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Linearly and nonlinearly bidirectionally coupled synchronization of hyperchaotic systems ☆

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Abstract

To date, there have been many results about unidirectionally coupled synchronization of chaotic systems. However, much less work is reported on bidirectionally-coupled synchronization. In this paper, we investigate the synchronization of two bidirectionally coupled Chen hyperchaotic systems, which are coupled linearly and nonlinearly respectively. Firstly, linearly coupled synchronization of two hyperchaotic Chen systems is investigated, and a theorem on how to choose the coupling coefficients are developed to guarantee the global asymptotical synchronization of two coupled hyperchaotic systems. Analysis shows that the choice of the coupling coefficients relies on the bound of the chaotic system. Secondly, the nonlinearly coupled synchronization is studied; a sufficient condition for the locally asymptotical synchronization is derived, which is independent of the bound of the hyperchaotic system. Finally, numerical simulations are included to verify the effectiveness and feasibility of the developed theorems.

1. Introduction

Hyperchaotic systems is usually classified as a chaotic system with more than one positive Lyapunov exponent, indicating that the chaotic dynamics of the system are expanded in more than one direction giving rise to a more complex attractor. In recent years, hyperchaos has been studied with increasing interests, in the fields of secure communication [1], multimode lasers [2], nonlinear circuits [3], biological networks [4], coupled map lattices [5,6], and so on.

Very recently, a new four-dimensional autonomous hyperchaotic system has been proposed, which is named as Chen hyperchaotic system [7]. The new system is described by

$$\begin{cases} \dot{x} = a(y - x) + u, \\ \dot{y} = dx - xz + cy, \\ \dot{z} = xy - bz, \\ \dot{u} = yz + ru, \end{cases} \tag{1}$$

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where a, b, c, d, r are positive constants. When $a = 35$, $b = 3$, $c = 12$, $d = 7$, $r = 0.58$, system (1) is hyperchaotic, since it has two positive Lyapunov exponents. Research on the synchronization of such systems is important due to its complex dynamics.

In the past years, most synchronizing techniques have been designed for unidirectionally coupled systems, such as impulsive control [8], adaptive synchronization [9], and so on [10–12]. While in recent years, bidirectionally coupled synchronization has attracted a great deal of attention due to its complicated dynamical behaviors. Some results have been reported about linearly coupled synchronization [13–16], while nonlinearly coupled synchronization has rarely been studied.

In this paper, we will investigate the bidirectionally coupled synchronization of two hyperchaotic Chen systems, which are coupled linearly and nonlinearly respectively. The paper is organized as follows. In Section 2, the model of our research and a lemma are introduced. Linearly and nonlinearly coupled synchronization criteria are proposed in Section 3. We will show that under certain conditions, the linearly coupled synchronization is globally stable but it relies on the bound of the hyperchaotic system. While the nonlinearly coupled synchronization is independent of the bound and the asymptotical stability is locally. Numerical simulations are provided for illustration and verification in Section 4. Finally, conclusions are given in Section 5.

2. Preliminaries

This section introduces the model of our problem and gives a preliminary lemma which will be used in Section 3.2.

2.1. The describing of the model

Consider two Chen hyperchaotic systems

$$\dot{X} = AX + f(X) + D(Y - X), \tag{2}$$

$$\dot{Y} = AY + f(Y) + D(X - Y), \tag{3}$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \quad A = \begin{pmatrix} -a & a & 0 & 1 \\ d & c & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad f(X) = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \\ x_2x_3 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix},$$

and d_i $(i = 1, \ldots, 4)$ are the coupling coefficients.

Our goal is to design an appropriate diagonal matrix D such that system (3) asymptotically synchronizes with system (2). That is, if $e = Y - X$, the goal is to achieve $\lim_{t\to\infty}\|e(t)\| = 0$, where $\|\cdot\|$ is the Euclidean norm.

2.2. Preliminary lemma

Lemma 1. *Suppose $A \in R^{n\times n}$ is a negative definite matrix, then,*

$$x^{\mathrm{T}}Ax + b\|x\|^3 < 0, \quad \forall x \in U \subset R^n, \ x \neq 0,$$

where $b \in R$, U is a domain that contains a neighbourhood of $x = 0$.

Proof. Since $A \in R^{n\times n}$ is negative definite, there exists an orthogonal matrix Γ such that

$$\Gamma^{\mathrm{T}}A\Gamma = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \triangleq \Lambda,$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n < 0$ are eigenvalues of matrix A.

Using the transformation $y = \Gamma^{\mathrm{T}}x$, one gets $x^{\mathrm{T}}Ax = y^{\mathrm{T}}\Lambda y$ and $\|x\| = \|y\|$. Since $x \neq 0$, one has $y \neq 0$, namely, $\|y\| \neq 0$. Then, one obtains

$$x^{\mathrm{T}}Ax + b\|x\|^3 = y^{\mathrm{T}}\Lambda y + b\|y\|^3 \leqslant \lambda_n\|y\|^2 + b\|y\|^3 = \|y\|^2(\lambda_n + b\|y\|).$$

Since $\lambda_n < 0$, there exists a domain $U(y)$ containing a neighbourhood of $y = 0$ which is small enough that $\lambda_n + b\|y\| < 0$. Consequently, there exists a domain $U = U(x)$ which contains a neighbourhood of $x = 0$ such that $x^{\mathrm{T}}Ax + b\|x\|^3 \leqslant \|y\|^2(\lambda_n + b\|y\|) < 0$ for any $b \in R$ and $x \in U \subset R^n$, $x \neq 0$. Thus the proof is completed. □

3. Synchronization of Chen hyperchaotic systems

3.1. Linearly coupled synchronization

Consider two Chen hyperchaotic systems (2) and (3). The linearly coupling coefficients d_i $(i = 1, \ldots, 4)$ are constants to be decided. From the two systems, we have

$$\dot{e} = Ae + f(Y) - f(X) - 2De = (A + B - 2D)e, \tag{4}$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -x_3 & 0 & -y_1 & 0 \\ x_2 & y_1 & 0 & 0 \\ 0 & x_3 & y_2 & 0 \end{pmatrix}.$$

Define a Lyapunov function $V = \frac{1}{2}e^{\mathrm{T}}e$, then its time derivative along system (4) is

$$\dot{V} = \frac{1}{2}(\dot{e}^{\mathrm{T}}e + e^{\mathrm{T}}\dot{e}) = e^{\mathrm{T}}\left(\frac{A + A^{\mathrm{T}}}{2} - 2D\right)e + e^{\mathrm{T}}\left(\frac{B + B^{\mathrm{T}}}{2}\right)e \triangleq e^{\mathrm{T}}(P_1 + P_2)e,$$

where

$$P_1 = \begin{pmatrix} -(a + 2d_1) & \frac{a+d}{2} & 0 & \frac{1}{2} \\ \frac{a+d}{2} & c - 2d_2 & 0 & 0 \\ 0 & 0 & -(b + 2d_3) & 0 \\ \frac{1}{2} & 0 & 0 & r - 2d_4 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -\frac{x_3}{2} & \frac{x_2}{2} & 0 \\ -\frac{x_3}{2} & 0 & 0 & \frac{x_3}{2} \\ \frac{x_2}{2} & 0 & 0 & \frac{y_2}{2} \\ 0 & \frac{x_3}{2} & \frac{y_2}{2} & 0 \end{pmatrix}.$$

Since the maximum eigenvalue of matrix P_2

$$\lambda_{\max}(P_2) = \frac{1}{4}\sqrt{2\sqrt{y_2^4 + 2y_2^2x_2^2 + x_2^4 + 4x_3^4 - 8x_3^2x_2y_2} + 2y_2^2 + 2x_2^2 + 4x_3^2} \leqslant M,$$

where $M \triangleq \max_{x_i \in (1), i=1,\ldots,4}\{|x_i|\} = \max_{y_i \in (1), i=1,\ldots,4}\{|y_i|\}$, one gets

$$\dot{V} \leqslant e^{\mathrm{T}}(P_1 + P_2)e \leqslant e^{\mathrm{T}}(P_1 + MI)e \triangleq -e^{\mathrm{T}}Pe,$$

where

$$P = \begin{pmatrix} a + 2d_1 - M & -\frac{a+d}{2} & 0 & -\frac{1}{2} \\ -\frac{a+d}{2} & 2d_2 - c - M & 0 & 0 \\ 0 & 0 & b + 2d_3 - M & 0 \\ -\frac{1}{2} & 0 & 0 & 2d_4 - r - M \end{pmatrix}.$$

Follow the following procedure,

(a) Choose a proper d_1 such that the first leading principal minor determinant of P satisfies $\Delta_1 = a + 2d_1 - M > 0$.
(b) Then, it is easy to find a constant d_2 satisfying $2d_2 - c - M > 0$ and $\Delta_2 = (a + 2d_1 - M)(2d_2 - c - M) - \frac{(a+d)^2}{4} > 0$.
(c) Next, one can choose d_3 such that $b + 2d_3 - M > 0$, i.e., $\Delta_3 = (b + 2d_3 - M)\Delta_2 > 0$.
(d) Finally, one can easily find a constant d_4 satisfying $2d_4 - r - M > 0$ and $\Delta_4 = |P| = (2d_4 - r - M)\Delta_3 - \frac{1}{4}(2d_2 - c - M)(b + 2d_3 - M) > 0$.

As a result, P is a positive definite matrix. By the Lyapunov stability theory, we have $\lim_{t\to\infty}\|e(t)\| = 0$. That is, systems (2) will synchronize with system (3). Consequently, we have the following theorem.

Theorem 1. *If the coupling coefficients* $d_i > 0$ $(i = 1, \ldots, 4)$ *satisfy the above conditions* (a)–(d), *systems* (2) *will synchronize with system* (3) *for any given initial value.*

3.2. Nonlinearly coupled synchronization

Consider two Chen hyperchaotic systems (2) and (3), where d_i $(i = 1, \ldots, 4)$ are nonlinear coupling coefficients which depend on the state variables. Linearizing the error system at the origin gives

$$\dot{e} = Ae + f(Y) - f(X) - 2De = (A + J(x) - 2D)e + (0, -e_1e_3, e_1e_2, e_2e_3)^{\mathrm{T}} \tag{5}$$

where

$$J(X) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -x_3 & 0 & -x_1 & 0 \\ x_2 & x_1 & 0 & 0 \\ 0 & x_3 & x_2 & 0 \end{pmatrix}$$

is the Jacobian matrix of $f(Y)$ along trajectory $X(t)$.

Similarly, define a Lyapunov function as $V = \frac{1}{2}e^{\mathrm{T}}e$, we have

$$\dot{V} = \frac{1}{2}(\dot{e}^{\mathrm{T}}e + e^{\mathrm{T}}\dot{e}) = e^{\mathrm{T}}\left(\frac{A + A^{\mathrm{T}}}{2} + \frac{J(X) + (J(X))^{\mathrm{T}}}{2} - 2D\right)e + e_2e_3e_4$$
$$\leqslant e^{\mathrm{T}}\left(\frac{A + A^{\mathrm{T}}}{2} - 2D + \lambda_{\max}\left(\frac{J(X) + (J(X))^{\mathrm{T}}}{2}\right)I\right)e + \|e\|^3.$$

Here,

$$\frac{J(X) + (J(X))^{\mathrm{T}}}{2} = \begin{pmatrix} 0 & -\frac{x_3}{2} & \frac{x_2}{2} & 0 \\ -\frac{x_3}{2} & 0 & 0 & \frac{x_3}{2} \\ \frac{x_2}{2} & 0 & 0 & \frac{x_2}{2} \\ 0 & \frac{x_3}{2} & \frac{x_2}{2} & 0 \end{pmatrix}.$$

By simple computation, we obtain $\lambda_{\max}\left(\frac{J(X)+(J(X))^{\mathrm{T}}}{2}\right) = \frac{1}{2}\sqrt{|x_2^2 - x_3^2| + x_2^2 + x_3^2}$. Since $|x_2^2 - x_3^2| \leqslant x_2^2 + x_3^2$, we have

$$\lambda_{\max}\left(\frac{J(X) + (J(X))^{\mathrm{T}}}{2}\right) \leqslant \frac{\sqrt{2}}{2}\sqrt{x_2^2 + x_3^2} \leqslant \frac{\sqrt{2}}{2}|x_2 + x_3| \triangleq \beta.$$

Then, the function $\dot{V}$ satisfies

$$\dot{V} \leqslant e^{\mathrm{T}}\left(\frac{A + A^{\mathrm{T}}}{2} - 2D + \beta I\right)e + \|e\|^3 \triangleq -e^{\mathrm{T}}Qe + \|e\|^3,$$

where

$$Q = \begin{pmatrix} a + 2d_1 - \beta & -\frac{a+d}{2} & 0 & -\frac{1}{2} \\ -\frac{a+d}{2} & 2d_2 - c - \beta & 0 & 0 \\ 0 & 0 & b + 2d_3 - \beta & 0 \\ -\frac{1}{2} & 0 & 0 & 2d_4 - r - \beta \end{pmatrix}.$$

By Lemma 1, $-e^{\mathrm{T}}Qe + \|e\|^3$ will be negative definite in a domain U that contains a neighbourhood of $e = 0$ if Q is a positive definite matrix. Let $d_1 = \frac{\beta}{2}$, $d_2 = \frac{\beta}{2} + \frac{(a+d)^2}{2a} + \frac{c}{2}$, $d_3 = \frac{\beta}{2}$, $d_4 = \frac{\beta}{2} + \frac{r}{2} + \frac{1}{a}$, thus the local asymptotical stability of the equilibrium point $e = 0$ of system (5) is guaranteed.

By topological transitivity of chaos, there exists a $T > 0$ such that $e(t)$ will enter U which contains a neighbourhood of $e = 0$ if $t > T$. It follows that $e_i(t) \to 0$ $(t \to \infty)$ $(i = 1, \ldots, 4)$. Thus we have

Theorem 2. *If the nonlinear coupling coefficients* d_i $(i = 1, \ldots, 4)$ *are set as*

$$d_1 = \frac{\beta}{2}, \qquad d_2 = \frac{\beta}{2} + \frac{(a + d)^2}{2a} + \frac{c}{2}, \qquad d_3 = \frac{\beta}{2}, \qquad d_4 = \frac{\beta}{2} + \frac{r}{2} + \frac{1}{a}, \tag{6}$$

where $\beta = \frac{\sqrt{2}}{2}|x_2 + x_3|$, *system* (2) *will synchronize with system* (3).

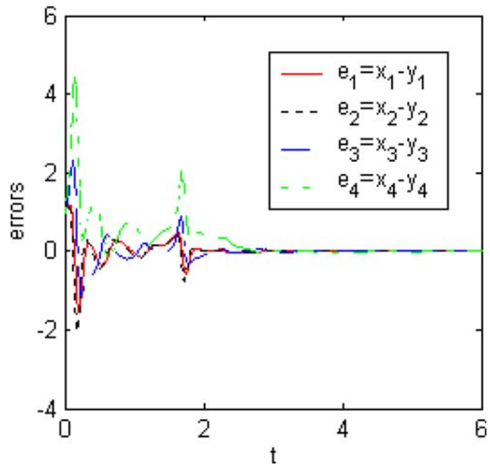

Fig. 1. Linearly coupled synchronization errors of system (4).

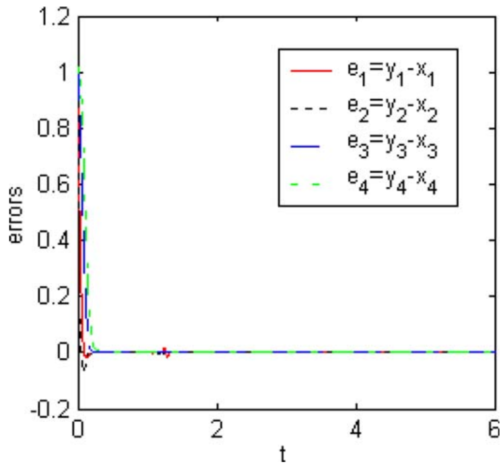

Fig. 2. Nonlinearly coupled synchronization errors of system (5).

4. Numerical simulations

In the simulations, we assume that $a = 35$, $b = 3$, $c = 12$, $d = 7$, $r = 0.58$, and initial values are $(x_1(0), x_2(0), x_3(0), x_4(0), y_1(0), y_2(0), y_3(0), y_4(0)) = (3, 5, 7, 9, 4, 6, 8, 10)$.

Fig. 1 displays the linearly coupled synchronizing errors of system (4) with $d_1 = 0.1$, $d_2 = d_3 = d_4 = 1$. Here, the coupling coefficients do not satisfy the conditions listed in Theorem 1, but synchronization is still reached, since the conditions are only sufficient but not necessary. Fig. 2 shows the nonlinearly coupled synchronizing errors of system (5), where coefficients d_1, d_2, d_3, d_4 are chosen as Eq. (6) in Theorem 2. It is obvious that the two systems soon reach synchronization.

5. Conclusions

In this paper, we have discussed the synchronization of two bidirectionally coupled Chen hyperchaotic systems, which are coupled linearly and nonlinearly respectively. For the linearly coupled synchronization of two hyperchaotic Chen systems, a theorem on how to choose the coupling coefficients has been developed to guarantee the global asymptotical stability, while the choice of the coupling coefficients relies on the bound of the hyperchaotic system. For the nonlinearly coupled synchronization, a sufficient condition for the locally asymptotical synchronization has been derived, which is independent of the bound of the hyperchaotic system. Numerical simulations are also included to verify the effectiveness and feasibility of the proposed theorems. In fact, these methods can be extended to three or more coupled chaotic systems.

References

[1] Udaltsov VS, Goedgebuer JP, Larger L, Cuenot JB, Levy P, Rhodes WT. Communicating with hyperchaos: the dynamics of a DNLF emitter and recovery of transmitted information. Opt Spectrosc 2003;95(1):114–8.
[2] Shahverdiev EM, Nuriev, RA, Hashimov RH, Shore KA. Adaptive time-delay hyperchaos synchronization in laser diodes subject to optical feedback. Available from: ArXiv:nlin.CD/0404053, v1, 29 April 2004.
[3] Barbara C, Silvano C. Hyperchaotic behavior of two bi-directionally Chua's circuits. Int J Circ Theory Appl 2002;30(6):625–37.
[4] Neiman A, Pei X, Russell D, Wojtenek W, Wilkens L, Moss F, et al. Synchronization of the noisy electrosensitive cells in the paddlefish. Phys Rev Lett 1999;82:660–3.
[5] Zhan M, Hu G, Yang J. Synchronization of chaos in coupled systems. Phys Rev E 2000;62:2963–6.
[6] Jinlan W, Guangzhi C, Tuanfa Q, Wansun N, Xuming W. Synchronizing spatiotemporal chaos in coupled map lattices via active-passive decomposition. Phys Rev E 1998;58:3017–21.
[7] Li Y, Tang Wallace KS, Chen G. Generating hyperchaos via state feedback control. Int J Bifur Chaos 2005;15(10).
[8] Itoh M, Yang T, Chua LO. Conditions for impulsive synchronization of chaotic and hyperchaotic system. Int J Bifur Chaos 2001;11(2):551–60.
[9] Chen SH, Lü JH. Synchronization of an uncertain unified chaotic system via adaptive control. Chaos, Solitons & Fractals 2002;14(4):643–7.
[10] Lu J, Wu X, Lü J. Synchronization of a unified chaotic system and the application in secure communication. Phys Lett A 2002;305:365–70.
[11] Tao C, Lu J, Lü J. The feedback synchronization of a unified chaotic system. Acta Phys Sinica 2002;52(7):1595–605.
[12] Lü J, Lu J. Controlling uncertain Lü system using linear feedback. Chaos, Solitons & Fractals 2003;17:127–33.
[13] Zhou T, Lü J, Chen G, Tang Y. Synchronization stability of chaotic array with linear coupling. Phys Lett A 2002;301:231–40.
[14] Lü J, Zhou T, Zhang S. Chaos synchronization between linearly coupled chaotic system. Chaos, Solitons & Fractals 2002;14:529–41.
[15] Li D, Lu J, Wu X. Linearly coupled synchronization of the unified chaotic systems and the Lorenz systems. Chaos, Solitons & Fractals 2005;23:79–85.
[16] Park Ju H. Stability criterion for synchronization of linearly coupled unified chaotic systems. Chaos, Solitons & Fractals 2005;23:1319–25.