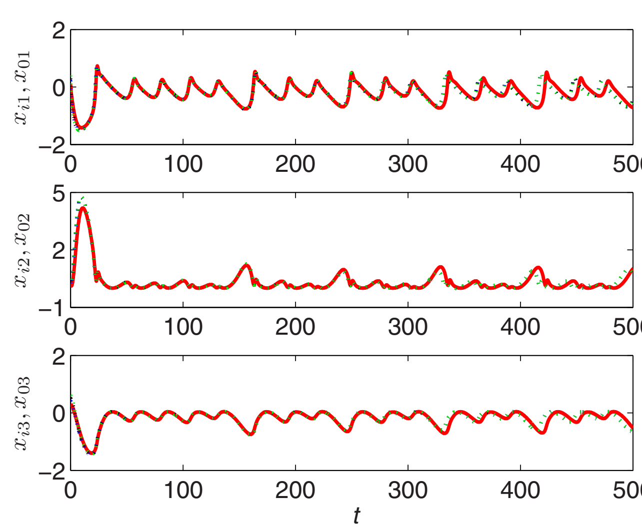

FIG. 1. (Color online) The state time evolutions of the neurons $x_{ir}(t)(1\leqslant i\leqslant 3, 1\leqslant r\leqslant 3)$ and of the desired synchronous state $x_{0r}(t)$ (red line) without control.

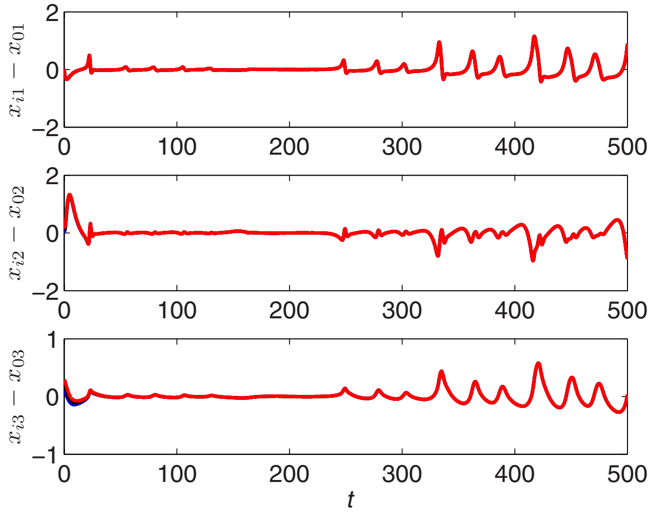

FIG. 2. (Color online) The state time evolutions of the neurons $x_{ir}(t)(1 \leqslant i \leqslant 3, 1 \leqslant r \leqslant 3)$ and of the desired synchronous state $x_{0r}(t)$ (red line) for pinning the first two neurons.

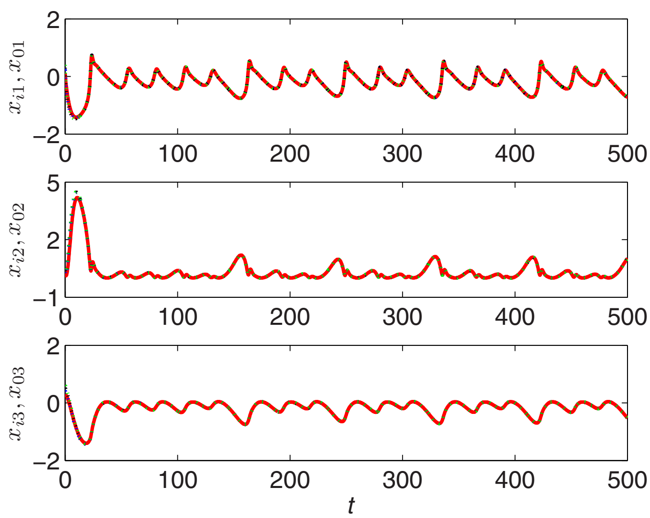

FIG. 3. (Color online) Synchronization errors $x_{ir} - x_{0r} (1 \leqslant i \leqslant 3, 1 \leqslant r \leqslant 3)$ without control.

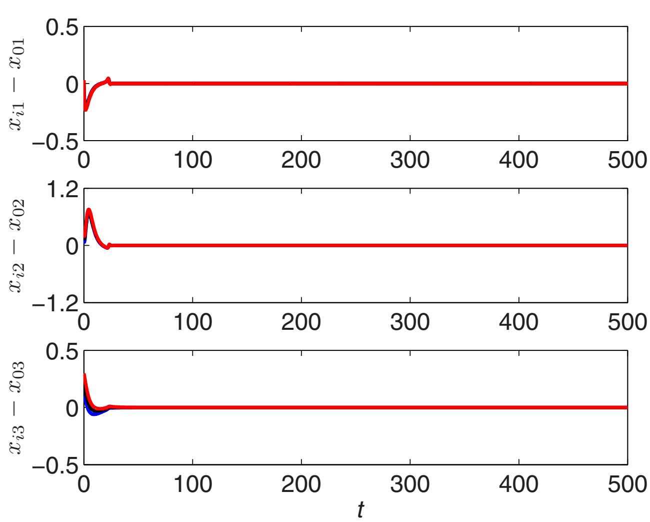

FIG. 4. (Color online) Synchronization errors $x_{ir} - x_{0r} (1 \leqslant i \leqslant 3, 1 \leqslant r \leqslant 3)$ for pinning the first two neurons.

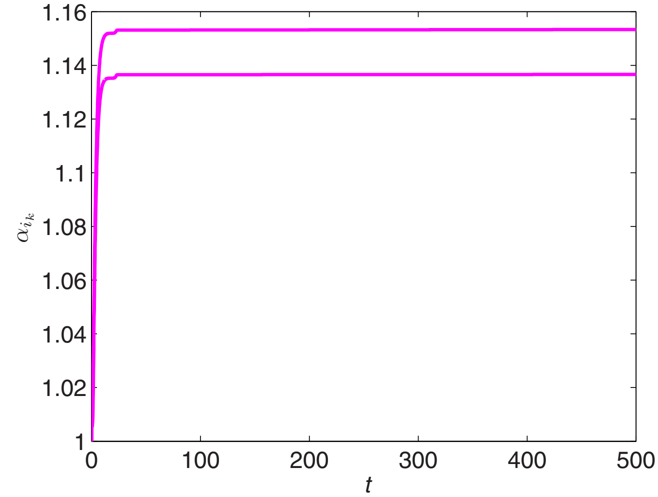

FIG. 5. (Color online) Adaptive feedback gains $\alpha_{i_k} (i_k \in \{1,2\})$ for pinning the first two neurons.

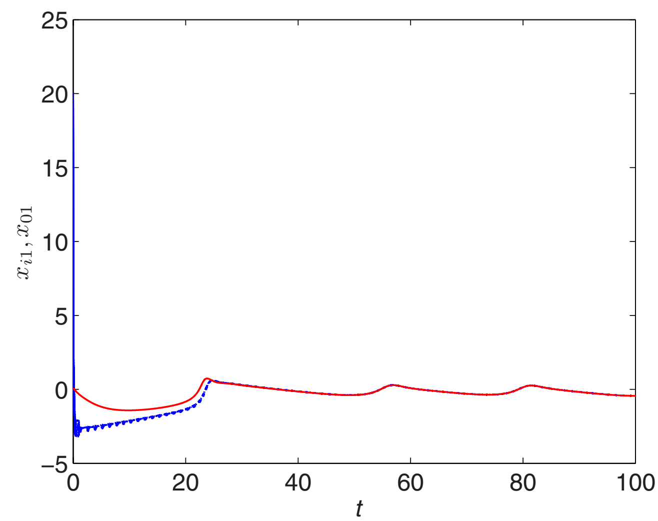

FIG. 6. (Color online) The state time evolutions of the neurons $x_{i1} (1 \leqslant i \leqslant 200)$ and of the desired synchronous state $x_{01}(t)$ (red line) for pinning the first 12 neurons.

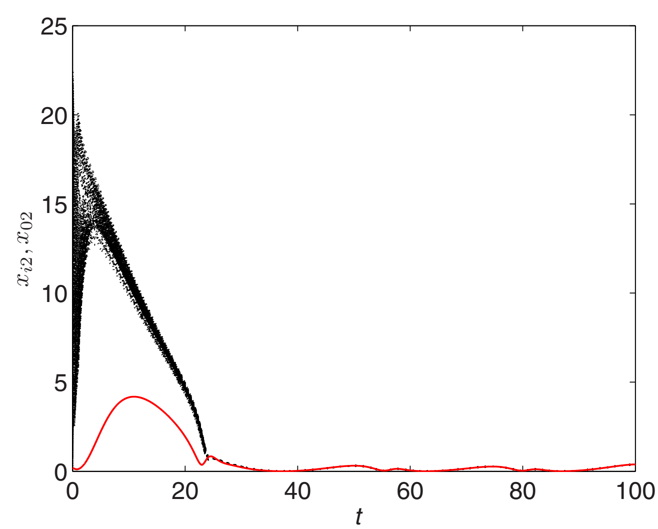

FIG. 7. (Color online) The state time evolutions of the neurons $x_{i2} (1 \leqslant i \leqslant 200)$ and of the desired synchronous state $x_{02}(t)$ (red line) for pinning the first 12 neurons.

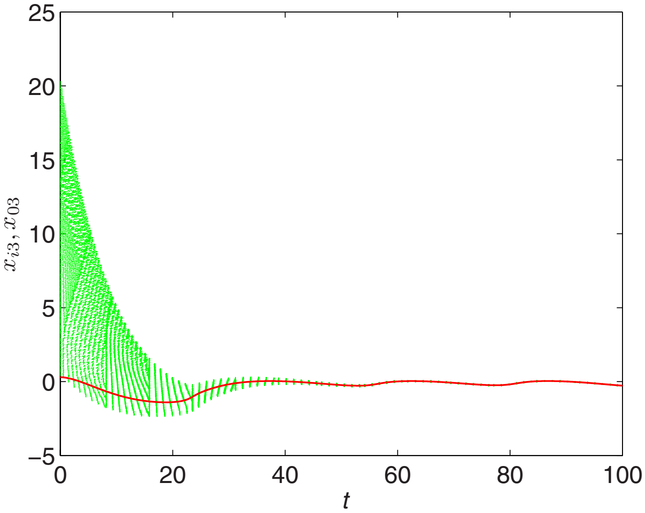

FIG. 8. (Color online) The state time evolutions of the neurons $x_{i3}(1 \leq i \leq 200)$ and of the desired synchronous state $x_{03}(t)$ (red line) for pinning the first 12 neurons.

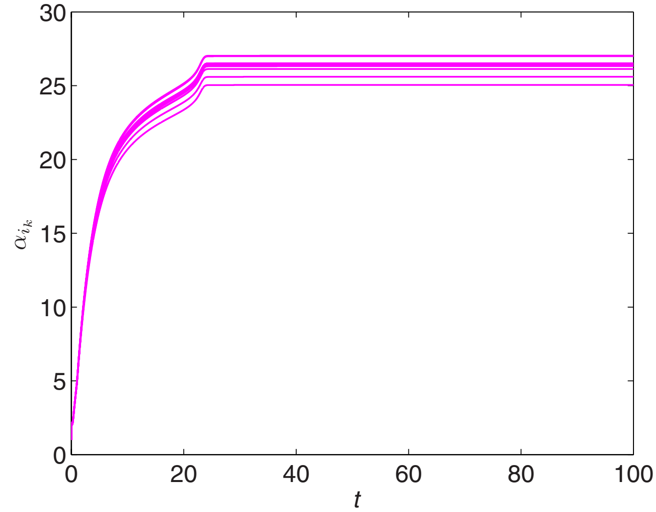

FIG. 9. (Color online) Adaptive feedback gains $\alpha_{i_k}(i_k \in \{1,2,\ldots,12\})$ for pinning the first 12 neurons.

applied, while the network approaches synchronization with the desired state asymptotically with the pinning technique presented in Sec. IV. In addition, the adaptive feedback gains $\alpha_{i_k}(i_k \in \{1,2\})$ are shown in Fig. 5, from which it is seen that α_{i_k} do not increase after a short period of time.

B. An example with large-scale neural network

Over the past two decades, small-world[20] and scale-free[21] networks have been intensively investigated as large-scale complex networks. Barabási and Albert (BA) model of preferential attachment has become the standard mechanism to explain the emergence of scale-free networks. Nodes are added to the network with a preferential bias toward attachment to nodes with already high degree. This naturally gives rise to hubs with a degree distribution following a power law.

In this subsection, a BA neural network consisting of 200 HR neurons with $m_0=5$ and $m=5$ are considered, where m_0 is the size of the initial network, m is the number of edges added in each step. Choose $\mathbf{A}=10\ \mathbf{A}^*$, $\mathbf{B}=0.5\ \mathbf{A}^*$, where $\mathbf{A}^*$ is the adjacency matrix in which $a_{ii}^*=-\Sigma_{j\in\mathcal{N}_i}a_{ij}^*$, $a_{ij}^*\in\{0,1\}$. Let the parameters and initial values in the network be the same as those in the previous subsection with $N=200$, $l=12$. After controlling the membrane potential of only 6% neurons with the largest degree, synchronization is achieved asymptotically. The state time evolutions of the neurons and of the desired synchronous state can be seen in Figs. 6–8. Besides, the adaptive parameters $\alpha_{i_k}(i_k \in \{1,2,\ldots,12\})$ are shown in Fig. 9. From that, it is obvious that α_{i_k} do not increase after a short interval of time.

VI. CONCLUSIONS

In this paper, we have studied the problem that which neurons in a delayed neurobiological network should be controlled to achieve adaptive pinning synchronization. In view of Schur complement and the Lyapunov function methods, we have proven that under a mild topology-based condition, synchronization of this general neural network can be reached using the criterion presented in Sec. III. Moreover, for a concrete neurobiological network consisting of identical HR neurons, we have proposed a specific pinning control technique to synchronize it. Finally, we have exhibited simple computational examples to illustrate the effectiveness of the proposed approach. In addition to the wide applications in neurobiological networks, this technique can also be applied to many other complex networks.

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APPENDIX: A PROPERTY OF HR MODEL

For any two vectors $\mathbf{z}_1(t)=(z_{11}(t),z_{12}(t),z_{13}(t))^{\mathsf{T}}$ and $\mathbf{z}_2(t)=(z_{21}(t),z_{22}(t),z_{23}(t))^{\mathsf{T}}$ in the HR equation, we have

$$\begin{aligned}(z_{21}(t)-z_{11}(t))(f_1(\mathbf{z}_2(t))-f_1(\mathbf{z}_1(t))) &= (z_{21}(t)-z_{11}(t))[\chi_1(z_{21}(t)+z_{11}(t))(z_{21}(t)-z_{11}(t))-(z_{21}(t)-z_{11}(t))(z_{21}^2(t)+z_{21}(t)z_{11}(t)\\ &\quad +z_{11}^2(t))-(z_{22}(t)-z_{12}(t))-(z_{23}(t)-z_{13}(t))]\\ &\leq 2\chi_1M(z_{21}(t)-z_{11}(t))^2+(z_{21}(t)-z_{11}(t))(z_{22}(t)-z_{12}(t))+(z_{21}(t)-z_{11}(t))(z_{23}(t)-z_{13}(t))\\ &\leq 2\chi_1M|z_{21}(t)-z_{11}(t)|^2+|z_{21}(t)-z_{11}(t)||z_{22}(t)-z_{12}(t)|+|z_{21}(t)-z_{11}(t)||z_{23}(t)-z_{13}(t)|,\end{aligned}$$

$$
\begin{aligned}
(z_{22}(t)-z_{12}(t))(f_2(\mathbf{z}_2(t))-f_2(\mathbf{z}_1(t))) &= (z_{22}(t)-z_{12}(t))[(\chi_1+\kappa_1)(z_{21}(t)+z_{11}(t))(z_{21}(t)-z_{11}(t))-(z_{22}(t)-z_{12}(t))] \\
&\leqslant 2(\chi_1+\kappa_1)M(z_{21}(t)-z_{11}(t))(z_{22}(t)-z_{12}(t))-(z_{22}(t)-z_{12}(t))^2 \\
&\leqslant 2(\chi_1+\kappa_1)M|z_{21}(t)-z_{11}(t)|z_{22}(t)-z_{12}(t)|-|z_{22}(t)-z_{12}(t)|^2
\end{aligned}
$$

and that

$$
\begin{aligned}
(z_{23}(t)-z_{13}(t))(f_3(\mathbf{z}_2(t))-f_3(\mathbf{z}_1(t))) &= \kappa_2(z_{23}(t)-z_{13}(t))[\chi_2(z_{21}(t)-z_{11}(t))-(z_{23}(t)-z_{13}(t))] \\
&\leqslant \kappa_2\chi_2|(z_{21}(t)-z_{11}(t))|z_{23}(t)-z_{13}(t)|-\kappa_2|z_{23}(t)-z_{13}(t)|^2,
\end{aligned}
$$

where M is a positive constant representing the boundary of the first variable in the HR equation. Thus the following inequality holds:

$$
\sum_{i=1}^{N}(\mathbf{x}_i-\mathbf{x}_0)^\top(\mathbf{f}(\mathbf{x}_i)-\mathbf{f}(\mathbf{x}_0)) \leqslant \sum_{i=1}^{N}\begin{pmatrix}|x_{i1}-x_{01}|\\|x_{i2}-x_{02}|\\|x_{i3}-x_{03}|\end{pmatrix}^\top \begin{pmatrix} 2\chi_2M & (\chi_1+\kappa_1)M+\frac{1}{2} & \frac{\kappa_2\chi_2+1}{2} \\ (\chi_1+\kappa_1)M+\frac{1}{2} & -1 & 0 \\ \frac{\kappa_2\chi_2+1}{2} & 0 & -\kappa_2 \end{pmatrix}\begin{pmatrix}|x_{i1}-x_{01}|\\|x_{i2}-x_{02}|\\|x_{i3}-x_{03}|\end{pmatrix}.
$$

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