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ABSTRACT

Though synchronization of complex dynamical systems has been widely studied in the past few decades, few studies pay attention to the impact of network parameters on synchronization in hypernetworks. In this paper, we focus on a specific hypernetwork model consisting of coupled Rössler oscillators and investigate the impact of inner-coupling and time delay on the synchronized region (SR). For the sake of simplicity, the inner-coupling matrix is chosen from three typical forms, which result in classical bounded, unbounded, and empty SR in a single-layer network, respectively. The impact of inner-couplings or time delays on unbounded SR is the most interesting one among the three types of SR. Once the SR of one subnetwork is unbounded, the SR of the whole hypernetwork is also unbounded with a different inner-coupling matrix. In a hypernetwork with unbounded SR, the time delays change not only the size but also the type of SR. In a hypernetwork with bounded or empty SR, the time delays have almost no effect on the type of SR.

In recent years, the synchronization of hypernetworks has attracted much attention. However, there is currently a lack of research on how the inner-coupling and the time delay affect the synchronization region of hypernetworks. Most of the previous work has focused on single-layer networks, while relatively little research has been done on hypernetworks. In addition, many real systems can be modeled as hypernetworks. Therefore, it is necessary to study the synchronization of hypernetworks. In this paper, the impact of inner-coupling and time delay on the synchronization region of both single-layer networks and hypernetworks is studied by numerical simulations, and some interesting results are obtained.

I. INTRODUCTION

In the past two decades, network science has developed rapidly, which helps us better understand various complex systems in real life, such as biological systems, social systems, and engineering systems. Previously, a common strategy for studying complex systems was to abstract them into single-layer networks. The rise and rapid development of single-layer networks provide new directions for better research on the real world.[1–5] Later, researchers found that the structure of many complex systems cannot be simply described as a single-layer network, but a multi-layer network.[6–10] A multi-layer network is the research front and hotspot of network science. Researchers have studied epidemic transmission,[11] diffusion processes,[12] evolutionary games, and social interactions in a multi-layered framework. This paper mainly studies the hypernetwork,[13–15] which is a specific type of multi-layer network.

In a hypernetwork, the nodes have two or more independent interaction modes corresponding to different network topologies. For example, fish use their visual perception and chemical sense to locate their mates to maintain coordinated movement of the entire school;[16] neurons communicate through two distinct interactions: chemical synaptic junctions and electrical gap junctions.[17] Computer communication networks, interdependent networks,[18] and networks of networks[19] can all be viewed as hypernetworks. In recent years, hypernetworks and its synchronization are becoming an important research topic. There are many types of synchronization, such as phase synchronization,[20] cluster synchronization,[21] and complete synchronization.[22,23] In this paper, the complete synchronization of hypernetworks is considered.

As one of the methods to study network synchronization, the master stability function[24,25] method has received extensive attention. According to the master stability function method, the factors that affect the synchronized region (SR) of networks include not only the node dynamics and the network topology, but also the inner-coupling functions and the coupling delays. In 2018, Tang *et al.*[9] studied the influence of three different types of inter-layer coupling functions on the intra-synchronized regions in a duplex network with coupling delays. Through numerical simulations, they found that there exists an inter-layer coupling function, which makes the inter-layer coupling strength neither increase nor decrease the intra-layer synchronizability. Shortly after, Wu *et al.*[10] investigated the impact of different inner-coupling functions on synchronization of duplex networks with identical and nonidentical intra-layer structures. They have shown that the intra-layer structural differences of the duplex network weaken both intra-layer and inter-layer synchronizability. However, focusing on duplex networks alone is not enough. Hypernetwork, as a typical multilayer network, becomes the object of the master stability function method in the paper.

In 2012, Sorrentino[13] derived the master stability equation for hypernetworks and then proposed an approach to decoupling the master stability equation for three cases. Shortly afterward, Irving and Sorrentino[14] proposed a general framework to study the stability of synchronous solutions for hypernetworks and showed that arbitrarily large networks can be reduced to a set of subsystems of no more than two dimensions by diagonalization of matrix blocks. In 2014, Bilal and Ramaswamy[15] analyzed the connection matrix of hypernetworks and found that a sufficiently small row sum of the connection matrix can lead to the phenomenon of amplitude or oscillation death. However, little attention has been paid to the impact of inner-coupling and time delay on the synchronization of hypernetworks. In reality, time delay on networks is generally unavoidable and has a significant impact on the stability of a synchronous solution.[26] Therefore, it is necessary and meaningful to investigate the impact of both the inner-coupling and the time delay on hypernetwork synchronization.

Motivated by above discussions, in this paper, we further investigate the impact of different inner-couplings and time delays on synchronization of a single-layer network and a hypernetwork, respectively. Particularly, the Rössler chaotic oscillator is employed to be the node dynamics. The rest of this paper is organized as follows. The model of a hypernetwork composed of two interaction layers is introduced in Sec. II. The impact of different inner-couplings and time delays is studied in detail in Sec. III. Finally, some conclusions are given in Sec. IV.

II. MODEL

Consider a hypernetwork consisting of N oscillators with two interaction layers,

$$\dot{x}_i(t) = f(x_i(t)) - c_1\sum_{j=1}^{N} l_{ij}^{(1)} H^{(1)}(x_j(t-\tau_1)) - c_2\sum_{j=1}^{N} l_{ij}^{(2)} H^{(2)}(x_j(t-\tau_2)), \tag{1}$$

where $i = 1, 2, \ldots, N$, and $x_i = (x_i^1, x_i^2, \ldots, x_i^m)^{\mathrm{T}}$ denotes the m-dimensional state of the ith node, $f: \mathbb{R}^m \to \mathbb{R}^m$ is a function describing the dynamics of an isolated oscillator, c_k $(k = 1, 2)$ represents the intra-coupling strength of the kth interaction layer, $L^{(k)} = (l_{ij}^{(k)})_{N\times N}$ is the Laplacian matrix describing the topology of the kth interaction layer and satisfying the property of zero row sum, $H^{(k)}: \mathbb{R}^m \to \mathbb{R}^m$ is the inner-coupling function determining how the oscillators are coupled, and τ_k is the time delay in the kth interaction layer.

The well-known Rössler chaotic oscillator is employed to be the node dynamics f due to the fact that the Rössler system is widely used for secure communication, which is described by

$$\begin{cases} \dot{x}^1 = -x^2 - x^3, \\ \dot{x}^2 = x^1 + ax^2, \\ \dot{x}^3 = x^3(x^1 - c) + b, \end{cases}$$

with $a = b = 0.2$ and $c = 9$.

III. MAIN RESULTS

In this section, the master stability function of the hypernetwork (1) is derived. The hypernetwork (1) satisfies the four conditions proposed by Pecora and Carroll.[24] Furthermore, the impact of inner-coupling and time delay on SR of a single-layer network and a hypernetwork is illustrated.

A. The master stability function

The hypernetwork (1) achieves complete synchronization if the states of all the oscillators are identical; i.e.,

$$x_1(t) = x_2(t) = \cdots = x_N(t) = s(t),$$

where $s(t)$ represents the synchronous state[27] described by $\dot{s}(t) = f(s(t))$.

Linearizing Eq. (1) at $s(t)$ gives

$$\begin{aligned} \delta\dot{x}_i(t) = {} & Jf(s(t))\delta x_i(t) \\ & - c_1\sum_{j=1}^{N} l_{ij}^{(1)} JH^{(1)}(s(t-\tau_1))\delta x_j(t-\tau_1) \\ & - c_2\sum_{j=1}^{N} l_{ij}^{(2)} JH^{(2)}(s(t-\tau_2))\delta x_j(t-\tau_2), \end{aligned} \tag{2}$$

where $\delta x_i(t) = x_i(t) - s(t)$. Jf represents the Jacobian matrix of the function f. Denoting $\delta X(t) = [\delta x_1(t)^T, \delta x_2(t)^T, \ldots, \delta x_N(t)^T]^T$, Eq. (2)

can be rewritten in the following compact form:

$$\begin{aligned}\delta\dot{X}(t) &= I_N \otimes Jf(s(t))\delta X(t)\\ &\quad - c_1 L^{(1)} \otimes JH^{(1)}(s(t-\tau_1))\delta X(t-\tau_1)\\ &\quad - c_2 L^{(2)} \otimes JH^{(2)}(s(t-\tau_2))\delta X(t-\tau_2),\end{aligned} \tag{3}$$

where $\otimes$ is the Kronecker product. For simplicity, $H^{(k)}(x) = H^{(k)}x$.

Assume that the two Laplacian matrices $L^{(1)}$ and $L^{(2)}$ are symmetric and commutable.[28] Then, there exists an orthogonal matrix P such that $P^T L^{(k)} P = D^{(k)}$, where $D_k = \mathrm{diag}\{\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_N^{(k)}\}$ and $0 = \lambda_1^{(k)} < \lambda_2^{(k)} \le \cdots \le \lambda_N^{(k)}$. Obviously, the main diagonal elements of the matrix $D^{(k)}$ are the eigenvalues of the matrix $L^{(k)}$. The columns of the matrix P are the corresponding eigenvectors.

By introducing $\xi(t) = (P^T \otimes I_m)\delta X(t)$, Eq. (3) transforms into

$$\begin{aligned}\dot{\xi}(t) &= I_N \otimes Jf(s(t))\xi(t)\\ &\quad - c_1 D^{(1)} \otimes JH^{(1)}(s(t-\tau_1))\xi(t-\tau_1)\\ &\quad - c_2 D^{(2)} \otimes JH^{(2)}(s(t-\tau_2))\xi(t-\tau_2).\end{aligned} \tag{4}$$

Denote $\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_N(t)]^T$. Equation (4) can be reduced to an N independent variational equation,

$$\begin{aligned}\dot{\xi}_p(t) &= Jf(s(t))\xi_p(t)\\ &\quad - c_1\lambda_p^{(1)} JH^{(1)}(s(t-\tau_1))\xi_p(t-\tau_1)\\ &\quad - c_2\lambda_p^{(2)} JH^{(2)}(s(t-\tau_2))\xi_p(t-\tau_2),\end{aligned} \tag{5}$$

where $p = 1, 2, \ldots, N$. Consequently, to investigate the stability of system (5), define $\alpha = c_1\lambda_p^{(1)}$, $\beta = c_2\lambda_p^{(2)}$ and then consider the following master stability equation:

$$\begin{aligned}\dot{y}(t) &= Jf(s)y(t)\\ &\quad - \alpha JH^{(1)}(s(t-\tau_1))y(t-\tau_1)\\ &\quad - \beta JH^{(2)}(s(t-\tau_2))y(t-\tau_2).\end{aligned} \tag{6}$$

Fixing the functions f and $H^{(k)}$, the largest Lyapunov exponent $LLE(\alpha, \beta, \tau_1, \tau_2)$ of Eq. (6) would be a function of α, β, τ_1, and τ_2, which is the so-called master stability function.

When adding a small perturbation to the synchronous state, the perturbation will decrease exponentially if $LLE(\alpha, \beta, \tau_1, \tau_2)$ is negative. That is, the synchronous solution is stable and the synchronization is achieved. Precisely, a network is more synchronizable if $LLE(\alpha, \beta, \tau_1, \tau_2)$ is smaller. On the contrary, the sychronous solution is unstable if $LLE(\alpha, \beta, \tau_1, \tau_2) > 0$. The SR of the hypernetwork (1) is denoted by $\{(\alpha, \beta, \tau_1, \tau_2) | LLE(\alpha, \beta, \tau_1, \tau_2) < 0\}$.

Fixing τ_1 and τ_2, the SRs of the hypernetwork (1) can be classified into three types[29] according to the master stability function:

Type I: $LLE(\alpha, \beta, \tau_1, \tau_2) < 0$ within a bounded area of the half plane ($\alpha \geqslant 0, \beta \geqslant 0$), meaning that the SR of the hypernetwork (1) is **bounded**.

Type II: $LLE(\alpha, \beta, \tau_1, \tau_2) < 0$ within an unbounded area of the half plane($\alpha \geqslant 0, \beta \geqslant 0$), meaning that the SR of the hypernetwork (1) is **unbounded**.

Type III: $LLE(\alpha, \beta, \tau_1, \tau_2)$ is always positive in the half plane ($\alpha \geqslant 0, \beta \geqslant 0$), meaning that the SR of the hypernetwork (1) is **empty**, or equivalently, the synchronous solution of the hypernetwork (1) is always unstable whatever the topology is.

B. Impact of inner-coupling

This subsection numerically investigates the impact of different inner-coupling matrices on SR of the hypernetwork (1). The inner-coupling matrices in all simulations are chosen from the following three typical ones:

$$I_{11} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ I_{22} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix},\ I_{33} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

To better demonstrate the impact of inner-coupling matrices, set time delay $\tau_1 = \tau_2 = 0$ in this subsection. Then, the hypernetwork (1) reduces to

$$\dot{x}_i = f(x_i) - c_1\sum_{j=1}^{N} l_{ij}^{(1)} H^{(1)}(x_j) - c_2\sum_{j=1}^{N} l_{ij}^{(2)} H^{(2)}(x_j), \tag{7}$$

where $i = 1, 2, \ldots, N$. The master stability equation (6) is simplified to

$$\dot{y} = \left[JF(s) - \alpha JH^{(1)}(s) - \beta JH^{(2)}(s)\right] y. \tag{8}$$

If $H^{(1)} = 0$ or $H^{(2)} = 0$, the network (7) becomes a single-layer network. Alternatively, if both of them are nonzero, then the network (7) is a hypernetwork.

1. Single-layer network

Consider that one of the matrices $H^{(1)}$ and $H^{(2)}$ is zero. Without loss of generality, assume that $H^{(2)} = 0$ and $H^{(1)} = I_{ii}, i = 1, 2, 3$, then the hypernetwork (7) transforms into a single-layer network, and the corresponding master stability equation is given by

$$\dot{y} = \left[JF(s) - \alpha JH^{(1)}(s)\right] y. \tag{9}$$

In general, different inner-coupling matrices result in different SRs. The variation of LLE of the hypernetwork (7) with parameter α is shown in Fig. 1. The small panel in Fig. 1 shows the case when parameter α is taken from 6 to 500. It is found that when $H^{(1)}$ is set as I_{11}, I_{22}, or I_{33}, the SR of the hypernetwork (7) is bounded, unbounded, or empty, respectively.

2. Hypernetwork

Consider that both $H^{(1)}$ and $H^{(2)}$ are nonzero. Different $H^{(1)}$-$H^{(2)}$ pairs lead to different SRs. There are in total six different $H^{(1)}$-$H^{(2)}$ pairs, and the corresponding SRs are demonstrated in Fig. 2.

When $\beta = 0$, the two-dimensional SR degenerate into one-dimensional ones of single-layer networks with $H^{(1)} = I_{ii}$ and $H^{(2)} = 0$. From Fig. 2(c), it is interesting to see the unbounded SR of the hypernetwork (7), even if the SR of the first subnetwork is empty and that of the second subnetwork is bounded. Besides, once the SRs of

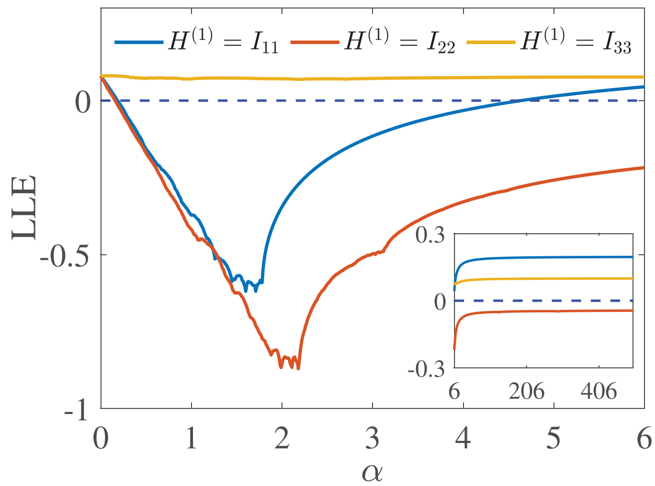

FIG. 1. The *LLE* of the hypernetwork (7) as a function of parameter α, where $H^{(2)} = 0$ and $H^{(1)}$ are set as I_{11}, I_{22}, and I_{33}, respectively.

one subnetwork is unbounded, the SR of the whole hypernetwork is unbounded too. To summarize, the type of SR of the hypernetwork (7) with different inner-coupling matrices is depicted in Table I. It is seen that the SR belongs to type I if and only if $H^{(1)} = H^{(2)} = I_{11}$, type III if and only if $H^{(1)} = H^{(2)} = I_{33}$, and type II for all the other cases. Therefore, the inner-coupling matrix has a significant impact on the type of SR.

C. Impact of time delay

In this subsection, we investigate the impact of different time delays on SR. For simplicity, let $\tau_1 = \tau_2 = \tau$. Then, Eq. (1) can be rewritten as

$$\dot{x}_i(t) = f(x_i(t)) - c_1 \sum_{j=1}^{N} l_{ij}^{(1)} H^{(1)}(x_j(t-\tau)) - c_2 \sum_{j=1}^{N} l_{ij}^{(2)} H^{(2)}(x_j(t-\tau)), \tag{10}$$

and the master stability equation (6) becomes

$$\dot{y} = JF(s)y - \left[\alpha J H^{(1)}(s) + \beta J H^{(2)}(s)\right] y(t-\tau). \tag{11}$$

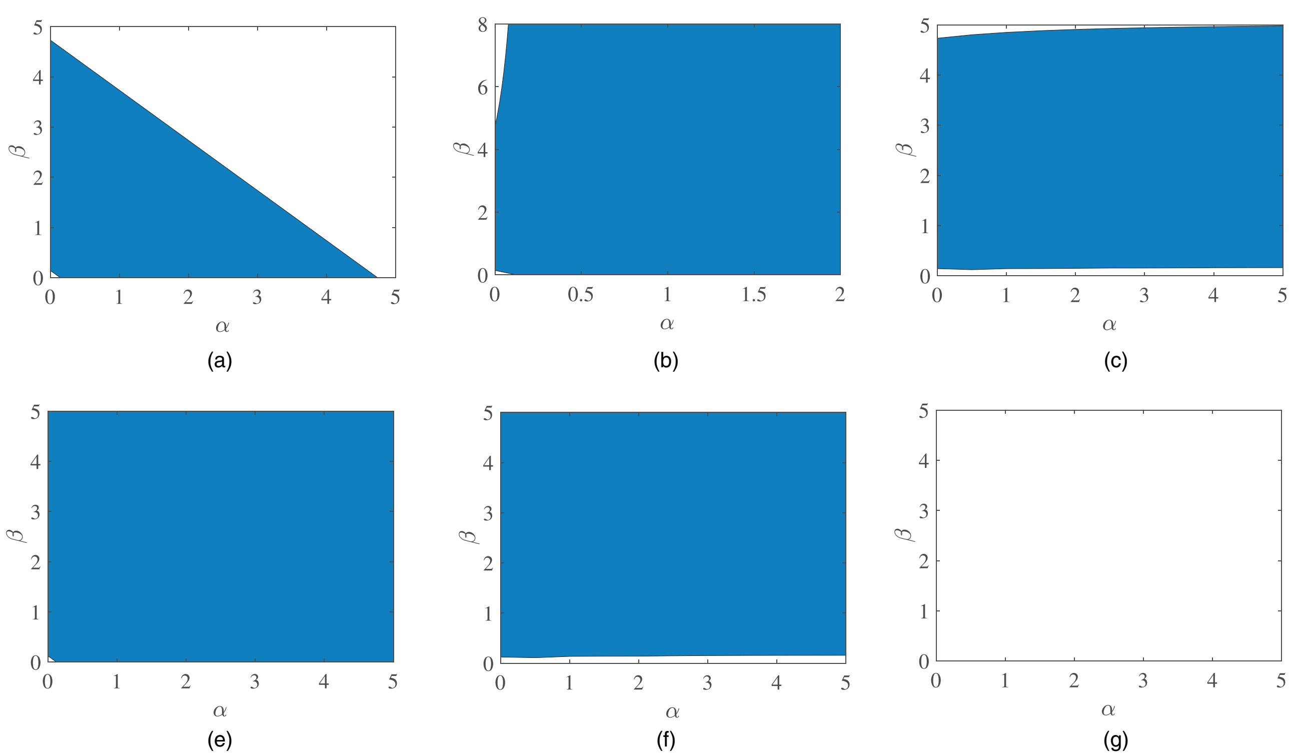

FIG. 2. SRs of the hypernetwork (7) in the parameter space of α and β, where $H^{(1)}$ and $H^{(2)}$ are chosen from $\{I_{11}, I_{22}, I_{33}\}$. The SR (i.e., $LLE < 0$) is shown in blue. (a) $H^{(1)} = I_{11}$, $H^{(2)} = I_{11}$. (b) $H^{(1)} = I_{22}$, $H^{(2)} = I_{11}$. (c) $H^{(1)} = I_{33}$, $H^{(2)} = I_{11}$. (d) $H^{(1)} = I_{22}$, $H^{(2)} = I_{22}$. (e) $H^{(1)} = I_{33}$, $H^{(2)} = I_{22}$. (f) $H^{(1)} = I_{33}$, $H^{(2)} = I_{33}$.

TABLE I. The type of SR of the hypernetwork (7) with different inner-coupling matrix combination. Type I, type II, and type III represent bounded, unbounded, and empty SR, respectively. The types in bold represent the typical bounded, unbounded, and empty SR.

$H^{(1)}$	$H^{(2)}$: 0	I_{11}	I_{22}	I_{33}
0	...	Type I	Type II	Type III
I_{11}	**Type I**	**Type I**	Type II	Type II
I_{22}	**Type II**	Type II	**Type II**	Type II
I_{33}	**Type III**	Type II	Type II	**Type III**

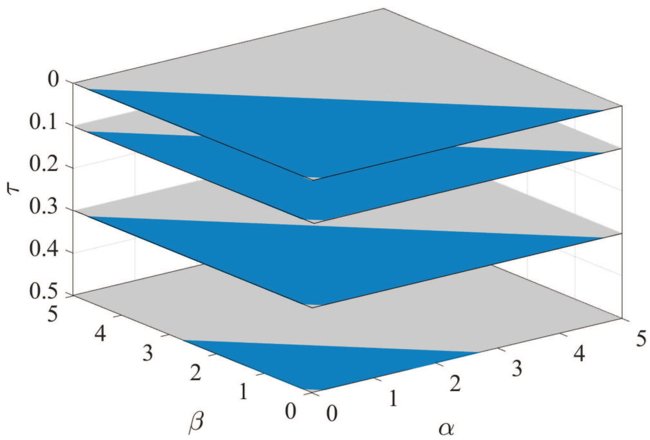

FIG. 4. The SRs of the hypernetwork (10) in the parameter space of α and β, where the inner-coupling matrices $H^{(1)} = H^{(2)} = I_{11}$ and the time delay $\tau = 0, 0.1, 0.3$, and 0.5, respectively. The SR and the ASR are shown in blue and gray, respectively.

1. Single-layer network

Without loss of generality, assume that $H^{(2)} = 0$. Then, Eq. (11) turns to

$$\dot{y} = JF(s)y - \alpha JH^{(1)}(s)y(t-\tau). \tag{12}$$

The variation of the *LLE* of the hypernetwork (10) with respect to α is plotted in Fig. 3, where coupling delay $\tau = 0, 0.1, 0.3$, and 0.5, respectively. When $H^{(1)} = I_{11}$ and $H^{(2)} = 0$, it is found from panel (a) of Fig. 3 that the function $LLE(\alpha)$ has two intersection points with the α-axis regardless of the value of τ, which are denoted as α_1 and α_2. Hence, the SR of the hypernetwork (10) is a bounded interval, and the interval is (α_1, α_2). The impact of coupling delay τ on the SR (α_1, α_2) is interesting. The relation of α_1 and α_2 with τ is displayed in the small panel of Fig. 3(a). It is seen that α_1 almost does not vary with τ; that is, the lower bound of SR is insensitive to time delay. On the contrary, time delay τ has a remarkable influence on the upper bound α_2 of SR. When $\tau \in [0, \tau_c]$ with $\tau_c = 0.32$, α_2 changes little with τ. When τ is bigger than the critical point τ_c, however, α_2 becomes smaller with the increase of τ. That is, a large enough delay could shrink SR of the hypernetwork (10). To sum up, SR is insensitive to the time delay τ if τ is smaller than the critical point τ_c, and SR shrinks if τ gets bigger than τ_c. When $H^{(1)} = I_{22}$ and $H^{(2)} = 0$, the SRs corresponding to different time delays are displayed in panel (b) of Fig. 3. When time delay $\tau = 0$, function $LLE(\alpha)$ has only one intersection point with the α-axis, which is denoted as α_3. This indicates that SR of the hypernetwork (10) is an unbounded interval $(\alpha_3, +\infty)$. When increasing time delay, the function $LLE(\alpha)$ has another intersection point with the α-axis, which is denoted as α_4, indicating that SR becomes bounded. The relation between α_4 and τ is displayed in the small panel of Fig. 3(b). When continue to increase the time delay, SR is still bounded, and the bounded interval shrinks if enlarging τ according to the small panel of Fig. 3(b). Therefore, the time delay can change not only the size but also the type of SR. Particularly, the lower bound α_3 of SR is insensitive to the time delay, which is the same as the case of $H^{(1)} = I_{11}$ and $H^{(2)} = 0$. When $H^{(1)} = I_{33}$ and $H^{(2)} = 0$, panel (c) of Fig. 3 demonstrates that SR of the hypernetwork (10) is empty whatever the time delay τ is; i.e., the SR is always of type III. Therefore, the hypernetwork (10) cannot achieve synchronization if $H^{(1)} = I_{33}$ and $H^{(2)} = 0$.

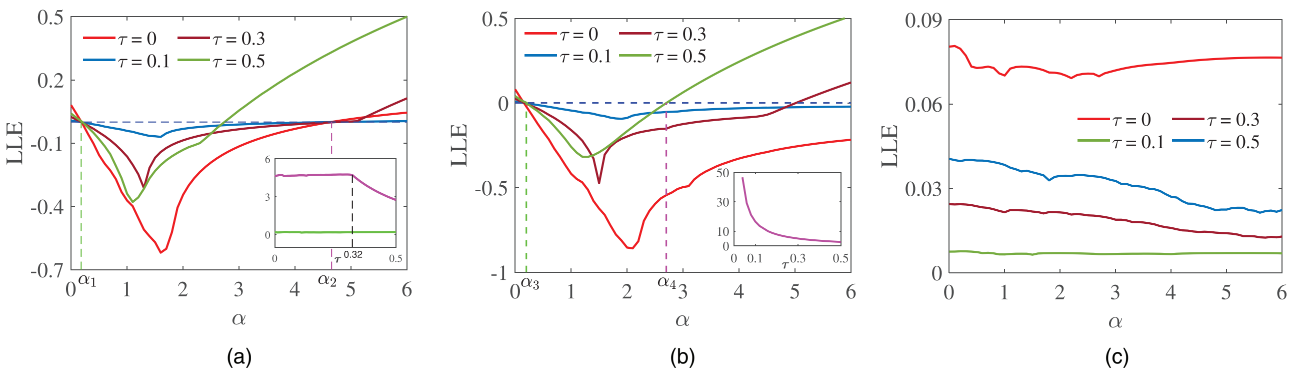

FIG. 3. The variation of *LLE* as a function of parameter α. The coupling delay $\tau = 0, 0.1, 0.3$, and 0.5, respectively. (a) $H^{(1)} = I_{11}$, $H^{(2)} = 0$. (b) $H^{(1)} = I_{22}$, $H^{(2)} = 0$. (c) $H^{(1)} = I_{33}$, $H^{(2)} = 0$.

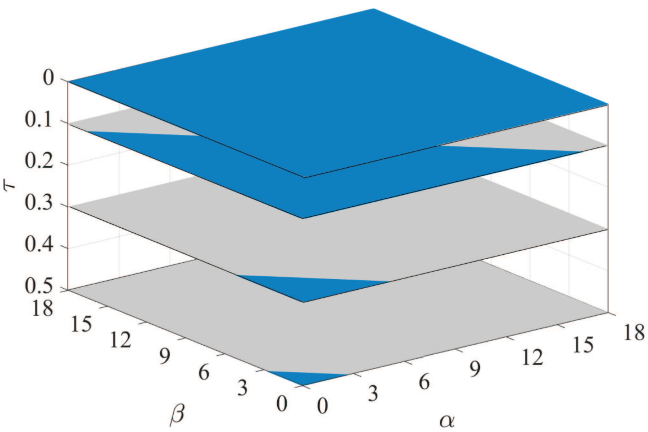

FIG. 5. The SRs of the hypernetwork (10) in the parameter space of α and β, where the inner-coupling matrices $H^{(1)} = H^{(2)} = I_{22}$ and the time delay $\tau = 0, 0.1, 0.3$, and 0.5, respectively. The SR and the ASR are shown in blue and gray, respectively.

2. Hypernetwork

Consider the hypernetwork (10) with nonzero inner-coupling matrices $H^{(1)}$ and $H^{(2)}$. It has been shown in Table I that when inner-coupling matrices are chosen as $H^{(1)} = H^{(2)} = I_{11}, H^{(1)} = H^{(2)} = I_{22}$ and $H^{(1)} = H^{(2)} = I_{33}$, the SRs are bounded, unbounded, and empty, respectively. Hence, the three cases are considered. SRs of the hypernetwork (10) corresponding to different coupling delays are illustrated in Figs. 4 and 5, where SR is in blue and the asynchronized region is in gray.

When $H^{(1)} = H^{(2)} = I_{11}$, Fig. 4 demonstrates that SR is a bounded area, and the ASR consists of two connected components whatever time delay τ is. Denote the connected component around 0 as D_1 and the other one as D_2. It is found that the component D_1 almost does not vary with the time delay, while the component D_2 keeps constant when $\tau \leq 0.3$ and enlarges when increasing τ to 0.5. Therefore, time delay τ has no impact on the type of SR, but shrinks the size of SR when τ is large enough. Particularly, the results here are consistent with those of the single-layer case.

When $H^{(1)} = H^{(2)} = I_{22}$, SR is shown in Fig. 5. It is found that SR of the hypernetwork (10) is an unbounded area only in the case of $\tau = 0$. However, when $\tau \neq 0$, SR of the hypernetwork (10) changes from an unbounded to a bounded region. The type of SR also changed from type II to type I. Moreover, SR shrinks with the increase of τ. In general, SR is sensitive to the time delay. The time delay can change not only the size but also the type of SR.

When $H^{(1)} = H^{(2)} = I_{33}$, SR of the hypernetwork (10) is always empty regardless of the change of time delay τ.

IV. CONCLUSIONS

In conclusion, this paper has studied the variation of SR of hypernetworks with three types of inner-coupling matrices and different time delays. The results show that the inner-coupling matrix has a significant impact on the type of SR of a hypernetwork, and the synchronization stability of hypernetworks can be achieved by coupling an unstable networked layer with a stable one. Once the SRs of the subnetworks are unbounded, the SR of the whole hypernetwork is unbounded too. The SR of the whole hypernetwork is empty (bounded) if and only if the SRs of all the subnetworks are empty (bounded). In addition, the time delay plays an important role in the synchronization of hypernetworks. With bounded or empty SR, there exists an interval in which the time delay has almost no effect on the stability of the synchronous state. With unbounded SR, the time delay can change not only the size but the type of SR. Future works include the hypernetworks consisting of other oscillators and coupling matrices.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Heng Guo: Writing – original draft (lead); Writing – review & editing (equal). **Jin Zhou:** Data curation (equal); Funding acquisition (lead); Supervision (equal); Writing – review & editing (equal). **Shuaibing Zhu:** Supervision (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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