# Finite Time Inter-Layer Synchronization of Duplex Networks via Event-Dependent Intermittent Control

Changjiang Song, Jin Zhou, and Jinshui Wang

***Abstract*—This brief is devoted to the finite time inter-layer synchronization of duplex networks by means of intermittent control. Different from most previous works, the intermittent control scheme devised is an event-dependent type, which relaxes the strict constraints of control time. To be specific, the work time of intermittent controller is no longer set in advance, and determined by the relation between the designed Lyapunov function and two preset boundary functions. Based on the proposed intermittent control mechanism, a sufficient condition is given to guarantee the finite time inter-layer synchronization of the duplex networks. Finally, numerical examples are presented to illustrate the validness of theoretical results.**

***Index Terms*—Duplex networks, inter-layer synchronization, finite time, intermittent control, event-dependent.**

## I. Introduction

In THE past decades, complex networks have received considerable attentions due to its ubiquitous existence in the real world, such as social networks [1], communication networks [2], neural networks [3], and epidemic spreading networks [4]. As one of the most important collective behaviors of complex networks, the synchronization problem has always been a hot issue in the research of complex networks, see [5]–[7].

Generally speaking, few complex networks can realize synchronization by the self connections of nodes. Therefore, in order to achieve synchronization, external controllers are often added to networks. Many effective control methods have been proposed in the literature [8]–[10]. And these controls can be divided into continuous controls and discontinuous controls based on their characters that whether the controllers work continuously or not. A typical example of discontinuous control is the intermittent control, where the time is decomposed as the rest time and the work time, and the control is only active in the work time. The synchronization problem in different scenes have been addressed via intermittent control. For example, in [11], aperiodically intermittent controllers with adaptive strategy were utilized to synchronize the coupled network. In [12] and [13], outer synchronization of complex networks was investigated via periodically and aperiodically intermittent control, respectively.

The synchronization considered above is the asymptotic synchronization, whose synchronization time is infinite. However, in reality, the networks might be expected to achieve synchronization as quickly as possible, particularly in engineering fields. To achieve faster synchronization of networks, finite time control technique is introduced in [14]. Combining with the intermittent control scheme, finite time synchronization, whose convergence time can be predicted, has been studied in [15]–[17].

It is well known that many real complex networks, including transportation networks and smart grid, have two-layer even multi-layer connections, where networks interact with other networks. Therefore, synchronization on multi-layer networks deserves more attentions. Fortunately, there are of increasing interest on the study of synchronization phenomena on two-layer networks (duplex networks) [18]–[20]. However, to the best of our knowledge, there are no results on the synchronization of duplex networks by intermittent control. More studies are expected to improve our understanding of duplex networks and reveal the complex phenomena in nature. Fig. 1 is an example of a duplex network with unidirectional connections.

From another point of view, the intermittent control strategy in the aforementioned studies is time-dependent. That is to say, the work time of the intermittent controller is preset and meets quite restrict conditions, which is not easily implemented in practice. To overcome this disadvantage, a novel intermittent control named event-dependent intermittent control was proposed in [21]. In the proposed intermittent control, whether the controller is working depends on the relation between the designed Lyapunov function and non-negative real regions. Such control can further save resources compared with the time-dependent intermittent control, yet few authors consider finite time synchronization on duplex networks via the new method.

Motivated by the above discussions, this brief investigates the finite time synchronization problem of duplex networks via event-dependent intermittent control. In addition, time delay and parametric uncertainties are taken into consideration, since these two factors are inevitable in practice and may lead to the failure of synchronization on networks.

*Notations:* Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ be the sets of $n$-dimensional real vectors and $n \times m$-dimensional real matrices respectively. The identity matrix is denoted by $I$. $M > 0 (M < 0)$ represents a positive definite (negative definite) matrix. $diag(\cdots)$ represents a block-diagonal matrix. The superscript $\top$ stands for

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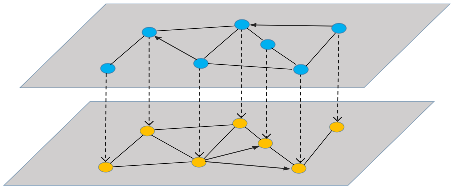

Fig. 1. A diagram of a duplex network, whose nodes are the same in the two layers. The inter-layer link is from each node in the upper layer (called drive layer) to its counterpart in the lower layer (called response layer). The solid and the dashed lines represent the intra-layer and the inter-layer links, respectively.

the transposition of vectors or matrices. $A^s = \frac{A^\top + A}{2}$ denotes the symmetrical part of matrix $A$ and $\otimes$ denotes the Kronecker product.

## II. Network Model and Preliminaries

Consider a directionally-coupled duplex network as shown in Fig. 1, in which each layer is consisted of $N$ identical nodes described by

$$\dot{x}_i(t) = \tilde{A}x_i(t) + \tilde{B}f(x_i(t-\tau_1(t))) + \sum_{j=1}^{N} c_{ij}^1 g(x_j(t)) + \sum_{j=1}^{N} d_{ij}^1 h(x_j(t-\tau_2(t))), \qquad i = 1, 2, \ldots, N, \quad (1)$$

$$\dot{y}_i(t) = \tilde{A}y_i(t) + \tilde{B}f(y_i(t-\tau_1(t))) + \sum_{j=1}^{N} c_{ij}^2 g(y_j(t)) + \sum_{j=1}^{N} d_{ij}^2 h(y_j(t-\tau_2(t))) + u_i(t), i = 1, 2, \ldots, N, \quad (2)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^\top \in \mathbb{R}^n$ and $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^\top \in \mathbb{R}^n$ are the state variables of the $i$th node in the drive and the response layer respectively. $f(\cdot) \in \mathbb{R}^n$ is a nonlinear vector valued function describing the dynamics of nodes with time-varying delay $\tau_1(t)$. $g(\cdot) \in \mathbb{R}^n$ and $h(\cdot) \in \mathbb{R}^n$ are the nonlinear coupling functions. The coupling delay $\tau_2(t)$ among different nodes is also time-varying. $\tilde{A} = A + \Delta A(t) \in \mathbb{R}^{n\times n}$ and $\tilde{B} = B + \Delta B(t) \in \mathbb{R}^{n\times n}$ stand for the parametric uncertainties. For $k = 1, 2$, $C^k \in R^{N\times N}$ and $D^k \in R^{N\times N}$ are the coupling configuration matrices which reflect the fundamental topology structure of each layer in the duplex network. If there is a connection between node $i$ and node $j (i \neq j)$, then $c_{ij}^k > 0$ and $d_{ij}^k > 0$; otherwise, $c_{ij}^k = 0$ and $d_{ij}^k = 0$. The diagonal elements of matrix $C^k$ and $D^k$ are defined as $c_{ii}^k = -\sum_{j=1, j\neq i}^{N} c_{ij}^k$ and $d_{ii}^k = -\sum_{j=1, j\neq i}^{N} d_{ij}^k$. $u_i(t) \in \mathbb{R}^n$ is the event-dependent intermittent controller to be designed. Furthermore, $C^k$ and $D^k$ are not necessarily assumed to be symmetric.

To obtain the main results, the following definitions, assumptions and lemmas are needed.

*Definition 1:* Let $e_i(t) = y_i(t) - x_i(t)$ be the inter-layer synchronization errors. The drive layer (1) and the response layer (2) are said to achieve finite time inter-layer synchronization if there exists a setting time $T^*$ such that $\lim_{t\to T^*} \|e_i(t)\| = 0$ and $\|e_i(t)\| = 0$ for all $t > T^*$.

*Definition 2:* For $\theta \in \mathbb{R}^n$, $r > 0$, the signal function $sign(\cdot)$ and $|\cdot|^r$ are defined as

$$sign(\theta) = diag(sign(\theta_1), sign(\theta_2), \ldots, sign(\theta_n)),$$

$$|\theta|^r = \left(|\theta_1|^r, |\theta_2|^r, \ldots, |\theta_n|^r\right)^\top.$$

*Assumption 1:* Time-varying delay $\tau_1(t)$ and $\tau_2(t)$ are bounded differential functions with $0 \leq \tau_1(t) \leq \tau_1^*$, $0 \leq \tau_2(t) \leq \tau_2^*$ and $\dot{\tau}_1(t) \leq \gamma < 1$, $\dot{\tau}_2(t) \leq \gamma < 1$ for all $t$, where $\tau_1^*$, $\tau_2^*$ and $\gamma$ are some positive constants.

*Assumption 2:* Nonlinear vector functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are Lipschitz functions, i.e., there exists positive constants $L_f$, $L_g$, $L_h$ such that for all $\mu, \nu \in R^n$,

$$\|f(\mu) - f(\nu)\| \leq L_f \|\mu - \nu\|,$$
$$\|g(\mu) - g(\nu)\| \leq L_g \|\mu - \nu\|,$$
$$\|h(\mu) - h(\nu)\| \leq L_h \|\mu - \nu\|.$$

*Assumption 3:* Uncertain parameter matrices $\Delta A(t)$ and $\Delta B(t)$ are assumed to be

$$[\Delta A(t), \Delta B(t)] = MF(t)[N_1, N_2],$$

where $M$, $N_1$ $N_2$ are known real matrices, $F(t)$ is an unknown time-varying real matrix satisfying $F^\top(t)F(t) \leq I$.

*Lemma 1 [15]:* If $X$ and $Y$ are real matrices, then the following inequality holds for any $\zeta > 0$

$$X^\top Y + Y^\top X \leq \zeta X^\top X + \frac{1}{\zeta} Y^\top Y.$$

*Lemma 2 [22]:* Let $z \in \mathbb{R}^n$, $0 < r < s$. The following norm equivalence property holds:

$$\left(\sum_{i=1}^{n} |z_i|^s\right)^{\frac{1}{s}} \leq \left(\sum_{i=1}^{n} |z_i|^r\right)^{\frac{1}{r}}.$$

*Lemma 3 [23]:* Assume that the positive Lyapunov function $V(x)$ is defined on a neighborhood $U$ of the origin, and

$$\dot{V}(x) \leq -pV^q(x),$$

where $p > 0$, $0 < q < 1$, then $V(x) \equiv 0$ for all $t \geq T(x_0)$ and the settling time $T(x_0) \leq \frac{V^{1-q}(x_0)}{p(1-q)}$.

## III. Main Results

In this section, we will give the event-dependent intermittent control scheme and the finite time synchronization criterion. According to system (1) and (2), the error state equations is written in the compact form

$$\dot{e}(t) = (I_N \otimes \tilde{A})e(t) + (I_N \otimes \tilde{B})F(e(\tau_1)) + (C \otimes I_n)G(e(t)) + (D \otimes I_n)H(e(\tau_2)) + u(t), \quad (3)$$

where $e(t) = (e_1^\top(t), e_2^\top(t), \ldots, e_N^\top(t))^\top$, $f(e_i(\tau_1)) = f(y_i(t - \tau_1(t))) - f(x_i(t - \tau_1(t)))$, $F(e(\tau_1)) = (f^\top(e_1(\tau_1)), f^\top(e_2(\tau_1)), \ldots, f^\top(e_N(\tau_1)))^\top$, $g(e_i(t)) = g(y_1(t)) - g(x_1(t))$, $G(e(t)) = (g^\top(e_1(t)), g^\top(e_2(t)), \ldots, g^\top(e_N(t)))^\top$, $h(e_i(\tau_2)) = h(y_i(t - \tau_2(t))) - h(x_i(t - \tau_2(t)))$, $H(e(\tau_2)) = (h^\top(e_1(\tau_2)), h^\top(e_2(\tau_2)), \ldots, h^\top(e_N(\tau_2)))^\top$, $C = C^2 - C^1$, $D = D^2 - D^1$ and $u(t) = (u_1^\top(t), u_2^\top(t), \ldots, u_N^\top(t))^\top$.

There usually exist serious constraints on work time or rest time in pervious time-dependent intermittent control papers, which is unrealistic in the applications. To reduce the control cost, the developed intermittent controller is designed as

$$u(t) = \begin{cases} u^*(t), & \text{if } V(t) \geq V_1(t), \\ u^*(t^-), & \text{if } V_2(t) < V(t) < V_1(t), \\ 0, & \text{if } V(t) \leq V_2(t), \end{cases} \quad (4)$$

$$u^*(t) = -\xi e(t) - \frac{\alpha}{2} sign(e(t))|e(t)|^\sigma$$

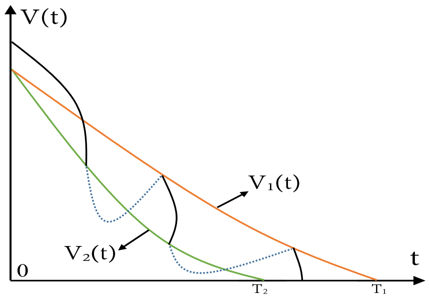


Fig. 2. Illustration of the proposed event-dependent intermittent mechanism. $T_1$ and $T_2$ represent the setting time of $V_1(t)$ and $V_2(t)$ respectively.

$$- \frac{\alpha}{2}\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t} e^{\top}(s)e(s)ds\right)^{\frac{1+\sigma}{2}} \frac{e(t)}{\|e(t)\|^2}$$
$$- \frac{\alpha}{2}\left(\frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t} e^{\top}(s)e(s)ds\right)^{\frac{1+\sigma}{2}} \frac{e(t)}{\|e(t)\|^2}, \quad (5)$$

where $\xi$, $\eta_1$ and $\eta_2$ are the positive control strengths, $\alpha$ and $\sigma$ are tunable positive constants with $0 \leq \sigma < 1$. $V(t)$ is a given Lyapunov function and $V_1(t)$, $V_2(t)$ are two non-negative continuous functions to be designed. $t^-$ indicates the previous instant of time $t$. From (4), it is easy to see that the work time of controller $u(t)$ is decided by the time-varying relation between $V(t)$ and $V_i(t)(i = 1, 2)$. In this sense, the developed intermittent control scheme can be classified into event-dependent intermittent control. For brevity, take $\beta = \frac{1+\sigma}{2}$. $V_i(t)$ is then given as

$$V_i(t) = \left((kV(t_0))^{1-\beta} - \alpha_i(1-\beta)(t-t_0)\right)^{\frac{1}{1-\beta}}, \quad i = 1, 2,$$

where $\alpha_i$ and $k$ are positive scalars satisfying $\alpha > \alpha_2 > \alpha_1$ and $0 < k < 1$, respectively.

Based on the relation between $V(t)$ and $V_i(t)$, the event-dependent intermittent mechanism is described as:

- If $V(t) \geq V_1(t)$, then the controller $u(t) = u^*(t)$ is active at instant $t$ and continuously working.
- If $V_2(t) < V(t) < V_1(t)$ and the controller $u(t) = u^*(t)$ is working at the instant $t^-$, then the controller $u(t) = u^*(t)$ is continuously working after the time $t$.
- If $V_2(t) < V(t) < V_1(t)$ and the controller $u(t) = 0$ is working at the instant $t^-$, then the controller $u(t) = 0$ is continuously working after the time $t$.
- If $V(t) \leq V_2(t)$, then the controller $u(t) = 0$ is active at instant $t$ and continuously working.

A graphical illustration of the proposed event-dependent intermittent mechanism is shown in Fig. 2.

*Remark 1:* The existence of the two boundary functions is necessary. Otherwise, suppose there is only one boundary function $V_1(t)$ in the control process, then the controller (4) is degenerated into

$$u(t) = \begin{cases} u^*(t), & \text{if } V(t) \geq V_1(t), \\ 0, & \text{if } V(t) < V_1(t). \end{cases}$$

In this case, the controller may switch rapidly between $u(t) = u^*(t)$ and $u(t) = 0$ with the trajectory of $V(t)$ touching the boundary function $V_1(t)$ from $V(t) \geq V_1(t)$ to $V(t) < V_1(t)$ or from $V(t) < V_1(t)$ to $V(t) \geq V_1(t)$, which usually gives rise to the chattering phenomenon and Zeno behavior. Compared with the methods in [8] and [10], our method can eliminate Zeno phenomenon by only introducing another boundary function $V_2(t)$, since the nonempty region $V_2(t) < V(t) < V_1(t)$ provides a dwell time when the trajectory of $V(t)$ reaches boundary function from $V(t) \geq V_1(t)$ to $V(t) < V_2(t)$ or from $V(t) < V_2(t)$ to $V(t) \geq V_1(t)$. If the scalars $\alpha_1$, $\alpha_2$, $\beta$ and $k$ are well defined, the size of region $V_2(t) < V(t) < V_1(t)$ that corresponds to the switching frequency of the controller between $u(t) = u^*(t)$ and $u(t) = 0$ will be larger. Generally, such switch frequency can be decreased by enlarging the values of $\alpha_2 - \alpha_1$.

*Remark 2:* The event-dependent intermittent control proposed is based on the Lyapunov function, where the designed controller contains two integral terms to deal with time-varying delay. Therefore, the designed Lyapunov function must also contain two integral terms related to time delay. Furthermore, it is necessary for the real-time calculation of the Lyapunov function to obtain its explicit analytical form, which limits the selection of the Lyapunov function.

Next, we give the finite time synchronization criterion with the Lyapunov function $V(t)$ designed as follows:

$$V(t) = e^{\top}(t)e(t) + \frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t} e^{\top}(s)e(s)ds$$
$$+ \frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t} e^{\top}(s)e(s)ds.$$

Then the finite time synchronization criterion is given.

*Theorem 1:* Suppose that Assumptions 1-3 hold, there exist tunable positive constants $\xi$, $\eta_1$, $\eta_2$ and any positive constants $s_1, \ldots, s_5$ such that:

1) $\Psi \leq 0$,
2) $s_2^{-1}L_f^2 + s_3^{-1}L_f^2\lambda_{max}(N_2^{\top}N_2) - \eta_1 \leq 0$,
3) $s_5^{-1}L_h^2 - \eta_2 \leq 0$,

where $\Psi = 2\lambda_{max}(A^s) + (s_1 + s_3)\lambda_{max}(MM^{\top}) + s_4^{-1}L_g^2 + s_1^{-1}\lambda_{max}(N_1^{\top}N_1) + s_2\lambda_{max}(BB^{\top}) + s_4\lambda_{max}(CC^{\top}) + s_5\lambda_{max}(DD^{\top}) + \frac{\eta_1+\eta_2}{1-\gamma} - 2\xi$. Then the subnetworks (1) and (2) with the event-dependent intermittent control scheme (4) and (5) will be synchronized in a finite time $T$, and $T < T_1 = \frac{(kV(t_0))^{1-\beta}}{\alpha_1(1-\beta)} + t_0$.

*Remark 3:* There are many results including [15]–[17] dealt with the finite time synchronization problem via intermittent control approach, in which the control time is time-dependent and determined artificially. The work time of controller in the proposed method, however, is no longer set in advance, and determined by the dynamic relation between the designed Lyapunov function and the two boundary functions. Besides, compared with the event-dependent intermittent control in [21], the proposed method has been improved to study the case of finite time synchronization from asymptotic synchronization.

*Proof:* Differentiating $V(t)$ along the solution of error system (3) gives

$$\dot{V}(t) \leq 2e^{\top}(t)\dot{e}(t) + \frac{\eta_1}{1-\gamma}e^{\top}(t)e(t) - \eta_1 e^{\top}(\tau_1)e(\tau_1)$$
$$+ \frac{\eta_2}{1-\gamma}e^{\top}(t)e(t) - \eta_2 e^{\top}(\tau_2)e(\tau_2).$$

Based on the error equation (3) and the control scheme (4) and (5), it is obtained that

$$\begin{aligned}2e^\top(t)\dot{e}(t) &= 2e^\top(t)(I_N\otimes\tilde{A})e(t)+2e^\top(t)(I_N\otimes\tilde{B})F(e(\tau_1))\\&\quad+2e^\top(t)(C\otimes I_n)G(e(t))+2e^\top(t)(D\otimes I_n)H(e(\tau_2))\\&\quad-2\xi e^\top(t)e(t)-\alpha e^\top(t)sign(e(t))|e(t)|^\sigma\\&\quad-\alpha\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e^\top(s)e(s)ds\right)^{\frac{1+\sigma}{2}}\\&\quad-\alpha\left(\frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t}e^\top(s)e(s)ds\right)^{\frac{1+\sigma}{2}}.\end{aligned}$$

Furthermore, by Lemma 1, one has the following estimation

$$\begin{aligned}&2e^\top(t)(I_N\otimes\tilde{A})e(t)+2e^\top(t)(I_N\otimes\tilde{B})F(e(\tau_1))\\&\le 2e^\top(t)(I_N\otimes A)e(t)+s_1e^\top(t)(I_N\otimes MM^\top)e(t)\\&\quad+s_1^{-1}e(t)(I_N\otimes N_1^\top N_1)e(t)+s_2e^\top(t)(I_N\otimes BB^\top)e(t)\\&\quad+s_2^{-1}F^\top(e(\tau_1))F(e(\tau_1))+s_3e^\top(t)(I_N\otimes MM^\top)e(t)\\&\quad+s_3^{-1}F^\top(e(\tau_1))(I_N\otimes N_2^\top N_2)F(e(\tau_1))\\&\le\Big(2\lambda_{max}(A^s)+s_1\lambda_{max}(MM^\top)+s_1^{-1}\lambda_{max}(N_1^\top N_1)\\&\qquad+s_2\lambda_{max}(BB^\top)+s_3\lambda_{max}(MM^\top)\Big)e^\top(t)e(t)\\&\quad+\Big(s_2^{-1}L_f^2+s_3^{-1}L_f^2\lambda_{max}(N_2^\top N_2)\Big)e^\top(\tau_1)e(\tau_1).\end{aligned}\tag{6}$$

Similarly, we have the estimation for another term:

$$\begin{aligned}&2e^\top(t)(C\otimes I_n)G(e(t))+2e^\top(t)(D\otimes I_n)H(e(\tau_2))\\&\le s_4e^\top(t)(CC^\top\otimes I_n)e(t)+s_4^{-1}G^\top(e(t))G(e(t))\\&s_5e^\top(t)(DD^\top\otimes I_n)e(t)+s_5^{-1}H^\top(e(\tau_2))H(e(\tau_2))\\&\le\Big(s_4\lambda_{max}(CC^\top)+s_4^{-1}L_g^2+s_5\lambda_{max}(DD^\top)\Big)e^\top(t)e(t)\\&\quad+s_5^{-1}L_h^2e^\top(\tau_2)e(\tau_2).\end{aligned}\tag{7}$$

It follows from Definition 2 and Lemma 2 that

$$-\alpha e^\top(t)sign(e(t))|e(t)|^\sigma=-\alpha|e^\top(t)||e(t)|^\sigma\le-\alpha\Big(e^\top(t)e(t)\Big)^{\frac{1+\sigma}{2}}.$$

Together with (6) and (7), it results in

$$\begin{aligned}\dot{V}(t)&\le\Psi e^\top(t)e(t)+\Big(s_5^{-1}L_h^2-\eta_2\Big)e^\top(\tau_2)e(\tau_2)\\&\quad+\Big(s_2^{-1}L_f^2+s_3^{-1}L_f^2\lambda_{max}(N_2^\top N_2)-\eta_1\Big)e^\top(\tau_1)e(\tau_1)\\&\quad-\alpha\Big(e^\top(t)e(t)\Big)^{\frac{1+\sigma}{2}}-\alpha\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e^\top(s)e(s)ds\right)^{\frac{1+\sigma}{2}}\\&\quad-\alpha\left(\frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t}e^\top(s)e(s)ds\right)^{\frac{1+\sigma}{2}}.\end{aligned}$$

Then, it follows from Lemma 2 that

$$\begin{aligned}\dot{V}(t)&\le-\alpha\Big(e^\top(t)e(t)\Big)^{\frac{1+\sigma}{2}}-\alpha\left(\frac{\eta_1}{1-\gamma}\int_{t-\tau_1(t)}^{t}e^\top(s)e(s)ds\right)^{\frac{1+\sigma}{2}}\\&\quad-\alpha\left(\frac{\eta_2}{1-\gamma}\int_{t-\tau_2(t)}^{t}e^\top(s)e(s)ds\right)^{\frac{1+\sigma}{2}}\le-\alpha V^\beta(t).\end{aligned}\tag{8}$$

Integrating (8) from $t_0$ to $t$, one gets

$$V(t)\le\Big((V(t_0))^{1-\beta}-\alpha(1-\beta)(t-t_0)\Big)^{\frac{1}{1-\beta}}.$$

Note that $0<k<1$, $V(t_0)>V_1(t_0)$. The controller $u(t)=u^*(t)$ is activated at $t_0$. Since $\alpha>\alpha_1$, the convergence rate of $V(t)$ is faster than that of the boundary function $V_1(t)$, and there exists an instant $t_0^*$ when the trajectory of $V(t)$ touches $V_1(t)$. Based on the proposed control mechanism, the controller $u(t)=u^*(t)$ is still active after $t_0^*$ and the trajectory of $V(t)$ will enter the region $V_2(t)<V(t)<V_1(t)$. Further, owing to $\alpha>\alpha_2$, there is an instant $t_1$ such that the trajectory of $V(t)$ touches the boundary $V_2(t)$, as well the controller $u(t)=0$ is activated at $t=t_1$. If there exists another instant $t_2$ such that $V(t)$ touches the boundary $V_1(t)$, the controller $u(t)=u^*(t)$ is activated again and the trajectory of $V(t)$ touches the boundary $V_2(t)$ and activates the controller $u(t)=0$ once again. Repeating the same operation, one gets the active instant series $t_{2m}$ and the stop instant series $t_{2m+1}$ determined by the relation between $V(t)$ and $V_i(t)$, where $m=0,1,2,\ldots,l$.

Based on the discussion, there may be an instant $t_l>T_2=t_0+\frac{(kV(t_0))^{1-\beta}}{\alpha_2(1-\beta)}$ such that the trajectory of $V(t)$ touches the boundary $V_1(t)$, i.e., $V(t_l)=V_1(t_l)$. For $t\in[t_l,T_1)$, $V(t)$ stays in the region $V(t)<V_1(t)$. Thus, the following inequality holds naturally:

$$V(t)\le V_1(t_l)=\Big((kV(0))^{1-\beta}-\alpha_1(1-\beta)(t_l-t_0)\Big)^{\frac{1}{1-\beta}},\quad t\ge t_l.$$

It yields directly $\|e(t)\|^2\le V(t)\le V_1(t),\ t\ge t_1$. Therefore, by Lemma 3, one obtains that the error system (3) is finite time stable with the setting time $T<T_1$, thus layer (1) synchronizes with layer (2). ■

*Remark 4:* If the controller $u(t)=u^*(t)$ works at the initial time, $V(t)$ will converge at a rate related to $\alpha$ and $\beta$. $V(t)$ may never touches the boundary function $V_1(t)$, even though $V(t)$ converges faster than $V_1(t)$. The reason lies in the fact that $T_0=t_0+\frac{V^{1-\beta}(t_0)}{\alpha(1-\beta)}>T_1=t_0+\frac{(kV(t_0))^{1-\beta}}{\alpha_1(1-\beta)}$ provided with $kV(t_0)<V(t_0)$. Actually, the intermittent mechanism does not occur in this case. The finite time synchronization is then achieved with the controller $u(t)=u^*(t)$ working continuously. This can be avoided if the boundary function $V_1(t)$ and $V_2(t)$ are appropriately designed.

## IV. Numerical Example

In this section, numerical examples are provided to demonstrate the effectiveness of our theoretical results. Consider the duplex network composed of four nodes. Nonlinear vector valued function $f(\cdot)$ in both of the subnetworks is described by $f(z(t))=(|z_1(t)+1|-|z_1(t)-1|,0,0)^\top$, the nonlinear inner connecting functions and the time-varying delays are taken as $g(\cdot)=h(\cdot)=tanh(\cdot)$ and $\tau_1(t)=\tau_2(t)=\frac{e^t}{1+e^t}$ respectively. Clearly, $L_f=2$, $L_h=L_g=1$, $\gamma=0.25$. Take

$$A=\begin{bmatrix}-0.1&-0.3&0\\0.2&-0.2&0.1\\0&0.1&0\end{bmatrix},B=\begin{bmatrix}-0.3&0.1&0.2\\0&-0.2&0.1\\-0.2&0&0.2\end{bmatrix},$$

$$C^1=\begin{bmatrix}-0.1&0.1&0&0\\0&-0.2&0.2&0\\0.1&0&-0.1&0\\0&0.1&0&-0.1\end{bmatrix},C^2=\begin{bmatrix}-0.3&0.1&0.1&0.1\\0.1&-0.5&0.2&0.2\\0.2&0&-0.2&0\\0&0.3&0&-0.3\end{bmatrix},$$

$$D^1=\begin{bmatrix}-0.2&0.1&0&0.1\\0.1&-0.2&0&0.1\\0.1&0&-0.2&0.1\\0&0.1&0&-0.1\end{bmatrix},D^2=\begin{bmatrix}-0.5&0.1&0.3&0.1\\0.4&-0.6&0.1&0.1\\0.1&0&-0.3&0.2\\0.2&0.3&0&-0.5\end{bmatrix}.$$

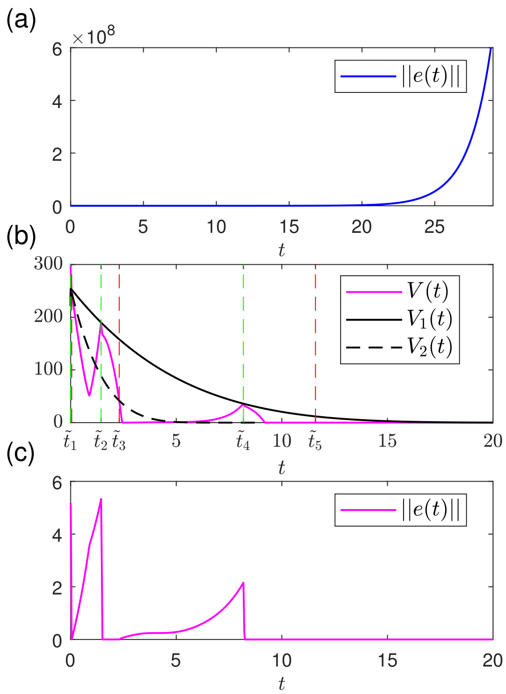


Fig. 3. (a). Evolution of synchronization error $\|e(t)\|$ in the absence of control. (b). Evolution of Lyapunov function $V(t)$. The green dashed line and red dashed line represent the instant when the intermittent controller is active and stopped respectively. Besides, $\tilde{t}_1 = 0.051$, $\tilde{t}_2 = 1.442$, $\tilde{t}_3 = 2.301$, $\tilde{t}_4 = 8.180$, $\tilde{t}_5 = 11.592$. It should be noted that $\tilde{t}_0 = 0$ is the first active instant of the controller. (c). Evolution of synchronization error $\|e(t)\|$ under event-dependent intermittent control.

It is clear that Assumption 3 holds for $M = N_1 = N_2 = I$, $F(t) = diag(0.8, sin(t), cos(t))$. Following the presented design scheme, one takes $s_i = 1 (i = 1, \ldots, 5)$ and other tunable parameters as $\xi = 10$, $\eta_1 = 8$, $\eta_2 = 1$, $\sigma = 0.6$, $\alpha = 5$, $\alpha_1 = 0.6$, $\alpha_2 = 2.5$ and $k = 0.9$. By computation, $\beta = 0.8$ and the conditions in Theorem 1 are verified. Moreover, the initial values are set as $x(t_0) = (0.04i, 0.02i, 0.03i)^\top$, $y(t_0) = (-0.2 - 0.2i, -0.5 - 0.3i, 0.8 - 0.2i)^\top$, $(i = 1, \ldots, 4)$. It is shown in Fig. 3. (a) that the synchronization error $\|e(t)\|$ diverges when the network is in the absence of control. Fig. 3. (b) and Fig. 3. (c) display the evolution of the Lyapunov function $V(t)$ and the synchronization error $\|e(t)\|$, respectively. Obviously, $\|e(t)\|$ approaches to zero with setting time $T = 11.592$, i.e., the response layer (2) and the drive layer (1) achieve finite time inter-layer synchronization finally. Besides, the whole work time of the proposed control is $T' = \tilde{t}_5 - \tilde{t}_4 + \tilde{t}_3 - \tilde{t}_2 + \tilde{t}_1 - \tilde{t}_0 = 4.322$.

## V. Conclusion

Finite time synchronization of duplex networks via a new type of intermittent control has been investigated in this brief. Compared with similar work, the work time or the rest time in our intermittent control is unnecessarily preset, and determined by the relation between the designed Lyapunov function $V(t)$ and the boundary functions $V_1(t)$ and $V_2(t)$. A criterion has been proposed to guarantee the synchronization within a finite time $T$. Numerical simulations show that the control scheme in this brief is of good performance. Future works include other intermittent control schemes and network models with time-varying topology.

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