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ABSTRACT

After the groundbreaking work by Gómez *et al.*, the superdiffusion phenomenon on multiplex networks begins to attract researchers' attention. The emergence of superdiffusion means that the time scale of the diffusion process of the multiplex network is shorter than that of each layer. Using the optimization theory, the manuscript studies the greatest impact of one edge on the network diffusion speed. It is proved that by deleting any edge from a given network, the drop of the second smallest eigenvalue of its Laplacian matrix is at most 2. Based on the conclusion, the relation between the complete structure and the superdiffusible network is studied, and, further, some superdiffusion criteria on general duplex networks are proposed. Interestingly, the theoretical results indicate that the emergence of superdiffusion depends on the complete structure rather than the overlap one. Some numerical examples are shown to verify the effectiveness of the theoretical results.

The relation between the superdiffusion phenomenon and the topology of a multilayer network has been an increasingly hot topic recently. Up to now, some superdiffusion criteria have been proposed. However, each criterion has its own advantages and disadvantages. Therefore, in this paper, we propose three superdiffusion criteria for different cases of network structures and conclude that superdiffusion is related to the complementarity rather than the overlap of the subnetworks. Numerical simulations verify the effectiveness of the theoretical results.

I. INTRODUCTION

In the last few decades, the diffusion problem on complex networks has received much attention.[1–3] In real life, one diffusion process is often influenced by other processes. Based on this fact, multiplex networks gradually come into researchers' sight.[4–8] Multiplex networks refer to a kind of networks with multiple layers in the same group of nodes. The layers are linked by interlayer couplings, which are from each node to itself. An example of a multiplex network is shown in Fig. 1. It provides us a more realistic framework for the study of dynamic behaviors, such as synchronization, diffusion, and propagation.[9,10]

As a common phenomenon, the diffusion dynamics is an important class of network dynamics. For a single network, the time scale of its diffusion to the steady state is inversely proportional to the second smallest eigenvalue of its Laplacian matrix.[11] In Ref. 12, Gómez *et al.* investigated the diffusion dynamics on multiplex networks and proposed the concept of superdiffusion, which means the time scale of the diffusion process on a multiplex network is shorter than that on each isolated layer.[12] From the eigenvalue's point of view, the emergence of superdiffusion requires that the second smallest eigenvalue of the Laplacian matrix, often called supra-Laplacian matrix, with respect to a multiplex network is greater than that in terms of each layer. Compared with the Laplacian matrix of a single layer, the supra-Laplacian matrix includes not only the topology information of each layer but also the intralayer and interlayer diffusion constants. Furthermore, Gómez *et al.* demonstrated that for fixed intralayer diffusion constants, when the interlayer diffusion constant is sufficiently large, the second smallest eigenvalue of the supra-Laplacian matrix is close to half of that of the sum of two Laplacian matrices corresponding to two layers. This finding, thus, provides a new direction to investigate the superdiffusion phenomenon by focusing on the Laplacian of the sum of two lower-dimension matrices rather than the supra-Laplacian matrix. After this groundbreaking work, some results have been developed. By numerical simulations, Ref. 13 found that the emergence of

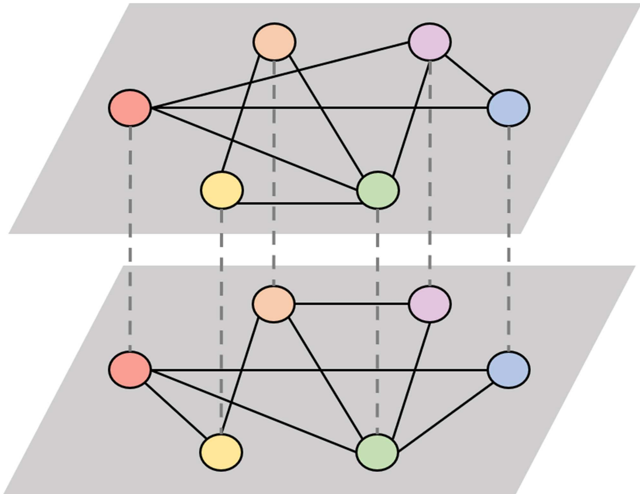

FIG. 1. An example of a multiplex network with two layers and six nodes. The solid lines represent the intralayer couplings. The dotted lines represent the interlayer couplings, which are from each node to itself.

superdiffusion on a multiplex network does not depend on the overlap between two isolated layers.[13] A special class of superdiffusible duplex networks was proposed in Ref. 14. In Ref. 15, some sufficient superdiffusible conditions for directed duplex networks were presented.[15] Due to the difficulty of solving the eigenvalues of large-scale matrices directly, it is very difficult to propose superdiffusion criteria applicable to all multiplex networks. This manuscript, from a more general perspective, develops some superdiffusion criteria for duplex networks.

Without loss of generality, the manuscript focuses on the unweighted and undirected duplex networks. By means of the optimization theory, the manuscript studies the greatest impact of one edge on the network diffusion speed. Adding (deleting) any one edge to (from) a given network, the increment (drop) of the second smallest eigenvalue of its Laplacian matrix is at most 2. Thus, based on this result, the relation between the number of deleted edges and the superdiffusible network is studied. Specifically, the union of the two layers can be obtained by deleting some edges from the complete graph with the same size. Their difference in the number of edges is the number of deleted edges mentioned earlier. Based on previous analysis, some superdiffusion criteria on duplex networks are proposed. From the theoretical results, we find that the emergence of superdiffusion depends on the complete structure rather than the overlap one. The method proposed can be extended to the study of general multiplex networks.

The remaining parts are organized as follows. In Sec. II, some mathematical preliminaries and the network model are given. In Sec. III, the greatest impact of adding (deleting) one edge on the network diffusion speed is studied. Based on the conclusions in Sec. III, Sec. IV gives some superdiffusion criteria on duplex networks. Numerical simulations are presented in Sec. V. Finally, the manuscript is concluded in Sec. VI.

II. NETWORK MODEL AND PRELIMINARIES

Consider a multiplex network with M layers and N nodes per layer. In the manuscript, the network is assumed to be unweighted and undirected. Let G_k be the graph of layer k, $A_k = (a_{ij}^k)$ the adjacency matrix of G_k, and $L_k = (d_{ij}^k)$ the Laplacian matrix of G_k.

Without loss of generality, we consider the simplest case that the multiplex network consists of two layers ($M = 2$). The diffusion dynamics on the duplex network is described as follows:

$$\frac{dx_i^k}{dt} = D_k \sum_{j=1}^{N} w_{ij}^k \left(x_j^k - x_i^k\right) + D_x \sum_{l=1}^{M} \left(x_i^l - x_i^k\right), \tag{1}$$

where $x_i^k \in R$ represents the state of node i ($i = 1, 2, \ldots, N$) in layer k ($k = 1, 2$) and w_{ij}^k the coupling strength between node i and j in layer k. Here, symbol D_k represents the diffusion constant of layer k and D_x the interlayer diffusion constant between layers. Since this manuscript only considers unweighted networks, w_{ij}^k in Eq. (1) is either 0 or 1.

Let $\boldsymbol{x}$ be $\left(x_1^1, \ldots, x_N^1 \mid x_1^2, \ldots, x_N^2\right)^T$, with I being the identity matrix. Then, Eq. (1) can be rewritten as $\dot{\boldsymbol{x}} = -\mathscr{L}\boldsymbol{x}$, where the supra-Laplacian matrix

$$\mathscr{L} = \begin{pmatrix} D_1 L_1 + D_x I & -D_x I \\ -D_x I & D_2 L_2 + D_x I \end{pmatrix}.$$

Without loss of generality, set $D_1 = D_2 = 1$. Denote λ_2^k as the second smallest eigenvalue of L_k, λ_N^k the largest eigenvalue of L_k, Λ_2 the second smallest eigenvalue of $\mathscr{L}$ and λ_s the second smallest eigenvalue of $\frac{L_1+L_2}{2}$. For any Laplacian matrix L, denote $\lambda_2(L)$ as its second smallest eigenvalue. The definitions for a duplex network are proposed.[14]

Definition 1: Superdiffusion emerges in a duplex network when $\Lambda_2 > \max\left\{\lambda_2^1, \lambda_2^2\right\}$.

The emergence of superdiffusion means the diffusion time scale of the multiplex network is shorter than that of each layer. Since the diffusion time scale of the network is controlled by the second smallest eigenvalue of its Laplacian matrix;[11] thus, when $\Lambda_2 > \max\left\{\lambda_2^1, \lambda_2^2\right\}$, the duplex network displays superdiffusion.

Definition 2: A duplex network is superdiffusible if $\lambda_s > \max\left\{\lambda_2^1, \lambda_2^2\right\}$.

It has been demonstrated that for fixed D_k, as D_x increases, Λ_2 will gradually increase and approach λ_s.[12] Therefore, whether a duplex network is superdiffusible depends on the relation between λ_s and $\max\left\{\lambda_2^1, \lambda_2^2\right\}$. If $\lambda_s \leq \max\left\{\lambda_2^1, \lambda_2^2\right\}$, it is impossible for a duplex network to achieve superdiffusion.

To study the relation between the number of deleted edges and the superdiffusible network, define $G_1 \cup G_2$ as the union of graphs G_1 and G_2,

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2),$$

where V_k and E_k represent the node set and edge set of graph G_k ($k = 1, 2$), respectively. In simple terms, $G_1 \cup G_2$ is a single graph with all edges in G_1 and G_2. For example, in Fig. 2, $G_1 \cup G_2$ is the six-order complete graph K_6. For any duplex network, $G_1 \cup G_2$ is a subgraph of the N-order complete graph K_N.

The smaller the difference in the number of edges between $G_1 \cap G_2$ and $G_1 \cup G_2$, the more overlap between graphs G_1 and G_2.

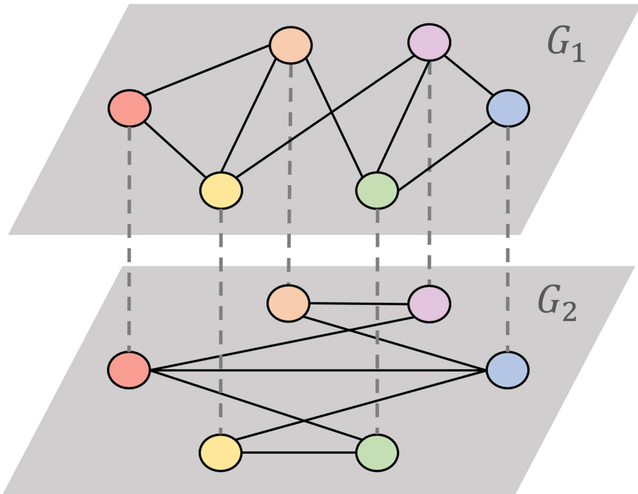

FIG. 2. A duplex network that satisfies the three conditions of Theorem 2: G_1 is the complement of G_2, $\delta(G_1) = 2 < \frac{N}{2} = 3$ and $\Delta(G_1) = 3 > \frac{N}{2} - 1 = 2$. Here, λ_2^1 and λ_2^2 are equal to 1, and λ_s is equal to 3. Thus, the duplex network is superdiffusible.

The smaller the difference in the number of edges between $G_1 \cup G_2$ and K_N, the more complete the graph $G_1 \cup G_2$. Denote $\varepsilon(G)$ as the number of edges of graph G. Then, the overlap index ω and the complete index ν of the network can be defined, respectively:[13,16]

$$\omega = \frac{\varepsilon(G_1 \cap G_2)}{\varepsilon(G_1 \cup G_2)}, \tag{2}$$

$$\nu = \frac{2\varepsilon(G_1 \cup G_2)}{N(N-1)}. \tag{3}$$

The two indices are real numbers between 0 and 1. The more the two networks overlap, the greater the value of ω. The closer the graph $G_1 \cup G_2$ is to the complete graph K_N, the greater the value of ν. Take the network in Fig. 1 as an example. Both layers have eight edges, five of which are in common, implying that $\varepsilon(G_1 \cap G_2) = 5$ and $\varepsilon(G_1 \cup G_2) = 11$. Thus, one has $\omega = \frac{5}{11} \approx 0.45$ and $\nu = \frac{22}{30} \approx 0.73$.

Furthermore, these two indices can also be represented by the elements in the Laplacian matrix or the adjacency matrix. Since

$$\varepsilon(G_1 \cap G_2) = \sum_{\substack{i,j=1\ldots N\\ j>i}} d_{ij}^1 d_{ij}^2 = \sum_{\substack{i,j=1\ldots N\\ j>i}} a_{ij}^1 a_{ij}^2,$$

$$\begin{aligned}\varepsilon(G_1 \cup G_2) &= \sum_{\substack{i,j=1\ldots N\\ j>i}} \left(-d_{ij}^1 - d_{ij}^2 - d_{ij}^1 d_{ij}^2\right)\\ &= \sum_{\substack{i,j=1\ldots N\\ j>i}} \left(a_{ij}^1 + a_{ij}^2 - a_{ij}^1 a_{ij}^2\right),\end{aligned}$$

Eqs. (2) and (3) can be rewritten as

$$\begin{aligned}\omega &= \frac{\sum_{\substack{i,j=1\ldots N\\ j>i}} d_{ij}^1 d_{ij}^2}{\sum_{\substack{i,j=1\ldots N\\ j>i}} \left(-d_{ij}^1 - d_{ij}^2 - d_{ij}^1 d_{ij}^2\right)}\\ &= \frac{\sum_{\substack{i,j=1\ldots N\\ j>i}} a_{ij}^1 a_{ij}^2}{\sum_{\substack{i,j=1\ldots N\\ j>i}} \left(a_{ij}^1 + a_{ij}^2 - a_{ij}^1 a_{ij}^2\right)},\end{aligned}$$

$$\begin{aligned}\nu &= \frac{2\sum_{\substack{i,j=1\ldots N\\ j>i}} \left(-d_{ij}^1 - d_{ij}^2 - d_{ij}^1 d_{ij}^2\right)}{N(N-1)}\\ &= \frac{2\sum_{\substack{i,j=1\ldots N\\ j>i}} \left(a_{ij}^1 + a_{ij}^2 - a_{ij}^1 a_{ij}^2\right)}{N(N-1)}.\end{aligned}$$

For the main results, the following lemmas are needed.[20]

Lemma 1: If $0 \le \lambda_2 \le \cdots \le \lambda_N$ are the Laplace eigenvalues of a single graph G, then $0 \le N - \lambda_N \le \cdots \le N - \lambda_2$ are the Laplace eigenvalues of the complement of G.

Lemma 2: Let G be a graph on N vertices with at least one edge and let $\Delta(G)$ be the maximum degree of G. Then,

$$\lambda_N(G) \ge 1 + \Delta(G).$$

Lemma 3: Let G be a non-complete graph on N vertices and let $\delta(G)$ be the minimum degree of G. Then,

$$\lambda_2(G) \le \delta(G).$$

It should be stressed that in Ref. 20, $\lambda_N(G)$ and $\lambda_2(G)$ represent the largest and second smallest eigenvalues of the Laplacian matrix of graph G, respectively.

III. THE GREATEST IMPACT OF AN EDGE ON THE NETWORK DIFFUSION SPEED

In this section, the greatest impact of an edge on the second smallest eigenvalue of the Laplacian matrix is studied.

Suppose the original graph is G and its corresponding Laplacian matrix is L. After adding an edge between nodes i and j, the new graph is G' and its corresponding Laplacian matrix is L'. In addition, assume there are N nodes in graph G. Then, Theorem 1 is obtained.

Theorem 1: After adding(deleting) an edge between nodes i and j, the Laplacian matrix changes from $L(L')$ to $L'(L)$. Then, $\lambda_2(L)$ and $\lambda_2(L')$ have the following relation:

$$\lambda_2(L') - \lambda_2(L) \le 2. \tag{4}$$

In the following, we give a detailed proof of the case of edge addition and a brief description of the case of edge deletion.

Proof. Due to the definition of the second smallest eigenvalue of the Laplacian matrix,[17–19] one obtains

$$\lambda_2(L) = \min_{z\perp\mathbf{1},\|z\|=1} z^T L z = \min_{z\perp\mathbf{1},\|z\|=1} \sum_{v_m v_n \in G} (z_m - z_n)^2, \tag{5}$$

where $\mathbf{1}$ is a column vector whose all N components be 1, z_m is the mth component of the vector z, and $v_m v_n$ represents the edge between nodes m and n. The solution of the second smallest eigenvalue of the Laplacian matrix can be regarded as an optimization problem, that is, how to distribute the values of components of z so that the sum of the quadratic terms is the minimum. Similar to Eq. (5), one obtains

$$\lambda_2(L') = \min_{z\perp\mathbf{1},\|z\|=1} \sum_{v_m v_n \in G'} (z_m - z_n)^2.$$

Assume that z^* is a unit vector orthogonal to $\mathbf{1}$ and satisfies that

$$z^* \in \left\{ \bar{z} \in R^N \,\middle|\, \sum_{v_m v_n \in G} (\bar{z}_m - \bar{z}_n)^2 = \min_{z\perp\mathbf{1},\|z\|=1} \sum_{v_m v_n \in G} (z_m - z_n)^2 \right\},$$

where $\bar{z}_m$ is the mth component of the column vector $\bar{z}$. From the existence of $\lambda_2(L)$, the existence of z^* is obvious. Then,

$$\lambda_2(L) = \sum_{v_m v_n \in G} (z_m^* - z_n^*)^2, \tag{6}$$

where z_m^* is the mth component of z^*. Moreover, for this z^*, one further has

$$\lambda_2(L') \le \sum_{v_m v_n \in G'} (z_m^* - z_n^*)^2. \tag{7}$$

Given that $G' = G \cup v_i v_j$, it is implied that

$$\sum_{v_m v_n \in G'} (z_m^* - z_n^*)^2 = \sum_{v_m v_n \in G} (z_m^* - z_n^*)^2 + (z_i^* - z_j^*)^2. \tag{8}$$

From Eq. (6), one obtains

$$\sum_{v_m v_n \in G} (z_m^* - z_n^*)^2 + (z_i^* - z_j^*)^2 = \lambda_2(L) + (z_i^* - z_j^*)^2. \tag{9}$$

It is derived from Eqs. (7)–(9) that

$$\lambda_2(L') \le \lambda_2(L) + (z_i^* - z_j^*)^2. \tag{10}$$

Now, we have known that $\lambda_2(L') - \lambda_2(L) \le (z_i^* - z_j^*)^2$, once the maximum value $(z_i^* - z_j^*)^2$ is acquired, the maximum increment of the second smallest eigenvalue is known.

For simplicity, replace z^* with $\boldsymbol{x_0} = (x_1, x_2, \ldots, x_N)^T$. The maximum problem of $(z_i^* - z_j^*)^2$ is equivalent to the following optimization problem:

$$\begin{aligned} \min \quad & -(x_i - x_j)^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 + \cdots + x_N^2 = 1, \\ & x_1 + x_2 + \cdots + x_N = 0. \end{aligned}$$

To solve this optimization problem, according to Karush–Kuhn–Tucker Conditions, we construct its Lagrangian function:[21]

$$\begin{aligned} L(\boldsymbol{x_0}, \mu_1, \mu_2) = {}& -(x_i - x_j)^2 - \mu_1(x_1^2 + x_2^2 + \cdots + x_N^2 - 1) \\ & - \mu_2(x_1 + x_2 + \cdots + x_N), \end{aligned}$$

where μ_1 and μ_2 are arbitrary constants. Because the vector $(2x_1, 2x_2, \ldots, 2x_N)^T$ is orthogonal to $(1, 1, \ldots, 1)^T$, Karush–Kuhn–Tucker conditions is applicable. If $\boldsymbol{x^*} = (x_1^*, x_2^*, \ldots, x_N^*)^T$ is the solution of the optimization problem, then μ_1^* and μ_2^* satisfy the following equations:

$$\begin{cases} \nabla_{\boldsymbol{x^*}} L(\boldsymbol{x^*}, \mu_1^*, \mu_2^*) = \mathbf{0}, \\ x_1^{*2} + x_2^{*2} + \cdots + x_N^{*2} = 1, \\ x_1^* + x_2^* + \cdots + x_N^* = 0, \end{cases} \tag{11}$$

where $\mathbf{0}$ is a column vector whose all N components are 0. For

$$\nabla_{\boldsymbol{x}}^* L(\boldsymbol{x}^*, \mu_1^*, \mu_2^*) = \begin{bmatrix} -2\mu_1^* x_1^* - \mu_2^* \\ \vdots \\ -2\mu_1^* x_{i-1}^* - \mu_2^* \\ -2x_i^* + 2x_j^* - 2\mu_1^* x_i^* - \mu_2^* \\ -2\mu_1^* x_{i+1}^* - \mu_2^* \\ \vdots \\ -2\mu_1^* x_{j-1}^* - \mu_2^* \\ 2x_i^* - 2x_j^* - 2\mu_1^* x_j^* - \mu_2^* \\ -2\mu_1^* x_{j+1}^* - \mu_2^* \\ \vdots \\ -2\mu_1^* x_N^* - \mu_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{12}$$

the sum of the N elements in the middle is equal to 0. Thus,

$$-2\mu_1^*(x_1^* + x_2^* + \cdots + x_N^*) - N\mu_2^* = 0. \tag{13}$$

Combine Eq. (13) with the third equation of Eq. (11) and get $\mu_2^* = 0$. Substitute $\mu_2^* = 0$ into Eq. (12) and consider two cases: $\mu_1^* = 0$ and $\mu_1^* \ne 0$.

If μ_1^* is equal to 0, substitute $\mu_1^* = 0$ into Eq. (12) and focus on the ith and jth rows. It can be seen that x_i^* is equal to x_j^*, or equivalently, $-(x_i^* - x_j^*)^2 = 0$. This solution is obviously not optimal. If μ_1^* is not equal to 0, focus on the rows of Eq. (12) except the ith and jth rows. It is seen that all the N components of $\boldsymbol{x^*}$ except the ith and jth components are 0. Solve Eq. (11) and get the optimal solution $\boldsymbol{x^*}$ as $(0, \ldots, 0, \frac{\sqrt{2}}{2}, 0, \ldots, 0, -\frac{\sqrt{2}}{2}, 0, \ldots, 0)^T$ or $(0, \ldots, 0, -\frac{\sqrt{2}}{2}, 0, \ldots, 0, \frac{\sqrt{2}}{2}, 0, \ldots, 0)^T$. In this case, $-(x_i^* - x_j^*)^2 = -2$.

Based on the above analysis, it is obtained that $\max\,(x_i - x_j)^2$ is equal to 2. Thus, from Eq. (10), one obtains

$$\lambda_2(L') - \lambda_2(L) \le \max\,(x_i - x_j)^2 = 2. \tag{14}$$

In other words, when an edge is added to the network, the maximum increment of the second smallest eigenvalue is 2. □

Similarly, delete one edge to make graph G' become G, Laplacian matrix L' become L. Conversely, graph G' can be regarded as graph G by adding an edge. From Eq. (14), then $\lambda_2(L') - \lambda_2(L) \le 2$. Therefore, when an edge is deleted from the network, the maximum drop of the second smallest eigenvalue is 2.

Actually, Theorem 1 can also be proved by the conclusion of Sec. 1.3 in Ref. 20. It is concluded that when an edge is added to (deleted from) the network, the increment (drop) of its second smallest eigenvalue is at most 2. The situation where the increment (drop) is equal to 2 exists. When G' is equal to K_N, $\lambda_2(L')$ is equal to N. From Lemma 3, $\lambda_2(L) \leq \delta(G) = N - 2$. Thus, $\lambda_2(L') - \lambda_2(L) \geq 2$. On the other hand, from Theorem 1, $\lambda_2(L') - \lambda_2(L) \leq 2$. To sum up, when G' is equal to K_N, $\lambda_2(L') - \lambda_2(L) = 2$.

IV. THE IMPACT OF COMPLETE STRUCTURE ON SUPERDIFFUSION

In this section, the relation between the number of deleted edges and the superdiffusible network is studied. Specifically, $G_1 \cup G_2$ can be obtained from the complete graph K_N by deleting some edges. According to different values of the complete index ν and the overlap index ω, we will present our discussion in three cases.

Case 1: When $\nu = 1$ and $\omega = 0$

When ν is equal to one and ω is equal to zero, graph G_1 is the complement of G_2, which means G_1 is equal to $(K_N - G_2)$.

In this case, λ_s is equal to $\frac{N}{2}$. If $0 \leq \lambda_2^1 \leq \cdots \leq \lambda_N^1$ are the Laplacian eigenvalues of G_1, then $0 \leq N - \lambda_N^1 \leq \cdots \leq N - \lambda_2^1$ are the Laplacian eigenvalues of G_2.[20] Therefore, the duplex network is superdiffusible provided that $\lambda_2^1 < \frac{N}{2}$ and $\lambda_N^1 > \frac{N}{2}$. Using Lemmas 2 and 3, Theorem 2 is derived.

Theorem 2: Assume that G_1 and G_2 are connected incomplete single graphs with N nodes. If

(a) G_1 is the complement of G_2,
(b) $\delta(G_1) < \frac{N}{2}$,
(c) $\Delta(G_1) > \frac{N}{2} - 1$,

where $\delta(G_1)$ and $\Delta(G_1)$ are the minimum and maximum degrees of G_1, respectively, then the duplex network composed of G_1 and G_2 is superdiffusible.

Proof. According to the assumption that G_1 is the complement of G_2, one obtains

$$\lambda_s = \lambda_2\left(\frac{L_1 + L_2}{2}\right) = \frac{N}{2}. \tag{15}$$

When G_1 is an incomplete graph and $\delta(G_1) < \frac{N}{2}$, from Lemma 3, one has

$$\lambda_2^1 \leq \delta(G_1) < \frac{N}{2}. \tag{16}$$

On the other hand, when G_1 is the complement of G_2 and G_2 is an incomplete graph, it implies that G_1 is a graph with at least one edge. Since $\Delta(G_1) > \frac{N}{2} - 1$, it is derived that

$$\lambda_N^1 \geq \Delta(G_1) + 1 > \frac{N}{2} \tag{17}$$

from Lemma 2. According to Lemma 1 and Eq. (17), then one obtains

$$\lambda_2^2 = N - \lambda_N^1 < \frac{N}{2}. \tag{18}$$

Combining Eqs. (15), (16), and (18), one gets $\lambda_s > \max\{\lambda_2^1, \lambda_2^2\}$. □

Remark 1: When G_1 is the complement of G_2, $\delta(G_1) < \frac{N}{2}$ and $\Delta(G_1) > \frac{N}{2} - 1$ are equivalent to $\Delta(G_2) > \frac{N}{2} - 1$ and $\delta(G_2) < \frac{N}{2}$, respectively. So, there are actually four different forms to state Theorem 2. One of them is consistent with Theorem 4.3.2 in Ref. 22.

From Theorem 2, it is concluded that if two layers of a duplex network are completely complementary, the smaller the minimum degree and the larger the maximum degree of one layer, the more likely the duplex network is to be superdiffusible. In addition, according to Remark 1, we realize when the minimum degrees of the two layers are small, the duplex network is likely to be superdiffusible. An example is given in Sec. V A.

Case 2: When $\nu \approx 1$ and $\omega = 0$

When ν is approximately equal to one and ω is equal to zero, $G_1 \cup G_2$ is close to the complete graph K_N. As a subgraph of K_N, $G_1 \cup G_2$ can be obtained by deleting some edges from K_N. Therefore, λ_s can be regarded as a disturbance of $\frac{N}{2}$. The estimated value of λ_s is related to the number of deleted edges. Define G_d as the graph $(K_N - G_1 \cup G_2)$. Then, $\varepsilon(G_d)$ is equal to $\left(\frac{N(N-1)}{2} - \varepsilon(G_1 \cup G_2)\right)$. From Theorem 1, it is seen that if one edge is deleted from the graph, the drop in the second smallest eigenvalues between the original and new Laplacian matrices will not exceed 2. Thus, Theorem 3 is obtained.

Theorem 3: Suppose the complete index ν is approximately equal to one, the overlap index ω is equal to zero, and both G_1 and G_2 are incomplete single graphs. If

$$\varepsilon(G_d) < \min\left\{\frac{N}{2} - \delta(G_1), \frac{N}{2} - \delta(G_2)\right\}, \tag{19}$$

then the duplex network composed of G_1 and G_2 is superdiffusible.

Proof. When ν is approximately equal to one, the graph $G_1 \cup G_2$ can be obtained by deleting some edges from K_N. Let $\varepsilon(G_d)$ represents the edge of the graph $(K_N - G_1 \cup G_2)$. From Theorem 1, it is seen that if one edge is deleted from the graph, the eigenvalue drop of the original Laplacian matrix is at most 2; thus,

$$\lambda_s \geq \frac{N - 2\varepsilon(G_d)}{2}.$$

For incomplete single graphs G_1 and G_2, one obtains $\lambda_2^1 \leq \delta(G_1)$ and $\lambda_2^2 \leq \delta(G_2)$ from Lemma 3.

If $\varepsilon(G_d) < \min\left\{\frac{N}{2} - \delta(G_1), \frac{N}{2} - \delta(G_2)\right\}$, then one has $\frac{N-2\varepsilon(G_d)}{2} > \max\{\delta(G_1), \delta(G_2)\}$, and, therefore, $\lambda_s > \max\{\lambda_2^1, \lambda_2^2\}$. □

In fact, Theorem 2 can be regarded as a special case of Theorem 3. By comparing Theorems 2 and 3, we find that if a duplex network satisfies the conditions of Theorem 2, then it is still superdiffusible after some of its edges are deleted. Equation (19) gives an estimate of the number of edges allowed to be deleted. However, this estimate is somewhat conservative, because each time we delete one edge, we make the drop of λ_2 equal to two.

Theorem 3 tells us that if the duplex network satisfies the conditions of Theorem 3, then the smaller the minimum degrees of the two layers are and the closer $G_1 \cup G_2$ is to K_N, the more likely the duplex network is to be superdiffusible.

Case 3: When $\nu \approx 1$

The condition that ω is equal to zero can be relaxed. When ω is not equal to zero, it means that there is an overlap part between G_1 and G_2. Then,

$$L_1 + L_2 = (L_1 - L_c + L_2 - L_c + L_c) + L_c, \tag{20}$$

where L_c represents the Laplacian matrix of $G_1 \cap G_2$. The term in the bracket is exactly the Laplacian matrix of $G_1 \cup G_2$. Actually, $G_1 \cup G_2$ can be obtained by deleting some edges from K_N.

Theorem 4: Suppose the complete index ν is approximately equal to one and both G_1 and G_2 are incomplete single graphs. If

$$\varepsilon(G_d) < \min\left\{\frac{N}{2} - \delta(G_1), \frac{N}{2} - \delta(G_2)\right\},$$

then the duplex network composed of G_1 and G_2 is superdiffusible.

Proof. Denote L_+ as the Laplacian matrix of graph $G_1 \cup G_2$. Considering L_c is a positive semi-definite matrix, one then obtains[20]

$$2\lambda_s = \lambda_2(L_1 + L_2) \geq \lambda_2(L_+)$$

from Eq. (20). It is inferred from Theorem 1 when $\varepsilon(G_d) = \frac{N(N-1)}{2} - \varepsilon(G_1 \cup G_2)$, then

$$\frac{\lambda_2(L_+)}{2} > \frac{N - 2\varepsilon(G_d)}{2}.$$

In addition, for incomplete single graphs G_1 and G_2, one gets $\lambda_2^1 \leq \delta(G_1)$ and $\lambda_2^2 \leq \delta(G_2)$ from Lemma 3.

If $\varepsilon(G_d) < \min\left\{\frac{N}{2} - \delta(G_1), \frac{N}{2} - \delta(G_2)\right\}$, then one has $\frac{N-2\varepsilon(G_d)}{2} > \max\{\delta(G_1), \delta(G_2)\}$, and, therefore, $\lambda_s > \max\left\{\lambda_2^1, \lambda_2^2\right\}$. □

By comparing Theorems 3 and 4, we find that, in some cases, whether there is an overlap between two layers has little influence on the emergence of superdiffusion, which is consistent with the numerical result in Ref. 13. For example, if one selects a network that satisfies the conditions of Theorem 3, and adds some edges in G_2 instead of G_1 to G_1 while ensuring $\delta(G_1)$ unchanged, then the network is still superdiffusible.

Theorem 4 reveals that, if the duplex network satisfies the conditions of Theorem 4, then the smaller the minimum degrees of the two layers are and the closer $G_1 \cup G_2$ is to K_N, the more likely the duplex network is to be superdiffusible.

The above results show that as long as $G_1 \cup G_2$ is approximately equal to K_N and $\max\{\delta(G_1), \delta(G_2)\} < \frac{N}{2}$, then the duplex network composed of G_1 and G_2 is superdiffusible. Thus, whether a duplex network is superdiffusible has no direct relation with the overlap of its two layers. In other words, the emergence of superdiffusion does not depend on the overlap but on the complete structure.

V. NUMERICAL SIMULATION

In this section, two examples are displayed to verify the effectiveness of our theoretical results.

A. Example 1: Verification of Theorem 2

A duplex network is shown in Fig. 2. It satisfies the three conditions of Theorem 2: G_1 is the complement of G_2, $\delta(G_1) = 2 < \frac{N}{2} = 3$, and $\Delta(G_1) = 3 > \frac{N}{2} - 1 = 2$.

The second smallest eigenvalues of the Laplacians of the network are plotted in Fig. 3. The dashed line represents the value of λ_s and the dashed-dotted line the value of λ_2^1 and λ_2^2. The blue solid line represents the value of Λ_2. Figure 3 shows that, when D_x increases, Λ_2 will gradually increase and approach λ_s. This phenomenon has been revealed previously in Ref. 12. Moreover, it can also be seen that when $D_x > D_c$, $\Lambda_2 > \max\left\{\lambda_2^1, \lambda_2^2\right\}$. Thus, the network is superdiffusible.

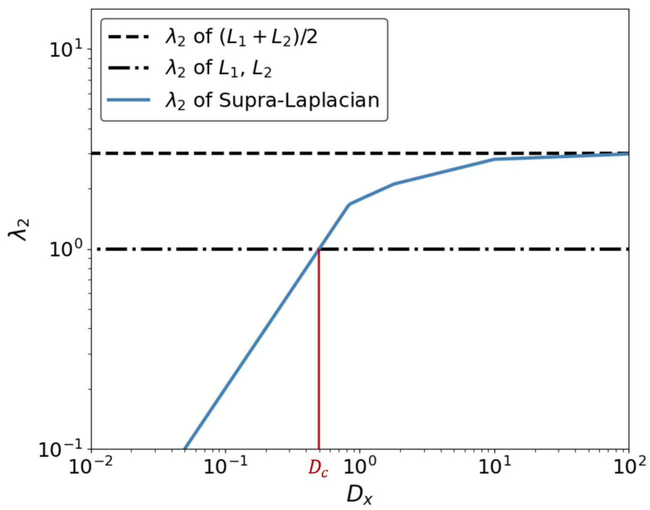

FIG. 3. Comparison of the second smallest eigenvalues of the Laplacians of the network in Fig. 2. Here, $\lambda_2^1 = \lambda_2^2 = 1$. It is seen that $\Lambda_2 > \max\left\{\lambda_2^1, \lambda_2^2\right\}$ when $D_x > D_c$. Thus, the duplex network is superdiffusible.

B. Example 2: Verification of Theorem 3

In order to verify the effectiveness of Theorem 3, a duplex network is employed: the first layer is shown in Fig. 4 and the second layer is the complement of the first layer. Obviously, $\delta(G_1) = 3, \delta(G_2) = 5$ and $N = 39$. From Theorem 3, if $\varepsilon < \min\left\{\frac{N}{2} - \delta(G_1), \frac{N}{2} - \delta(G_2)\right\} = 14.5$, the network is superdiffusible. In other words, if the number of deleted edges from the

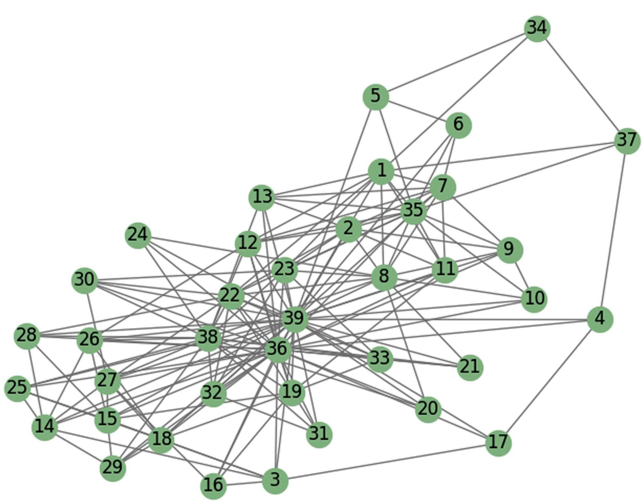

FIG. 4. A network with 39 nodes and 171 edges.

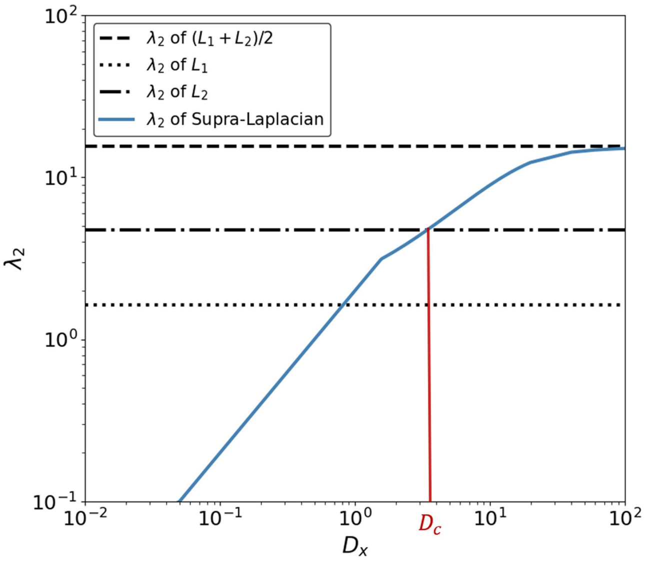

FIG. 5. Comparison of the second smallest eigenvalues of the Laplacians of a network satisfying the conditions of Theorem 3. When $D_x > D_c$, $\Lambda_2 > \max\{\lambda_2^1, \lambda_2^2\}$ is satisfied; thus, the duplex network is superdiffusible.

duplex network is smaller than 14.5, the duplex network keeps superdiffusible.

Randomly delete 14 edges while keeping $\delta(G_1) = 3$ and $\delta(G_2) = 5$ unchanged. Figure 5 displays the second smallest eigenvalues of the Laplacians of the network. It is seen that $\lambda_s > \max\{\lambda_2^1, \lambda_2^2\}$, and, thus, the duplex network is superdiffusible. Each of the 100 stochastic simulations is similar to one shown in Fig. 5.

VI. CONCLUSIONS

Based on the optimization theory, this manuscript theoretically analyzes the greatest impact of one edge on the network diffusion speed and gives a quantitative conclusion. Further, the relation between the number of deleted edges and the superdiffusible network is studied and some superdiffusion criteria on duplex networks are proposed. It is proved that if $G_1 \cup G_2$ is approximately equal to K_N and $\max\{\delta(G_1), \delta(G_2)\} < \frac{N}{2}$, the duplex network composed of G_1 and G_2 is superdiffusible. The results supplement the proof for the numerical results in Ref. 13: the emergence of superdiffusion does not depend on the overlap. In addition, it is interesting to note that the emergence of superdiffusion depends on the complete structure. The result can be used to discover and construct a superdiffusible duplex network. In addition, a method has been developed to judge whether a multiplex network is superdiffusible, as shown in the proof of Theorem 4 in detail. Obviously, the method can be extended to the research of the multiplex network with more than two layers. Compared with the previous studies on superdiffusible multiplex networks, the manuscript provides a new angle from which we can use disturbance theory to estimate λ_s based on the known quantities.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Yanqi Zhang: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Validation (equal); Writing – original draft (equal). **Jin Zhou:** Conceptualization (equal); Funding acquisition (equal); Methodology (equal); Validation (equal); Writing – review & editing (equal). **Jun-an Lu:** Funding acquisition (equal); Methodology (equal); Validation (equal); Writing – review & editing (equal). **Weiqiang Li:** Methodology (equal); Validation (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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