Communications in Nonlinear Science and Numerical Simulation 126 (2023) 107504

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Short communication

Identifying partial topology of complex dynamical networks with distributed delay

Shuaibing Zhu [a,b], Xueyi Zhao [b,c], Na Li [d], Jin Zhou [e,*], Jun-An Lu [e]

[a] MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, China
[b] Hubei Key Laboratory of Applied Mathematics, Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, China
[c] School of Mathematics and Computer Science, Hanjiang Normal University, Shiyan 442000, China
[d] School of Mathematics and Statistics, Henan University, Kaifeng 475004, China
[e] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

A R T I C L E I N F O

Article history:
Received 25 April 2023
Received in revised form 27 July 2023
Accepted 15 August 2023
Available online 18 August 2023

Keywords:
Complex network
Partial topology identification
Distributed delay
Synchronization

A B S T R A C T

Though the discrete delay has been extensively studied in the field of network topology identification, little attention has been paid to the distributed delay. In this paper, we address the partial topology identification problem of complex dynamical networks with distributed delay. By constructing a response network having fewer nodes than the drive network, partial topology of networks with distributed delay can be identified under the widely-used linear independence condition. It is the first time that the distributed delay has been successfully addressed in the field of topology identification. Finally, we provide two numerical examples to verify the effectiveness of the theoretical results.

1. Introduction

Complex networks are usually used to describe large-scale interconnected systems in the real world, including protein networks, biological neural networks, social networks, and citation networks. Since the proposal of the small-world network model [1] and the scale-free network model [2], there has been a surge of interest in the study of complex networks [3–11].

At present, most research on complex networks focus on the condition that the network topology is precisely known, discussing the synchronization conditions [12–14], region-of-attraction estimation [15], disease transmission [16], information processing [17], etc. However, in the real world and engineering practice, the topology of a network is often unknown, or changes suddenly at some time. For example, to study the operation of the cerebral motor cortex in the learning process, it is necessary to detect and analyze the topology of the neural network and its real-time changes. In the Internet of Things, a sensor may lose communication with the other sensors due to power failure or other reasons, resulting in sudden changes in the topology of the sensor network. In addition to neural networks and the Internet of Things, such problems also widely exist in power networks, social networks, information networks, and other practical networks. Therefore, the topology identification problem of complex networks has both theoretical and practical significance.

In the past two decades, considerable effort has been devoted to the synchronization-based topology identification of complex networks. In 2007, Zhou et al. [18] studied the topology identification problem of weighted networks for the first time via an adaptive feedback control method. Later, the topology identification problem of various network

* Corresponding author.
E-mail address: jzhou@whu.edu.cn (J. Zhou).

https://doi.org/10.1016/j.cnsns.2023.107504
1007-5704/

models has been studied, including networks with coupling delay [19,20], networks with unknown node parameter [21], multi-link networks [22], multiplex delayed networks [23], etc. It is noteworthy that the aforementioned studies identified the whole topology of networks by constructing a response network having the same number of nodes with the drive network (network to be identified). Sometimes, we are interested in the connections concerning only several important nodes. That is, we are only interested in partial topology of the drive network. In 2018, Zhu et al. [24] proposed a method of identifying partial network topology, where the response network has fewer nodes than the drive network. Afterwards, the partial topology identification problem for fractional-order networks [25] and delayed networks with stochastic perturbations [26] was studied.

In general, time delay is not negligible when dealing with large-scale networks as the speed of signals is finite. The delay can be classified into two basic types: *discrete* and *distributed*. In the field of topology identification, the discrete delay has been well explored [19–21], while the distributed delay draws relatively less attention. In fact, the distributed delay exists since a network may have many parallel pathways with different axon sizes and length [27–29]. In [30], the topology identification problem of networks with distributed delay was considered. As to the results in [30], there are two problems worthy of highlight. First, the constructed response network in [30] is of the same dimension with the drive network, thus it is not cost-effective when only partial topology needs to be identified. Second, the criteria established therein relies on a condition expressed in terms of LMI. It is found that the network topology must be known *a priori* when solving the LMI, but the topology is to be identified and should be unknown. Hence, the topology identification problem of networks with distributed delay has not yet really been solved, though it has been studied.

Enlightened by the above discussions, this paper aims to identify partial topology of complex dynamical networks with distributed delay. The main contributions are as follows: (1) By constructing a response network having fewer nodes than the drive network, partial topology of networks with distributed delay can be identified. (2) It is the first time that the distributed delay has been successfully addressed in the field of topology identification.

Notations: Denote the n-dimensional Euclidean space and the set of all the $(n\times m)$-dimensional real matrices as $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively. $\otimes$ represents the Kronecker product. For a vector or a matrix Ψ, Ψ^T and $\|\Psi\|$ represent the transpose and the 2-norm of Ψ, respectively.

2. Preliminaries

2.1. Problem formulation

Consider a complex dynamical network of N nodes with distributed coupling delay:

$$\begin{cases} \dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j=1}^{N} a_{ij}\int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta, \ \ t\geq 0 \\ x_i(t) = \phi_i(t), \qquad t\in[-\tau,0] \end{cases} \tag{1}$$

where $1\leq i\leq N,\ x_i\in\mathbb{R}^n$ represents the state vector of the ith node, function $f_i:\mathbb{R}^n\to\mathbb{R}^n$ describes the dynamics of the ith node, $g:\mathbb{R}^n\to\mathbb{R}^n$ stands for the inner coupling function, $c>0$ is the coupling strength, $\tau>0$ denotes the constant distributed coupling delay, $A=(a_{ij})_{N\times N}$ represents the outer coupling matrix, and function $\phi_i:[-\tau,0]\to\mathbb{R}^n$ describes the initial condition for the ith node. If there is an edge from node j to node i ($i\neq j$), then $a_{ij}>0$; otherwise, $a_{ij}=0$; the diagonal entries of A are defined by $a_{ii}=-\sum_{j=1,j\neq i}^{N}a_{ij}$.

Hereafter, network (1) is referred to as the drive network, and the states $x_1,\dots,x_N$ of network (1) are assumed to be bounded. The topology A describes the relationship among all the nodes of the drive network. In reality, we may be interested in the relationship between some nodes and their neighbors. Without loss of generality, suppose that these nodes are the first r nodes of the drive network, where $1\leq r\leq N$. Then, the main aim of this paper is to identify partial topology $A_{r\times N}=(a_{ij})_{r\times N}$. Particularly, the whole topology A is to be identified when $r=N$.

In order to realize partial topology identification, construct a response network of r nodes:

$$\begin{cases} \dot{y}_i(t) = f_i(y_i(t)) + c\sum_{j=1}^{r}\hat{a}_{ij}(t)\int_{t-\tau}^{t} g(y_j(\theta))\mathrm{d}\theta \\ \qquad + c\sum_{j=r+1}^{N}\hat{a}_{ij}(t)\int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta + u_i(t), \ \ t\geq 0 \\ y_i(t) = \psi_i(t), \qquad t\in[-\tau,0] \end{cases} \tag{2}$$

where $1\leq i\leq r$, $y_i\in\mathbb{R}^n$ is state vector of the ith node of the response network, u_i is the control to be designed, $\hat{a}_{ij}(t)$ is the estimation for a_{ij}. Then, the topology identification is achieved if one has $\lim_{t\to+\infty}\hat{a}_{ij}(t)=a_{ij}$, where $1\leq i\leq r$ and $1\leq j\leq N$.

2.2. Mathematical preliminaries

To proceed, we need the following assumptions and lemmas.

Assumption 1 (*A1*)**.** The functions f_i and g are Lipschitz continuous, i.e., there exists positive constants ζ_i and η such that

$$\|f_i(x) - f_i(y)\| \le \zeta_i \|x - y\|$$
$$\|g(x) - g(y)\| \le \eta \|x - y\|,$$

where $x, y \in \mathbb{R}^n$ and $1 \le i \le N$.

Assumption 2 (*A2*)**.** $\{\int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta\}_{i=1}^{N}$ are linearly independent on the orbit $\{x_i(t)\}_{i=1}^{N}$ of the outer synchronization manifold $\{x_i(t) = y_i(t)\}_{i=1}^{r}$.

Assumption 3 (*A3*)**.** $\{g(x_j(t)) - g(x_j(t-\tau))\}_{i=1}^{N}$ are linearly independent on the orbit $\{x_i(t)\}_{i=1}^{N}$ of the outer synchronization manifold $\{x_i(t) = y_i(t)\}_{i=1}^{r}$.

Lemma 1 (*Barbălat Lemma [31, Lemma 8.2]*)**.** *Let $h : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $[0, +\infty)$. Then, $\lim_{t\to+\infty} h(t) = 0$ if $\int_0^{+\infty} h(\omega)\mathrm{d}\omega$ converges.*

Lemma 2 (*[20]*)**.** *Let Ξ be of the form $(\alpha, +\infty)$ or $[\alpha, +\infty)$, where α is a constant or $-\infty$. Let m be a positive integer, and*

$$\dot{a}(t) = h_1(t) + h_2(t), \tag{3}$$

where $a, h_2 : \Xi \to \mathbb{R}^m$ are differentiable functions and $h_1 : \Xi \to \mathbb{R}^m$ is a function. If $\lim_{t\to+\infty} a(t)$ exists, $\lim_{t\to+\infty} h_1(t) = 0$, and $\dot{h}_2$ is bounded, then $\lim_{t\to+\infty} \dot{a}(t) = 0$.

3. Main results

It is now ready to present the results on how to achieve partial topology identification of drive network (1) via response network (2).

Define the synchronization error $e_i(t) = y_i(t) - x_i(t)$ and the estimation error $\tilde{a}_{ij}(t) = \hat{a}_{ij}(t) - a_{ij}$, and denote

$$e(t) = [e_1^T(t), e_2^T(t), \ldots, e_r^T(t)]^T.$$

Then, one obtains from (1) and (2) the following error system:

$$\begin{aligned} \dot{e}_i(t) =& \bar{f}_i(e_i(t)) + c\sum_{j=1}^{r} a_{ij}\int_{t-\tau}^{t} \bar{g}(e_j(\theta))\mathrm{d}\theta + c\sum_{j=1}^{r} \tilde{a}_{ij}(t)\int_{t-\tau}^{t} g(y_j(\theta))\mathrm{d}\theta \\ &+ c\sum_{j=r+1}^{N} \tilde{a}_{ij}(t)\int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta + u_i(t) \end{aligned} \tag{4}$$

where $1 \le i \le r$, $\bar{f}_i(e_i(t)) = f_i(y_i(t)) - f_i(x_i(t))$, and $\bar{g}(e_i(t)) = g(y_i(t)) - g(x_i(t))$.

Theorem 1. *Under Assumptions (A1) and (A2), the partial topology $A_{r\times N}$ of drive network (1) can be estimated via response network (2) if the control $u_i(t)$ and the estimation $\hat{a}_{ij}(t)$ are designed as*

$$\begin{cases} u_i(t) = -d_i(t)e_i(t) \\ \dot{d}_i(t) = \alpha_i e_i^T(t)e_i(t) \\ \dot{\hat{a}}_{ij}(t) = -\beta_{ij}e_i^T(t)\int_{t-\tau}^{t} g(y_j(\theta))\mathrm{d}\theta, \ \ 1 \le j \le r \\ \dot{\hat{a}}_{ij}(t) = -\beta_{ij}e_i^T(t)\int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta, \ \ r < j \le N \end{cases} \tag{5}$$

that is,

$$\lim_{t\to+\infty} \hat{a}_{ij}(t) = a_{ij}, \tag{6}$$

where $1 \le i \le r$, $1 \le j \le N$, and α_i, β_{ij} can be any positive constants.

Proof. Consider the function $V(t) = V_1(t) + V_2(t)$, where

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{r} e_i^T(t)e_i(t) + \sum_{i=1}^{r}\sum_{j=1}^{N}\frac{c}{2\beta_{ij}}\tilde{a}_{ij}^2(t) + \sum_{i=1}^{r}\frac{(d_i(t)-d^*)^2}{2\alpha_i}$$

$$V_2(t) = \gamma\sum_{i=1}^{r}\int_0^{\tau} \mathrm{d}s\int_{t-s}^{t} e_i^T(\theta)e_i(\theta)\mathrm{d}\theta$$

with d^* and γ being positive constants to be determined. Then, differentiating $V_1(t)$ along the solution of (4) gives

$$\begin{aligned}
\dot{V}_1(t) &= \sum_{i=1}^{r} e_i^T(t)\dot{e}_i(t) + \sum_{i=1}^{r}\sum_{j=1}^{N}\frac{c}{\beta_{ij}}\tilde{a}_{ij}(t)\dot{\tilde{a}}_{ij}(t) + \sum_{i=1}^{r}\frac{(d_i(t)-d^*)\dot{d}_i(t)}{\alpha_i} \\
&= \sum_{i=1}^{r} e_i^T(t)\bar{f}_i(e_i(t)) + c\sum_{i=1}^{r}\sum_{j=1}^{r} a_{ij}e_i^T(t)\int_{t-\tau}^{t}\bar{g}(e_j(\theta))\mathrm{d}\theta \\
&\quad + c\sum_{i=1}^{r}\sum_{j=1}^{r}\tilde{a}_{ij}(t)e_i^T(t)\int_{t-\tau}^{t} g(y_j(\theta))\mathrm{d}\theta \\
&\quad + c\sum_{i=1}^{r}\sum_{j=r+1}^{N}\tilde{a}_{ij}(t)e_i^T(t)\int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta - \sum_{i=1}^{r} d_i(t)e_i^T(t)e_i(t) \\
&\quad + \sum_{i=1}^{r}\sum_{j=1}^{r}\frac{c}{\beta_{ij}}\tilde{a}_{ij}(t)\left(-\beta_{ij}e_i^T(t)\int_{t-\tau}^{t} g(y_j(\theta))\mathrm{d}\theta\right) \\
&\quad + \sum_{i=1}^{r}\sum_{j=r+1}^{N}\frac{c}{\beta_{ij}}\tilde{a}_{ij}(t)\left(-\beta_{ij}e_i^T(t)\int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta\right) \\
&\quad + \sum_{i=1}^{r}\frac{d_i(t)-d^*}{\alpha_i}\alpha_i e_i^T(t)e_i(t) \\
&= \sum_{i=1}^{r} e_i^T(t)\bar{f}_i(e_i(t)) + c\sum_{i=1}^{r}\sum_{j=1}^{r} a_{ij}e_i^T(t)\int_{t-\tau}^{t}\bar{g}(e_i(\theta))\mathrm{d}\theta - d^*\sum_{i=1}^{r} e_i^T(t)e_i(t).
\end{aligned} \tag{7}$$

According to Assumption (A1), one gets

$$\sum_{i=1}^{r} e_i^T(t)\bar{f}_i(e_i(t)) \le \sum_{i=1}^{r}\zeta_i e_i^T(t)e_i(t) \le \zeta^*\|e(t)\|^2 \tag{8}$$

with $\zeta^* = \max\{\zeta_1, \zeta_2, \dots, \zeta_r\}$. Denoting

$$A_r = (a_{ij})_{r\times r}$$
$$G(e(t)) = [\bar{g}(e_1(t))^T,\ \bar{g}(e_2(t))^T,\ \dots,\ \bar{g}(e_r(t))^T]^T,$$

one has

$$\begin{aligned}
c\sum_{i=1}^{r}\sum_{j=1}^{r} a_{ij}e_i^T(t)\int_{t-\tau}^{t}\bar{g}(e_i(\theta))\mathrm{d}\theta &= ce^T(t)(A_r\otimes I_n)\int_{t-\tau}^{t} G(e(\theta))\mathrm{d}\theta \\
&\le \frac{1}{4}c^2\|e^T(t)(A_r\otimes I_n)\|^2 + \left\|\int_{t-\tau}^{t} G(e(\theta))\mathrm{d}\theta\right\|^2 \\
&\le \frac{1}{4}c^2\|A_r\|^2\|e(t)\|^2 + \left\|\int_{t-\tau}^{t} G(e(\theta))\mathrm{d}\theta\right\|^2.
\end{aligned} \tag{9}$$

Recalling Assumption (A1), it is derived that

$$\|\bar{g}(e_i(t))\| \le \eta\|e_i(t)\|,\ \ 1 \le i \le r$$

which further yields $\|G(e(t))\| \le \eta\|e(t)\|$. It then follows that

$$\left\|\int_{t-\tau}^{t} G(e(\theta))\mathrm{d}\theta\right\|^2 \le \tau\int_{t-\tau}^{t}\|G(e(\theta))\|^2\mathrm{d}\theta \le \tau\eta^2\int_{t-\tau}^{t}\|e(\theta)\|^2\mathrm{d}\theta. \tag{10}$$

Combining (7)–(10), one deduces that

$$\dot{V}_1(t) \leq (\zeta^* + \frac{1}{4}c^2\|A_r\|^2 - d^*)\|e(t)\|^2 + \tau\eta^2 \int_{t-\tau}^{t} \|e(\theta)\|^2 \mathrm{d}\theta. \tag{11}$$

Differentiating $V_2(t)$ yields

$$\begin{aligned}\dot{V}_2(t) =& \gamma \sum_{i=1}^{r} \int_0^{\tau} [e_i^T(t)e_i(t) - e_i^T(t-s)e_i(t-s)]\mathrm{d}s \\ =& \gamma\tau\|e(t)\|^2 - \gamma \int_0^{\tau} \|e(t-s)\|^2 \mathrm{d}s \\ =& \gamma\tau\|e(t)\|^2 - \gamma \int_{t-\tau}^{t} \|e(\theta)\|^2 \mathrm{d}\theta.\end{aligned} \tag{12}$$

Then, one has

$$\dot{V}(t) \leq (\zeta^* + \frac{1}{4}c^2\|A_r\|^2 + \gamma\tau - d^*)\|e(t)\|^2 + (\tau\eta^2 - \gamma)\int_{t-\tau}^{t} \|e(\theta)\|^2 \mathrm{d}\theta. \tag{13}$$

Taking $\gamma = \tau\eta^2$ and $d^* = \zeta^* + \frac{1}{4}c^2\|A_r\|^2 + \gamma\tau + 1$ leads to

$$\dot{V}(t) \leq -\|e(t)\|^2, \tag{14}$$

which further gives $V(t) \leq V(0)$. That is, $e_i(t), \tilde{a}_{ij}(t), d_i(t)$ are bounded for $1 \leq i \leq r$ and $1 \leq j \leq N$.

Since $d_i(t)$ is bounded and nondecreasing, the limit $d_i(+\infty)$ exists and is finite. By (5), it is clear that

$$\int_0^{+\infty} \|e(\theta)\|^2 \mathrm{d}\theta = \sum_{i=1}^{r} \frac{d_i(+\infty) - d_i(0)}{\alpha_i},$$

which indicates that $\int_0^{+\infty} \|e(\theta)\|^2 \mathrm{d}\theta$ is finite. Since $x_i(t), e_i(t), \tilde{a}_{ij}(t)$ are bounded, it is derived that $y_i(t) = x_i(t) + e_i(t)$ and $\hat{a}_{ij}(t) = \tilde{a}_{ij}(t) + a_{ij}$ are bounded. The Lipschitz continuity of g accounts for the boundedness of $g(x_i(t))$ and $g(y_i(t))$. According to error system (4), $\dot{e}_i(t)$ is bounded. Then, it is easy to prove that the derivative of $\|e(t)\|^2$ is bounded, thus $\|e(t)\|^2$ is uniformly continuous. In light of Lemma 1, one has

$$\lim_{t\to+\infty} e(t) = 0. \tag{15}$$

Rewrite (4) as $\dot{e}_i(t) = h_{i1}(t) + h_{i2}(t)$, where

$$\begin{aligned}h_{i1}(t) =& \bar{f}_i(e_i(t)) + c\sum_{j=1}^{r} a_{ij} \int_{t-\tau}^{t} \bar{g}(e_j(\theta))\mathrm{d}\theta + u_i(t) \\ h_{i2}(t) =& c\sum_{j=1}^{r} \tilde{a}_{ij}(t) \int_{t-\tau}^{t} g(y_j(\theta))\mathrm{d}\theta + c\sum_{j=r+1}^{N} \tilde{a}_{ij}(t) \int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta \\ \triangleq& h_{i3}(t) + h_{i4}(t).\end{aligned}$$

Since $\|\bar{f}_i(e_i(t))\| \leq \zeta_i\|e_i(t)\|$ and $\|\bar{g}(e_i(t))\| \leq \eta\|e_i(t)\|$, one has $\lim_{t\to+\infty} \bar{f}_i(e_i(t)) = 0$ and $\lim_{t\to+\infty} \bar{g}(e_i(t)) = 0$. Together with $\lim_{t\to+\infty} u_i(t) = 0$, it suffices to derive that

$$\lim_{t\to+\infty} h_{i1}(t) = 0.$$

Differentiating $h_{i3}(t)$ gives

$$\dot{h}_{i3}(t) = c\sum_{j=1}^{r} \dot{\hat{a}}_{ij}(t) \int_{t-\tau}^{t} g(y_j(\theta))\mathrm{d}\theta + c\sum_{j=1}^{r} \tilde{a}_{ij}(t)[g(y_j(t)) - g(y_j(t-\tau))]. \tag{16}$$

According to (5), one gets $\lim_{t\to+\infty} \dot{\hat{a}}_{ij}(t) = 0$. Noticing that the boundedness of $\tilde{a}_{ij}(t)$ and $g(y_j(t))$ has been proved, it can be concluded that $\dot{h}_{i3}(t)$ is bounded. Similarly, $\dot{h}_{i4}(t)$ is also bounded, implying that $\dot{h}_{i2}(t)$ is bounded. In view of Lemma 2, one has

$$\lim_{t\to+\infty} \dot{e}_i(t) = 0, \ \ 1 \leq i \leq r. \tag{17}$$

Therefore,

$$\lim_{t\to+\infty} h_{i2}(t) = \lim_{t\to+\infty} \dot{e}_i(t) - \lim_{t\to+\infty} h_{i1}(t) = 0, \ \ 1 \leq i \leq r. \tag{18}$$

Recalling that $\lim_{t\to+\infty} \bar{g}(e_i(t)) = 0$, it follows from (18) that

$$\lim_{t\to+\infty} \sum_{j=1}^{N} \tilde{a}_{ij}(t) \int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta = 0, \ \ 1 \le i \le r. \tag{19}$$

By using Assumption (A2), it is derived that $\lim_{t\to+\infty} \tilde{a}_{ij}(t) = 0$. That is, $\lim_{t\to+\infty} \hat{a}_{ij}(t) = a_{ij}$. $\square$

Remark 1. In Theorem 1, a scheme for partial topology identification of networks with distributed delay is proposed, which is the first time that the distributed delay has been successfully addressed in the field of network topology identification. Compared to much of the literature wherein the proposed methods identify the whole topology of networks, the scheme proposed here is more cost-effective as there are less nodes in the response network. Particularly, the whole topology of network (1) can be identified when $r = N$.

Remark 2. Though the discrete delay has been widely studied in the literature [19–23,26], little attention has been paid to the distributed delay. In [30], the topology identification problem of networks with distributed delay was considered. However, there are two problems worthy of highlight. First, the response network in [30] is of the same dimension with the drive network, thus it is not cost-effective when only partial topology needs to be identified. Second, the criteria established therein relies on a condition expressed in terms of LMI. It is found that the topology A should be known *a priori* when solving the LMI, but A is in fact unknown and to be identified. Hence, the topology identification problem of networks with distributed delay has not been successfully solved until now, though it has been studied. In Theorem 1, the distributed delay has been successfully addressed for the first time in the field of topology identification.

Remark 3. In the literature [21,23–26,32–34], the linear independence condition (LIC) has been extensively used to ensure the success of topology identification. Here, the LIC is expressed as Assumption (A2) to address the distributed delay. Despite the importance of the LIC, there is still no effective method to verify it until now.

To deal with the distributed delay, we have to compute the integral $\int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta$ in Assumption (A2), which seems to be troublesome. Next, another scheme for topology identification is to be proposed by using Assumption (A3), which is simpler than (A2).

Theorem 2. *Under Assumptions (A1) and (A3), the partial topology $A_{r\times N}$ of drive network (1) can be estimated via response network (2) if the control $u_i(t)$ and the estimation $\hat{a}_{ij}(t)$ are designed as (5). That is,*

$$\lim_{t\to+\infty} \hat{a}_{ij}(t) = a_{ij}, \tag{20}$$

where $1 \le i \le r$ and $1 \le j \le N$.

Proof. Denoting

$$\omega_i(t) = \sum_{j=1}^{N} \tilde{a}_{ij}(t) \int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta, \ \ 1 \le i \le r$$

it follows from (19) that $\lim_{t\to+\infty} \omega_i(t) = 0$. Similar to proving the boundedness of $\dot{h}_{i3}(t)$ in (16), it can be proved that $\dot{\omega}_i(t)$ is bounded, which further implies that $\omega_i(t)$ is uniformly continuous. Then, applying Lemma 1 yields

$$\lim_{t\to+\infty} \dot{\omega}_i(t) = 0, \ \ 1 \le i \le r.$$

By simple calculation, one has

$$\dot{\omega}_i(t) = c \sum_{j=1}^{N} \dot{\hat{a}}_{ij}(t) \int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta + c \sum_{j=1}^{N} \tilde{a}_{ij}(t)[g(x_j(t)) - g(x_j(t-\tau))].$$

It has been proved in Theorem 1 that $\lim_{t\to+\infty} \dot{\hat{a}}_{ij}(t) = 0$ and that $g(x_j(t))$ is bounded. Then,

$$\lim_{t\to+\infty} \sum_{j=1}^{N} \tilde{a}_{ij}(t)[g(x_j(t)) - g(x_j(t-\tau))] = 0, \ \ 1 \le i \le r.$$

From Assumption (A3), it is derived that $\lim_{t\to+\infty} \tilde{a}_{ij}(t) = 0$, i.e., $\lim_{t\to+\infty} \hat{a}_{ij}(t) = a_{ij}$. $\square$

In most existing studies, g is taken as a linear function in the form of $g(x(t)) = \Phi x(t)$, where $\Phi \in \mathbb{R}^{n\times n}$ is a constant matrix. If $g(x(t)) = \Phi x(t)$, we have the following corollary.

Corollary 1. *Let f_i be Lipschitz continuous and let Assumption (A3) hold. If the inner coupling function $g(x(t)) = \Phi x(t)$, then the partial topology $A_{r\times N}$ of drive network (1) can be estimated via response network (2) if the control $u_i(t)$ and the estimation $\hat{a}_{ij}(t)$ are designed as*

$$\begin{cases} u_i(t) = -d_i(t)e_i(t) \\ \dot{d}_i(t) = \alpha_i e_i^T(t)e_i(t) \\ \dot{\hat{a}}_{ij}(t) = -\beta_{ij}e_i^T(t)\Phi\int_{t-\tau}^{t} y_j(\theta)\mathrm{d}\theta, \ 1 \le j \le r \\ \dot{\hat{a}}_{ij}(t) = -\beta_{ij}e_i^T(t)\Phi\int_{t-\tau}^{t} x_j(\theta)\mathrm{d}\theta, \ r < j \le N \end{cases} \tag{21}$$

that is,

$$\lim_{t\to+\infty} \hat{a}_{ij}(t) = a_{ij}, \tag{22}$$

where $1 \le i \le r$, $1 \le j \le N$, and α_i, β_{ij} can be any positive constants.

4. Numerical simulations

The well-known chaotic Lorenz system [35] is describe by

$$\dot{s}(t) = f(s(t)) = Hs(t) + W(t), \tag{23}$$

where

$$H = \begin{bmatrix} -c_1 & c_1 & 0 \\ c_3 & -1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix}, \ W(t) = \begin{bmatrix} 0 \\ -s_1(t)s_3(t) \\ s_1(t)s_2(t) \end{bmatrix}$$

with $c_1 = 10$, $c_2 = 8/3$, $c_3 = 28$. Add perturbation

$$w_i(s(t)) = \rho[\cos(is_1(t)), \cos(is_2(t)), \cos(is_3(t))]^T, \ i \ge 1$$

to the Lorenz system, where $\rho > 0$ represents the strength of perturbation. Then, a class of nonidentical node systems are generated, described by

$$\dot{s}_i(t) = f_i(s_i(t)) = f(s_i(t)) + w_i(s_i(t)), \ i \ge 1. \tag{24}$$

The inner coupling function is taken as

$$g(z(t)) = [z_1(t) + \sin z_2(t), z_2(t), z_3(t)]^T. \tag{25}$$

Since the Lorenz system is ultimately bounded [18], it can be easily verified that f is Lipschitz continuous. Considering that w_i is Lipschitz continuous, it is concluded that f_i is Lipschitz continuous. It is clear that g is also Lipschitz continuous, thus Assumption (A1) is satisfied. In addition, the linear independence condition expressed in Assumption (A2) and (A3) naturally holds for many networks with nonidentical nodes [21].

Next, we present two examples to demonstrate the effectiveness of our theoretical results. For simplicity, we take $r = 3$ in both the two examples.

Example 1. Consider a network of $N = 6$ nonidentical nodes:

$$\dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j=1}^{6} a_{ij}\int_{t-\tau}^{t} g(x_j(\theta))\mathrm{d}\theta, \tag{26}$$

where $1 \le i \le 6$, f_i and g are as defined in (24) and (25), $\rho = 2$, $c = 5$, $\tau = 0.3$, and

$$A = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 2 \\ 1 & -3 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 2 & -5 & 1 & 1 \\ 1 & 2 & 2 & 0 & -6 & 1 \\ 0 & 0 & 1 & 2 & 1 & -4 \end{bmatrix}.$$

According to Theorem 1, partial topology $A_{3\times 6}$ of network (26) can be identified under Assumption (A2). The identified partial topology is demonstrated in Fig. 1, where only a part of $(\hat{a}_{ij}(t))_{3\times 6}$ is shown for brevity. It is found that the presented couplings $\hat{a}_{11}(t)$, $\hat{a}_{16}(t)$, $\hat{a}_{21}(t)$, $\hat{a}_{22}(t)$, $\hat{a}_{33}(t)$, $\hat{a}_{35}(t)$ converge to the real values, that is, partial topology $A_{3\times 6}$ of network (26) is successfully identified.

The initial values and some parameters are taken as $\psi_i(t) = 0.5i + [1, 0, -1]^T$, $\phi_j(t) = 0.5(j + [1, 2, 3]^T)$, $d_i(0) = 1$, $\hat{a}_{ij}(0) = 1$, $\alpha_i = 1$, $\beta_{ij} = 2$, where $1 \le i \le 3$, $1 \le j \le 6$, and $t \in [-0.3, 0]$.

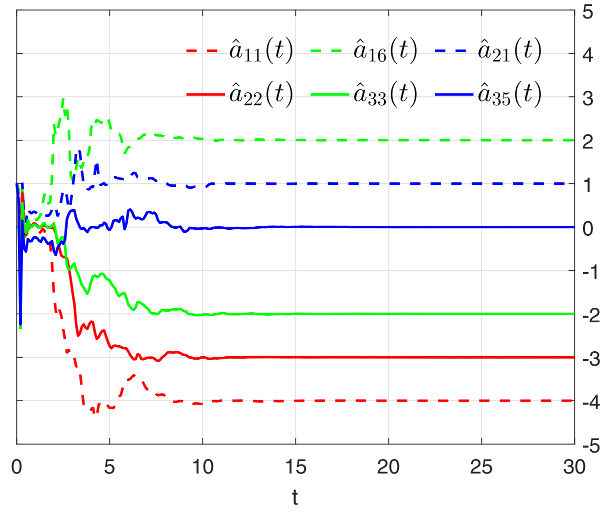

Fig. 1. Partial topology identification of network (26): evolution of the couplings $\hat{a}_{11}(t)$, $\hat{a}_{16}(t)$, $\hat{a}_{21}(t)$, $\hat{a}_{22}(t)$, $\hat{a}_{33}(t)$, $\hat{a}_{35}(t)$.

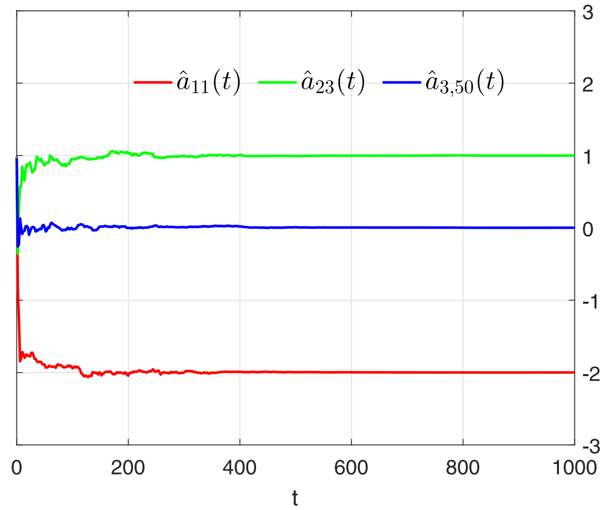

Fig. 2. Partial topology identification of network (27): the evolution of the couplings $\hat{a}_{11}(t)$, $\hat{a}_{23}(t)$, $\hat{a}_{3,50}(t)$.

Example 2. Consider a network of $N = 50$ identical nodes:

$$\dot{x}_i(t) = f(x_i(t)) + c\int_{t-\tau}^{t} \Phi x_{i-1}(\theta)\mathrm{d}\theta - 2c\int_{t-\tau}^{t} \Phi x_i(\theta)\mathrm{d}\theta + c\int_{t-\tau}^{t} \Phi x_{i+1}(\theta)\mathrm{d}\theta, \tag{27}$$

where $1 \le i \le 50$, $x_0 \equiv x_{50}$, $x_{51} \equiv x_1$, f is as defined in (23), $c = 5$, $\tau = 1$, and $\Phi = I_3$. Fig. 2 presents the evolution of the couplings $\hat{a}_{11}(t)$, $\hat{a}_{23}(t)$, $\hat{a}_{3,50}(t)$. It is shown that partial topology $A_{3\times 50}$ of network (27) is correctly identified.

The initial values and some parameters are taken as $\psi_i(t) = 0.5i + [1, 0, -1]^T$, $\phi_j(t) = 0.05j + [0.5, 1, 1.5]^T$, $d_i(0) = 1$, $\hat{a}_{ij}(0) = 1$, $\alpha_i = 2$, $\beta_{ij} = 5$, where $1 \le i \le 3$, $1 \le j \le 50$, and $t \in [-1, 0]$.

5. Conclusion

In this paper, the partial topology identification problem of complex dynamical networks with distributed delay has been investigated. By using the synchronization theory and the adaptive feed control, some theorems and corollaries have been rigorously established, which show that partial topology of networks with distributed delay can be identified. It is noteworthy that the distributed delay is addressed for the first time for the topology identification problem. Numerical examples show that partial topology identification succeeds in the presence of distributed delay. In addition, it should be highlighted that the verification of the linear independence condition remains to be challenging and deserves further study.

CRediT authorship contribution statement

Shuaibing Zhu: Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft, Funding acquisition. **Xueyi Zhao:** Software, Formal analysis, Writing – original draft. **Na Li:** Software, Formal analysis, Writing – original draft. **Jin Zhou:** Methodology, Writing – original draft, Funding acquisition. **Jun-An Lu:** Methodology, Data curation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant 62203166, Grant 62173254, and Grant 61773294, the Hunan Provincial Natural Science Foundation of China under Grant 2022JJ40281, and the Open Foundation of Hubei Key Laboratory of Applied Mathematics (Hubei University) under Grant HBAM202103.

References

[1] Watts DJ, Strogatz SH. Collective dynamics of 'small-world' networks. Nature 1998;393:440–2.
[2] Barabási AL, Albert R. Emergence of scaling in random networks. Science 1999;286(5439):509–12.
[3] Newman MEJ. The structure and function of complex networks. SIAM Rev 2003;45(2):167–256.
[4] Gao H, Dong H, Wang Z, Han F. An event-triggering approach to recursive filtering for complex networks with state saturations and random coupling strengths. IEEE Trans Neural Netw Learn Syst 2020;31(10):4279–89.
[5] Zhu S, Zhou J, Yu X, Lu JA. Bounded synchronization of heterogeneous complex dynamical networks: A unified approach. IEEE Trans Autom Control 2021;66(4):1756–62.
[6] Zhang C, Zhang C, Zhang X, Wang F, Liang Y. Dynamic event-triggered control for intra/inter-layer synchronization in multi-layer networks. Commun Nonlin Sci Numer Simulat 2023;119:107124.
[7] Strogatz SH. Exploring complex networks. Nature 2001;410(6825):268–76.
[8] Wang P, Li X, Su H. Asynchronously intermittent decentralized control for synchronization of stochastic delayed networks. Commun Nonlin Sci Numer Simulat 2023;117:106964.
[9] Albert R, Barabási AL. Statistical mechanics of complex networks. Rev Modern Phys 2002;74(1):47–97.
[10] Zhu S, Zhou J, Chen G, Lu JA. Estimating the region of attraction on a complex dynamical network. SIAM J Control Optim 2019;57(2):1189–208.
[11] Boccaletti S, Bianconi G, Criado R, Genio CID, Gómez-Gardeñes J, Romance M, et al. The structure and dynamics of multilayer networks. Phys Rep 2014;544(1):1–122.
[12] Liang T, Zhang W, Yang D. Fixed-time synchronization of switched duplex networks with stochastic disturbances and limited communication. Commun Nonlin Sci Numer Simulat 2023;121:107227.
[13] Li N, Wu X, Feng J, Xu Y, Lü J. Fixed-time synchronization of coupled neural networks with discontinuous activation and mismatched parameters. IEEE Trans Neural Netw Learn Syst 2021;32(6):2470–82.
[14] Shen Y, Liu X. Generalized synchronization of delayed complex-valued dynamical networks via hybrid control. Commun Nonlin Sci Numer Simulat 2023;118:107057.
[15] Zhu S, Zhou J, Lu JA. Estimating the region of attraction on controlled complex networks with time-varying delay. IEEE Trans Automat Control 2023;68(1):516–23.
[16] Salathe M, Kazandjieva M, Lee JW, Levis P, Feldman MW, Jones JH. A high-resolution human contact network for infectious disease transmission. Proc Nat Acad Sci USA 2010;107(51):22020–5.
[17] Lynn CW, Papadopoulos L, Kahn AE, Bassett DS. Human information processing in complex networks. Nat Phys 2020;16(9):965–73.
[18] Zhou J, Lu JA. Topology identification of weighted complex dynamical networks. Physica A, Stat Mech Appl 2007;386(1):481–91.
[19] Wu X. Synchronization-based topology identification of weighted general complex dynamical networks with time-varying coupling delay. Physica A, Stat Mech Appl 2008;387(4):997–1008.
[20] Zhu S, Zhou J, Chen G, Lu JA. A new method for topology identification of complex dynamical networks. IEEE Trans Cybern 2021;51(4):2224–31.
[21] Liu H, Lu JA, Lü J, Hill DJ. Structure identification of uncertain general complex dynamical networks with time delay. Automatica 2009;45(8):1799–807.
[22] Xu Y, Zhou W, Fang J. Topology identification of the modified complex dynamical network with non-delayed and delayed coupling. Nonlin Dyn 2012;68(1–2):195–205.
[23] Zhao X, Zhou J, Zhu S, Ma C, Lu JA. Topology identification of multiplex delayed networks. IEEE Trans Circuits Syst II, Exp Briefs 2020;67(2):290–4.
[24] Zhu S, Zhou J, Lu JA. Identifying partial topology of complex dynamical networks via a pinning mechanism. Chaos Interdiscipl J Nonlin Sci 2018;28(4):043108.
[25] Li Z, Ma W, Ma N. Partial topology identification of tempered fractional-order complex networks via synchronization method. Math Methods Appl Sci 2021.
[26] Chen C, Zhou J, Qu F, Song C, Zhu S. Identifying partial topology of complex networks with stochastic perturbations and time delay. Commun Nonlin Sci Numer Simulat 2022;115:106779.
[27] Liu Y, Wang Z, Liu X. Exponential synchronization of complex networks with markovian jump and mixed delays. Phys Lett A 2008;372(22):3986–98.
[28] Peng D, Li J, Xu W, Li X. Finite-time synchronization of coupled cohen-grossberg neural networks with mixed time delays. J Franklin Inst 2020;357(16):11349–67.

[29] Long C, Zhang G, Zeng Z, Hu J. Finite-time lag synchronization of inertial neural networks with mixed infinite time-varying delays and state-dependent switching. Neurocomputing 2021;433(6):50–8.
[30] Zhao-Yan W, Xin-Chu F. Topology identification of general dynamical network with distributed time delays. Chin Phys Lett 2009;26(7):070201.
[31] Khalil HK. Nonlinear systems. 3rd ed. Upper Saddle River, NJ: Prentice Hall; 2002.
[32] Xu Y, Zhou W, Fang J, Sun W, Pan L. Topology identification and adaptive synchronization of uncertain complex networks with non-derivative and derivative coupling. J Franklin Inst 2010;347(8):1566–76.
[33] Wu X, Zhao X, Lü J, Tang L, Lu JA. Identifying topologies of complex dynamical networks with stochastic perturbations. IEEE Trans Control Netw Syst 2017;3(4):379–89.
[34] Chen X, Zhang J, Ma T. Parameter estimation and topology identification of uncertain general fractional-order complex dynamical networks with time delay. IEEE/CAA J Autom Sinica 2016;3(3):295–303.
[35] Lorenz EN. Deterministic nonperiodic flow. J Atmos Sci 1963;20(2):130–48.