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SURVEY ARTICLE

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On vacuum free boundary problems in ideal compressible magnetohydrodynamics

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Abstract

We survey some recent results on vacuum free boundary problems in three-dimensional ideal compressible magnetohydrodynamics, restate the main theorems in our works (Secchi and Trakhinin, *Nonlinearity* **27**(1) (2014) 105–169; Trakhinin and Wang, *Arch. Ration. Mech. Anal.* **239**(2) (2021) 1131–1176; Trakhinin and Wang, *Math. Ann.* **383**(1–2) (2022) 761–808; Trakhinin and Wang, *SIAM J. Math. Anal.* **54**(6) (2022) 5888–5921), and provide an alternative proof for the linear well-posedness of the full plasma–vacuum interface problem under the noncollinearity condition.

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Contents

1 | INTRODUCTION

We survey some recent results on vacuum free boundary problems in three-dimensional (3D) ideal compressible magnetohydrodynamics (MHD), which can describe the evolution of an inviscid perfectly conducting fluid (e.g., plasma, liquid metal) interacting with a magnetic field and separated from a vacuum.

Let $\Omega \subset \mathbb{R}^3$ be the reference domain occupied by the fluid and the vacuum. In the fluid region $\Omega^+(t) \subset \Omega$, the motion is governed by the following ideal compressible MHD equations (see Landau–Lifshitz [31, section 65]):

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0, \\ \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v - H \otimes H) + \nabla q = 0, \\ \partial_t H - \nabla \times (v \times H) = 0, \\ \partial_t \left(\rho E + \frac{1}{2}|H|^2 \right) + \nabla \cdot (\rho E v + p v + H \times (v \times H)) = 0, \end{cases} \tag{1.1}$$

together with the divergence-free equation

$$\nabla \cdot H = 0. \tag{1.2}$$

Here density ρ, fluid velocity $v = (v_1, v_2, v_3)^\top$, magnetic field $H = (H_1, H_2, H_3)^\top$, and pressure p are unknown functions of time t and space variable $x = (x_1, x_2, x_3)$. We denote by $q = p + \frac{1}{2}|H|^2$ the total pressure and by $E = \mathfrak{e} + \frac{1}{2}|v|^2$ the specific total pressure, where $\mathfrak{e}$ is the specific internal energy. It is known from thermodynamics that the density ρ and the internal energy $\mathfrak{e}$ are given functions of the pressure p and the specific entropy S, which renders the system of equations (1.1) closed for the primary unknowns $U := (p, v, H, S)^\top \in \mathbb{R}^8$. The constitutive relations $\rho = \rho(p, S)$ and $\mathfrak{e} = \mathfrak{e}(p, S)$ are assumed to be smooth and satisfy the physical condition that the sound speed $a = a(\rho, S)$ is positive:

$$a(\rho, S) := \sqrt{\frac{\partial p}{\partial \rho}(\rho, S)} > 0 \qquad \text{for all } \rho \in (\rho_*, \rho^*), \tag{1.3}$$

where ρ_* and ρ^* are some nonnegative constants. Our constitutive relations are very general and include the polytropic and barotropic cases as special examples.

To symmetrize equations (1.1), we take into account the Gibbs relation $\vartheta\,\mathrm{d}S = \mathrm{d}\mathfrak{e} + p\,\mathrm{d}(1/\rho)$, where $\vartheta > 0$ is the absolute temperature. In view of (1.2)–(1.3), we find that smooth solutions of (1.1) with $\rho_* < \rho < \rho^*$ satisfy the equivalent symmetric hyperbolic system

$$A_0^+(U)\partial_t U + \sum_{j=1}^{3} A_j^+(U)\partial_j U = 0 \qquad \text{in } \Omega^+(t), \tag{1.4}$$

where $A_0^+(U) := \mathrm{diag}(1/(\rho a^2), \rho, \rho, \rho, 1, 1, 1, 1)$ and

$$A_j^+(U) := \begin{pmatrix} \dfrac{v_j}{\rho a^2} & \mathbf{e}_j^\top & 0 & 0 \\ \mathbf{e}_j & \rho v_j I_3 & \mathbf{e}_j \otimes H - H_j I_3 & 0 \\ 0 & H \otimes \mathbf{e}_j - H_j I_3 & v_j I_3 & 0 \\ 0 & 0 & 0 & v_j \end{pmatrix} \quad \text{for } j = 1, 2, 3.$$

Here and below, $\mathbf{e}_j := (\delta_{1j}, \delta_{2j}, \delta_{3j})^\top$ and $I_m := (\delta_{ij})_{m\times m}$ with δ_{ij} being the Kronecker delta. In [51, 52, 63–65], we adopt a different symmetrization by taking the total pressure q instead of the pressure p as one of the primary unknowns. According to the local existence results in [30] or [37, chapter 2] for general symmetric hyperbolic systems, the Cauchy problem of the compressible MHD equations (1.1)–(1.3) allows smooth nonvacuum solutions within a short time.

In the vacuum region $\Omega^-(t) \subset \Omega$, for vacuum magnetic field $h = (h_1, h_2, h_3)^\top$ and vacuum electric field $e = (e_1, e_2, e_3)^\top$, we consider the pre-Maxwell equations

$$\nabla \times h = 0, \qquad \nabla \cdot h = 0, \tag{1.5}$$

$$\nabla \times e = -\partial_t h, \qquad \nabla \cdot e = 0, \tag{1.6}$$

where we have neglected the displacement current from Maxwell's equations in vacuum as in nonrelativistic MHD. The vacuum electric field e in (1.5)–(1.6) is a secondary variable, so that the dynamics in $\Omega^-(t)$ can be described by the elliptic (div-curl) system (1.5), or equivalently,

$$\sum_{j=1}^{3} A_j^- \partial_j h = 0 \qquad \text{in } \Omega^-(t), \tag{1.7}$$

where the constant matrices A_1^-, A_2^-, and A_3^- are defined by

$$A_1^- := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2^- := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_3^- := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We refer the interested reader to [6, 7, 38, 42, 61] for the stability of nonrelativistic or relativistic plasma–vacuum interfaces with displacement current in vacuum.

For technical simplicity, we assume that $\Omega^{\pm}(t) = \{x \in \Omega : x_1 \gtrless \varphi(t,x')\}$ and the plasma–vacuum interface is given by the form of a graph

$$\Sigma(t) := \{x \in \Omega : x_1 = \varphi(t,x')\} \quad \text{with } x' = (x_2, x_3),$$

where the interface function φ is to be determined. The case of more general interfaces can be dealt with by standard but technically involved arguments as in [59, Remark 2.3]. To unify the presentation, we focus on the case of $\Omega = (-1,1)\times\mathbb{T}^2$ with boundaries $\Sigma^{\pm} := \{\pm 1\}\times\mathbb{T}^2$, where $\mathbb{T}^2$ denotes the 2-torus and can be thought of as the unit square with periodic boundary conditions. For the plasma–vacuum system the boundary conditions read as

$$q - \tfrac{1}{2}|h|^2 = \mathfrak{s}\mathcal{H}(\varphi), \quad \partial_t\varphi = v\cdot N \qquad \text{on } \Sigma(t), \tag{1.8a}$$

$$H\cdot N = 0, \quad h\cdot N = 0 \qquad \text{on } \Sigma(t), \tag{1.8b}$$

$$H_1 = 0, \quad v_1 = 0 \qquad \text{on } \Sigma^+, \tag{1.8c}$$

$$h \times \mathbf{e}_1 = \boldsymbol{j}_{\mathrm{c}} \qquad \text{on } \Sigma^-, \tag{1.8d}$$

where $\mathfrak{s} \geqslant 0$ is the constant surface-tension coefficient, $\mathcal{H}(\varphi)$ is twice the mean curvature of $\Sigma(t)$ defined by

$$\mathcal{H}(\varphi) := \mathrm{D}_{x'}\cdot\left(\frac{\mathrm{D}_{x'}\varphi}{\sqrt{1+|\mathrm{D}_{x'}\varphi|^2}}\right) \quad \text{with } \mathrm{D}_{x'} := \begin{pmatrix}\partial_2\\ \partial_3\end{pmatrix},$$

and $N := (1, -\partial_2\varphi, -\partial_3\varphi)^{\mathsf{T}}$ is the normal to $\Sigma(t)$. The vector function $\boldsymbol{j}_{\mathrm{c}}$ represents a given surface current that forces oscillations onto the plasma–vacuum system. For laboratory plasmas, this external excitation may be caused by a system of coils; see [19, section 4.6] for a thorough discussion of the condition (1.8d). The first condition in (1.8a) comes from the balance of the normal stresses at the interface [17], while the second condition in (1.8a) means that the interface moves with the motion of the fluid. Note that the effect of surface tension becomes especially important in modeling the flows of liquid metals [45]. Conditions (1.8b) state that the fluid and vacuum magnetic fields are tangential to the interface, while conditions (1.8c) are the perfectly conducting wall and impermeable conditions.

We supplement (1.4) and (1.7)–(1.8) with the initial conditions

$$\varphi|_{t=0} = \varphi_0, \qquad U|_{t=0} = U_0, \tag{1.9}$$

where $\|\varphi_0\|_{L^\infty(\mathbb{T}^2)} < 1$. Note that the vacuum magnetic field $h \in \mathbb{R}^3$ can be uniquely determined from the elliptic problem consisting of (1.7), the second condition in (1.8b), and (1.8d) when the interface function φ is given. It is worth mentioning that system (1.4), (1.7)–(1.9) is a nonlinear hyperbolic–elliptic coupled problem with a characteristic free boundary.

The absence of the magnetic field ($H = h \equiv 0$) reduces the system into the compressible Euler equations with free boundary. For zero surface tension ($\mathfrak{s} = 0$), Ebin [18] showed that the free-boundary incompressible Euler equations is ill-posed provided the following Taylor sign condition fails:

$$\boldsymbol{n}\cdot\nabla p \leqslant -\kappa < 0 \quad \text{on } \Sigma(t), \tag{1.10}$$

where $\boldsymbol{n}$ denotes the outward unit normal to the interface $\Sigma(t)$ and κ is a positive constant. The local well-posedness for two-dimensional (2D) and 3D irrotational ideal flows was established in the seminal works of Wu [66, 67]. Without irrotationality, the local existence was proved in [15, 33, 69] and [34, 40, 59], respectively, for incompressible and compressible liquids ($\rho|_{\Sigma(t)} > 0$) under the Taylor sign condition (1.10), and in [16, 29, 36] for gases ($\rho|_{\Sigma(t)} = 0$) under the physical vacuum condition. For positive surface tension ($\mathfrak{s} > 0$), the local existence was obtained in [3, 4, 14, 15, 55] for liquids without imposing the Taylor sign condition (1.10). The results of [3, 4, 14, 15, 55] indicate that surface tension provides a regularizing effect on the moving vacuum boundary. We refer to [3, 4, 14, 54] for the zero surface tension limit of the free-boundary Euler equations.

The full plasma–vacuum interface problem (1.4), (1.7)–(1.9) appears in the mathematical modeling of plasma confinement by magnetic fields [19, section 4.6]. In this model, the plasma is confined inside a rigid, perfectly conducting wall and separated from a vacuum. The extensive study of this model began from the middle of the last century; see Bernstein et al. [5] for the stability criteria of equilibrium states and Grad [20] for a survey of some early works.

For model (1.4), (1.7)–(1.9) with zero surface tension ($\mathfrak{s} = 0$), the second author [60] proposed two different well-posedness conditions for the linearization. The first one is the noncollinearity condition, stating that the magnetic fields on either side of the interface are not collinear:

$$|H \times h| \geqslant \kappa > 0 \quad \text{on } \Sigma(t), \tag{1.11}$$

which enhances the regularity of the moving interface and stems from the study of compressible current-vortex sheets in [57, 58] (see also [9, 10]). The second one is the MHD counterpart of the Taylor sign condition (1.10), which reads as

$$\boldsymbol{n} \cdot \nabla \left(q - \frac{1}{2}|h|^2 \right) \leqslant -\kappa < 0 \quad \text{on } \Sigma(t). \tag{1.12}$$

In [60], basic a priori estimates were derived, respectively, for the *variable coefficient* linearized problem under the noncollinearity condition (1.11) and for the *constant coefficient* linearized problem under the Taylor-type sign condition (1.12).

Based on the linear results in [51, 60], the first two authors [52] proved the first local well-posedness theorem for the full plasma–vacuum interface problem (1.4), (1.7)–(1.9) with $\mathfrak{s} = 0$ under condition (1.11). However, the noncollinearity condition (1.11) excludes the important case with zero magnetic field. Motivated by this fact, the second and third authors [63] studied the free boundary problem (1.4), (1.8)–(1.9) for $h \equiv 0$ and showed the first local well-posedness result under the Taylor-type sign condition (1.12). Recently Lindblad–Zhang [35] established the a priori estimates without loss of anisotropic regularity and hence improved the nonrelativistic result in [63]. Even so, it is still an open problem to extend the results in [35, 63] to the case of gases (cf. [16, 29, 36]). Moreover, the local well-posedness for the plasma–vacuum interface problem (1.4), (1.7)–(1.9) is still unknown for nontrivial vacuum magnetic field without the noncollinearity condition (1.11); see [62] for a thorough discussion of this issue.

In view of the works [3, 4, 14, 15, 55], we would expect that surface tension could have a stabilizing effect also on the motion of plasma–vacuum interfaces. This expectation was confirmed rigorously by the second and third authors in [64, 65], where they proved the local existence of solutions to the nonlinear problem (1.4), (1.7)–(1.9) with surface tension replacing the Taylor-type sign condition (1.12) and the noncollinearity condition (1.11). Regarding the incompressible plasma–vacuum interfaces, we refer to [49, 53] for the qualitative behavior of surface waves, [23–25, 27, 43, 56] and [26], respectively, for the well-posedness and ill-posedness without surface tension, [21, 32] for the well-posedness with surface tension, [22, 32] for the zero surface tension limit. It will be interesting to investigate the zero surface tension limit for compressible plasma–vacuum interface problems.

The approach for solving the nonlinear problem (1.4), (1.7)–(1.9) in our works [52, 63–65] involves the reduction to a fixed domain, the application of Alinhac's good unknowns, the existence and tame estimates in certain functional spaces for the linearized problem, and an appropriate Nash–Moser iteration scheme. We work in the anisotropic Sobolev spaces H^m_* first introduced by Chen [11] with different regularity in the normal and tangential directions. See [47] for the well-posedness of general characteristic symmetric hyperbolic systems and [9, 46, 58, 68] for the study of other characteristic boundary problems in ideal compressible MHD.

To solve the linearized problem of the hyperbolic–elliptic coupled system (1.4), (1.7)–(1.9) for $\mathfrak{s} = 0$, the first two authors [51, 52] introduced a fully hyperbolic ε-regularization and proved the existence of solutions by the fixed point argument with the noncollinearity condition (1.11) satisfied for the basic state. Then passing to the limit as $\varepsilon \to 0$ yields the linear well-posedness. For the case of vanishing magnetic field and the case of positive surface tension, the second and third authors [63–65] proved the existence of the linearized problem by the duality argument. In [52, 63–65], the high-order energy estimates for the linearized problem exhibit a *fixed* loss of regularity from the basic state and source terms to the solution, and the local solutions for the nonlinear problem are constructed by an appropriate modification of the Nash–Moser iteration scheme developed by Hörmander [28] and Coulombel–Secchi [13]. In particular, a smooth intermediate state is introduced and estimated, so that the state around which we linearize at each iteration step can satisfy certain constraints for the linear solvability. See Alinhac–Gérard [2] and Secchi [50] for a more general presentation of the Nash–Moser method.

The plan of the rest of this paper is as follows. In Section 2, we reduce the system (1.4), (1.7)–(1.9) to an equivalent fixed-boundary problem and restate the main theorems in our works [52, 63–65]. In Section 3, we sketch the proof of linear well-posedness in the case of zero surface tension under the noncollinearity condition by employing the duality argument as in [63–65], which is alternative to the method used in [51, 52].

2 | NONLINEAR WELL-POSEDNESS THEOREMS

In this section, we reduce the nonlinear free boundary problem (1.4), (1.7)–(1.9) to an equivalent fixed-boundary problem, introduce anisotropic Sobolev spaces for later use, and restate the main theorems in our works [52, 63–65].

2.1 | Equivalent fixed-boundary problems

Let us reformulate the free boundary problem (1.4), (1.7)–(1.9) into an equivalent fixed-boundary problem by introducing $U_\sharp(t,x) := U(t,\Phi(t,x),x')$ and $h_\sharp(t,x) := h(t,\Phi(t,x),x')$. We choose the lifting function Φ as

$$\Phi(t,x) := x_1 + \chi(x_1)\varphi(t,x'),$$

where $\chi \in C_0^\infty(-1,1)$ is the cut-off function that satisfies $\|\chi'\|_{L^\infty(\mathbb{R})} < 4/(\|\varphi_0\|_{L^\infty(\mathbb{T}^2)} + 3)$ and equals to 1 on a small neighborhood of the origin. See [12, 52] for another change of variables, which can gain one half derivative. The free boundary problem (1.4), (1.7)–(1.9) can be reduced to the following nonlinear fixed boundary problem:

$$\mathbb{L}_+(U,\Phi) := L_+(U,\Phi)U = 0 \qquad \text{in } \Omega^+ := (0,1)\times\mathbb{T}^2, \tag{2.1a}$$

$$\mathbb{L}_-(h,\Phi) := L_-(\Phi)h = 0 \qquad \text{in } \Omega^- := (-1,0)\times\mathbb{T}^2, \tag{2.1b}$$

$$\mathbb{B}(U,h,\varphi) = 0 \qquad \text{on } \Sigma^3\times\Sigma^+\times\Sigma^-, \tag{2.1c}$$

$$U|_{t=0} = U_0, \qquad \varphi|_{t=0} = \varphi_0, \tag{2.1d}$$

where we have dropped the subscript "♯" for convenience, $\Sigma := \{0\}\times\mathbb{T}^2$, and

$$L_+(U,\Phi) := A_0^+(U)\partial_t + \widetilde{A}_1^+(U,\Phi)\partial_1 + A_2^+(U)\partial_2 + A_3^+(U)\partial_3, \tag{2.2}$$

$$L_-(\Phi) := \widetilde{A}_1^-(\Phi)\partial_1 + A_2^-\partial_2 + A_3^-\partial_3, \tag{2.3}$$

$$\mathbb{B}(U,h,\varphi) := \begin{pmatrix} \partial_t\varphi - v\cdot N \\ q - \frac{1}{2}|h|^2 - \mathfrak{s}\mathcal{H}(\varphi) \\ h\cdot N \\ v_1 \\ h\times\mathbf{e}_1 - \boldsymbol{j}_{\mathrm{c}} \end{pmatrix}, \tag{2.4}$$

with $\widetilde{A}_1^-(\Phi) := (A_1^- - \partial_2\Phi A_2^- - \partial_3\Phi A_3^-)/\partial_1\Phi$ and

$$\widetilde{A}_1^+(U,\Phi) := \frac{1}{\partial_1\Phi}\big(A_1^+(U) - \partial_t\Phi A_0^+(U) - \partial_2\Phi A_2^+(U) - \partial_3\Phi A_3^+(U)\big).$$

In (2.1c), we employ the notation $\Sigma^3\times\Sigma^+\times\Sigma^-$ to denote that the first three components of this vector equation are taken on Σ, the fourth one on Σ^+, and the fifth one on Σ^-. The equations for H contained in (2.1a) can be written as

$$\mathbb{H}(H,v,\Phi) := \big(\partial_t^\Phi + v\cdot\nabla^\Phi\big)H - (H\cdot\nabla^\Phi)v + H\nabla^\Phi\cdot v = 0 \quad \text{in } \Omega^+, \tag{2.5}$$

where

$$\partial_t^\Phi := \partial_t - \frac{\partial_t\Phi}{\partial_1\Phi}\partial_1,\ \nabla^\Phi := \big(\partial_1^\Phi,\partial_2^\Phi,\partial_3^\Phi\big)^{\mathsf{T}},\ \partial_1^\Phi := \frac{\partial_1}{\partial_1\Phi},\ \partial_j^\Phi := \partial_j - \frac{\partial_j\Phi}{\partial_1\Phi}\partial_1 \tag{2.6}$$

for $j = 2,3$. In the new variables, Equation (1.2) and first conditions in (1.8b)–(1.8c) become

$$\nabla^\Phi\cdot H = 0 \quad \text{in } \Omega^+, \qquad H\cdot N = 0 \quad \text{on } \Sigma, \qquad H_1 = 0 \quad \text{on } \Sigma^+, \tag{2.7}$$

which can be regarded as initial constraints, meaning that they hold for $t > 0$ as long as they are satisfied initially; see [58, appendix A] for the detailed proof.

2.2 | Anisotropic Sobolev spaces

Let us introduce the anisotropic Sobolev spaces to be used in this paper. We denote

$$\mathrm{D}_*^\alpha := \partial_t^{\alpha_0}(\sigma\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}\partial_1^{\alpha_4} \qquad \text{for } \alpha := (\alpha_0, \dots, \alpha_4) \in \mathbb{N}^5, \tag{2.8}$$

where $\sigma = \sigma(x_1)$ is a positive C^∞–function on (0,1) such that $\sigma(x_1) = x_1$ in a neighborhood of the origin and $\sigma(x_1) = 1 - x_1$ in a neighborhood of $x_1 = 1$. For $m \in \mathbb{N}$ and $I \subset \mathbb{R}$, the anisotropic Sobolev space $H_*^m(I \times \Omega^+)$ is defined as

$$H_*^m(I \times \Omega^+) := \{u \in L^2(I \times \Omega^+) : \ \mathrm{D}_*^\alpha u \in L^2(I \times \Omega^+) \text{ for } \langle\alpha\rangle \leqslant m\},$$

and equipped with the norm $\|\cdot\|_{H_*^m(I\times\Omega^+)}$, where

$$\langle\alpha\rangle := \sum_{i=0}^{3} \alpha_i + 2\alpha_4, \qquad \|u\|^2_{H_*^m(I\times\Omega^+)} := \sum_{\langle\alpha\rangle\leqslant m} \|\mathrm{D}_*^\alpha u\|^2_{L^2(I\times\Omega^+)}.$$

By definition, $H^m(I \times \Omega^+) \hookrightarrow H_*^m(I \times \Omega^+) \hookrightarrow H^{\lfloor m/2\rfloor}(I \times \Omega^+)$ for all $m \in \mathbb{N}$ and $I \subset \mathbb{R}$, where $\lfloor s\rfloor$ denotes the floor function of $s \in \mathbb{R}$ that maps s to the greatest integer less than or equal to s. We refer to [11, 41, 48], and references therein for an extensive study of anisotropic Sobolev spaces.

2.3 | Main theorems

To present the main theorems, we first introduce the compatibility conditions on the initial data. For this purpose, we assume that for some integer $m \geqslant 3$, the initial data $U_0 \in H^{m+3/2}(\Omega^+)$ and $\varphi_0 \in H^{m+2}(\mathbb{T}^2)$ satisfy $\|\varphi_0\|_{L^\infty(\mathbb{T}^2)} < 1$ and the hyperbolicity condition

$$\rho_* < \inf_{\Omega^+} \rho(U_0) \leqslant \sup_{\Omega^+} \rho(U_0) < \rho^*, \tag{2.9}$$

where the nonnegative constants ρ_*, ρ^* are specified in (1.3). Then

$$\partial_1\Phi_0 \geqslant \frac{3\big(1 - \|\varphi_0\|_{L^\infty(\mathbb{T}^2)}\big)}{3 + \|\varphi_0\|_{L^\infty(\mathbb{T}^2)}} > 0 \quad \text{for } \Phi_0(x) := x_1 + \chi(x_1)\varphi_0(x').$$

The initial vacuum magnetic field h_0 can be uniquely determined by the div-curl system

$$L_-(\Phi_0)h_0 = 0 \text{ in } \Omega^-, \quad h_0 \cdot N_0 = 0 \text{ on } \Sigma, \quad h_0 \times \mathbf{e}_1 = \boldsymbol{j}_{\mathrm{c}}(0) \text{ on } \Sigma^-,$$

where L_- is the operator given by (2.3) and $N_0 := (1, -\partial_2\varphi_0, -\partial_3\varphi_0)^\top$. Define $U_{(j)} := \partial_t^j U|_{t=0}$ and $\varphi_{(j)} := \partial_t^j \varphi|_{t=0}$ for any $j \in \mathbb{N}$. Taking j time derivatives of the interior equations (2.1a) and the first condition in (2.1c), we evaluate the resulting identities at the initial time to determine $U_{(j)}$ and $\varphi_{(j)}$ inductively. Then we set $h_{(j)} := \partial_t^j h|_{t=0}$ as the unique solution of the elliptic problem that results from taking j time derivatives of the equations (2.1b), the third condition in (2.1c), and the second condition in (2.1d). More precisely, we have the following result (see [52, section 9] for the proof).

Lemma 2.1. *Let $m \in \mathbb{N}$ with $m \geqslant 3$ and $\boldsymbol{j}_{\mathrm{c}} \in H^{m+3/2}([0, T_0] \times \Sigma^-)$ for some $T_0 > 0$. Assume that $(U_0, \varphi_0) \in H^{m+3/2}(\Omega^+) \times H^{m+2}(\mathbb{T}^2)$ satisfy $\|\varphi_0\|_{L^\infty(\mathbb{T}^2)} < 1$ and (2.9). Then the procedure described above determines $U_{(j)} \in H^{m+3/2-j}(\Omega^+)$, $\varphi_{(j)} \in H^{m+2-j}(\mathbb{T}^2)$, and $h_{(j)} \in H^{m+3/2-j}(\Omega^-)$, for $j = 0, 1, \dots, m$, satisfying*

$$\sum_{j=0}^{m} \left(\|U_{(j)}\|_{H^{m+3/2-j}(\Omega^+)} + \|\varphi_{(j)}\|_{H^{m+2-j}(\mathbb{T}^2)} + \|h_{(j)}\|_{H^{m+3/2-j}(\Omega^-)} \right) \leqslant C(M_0),$$

where $M_0 := \|U_0\|_{H^{m+3/2}(\Omega^+)} + \|\varphi_0\|_{H^{m+2}(\mathbb{T}^2)} + \|\boldsymbol{j}_{\mathrm{c}}\|_{H^{m+3/2}([0,T_0]\times\Sigma^-)}$ and $C(M_0) > 0$ is some constant depending on M_0.

Taking j time derivatives of the second condition in (2.1c) yields the following terminology for the compatibility conditions on the initial data.

Definition 2.1. Assume that all the conditions of Lemma 2.1 are satisfied. The initial data (U_0, φ_0) are said to fulfill the compatibility conditions up to order m, if $U_{(j)}$, $\varphi_{(j)}$, and $h_{(j)}$ satisfy the boundary conditions $v_{1(j)}|_{\Sigma^+} = 0$ and

$$\begin{aligned} q_{(j)} = {} & \sum_{i=0}^{j-1} \binom{j-1}{i} h_{(i)} \cdot h_{(j-i)} \\ & + \mathfrak{s} \sum_{\substack{\alpha_i \in \mathbb{N}^2 \\ |\alpha_1| + \cdots + j|\alpha_j| = j}} \mathrm{D}_{x'} \cdot \left(\mathrm{D}_{\zeta}^{\alpha_1 + \cdots + \alpha_j} \mathfrak{f}(\zeta_{(0)}) \, j! \prod_{i=1}^{j} \frac{1}{\alpha_i!} \left(\frac{\zeta_{(i)}}{i!} \right)^{\alpha_i} \right) \quad \text{on } \Sigma, \end{aligned}$$

for $j = 0, \dots, m$, where $\zeta_{(i)} := \mathrm{D}_{x'} \varphi_{(i)} \in \mathbb{R}^2$ and $\mathfrak{f}(\zeta) := \zeta / \sqrt{1 + |\zeta|^2}$.

Now we are ready to restate the main theorems in [52, 63–65].

Theorem 2.1 [52, Theorem 5]. *Assume that $\mathfrak{s} = 0$ and $\boldsymbol{j}_{\mathrm{c}} \in H^{m+3/2}([0, T_0] \times \Sigma^-)$ for some $T_0 > 0$ and $m \in \mathbb{N}$ with $m \geqslant 20$. Assume further that the initial data $(U_0, \varphi_0) \in H^{m+3/2}(\Omega^+) \times H^{m+2}(\mathbb{T}^2)$ satisfy $\|\varphi_0\|_{L^\infty(\mathbb{T}^2)} < 1$, the constraints (2.7), the hyperbolicity condition (2.9), the compatibility conditions up to order m, and the noncollinearity condition*

$$|H_0 \times h_0| \big|_{\Sigma} \geqslant \delta_0 > 0 \tag{2.10}$$

for some fixed constant δ_0. Then problem (2.1) admits a unique solution (U, h, φ) in $H_*^{m-9}([0, T] \times \Omega^+) \times H^{m-9}([0, T] \times \Omega^-) \times H^{m-9}([0, T] \times \mathbb{T}^2)$ for some $T > 0$.

Theorem 2.2 [63, Theorem 2.1]. *Let $\mathfrak{s} = 0$ and $\boldsymbol{j}_{\mathrm{c}} \equiv 0$. Suppose that the initial data $(U_0, \varphi_0) \in H^{m+3/2}(\Omega^+) \times H^{m+2}(\mathbb{T}^2)$ satisfy $\|\varphi_0\|_{L^\infty(\mathbb{T}^2)} < 1$, the constraints (2.7), the hyperbolicity condition (2.9), the compatibility conditions up to order m, and the sign condition*

$$\partial_1 q_0 \big|_{\Sigma} \geqslant \delta_0 > 0 \tag{2.11}$$

for some fixed constant δ_0. Then problem (2.1) admits a unique solution $(U, 0, \varphi)$ on $[0, T]$ for some $T > 0$ satisfying $U \in H_*^{m-9}([0, T] \times \Omega^+)$ and $\varphi \in H^{m-9}([0, T] \times \mathbb{T}^2)$.

Remark 2.1. In our case of a simply connected vacuum domain Ω^-, the assumption $\boldsymbol{j}_{\rm c} \equiv 0$ implies that the vacuum magnetic field h vanishes on the whole domain. For zero vacuum magnetic field, we refer to [35] for a priori estimates without loss of anisotropic regularity and [63] for existence of solutions in the relativistic case.

Theorem 2.3 [65, Theorem 2.1]. *Assume that* $\mathfrak{s} > 0$ *and* $\boldsymbol{j}_{\rm c} \in H^{m+3/2}([0,T_0]\times\Sigma^-)$ *for some* $T_0 > 0$ *and* $m \in \mathbb{N}$ *with* $m \geqslant 20$. *Assume further that* $(U_0, \varphi_0) \in H^{m+3/2}(\Omega^+)\times H^{m+2}(\mathbb{T}^2)$ *satisfy* $\|\varphi_0\|_{L^\infty(\mathbb{T}^2)} < 1$, *the constraints (2.7), the hyperbolicity condition (2.9), and the compatibility conditions up to order m. Then there exists a small time* $T > 0$, *such that problem (2.1) has a unique solution* (U,h,φ) *in* $H^{m-9}_*([0,T]\times\Omega^+)\times H^{m-9}([0,T]\times\Omega^-)\times H^{m-9}([0,T]\times\mathbb{T}^2)$ *satisfying* $\mathrm{D}_{x'}\varphi \in H^{m-9}([0,T]\times\mathbb{T}^2)$.

Remark 2.2. The main result of [64] gives the existence of solutions to the problem (2.1) (or equivalently, the free boundary problem (1.4), (1.7)–(1.9)) in the case of positive surface tension and zero vacuum magnetic field, which corresponds to Theorem 2.3 with $\boldsymbol{j}_{\rm c} \equiv 0$.

3 | WELL-POSEDNESS FOR THE LINEARIZED PROBLEM

This section is devoted to sketching the proof of existence and high-order estimates for the linearized problem with zero surface tension $\mathfrak{s} = 0$ under the noncollinearity condition (2.10) by means of the duality argument in [63–65], which is alternative to the method used in [51, 52].

3.1 | Linearization

We first perform the linearization of the nonlinear problem (2.1) with $\mathfrak{s} = 0$ around a suitable basic state.

3.1.1 | Basic state

Assume that the basic state $(\mathring{U}(t,x), \mathring{h}(t,x), \mathring{\varphi}(t,x'))$ is a sufficiently smooth vector-function defined on $\Omega_T^+\times\Omega_T^-\times\Sigma_T$ and satisfies

$$\|\mathring{\varphi}\|_{L^\infty(\Sigma_T)} \leqslant \tfrac{1}{2}(\|\varphi_0\|_{L^\infty(\mathbb{T}^2)} + 1), \tag{3.1}$$

$$\rho_* < \rho(\mathring{U}) < \rho^* \qquad \text{on } \overline{\Omega_T^+}, \tag{3.2}$$

$$\|\mathring{U}\|_{H^{10}_*(\Omega_T^+)} + \|\mathring{h}\|_{H^{10}(\Omega_T^-)} + \|\mathring{\varphi}\|_{H^{10}(\Sigma_T)} \leqslant K \tag{3.3}$$

for some constant $K > 0$, where $\mathring{U} = (\mathring{p}, \mathring{v}, \mathring{H}, \mathring{S})^{\mathsf{T}} \in \mathbb{R}^8$, $\mathring{h} = (\mathring{h}_1, \mathring{h}_2, \mathring{h}_3)^{\mathsf{T}} \in \mathbb{R}^3$, $\Omega_T^\pm := (-\infty, T)\times\Omega^\pm$, and $\Sigma_T := (-\infty,T)\times\Sigma$. Then we have $\partial_1\mathring{\Phi} > 0$ on $\overline{\Omega_T}$, where $\Omega_T := (-\infty,T)\times\Omega$ and $\mathring{\Phi}(t,x) := x_1 + \mathring{\Psi}(t,x)$ with $\mathring{\Psi}(t,x) := \chi(x_1)\mathring{\varphi}(t,x')$. We also assume that

$$\mathbb{H}(\mathring{H}, \mathring{v}, \mathring{\Phi}) = 0 \qquad \text{in } \Omega_T^+, \tag{3.4}$$

$$\partial_t \mathring{\varphi} = \mathring{v} \cdot \mathring{N}, \quad \mathring{h} \cdot \mathring{N} = 0 \qquad \text{on } \Sigma_T, \tag{3.5}$$

$$\partial_1 \mathring{h} \cdot \mathring{N} + \partial_2 \mathring{h}_2 + \partial_3 \mathring{h}_3 = 0 \qquad \text{on } \Sigma_T, \tag{3.6}$$

$$\mathring{v}_1 = 0 \quad \text{on } \Sigma_T^+, \qquad \mathring{h} \times \mathbf{e}_1 = \boldsymbol{j}_{\mathrm{c}} \qquad \text{on } \Sigma_T^-, \tag{3.7}$$

where $\mathbb{H}$ is the operator defined in (2.5), $\mathring{N} := (1, -\partial_2\mathring{\Phi}, -\partial_3\mathring{\Phi})^{\mathsf{T}}$, and $\Sigma_T^{\pm} := (-\infty, T) \times \Sigma^{\pm}$. It follows from (3.4) that the identities

$$\nabla^{\mathring{\Phi}} \cdot \mathring{H}|_{\Omega_T^+} = 0 \qquad \mathring{H} \cdot \mathring{N}|_{\Sigma_T} = 0, \qquad \mathring{H}_1|_{\Sigma_T^+} = 0 \tag{3.8}$$

are satisfied if they hold at the initial time (see [58, appendix B] for the proof). As such, we require that the conditions (3.8) are satisfied at $t = 0$. Moreover, we assume that the noncollinearity condition holds for the basic state (cf. (2.10)):

$$|\mathring{H} \times \mathring{h}| \geqslant \frac{\delta_0}{2} > 0 \qquad \text{on } \Sigma_T. \tag{3.9}$$

3.1.2 | Linearized problem

Introduce the good unknowns of Alinhac [1]:

$$\dot{V} := V - \frac{\Psi}{\partial_1 \mathring{\Phi}} \partial_1 \mathring{U}, \quad \dot{h} := h - \frac{\Psi}{\partial_1 \mathring{\Phi}} \partial_1 \mathring{h}, \tag{3.10}$$

for $V := (p, v, H, S)^{\mathsf{T}}$ and $\Psi(t, x) := \chi(x_1)\psi(t, x')$. Then the linearized operators for Equations (2.1a)–(2.1b) around the basic state $(\mathring{U}, \mathring{h}, \mathring{\varphi})$ are defined and simplified as (cf. [39, Proposition 1.3.1])

$$\begin{aligned} \mathbb{L}'_+(\mathring{U}, \mathring{\Phi})(V, \Psi) &:= \frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{L}_+(\mathring{U} + \theta V, \mathring{\Phi} + \theta \Psi)\Big|_{\theta=0} \\ &= L_+(\mathring{U}, \mathring{\Phi})\dot{V} + \mathcal{C}_+(\mathring{U}, \mathring{\Phi})\dot{V} + \frac{\Psi}{\partial_1 \mathring{\Phi}} \partial_1 \mathbb{L}_+(\mathring{U}, \mathring{\Phi}), \end{aligned} \tag{3.11}$$

$$\mathbb{L}'_-(\mathring{h}, \mathring{\Phi})(h, \Psi) := \frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{L}_-(\mathring{h} + \theta h, \mathring{\Phi} + \theta \Psi)\Big|_{\theta=0} = L_-(\mathring{\Phi})\dot{h} + \frac{\Psi}{\partial_1 \mathring{\Phi}} \partial_1 \mathbb{L}_-(\mathring{h}, \mathring{\Phi}), \tag{3.12}$$

where $L_{\pm}$ are the operators defined in (2.2)–(2.3) and

$$\mathcal{C}_+(U, \Phi)V := \sum_{k=1}^{8} V_k \left(\frac{\partial \widetilde{A}_1^+}{\partial U_k}(U, \Phi)\partial_1 U + \sum_{i=0,2,3} \frac{\partial A_i^+}{\partial U_k}(U)\partial_i U \right).$$

For the boundary operator $\mathbb{B}$ defined by (2.4), we have

$$\mathbb{B}'(\mathring{U}, \mathring{h}, \mathring{\varphi})(V, h, \psi) := \frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{B}(\mathring{U} + \theta V,\ \mathring{h} + \theta h,\ \mathring{\varphi} + \theta \psi)\Big|_{\theta=0}$$

$$= \left((\partial_t + \mathring{v}' \cdot \mathrm{D}_{x'})\psi - v \cdot \mathring{N},\ p + \mathring{H} \cdot H - \mathring{h} \cdot h,\ h \cdot \mathring{N} - \mathring{h}' \cdot \mathrm{D}_{x'}\psi,\ v_1,\ h \times \mathbf{e}_1\right)^{\mathsf{T}},$$

where we denote $z' := (z_2, z_3)^{\mathsf{T}}$ for any vector $z := (z_1, z_2, z_3)^{\mathsf{T}}$.

We apply the good unknowns (3.10) and neglect the last terms in (3.11)–(3.12) to study the following effective linear problem:

$$\mathbb{L}'_{e+}(\mathring{U},\mathring{\Phi})\dot{V} := L_+(\mathring{U},\mathring{\Phi})\dot{V} + \mathcal{C}_+(\mathring{U},\mathring{\Phi})\dot{V} = f^+ \qquad \text{in } \Omega_T^+, \tag{3.13a}$$

$$L_-(\mathring{\Phi})\dot{h} = f^- \qquad \text{in } \Omega_T^-, \tag{3.13b}$$

$$\mathbb{B}'_e(\mathring{U},\mathring{h},\mathring{\varphi})(\dot{V},\dot{h},\psi) = g \qquad \text{on } \Sigma_T^3\times\Sigma_T^+\times\Sigma_T^-, \tag{3.13c}$$

$$(\dot{V},\psi)\big|_{t<0} = 0, \qquad \dot{h}\big|_{t<0} = 0, \tag{3.13d}$$

where

$$\mathbb{B}'_e(\mathring{U},\mathring{h},\mathring{\varphi})(\dot{V},\dot{h},\psi) := \begin{pmatrix} (\partial_t + \mathring{v}'\cdot \mathrm{D}_{x'} + \mathring{b}_1)\psi - \dot{v}\cdot\mathring{N} \\ \dot{p} + \mathring{H}\cdot\dot{H} - \mathring{h}\cdot\dot{h} + \mathring{b}_2\psi \\ \dot{h}\cdot\mathring{N} - \mathrm{D}_{x'}\cdot(\mathring{h}'\psi) \\ \dot{v}_1 \\ \dot{h}\times\mathbf{e}_1 \end{pmatrix} \tag{3.14}$$

with $\mathring{b}_1 := -\partial_1\mathring{v}\cdot\mathring{N}$ and $\mathring{b}_2 := \partial_1\mathring{p} + \mathring{H}\cdot\partial_1\mathring{H} - \mathring{h}\cdot\partial_1\mathring{h}$, thanks to the identity $\mathbb{B}'_e(\mathring{U},\mathring{h},\mathring{\varphi})(\dot{V},\dot{h},\psi) = \mathbb{B}'(\mathring{U},\mathring{h},\mathring{\varphi})(V,h,\psi)$, the definitions (3.10), and the constraint (3.6). The last terms in (3.11)–(3.12) will be considered as error terms at each iteration step.

The source terms $f^\pm$ and g are supposed to vanish in the past, so that the second equation in (3.13d) follows from (3.13b), the third and fifth equations in (3.13c), and the first equation in (3.13d). In particular, the fifth, sixth, and seventh components of (3.13a) read as

$$\begin{aligned}\mathbb{H}'_e(\mathring{H},\mathring{v},\mathring{\Phi})(\dot{H},\dot{v}) :=\ & \big(\partial_t^{\mathring{\Phi}} + \mathring{v}\cdot\nabla^{\mathring{\Phi}}\big)\dot{H} - (\mathring{H}\cdot\nabla^{\mathring{\Phi}})\dot{v} + \mathring{H}\nabla^{\mathring{\Phi}}\cdot\dot{v} + (\dot{v}\cdot\nabla^{\mathring{\Phi}})\mathring{H} \\ & - (\dot{H}\cdot\nabla^{\mathring{\Phi}})\mathring{v} + \dot{H}\nabla^{\mathring{\Phi}}\cdot\mathring{v} = \big(f_5^+, f_6^+, f_7^+\big)^{\mathsf{T}},\end{aligned} \tag{3.15}$$

where $\partial_t^{\mathring{\Phi}}$ and $\nabla^{\mathring{\Phi}}$ are defined by (2.6).

3.1.3 | Reformulation

Let us reformulate the effective linear problem (3.13) for obtaining its solvability. First, we transform the equations (3.13b) to

$$\begin{pmatrix} \nabla\times(\partial_1\mathring{\Phi}\mathring{\eta}^{-\mathsf{T}}\dot{h}) \\ \nabla\cdot(\mathring{\eta}\dot{h}) \end{pmatrix} = \begin{pmatrix} \mathring{\eta} & 0 \\ 0 & \partial_1\mathring{\Phi} \end{pmatrix} f^- =: \tilde{f}^- \qquad \text{in } \Omega_T^-, \tag{3.16}$$

where

$$\mathring{\eta} := \begin{pmatrix} 1 & -\partial_2\mathring{\Phi} & -\partial_3\mathring{\Phi} \\ 0 & \partial_1\mathring{\Phi} & 0 \\ 0 & 0 & \partial_1\mathring{\Phi} \end{pmatrix}.$$

Then we decompose $\dot{h}$ as $\dot{h} = h_\flat + h_\natural$ with $h_\natural$ solving the div-curl boundary value problem (cf. (3.16), (3.13c), and (3.14))

$$\begin{cases} \begin{pmatrix} \nabla \times (\partial_1 \mathring{\Phi} \mathring{\eta}^{-\top} h_\natural) \\ \nabla \cdot (\mathring{\eta} h_\natural) \end{pmatrix} = \tilde{f}^- \quad \text{in } \Omega_T^-, \\ h_\natural \cdot \mathring{N} = g_3 \quad \text{on } \Sigma_T, \qquad h_\natural \times \mathbf{e}_1 = g_5 \quad \text{on } \Sigma_T^-, \\ (x_2, x_3) \to h_\natural(t, x_1, x_2, x_3) \quad \text{is 1-periodic.} \end{cases} \tag{3.17}$$

The next lemma gives the H^m-estimate for the above problem (3.17) with $m \geqslant 6$ (see [52, section 6] for the proof of the resolution and H^2-estimate). We denote by C some universal positive constant and by $C(\cdot)$ some positive constant depending on the quantities listed in the parenthesis. We use $A \lesssim_{a_1,\dots,a_m} B$ to denote that $A \leqslant C(a_1, \dots, a_m)B$ for given parameters $a_1, \dots, a_m$.

Lemma 3.1. *Let $(\tilde{f}^-, g_3, g_5)$ belong to $H^{m-1}(\Omega_T^-) \times H^{m-1/2}(\Sigma_T) \times H^{m-1/2}(\Sigma_T^-)$ for some integer $m \geqslant 6$. Assume that the compatibility conditions*

$$g_5 \cdot \mathbf{e}_1|_{\Sigma^-} = 0, \qquad \int_{\Sigma^-} u \cdot g_5 = \int_{\Omega^-} u \cdot \tilde{f}^-$$

hold for all vectors $u \in H^1(\Omega^-)$ satisfying $u_2 = u_3 = 0$ on Σ and $\nabla \times u = 0$ in Ω^-. Then the problem (3.17) has a unique solution $h_\natural$ in $H^m(\Omega_T^-)$ and

$$\begin{aligned} \|h_\natural\|_{H^m(\Omega_T^-)} \lesssim_K & \left(1 + \|\mathring{\varphi}\|_{H^{m+1}(\Sigma_T)}\right) \|(\tilde{f}^-, g_3, g_5)\|_{H^5(\Omega_T^-) \times H^5(\Sigma_T) \times H^5(\Sigma_T^-)} \\ & + \|\tilde{f}^-\|_{H^{m-1}(\Omega_T^-)} + \|g_3\|_{H^{m-1/2}(\Sigma_T)} + \|g_5\|_{H^{m-1/2}(\Sigma_T^-)}, \end{aligned} \tag{3.18}$$

where K is the upper bound given in (3.3).

Next we reduce (3.13) into a problem with homogeneous boundary conditions. To this end, we introduce the decomposition $\dot{V} = V_\flat + V_\natural$ for $V_\natural = (q_\natural, v_\natural, H_\natural, 0)^\top \in \mathbb{R}^8$, where

$$v_\natural = \left(v_1^\natural, 0, 0\right)^\top \quad \text{with } v_1^\natural := \chi(x_1)\mathfrak{R}_T(-g_1) + \chi(1 - x_1)\widetilde{\mathfrak{R}}_T g_4, \tag{3.19}$$

for $\mathfrak{R}_T : H^m(\Sigma_T) \to H_*^{m+1}(\Omega_T^+)$ and $\widetilde{\mathfrak{R}}_T : H^m(\Sigma_T^+) \to H_*^{m+1}(\Omega_T^+)$ being the continuous extension operators [44], $H_\natural$ is the unique solution of

$$\mathbb{H}'_e(\mathring{H}, \mathring{v}, \mathring{\Phi})(H_\natural, v_\natural) = (f_5^+, f_6^+, f_7^+)^\top \qquad \text{in } \Omega_T^+ \tag{3.20}$$

for $\mathbb{H}'_e(\mathring{H}, \mathring{v}, \mathring{\Phi})$ being defined by (3.15), and

$$p_\natural := \mathfrak{R}_T\left(g_2 + \mathring{h} \cdot h_\natural - \mathring{H} \cdot H_\natural\right). \tag{3.21}$$

It follows from (3.13), (3.17), (3.19), and (3.21) that vectors $V_\flat = \dot{V} - V_\natural$ and $h_\flat = \dot{h} - h_\natural$ solve the problem

$$\mathbb{L}'_{e+}(\mathring{U}, \mathring{\Phi})V = \tilde{f}^+ := f^+ - \mathbb{L}'_{e+}(\mathring{U}, \mathring{\Phi})V_\natural \qquad \text{in } \Omega_T^+, \tag{3.22a}$$

$$\nabla \times \left(\partial_1 \mathring{\Phi} \mathring{\eta}^{-\top} h\right) = 0, \quad \nabla \cdot (\mathring{\eta} h) = 0 \qquad \text{in } \Omega_T^-, \tag{3.22b}$$

$$\mathbb{B}_e'(\mathring{U}, \mathring{h}, \mathring{\varphi})(V, h, \psi) = 0 \qquad \text{on } \Sigma_T^3 \times \Sigma_T^+ \times \Sigma_T^-, \tag{3.22c}$$

$$(V, h, \psi) = 0 \qquad \text{if } t < 0, \tag{3.22d}$$

where we have dropped the subscript "♭" for notational simplicity. Utilize (3.15) and (3.20) to get $(\tilde{f}_5^+, \tilde{f}_6^+, \tilde{f}_7^+) = 0$, which implies that solutions of (3.22) satisfy

$$\partial_1(H \cdot \mathring{N}) + \partial_2(\partial_1 \mathring{\Phi} H_2) + \partial_3(\partial_1 \mathring{\Phi} H_3) = 0 \qquad \text{in } \Omega_T^+, \tag{3.23}$$

$$H \cdot \mathring{N} = \mathring{H}_2 \partial_2 \psi + \mathring{H}_3 \partial_3 \psi - \partial_1 \mathring{H} \cdot \mathring{N} \psi \qquad \text{on } \Sigma_T, \tag{3.24}$$

$$H_1 = 0 \qquad \text{on } \Sigma_T^+, \tag{3.25}$$

thanks to [58, Proposition 2].

By virtue of (3.22b), we can rewrite the problem (3.22) in terms of the scalar potential ξ determined by

$$\nabla \xi = \partial_1 \mathring{\Phi} \mathring{\eta}^{-\top} h \qquad \text{in } \Omega_T^-. \tag{3.26}$$

Then $\nabla \cdot (\mathring{A} \nabla \xi) = \nabla \cdot (\mathring{\eta} h) = 0$ in Ω_T^-, where

$$\mathring{A} := \frac{1}{\partial_1 \mathring{\Phi}} \mathring{\eta} \mathring{\eta}^\top. \tag{3.27}$$

Moreover, we set

$$W := (p + \mathring{H} \cdot H, v \cdot \mathring{N}, v_2, v_3, H \cdot \mathring{N}, H_2, H_3, S)^\top = J(\mathring{\Phi})^{-1} V,$$

where

$$J(\mathring{\Phi}) := \begin{pmatrix} 1 & 0 & 0 & 0 & -\mathring{H}_1 & -\mathring{H}_1 \partial_2 \mathring{\Phi} - \mathring{H}_2 & -\mathring{H}_1 \partial_3 \mathring{\Phi} - \mathring{H}_3 & 0 \\ 0 & 1 & \partial_2 \mathring{\Phi} & \partial_3 \mathring{\Phi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \partial_2 \mathring{\Phi} & \partial_3 \mathring{\Phi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the problem (3.22) can be reduced to

$$\mathbf{L} W := \sum_{i=0}^{3} \boldsymbol{A}_i \partial_i W + \boldsymbol{A}_4 W = \boldsymbol{f} := J(\mathring{\Phi})^\top \tilde{f}^+ \qquad \text{in } \Omega_T^+, \tag{3.28a}$$

$$\nabla \cdot (\mathring{A}\nabla\xi) = 0 \qquad \text{in } \Omega_T^-, \tag{3.28b}$$

$$W_2 = (\partial_t + \mathring{v}' \cdot \mathrm{D}_{x'} + \mathring{b}_1)\psi \qquad \text{on } \Sigma_T, \tag{3.28c}$$

$$W_1 = \mathring{h}' \cdot \mathrm{D}_{x'}\xi - \mathring{b}_2\psi \qquad \text{on } \Sigma_T, \tag{3.28d}$$

$$(\mathring{A}\nabla\xi)_1 = \mathrm{D}_{x'} \cdot (\mathring{h}'\psi) \qquad \text{on } \Sigma_T, \tag{3.28e}$$

$$W_2 = 0 \quad \text{on } \Sigma_T^+, \qquad \xi = 0 \quad \text{on } \Sigma_T^-, \qquad (W, \xi, \psi)|_{t<0} = 0, \tag{3.28f}$$

where $\boldsymbol{A}_i := J(\mathring{\Phi})^\top A_i^+(\mathring{U})J(\mathring{\Phi})$ for $i = 0, 2, 3$, $\boldsymbol{A}_1 := J(\mathring{\Phi})^\top \widetilde{A}_1^+(\mathring{U}, \mathring{\Phi})J(\mathring{\Phi})$, and $\boldsymbol{A}_4 := J(\mathring{\Phi})^\top \mathbb{L}'_{e+}(\mathring{U}, \mathring{\Phi})J(\mathring{\Phi})$. Besides, the identities (3.23)–(3.25) become

$$\partial_1 W_5 + \partial_2(\partial_1\mathring{\Phi}W_6) + \partial_3(\partial_1\mathring{\Phi}W_7) = 0 \qquad \text{in } \Omega_T^+, \tag{3.29}$$

$$W_5 = \mathring{H}_2\partial_2\psi + \mathring{H}_3\partial_3\psi - \partial_1\mathring{H} \cdot \mathring{N}\psi \qquad \text{on } \Sigma_T, \tag{3.30}$$

$$W_5 = 0 \qquad \text{on } \Sigma_T^+. \tag{3.31}$$

It follows from (3.5) and (3.7)–(3.8) that

$$\boldsymbol{A}_1|_\Sigma = \boldsymbol{A}_1|_{\Sigma^+} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & O_6 \end{pmatrix} =: \boldsymbol{A}_1^{(1)}, \tag{3.32}$$

where O_m denotes the zero matrix of order m. We introduce $\boldsymbol{A}_1^{(0)} := \boldsymbol{A} - \boldsymbol{A}_1^{(1)}$ so that $\boldsymbol{A}_1^{(0)}|_\Sigma = \boldsymbol{A}_1^{(0)}|_{\Sigma^+} = 0$.

3.2 | Existence for the ε-regularization

Let us introduce for the problem (3.28) a suitable ε-regularization so that we can close the L^2 estimates of the ε-regularization and its dual problem, which allows us to derive the existence of solutions to the regularized problem for any small fixed $\varepsilon > 0$ by using the duality argument.

3.2.1 | Regularized problem

To solve the problem (3.28), we introduce the following ε-regularization:

$$\sum_{i=0}^{3} \boldsymbol{A}_i\partial_i W - \varepsilon \boldsymbol{J}\partial_1 W + \boldsymbol{A}_4 W = \boldsymbol{f} \qquad \text{in } \Omega_T^+, \tag{3.33a}$$

$$\nabla \cdot (\mathring{A}\nabla\xi) = 0 \qquad \text{in } \Omega_T^-, \tag{3.33b}$$

$$W_2 = (\partial_t + \mathring{v}' \cdot \mathrm{D}_{x'} + \mathring{b}_1)\psi \qquad \text{on } \Sigma_T, \tag{3.33c}$$

$$W_1 = \mathring{h}' \cdot \mathrm{D}_{x'}\xi - \mathring{b}_2\psi \qquad \text{on } \Sigma_T, \tag{3.33d}$$

$$(\mathring{A}\nabla\xi)_1 = \mathrm{D}_{x'} \cdot (\mathring{h}'\psi) + \varepsilon\Delta_{x'}\xi \qquad \text{on } \Sigma_T, \tag{3.33e}$$

$$W_2 = 0 \quad \text{on } \Sigma_T^+, \qquad \xi = 0 \quad \text{on } \Sigma_T^-, \qquad (W, \xi, \psi)|_{t<0} = 0, \tag{3.33f}$$

where $\boldsymbol{J} := \mathrm{diag}\,(0,1,0,0,0,0,0,0)$, the matrix $\mathring{A}$ is defined in (3.27), and $\Delta_{x'} := \mathrm{D}_{x'} \cdot \mathrm{D}_{x'}$. As in [64, section 2.3] for the problem with positive surface tension and zero vacuum magnetic field, we add the term $-\varepsilon\boldsymbol{J}\partial_1 W$ in (3.33a) to derive the L^2 estimate for regularization (3.33). The term $\varepsilon\Delta_{x'}\xi$ contained in (3.33e) helps us to obtain the L^2 estimate for the dual problem of (3.33). It is important to note that (3.29)–(3.31) hold also for solutions of the regularized problem (3.33).

3.2.2 | L^2 estimate for the regularized problem

Let us first show the L^2 a priori estimate for the regularized problem (3.33). Taking the scalar product of (3.33a) by W, we utilize (3.32), (3.33d), and (3.33f) to discover

$$\int_{\Omega^+} \boldsymbol{A}_0 W \cdot W + \frac{\varepsilon}{2}\|W_2\|^2_{L^2(\Sigma_t)} \lesssim_{K,\varepsilon} \|(\boldsymbol{f}, W)\|^2_{L^2(\Omega_t^+)} + \|(\psi, \mathrm{D}_{x'}\xi)\|^2_{L^2(\Sigma_t)}. \tag{3.34}$$

To control the last term in (3.34), we multiply (3.33c) with ψ to infer

$$\|\psi(t)\|^2_{L^2(\Sigma)} \leqslant \epsilon\varepsilon\|W_2\|^2_{L^2(\Sigma_t)} + C(K, \epsilon\varepsilon)\|\psi\|^2_{L^2(\Sigma_t)} \tag{3.35}$$

for all $\epsilon > 0$. Multiplying (3.33e) by ξ leads to

$$\int_{\Sigma_t} \xi(\mathring{A}\nabla\xi)_1 + \frac{\varepsilon}{2}\|\mathrm{D}_{x'}\xi\|^2_{L^2(\Sigma_t)} \lesssim_{K,\varepsilon} \|\psi\|^2_{L^2(\Sigma_t)}. \tag{3.36}$$

Combine the estimates (3.34)–(3.36), utilize the identity

$$\int_{\Sigma_t} \xi(\mathring{A}\nabla\xi)_1 = \int_{\Omega_t^-} \nabla \cdot (\xi(\mathring{A}\nabla\xi)) = \int_{\Omega_t^-} \mathring{A}\nabla\xi \cdot \nabla\xi,$$

and take $\epsilon > 0$ small enough to deduce

$$\begin{aligned} &\|W(t)\|^2_{L^2(\Omega^+)} + \|\psi(t)\|^2_{L^2(\Sigma)} + \|\nabla\xi\|^2_{L^2(\Omega_t^-)} + \|(W_2, \mathrm{D}_{x'}\xi)\|^2_{L^2(\Sigma_t)} \\ &\quad \lesssim_{K,\varepsilon} \|(\boldsymbol{f}, W)\|^2_{L^2(\Omega_t^+)} + \|\psi\|^2_{L^2(\Sigma_t)}. \end{aligned}$$

Applying Grönwall's and Poincaré's inequalities to the last estimate yields

$$\begin{aligned} \|W(t)\|^2_{L^2(\Omega^+)} + \|\psi(t)\|^2_{L^2(\Sigma)} + \|(\xi, \nabla\xi)\|^2_{L^2(\Omega_t^-)} \\ + \|(W_2, \mathrm{D}_{x'}\xi)\|^2_{L^2(\Sigma_t)} \lesssim_{K,\varepsilon} \|\boldsymbol{f}\|^2_{L^2(\Omega_t^+)}, \end{aligned} \tag{3.37}$$

which is the desired ε-dependent L^2 estimate for the regularization (3.33).

3.2.3 | L^2 estimate for the dual problem

We will apply the duality argument to show the existence of solutions of the regularized problem (3.33). For the dual problem of (3.33), we pass to the back time $\tilde{t} := T - t$ and obtain

$$\left(\boldsymbol{A}_0 \partial_t - \sum_{i=1}^{3} \boldsymbol{A}_i \partial_i + \varepsilon \boldsymbol{J} \partial_1 + \boldsymbol{A}_4^{\mathsf{T}} - \sum_{i=0}^{3} \partial_i \boldsymbol{A}_i \right) W^* = \boldsymbol{f}^* \qquad \text{in } \Omega^+, \tag{3.38a}$$

$$\nabla \cdot (\mathring{A} \nabla \xi^*) = 0 \qquad \text{in } \Omega^-, \tag{3.38b}$$

$$\partial_t w^* - \mathrm{D}_{x'} \cdot (\mathring{v}' w^*) + \mathring{b}_1 w^* + \mathring{h}' \cdot \mathrm{D}_{x'} \xi^* - \mathring{b}_2 W_2^* = 0 \qquad \text{on } \Sigma, \tag{3.38c}$$

$$(\mathring{A} \nabla \xi^*)_1 = \mathrm{D}_{x'} \cdot (\mathring{h}' W_2^*) + \varepsilon \Delta_{x'} \xi^* \qquad \text{on } \Sigma, \tag{3.38d}$$

$$W_2^* = 0 \quad \text{on } \Sigma^+, \qquad \xi^* = 0 \quad \text{on } \Sigma^-, \qquad (W^*, \xi^*)|_{t<0} = 0, \tag{3.38e}$$

with $w^* := W_1^* - \varepsilon W_2^*$, where we have dropped the tildes for convenience.

Taking the scalar product of (3.38a) with W^*, we use (3.32) and (3.38e) to derive

$$\int_{\Omega^+} \boldsymbol{A}_0 W^* \cdot W^* + \frac{\varepsilon}{2} \|W_2^*\|^2_{L^2(\Sigma_t)} \lesssim_{K,\varepsilon} \|(\boldsymbol{f}^*, W^*)\|^2_{L^2(\Omega_t^+)} + \|w^*\|^2_{L^2(\Sigma_t)}. \tag{3.39}$$

From (3.38b) and (3.38d)–(3.38e), we have

$$\int_{\Omega_t^-} \mathring{A} \nabla \xi^* \cdot \nabla \xi^* = \int_{\Sigma_t} \xi^* (\mathring{A} \nabla \xi^*)_1 = - \int_{\Sigma_t} W_2^* \mathring{h}' \cdot \mathrm{D}_{x'} \xi^* - \varepsilon \int_{\Sigma_t} |\mathrm{D}_{x'} \xi^*|^2,$$

which implies

$$\|\nabla \xi^*\|^2_{L^2(\Omega_t^-)} + \|\mathrm{D}_{x'} \xi^*\|^2_{L^2(\Sigma_t)} \lesssim_{K,\varepsilon} \|W_2^*\|^2_{L^2(\Sigma_t)}. \tag{3.40}$$

Moreover, multiplying the boundary condition (3.38c) by w^* leads to

$$\|w^*(t)\|^2_{L^2(\Sigma)} \lesssim_K \|(w^*, W_2^*, \mathrm{D}_{x'} \xi^*)\|^2_{L^2(\Sigma_t)}. \tag{3.41}$$

In view of (3.39)–(3.41), we use Grönwall's and Poincaré's inequalities to deduce

$$\begin{aligned} &\|W^*(t)\|^2_{L^2(\Omega^+)} + \|w^*(t)\|^2_{L^2(\Sigma)} + \|(\xi^*, \nabla \xi^*)\|^2_{L^2(\Omega_t^-)} \\ &\qquad + \|(W_2^*, \mathrm{D}_{x'} \xi^*)\|^2_{L^2(\Sigma_t)} \lesssim_{K,\varepsilon} \|\boldsymbol{f}^*\|^2_{L^2(\Omega_t^+)}. \end{aligned} \tag{3.42}$$

With the ε-dependent L^2 estimates (3.37) and (3.42), we can deduce the existence of weak solutions $(W, \xi) \in L^2(\Omega_T^+) \times L^2(\Omega_T^-)$ to the regularization (3.33) for any small fixed parameter $\varepsilon \in (0, 1)$ by the standard duality argument in [8]. For the given source term $W_2|_{x_1=0} \in L^2(\Sigma_T)$ and zero initial data $\psi|_{t=0} = 0$, we can derive that the Cauchy problem for the transport equation (3.33c) for ψ has a unique solution $\psi \in C([0, T], L^2(\mathbb{T}^2))$. Therefore, for any small and fixed parameter $\varepsilon > 0$, we obtain the existence of solutions $(W, \xi, \psi) \in L^2(\Omega_T^+) \times L^2(\Omega_T^-) \times L^2((-\infty, T]; L^2(\mathbb{T}^2))$ to the regularized problem (3.33).

3.3 Uniform-in-ε estimates

Let us derive the uniform-in-ε high-order energy estimates for the regularized problem (3.33) in order to establish the existence of solutions for the linearized problem (3.28) by passing to the limit.

Apply the operator D_*^α (cf. (2.8)) with $\langle\alpha\rangle := \sum_{i=0}^3 \alpha_i + 2\alpha_4 \leqslant m$ to (3.33a) and take the scalar product of the resulting equations with $\mathrm{D}_*^\alpha W$ to get (cf. [65, section 3.5])

$$\int_{\Omega^+} \boldsymbol{A}_0 \mathrm{D}_*^\alpha W \cdot \mathrm{D}_*^\alpha W + \varepsilon \|\mathrm{D}_*^\alpha W_2\|_{L^2(\Sigma_t)}^2 \leqslant \mathcal{Q}_\alpha(t) + C(K)\mathcal{M}_1(t), \tag{3.43}$$

where

$$\mathcal{Q}_\alpha(t) := 2\int_{\Sigma_t} \mathrm{D}_*^\alpha W_1 \mathrm{D}_*^\alpha W_2 + \int_{\Sigma_t^+} \left(\varepsilon|\mathrm{D}_*^\alpha W_2|^2 - 2\mathrm{D}_*^\alpha W_1 \mathrm{D}_*^\alpha W_2\right),$$

$$\mathcal{M}_1(t) := \|(\boldsymbol{f}, W)\|_{H_*^m(\Omega_t^+)}^2 + \mathring{\mathrm{C}}_{m+4}\|(\boldsymbol{f}, W)\|_{W_*^{2,\infty}(\Omega_t^+)}^2 \tag{3.44}$$

with $\|u\|_{W_*^{2,\infty}(\Omega_t^+)} := \sum_{\langle\alpha\rangle\leqslant 1} \|\mathrm{D}_*^\alpha u\|_{W^{1,\infty}(\Omega_t^+)}$ and

$$\mathring{\mathrm{C}}_m := 1 + \|\mathring{U}\|_{H_*^m(\Omega_T^+)} + \|\mathring{h}\|_{H^m(\Omega_T^-)} + \|\mathring{\varphi}\|_{H^m(\Sigma_T)}.$$

As in [65, section 3.5], we can obtain

$$\sum_{\langle\alpha\rangle\leqslant m,\, \alpha_1+\alpha_4>0} \left(\|\mathrm{D}_*^\alpha W(t)\|_{L^2(\Omega^+)}^2 + \varepsilon\|\mathrm{D}_*^\alpha W_2\|_{L^2(\Sigma_t)}^2\right) \lesssim_K \mathcal{M}_1(t). \tag{3.45}$$

Let us focus on the case of $\alpha_1 = \alpha_4 = 0$ so that $\mathrm{D}_*^\alpha = \partial_t^{\alpha_0}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ and $\alpha_0 + \alpha_2 + \alpha_3 \leqslant m$. By virtue of the boundary conditions (3.33f) and (3.33d), we infer

$$\mathcal{Q}_\alpha(t) = 2\int_{\Sigma_t} \mathrm{D}_*^\alpha W_1 \mathrm{D}_*^\alpha W_2 = \int_{\Sigma_t} Q_1 + \int_{\Sigma_t} Q_2 + \int_{\Sigma_t} Q_4 \tag{3.46}$$

for $Q_1 := 2[\mathrm{D}_*^\alpha, \mathring{h}' \cdot \mathrm{D}_{x'}]\xi \mathrm{D}_*^\alpha W_2$, $Q_2 := 2\mathring{h}' \cdot \mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi \mathrm{D}_*^\alpha W_2$, $Q_4 := -2\mathrm{D}_*^\alpha(\mathring{b}_2\psi)\mathrm{D}_*^\alpha W_2$. As in [65, section 3.5], we have

$$\int_{\Sigma_t} Q_1 \lesssim_K \epsilon \sum_{\langle\beta\rangle\leqslant m} \|\mathrm{D}_*^\beta W(t)\|_{L^2(\Omega^+)}^2 + C(\epsilon)\mathcal{M}_1(t) + C(\epsilon)\mathcal{M}_2(t) \tag{3.47}$$

for all $\epsilon > 0$, where $\mathcal{M}_2(t)$ is defined by

$$\mathcal{M}_2(t) := \|\nabla\xi\|_{H^m(\Omega_t^-)}^2 + \mathring{\mathrm{C}}_{m+4}\|\nabla\xi\|_{L^\infty(\Omega_t^-)}^2. \tag{3.48}$$

It follows from (3.33c) that

$$\int_{\Sigma_t} Q_4 = -2\int_{\Sigma_t} \mathrm{D}_*^\alpha(\mathring{b}_2\psi) \cdot \left\{(\partial_t + \mathring{v}' \cdot \mathrm{D}_{x'})\mathrm{D}_*^\alpha\psi + [\mathrm{D}_*^\alpha, \mathring{v}' \cdot \mathrm{D}_{x'}]\psi + \mathrm{D}_*^\alpha(\mathring{b}_1\psi)\right\},$$

which can be estimated as for the integrals $\mathcal{Q}_\alpha^{(1)}(t)$ and $\mathcal{Q}_\alpha^{(3)}(t)$ defined in [64, section 2.4]. Precisely, we can get

$$\int_{\Sigma_t} Q_4 \lesssim_K \|\mathrm{D}_*^\alpha \psi(t)\|_{L^2(\Sigma)}^2 + \mathcal{M}_3(t), \tag{3.49}$$

where

$$\mathcal{M}_3(t) := \|\psi\|_{H^m(\Sigma_t)}^2 + \mathring{\mathrm{C}}_{m+4}\|\psi\|_{L^\infty(\Sigma_t)}^2. \tag{3.50}$$

It remains to estimate the integral of Q_2 that can be decomposed as

$$Q_2 = \underbrace{2\mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi \cdot \mathrm{D}_*^\alpha \partial_t(\mathring{h}'\psi)}_{Q_{2a}} + \underbrace{2\mathring{h}' \cdot \mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi(\mathring{v}' \cdot \mathrm{D}_{x'})\mathrm{D}_*^\alpha \psi}_{Q_{2b}} + Q_{2d} \tag{3.51}$$

with $Q_{2d} := -2\mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi \cdot [\mathrm{D}_*^\alpha \partial_t, \mathring{h}']\psi + 2\mathring{h}' \cdot \mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi\{[\mathrm{D}_*^\alpha, \mathring{v}' \cdot \mathrm{D}_{x'}]\psi + \mathrm{D}_*^\alpha(\mathring{b}_1\psi)\}$. Similar to [65, (3.80)], we can deduce

$$\int_{\Sigma_t} Q_{2a} + \int_{\Omega^-} \mathring{A}\, \mathrm{D}_*^\alpha \nabla\xi \cdot \mathrm{D}_*^\alpha \nabla\xi + \varepsilon\|\mathrm{D}_*^\alpha \mathrm{D}_{x'}\xi(t)\|_{L^2(\Sigma)}^2 \lesssim_K \mathcal{M}_2(t), \tag{3.52}$$

where $\mathcal{M}_2(t)$ is defined by (3.48). Moreover, we have

$$\begin{aligned}\int_{\Sigma_t} Q_{2b} = {} & \underbrace{2\int_{\Sigma_t} \mathring{v}' \cdot \mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi \mathrm{D}_*^\alpha(\mathring{A}\nabla\xi)_1}_{\mathcal{J}_{3a}} \underbrace{-2\varepsilon\int_{\Sigma_t} \mathring{v}' \cdot \mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi \mathrm{D}_*^\alpha \Delta_{x'}\xi}_{\mathcal{J}_{3b}} \\ & \underbrace{-2\sum_{i=2,3}\int_{\Sigma_t} \mathring{v}' \cdot \mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi\, [\mathrm{D}_*^\alpha \partial_i, \mathring{h}_i]\psi}_{\mathcal{J}_4} + \underbrace{\int_{\Sigma_t} \mathring{\mathrm{c}}_1 \mathrm{D}_*^\alpha \psi \mathrm{D}_*^\alpha \mathrm{D}_{x'}\xi}_{\mathcal{J}_5},\end{aligned}$$

where $\mathring{\mathrm{c}}_m$ denotes a generic and smooth matrix-valued function of $\{(\mathrm{D}^\alpha \mathring{U}, \mathrm{D}^\alpha \mathring{h}, \mathrm{D}^\alpha \mathring{\Psi}) : |\alpha| \leqslant m\}$ and $\mathrm{D}^\alpha := \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for $\alpha := (\alpha_0, \dots, \alpha_3) \in \mathbb{N}^4$. It follows from (3.33b) and integration by parts that

$$\mathcal{J}_{3a} = \int_{\Omega_t^-} \mathring{\mathrm{c}}_2 \nabla \mathrm{D}_*^\alpha \xi \cdot \left\{\mathrm{D}_*^\alpha(\mathring{\mathrm{c}}_1 \nabla\xi) + [\mathrm{D}_{x'}\mathrm{D}_*^\alpha, \mathring{\mathrm{c}}_1]\nabla\xi\right\} \lesssim_K \mathcal{M}_2(t), \tag{3.53}$$

where $\mathcal{M}_2(t)$ is defined by (3.48). The integral $\mathcal{J}_{3b}$ can be estimated as

$$\mathcal{J}_{3b} = 2\varepsilon\int_{\Sigma_t} \mathrm{D}_{x'}(\mathring{v}' \cdot \mathrm{D}_{x'})\mathrm{D}_*^\alpha \xi \cdot \mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi \lesssim_K \varepsilon\|\mathrm{D}_{x'}\xi\|_{H^m(\Sigma_t)}^2. \tag{3.54}$$

A direct calculation shows

$$\mathcal{J}_4 + \mathcal{J}_5 + \int_{\Sigma_t} Q_{2d} = \underbrace{\int_{\Sigma_t} \mathring{\mathrm{c}}_1 \mathrm{D}_*^\alpha \psi \mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi}_{\mathcal{I}_1} + \underbrace{\sum_{i=0,2,3}\sum_{\beta\leqslant\alpha,\, |\beta|=1}\int_{\Sigma_t} \mathring{\mathrm{c}}_1 \mathrm{D}_*^{\alpha-\beta}\partial_i\psi \mathrm{D}_{x'}\mathrm{D}_*^\alpha \xi}_{\mathcal{I}_2}$$

$$+\underbrace{\sum_{0<\beta\leqslant\alpha}\int_{\Sigma_t}\mathring{c}_1\mathrm{D}_*^{\beta}\mathring{c}_1\mathrm{D}_*^{\alpha-\beta}\psi\mathrm{D}_{x'}\mathrm{D}_*^{\alpha}\xi}_{\mathcal{I}_3}+\underbrace{\sum_{i=0,2,3}\sum_{\beta\leqslant\alpha,\,|\beta|>1}\int_{\Sigma_t}\mathring{c}_1\mathrm{D}_*^{\beta}\mathring{c}_0\mathrm{D}_*^{\alpha-\beta}\partial_i\psi\mathrm{D}_{x'}\mathrm{D}_*^{\alpha}\xi}_{\mathcal{I}_4}.$$

Utilize the trace theorem and the Moser-type calculus inequalities to obtain

$$\begin{aligned}\mathcal{I}_3&\lesssim_K\sum_{0<\beta\leqslant\alpha}\|\mathring{c}_1\mathrm{D}_*^{\beta}\mathring{c}_1\mathrm{D}_*^{\alpha-\beta}\psi\|_{H^1(\Sigma_t)}\|\mathrm{D}_{x'}\mathrm{D}_*^{\alpha}\xi\|_{H^{-1}(\Sigma_t)}\\&\lesssim_K\sum_{0<\beta\leqslant\alpha}\|\mathrm{D}_*^{\beta}\mathring{c}_1\mathrm{D}_*^{\alpha-\beta}\psi\|_{H^1(\Sigma_t)}\|\mathrm{D}_{x'}\xi\|_{H^m(\Omega_t^-)}\lesssim_K\mathcal{M}_2(t)+\mathcal{M}_3(t),\end{aligned}\tag{3.55}$$

where $\mathcal{M}_2(t)$ and $\mathcal{M}_3(t)$ are defined by (3.48) and (3.50), respectively. The integral term $\mathcal{I}_4$ can be estimated in an entirely similar way. The noncollinearity condition (3.9) is useful in controlling the terms $\mathcal{I}_1$ and $\mathcal{I}_2$. More precisely, thanks to (3.9), we get from (3.30), (3.33e), and (3.33c) that

$$(\partial_t\psi,\partial_2\psi,\partial_3\psi)=\mathring{c}_0\mathbf{u}+\mathring{c}_1\psi+\varepsilon\mathring{c}_0\Delta_{x'}\xi\quad\text{on }\Sigma_T,\tag{3.56}$$

where $\mathbf{u}:=(W_2,W_5,(\mathring{A}\nabla\xi)_1)^{\mathsf{T}}$. For $\alpha>0$, we take $\alpha'<\alpha$ with $|\alpha'|=|\alpha|-1$ and use (3.56) to deduce

$$\mathcal{I}_1=\underbrace{\int_{\Sigma_t}\mathring{c}_1\mathrm{D}_*^{\alpha'}(\mathring{c}_0\mathbf{u})\mathrm{D}_{x'}\mathrm{D}_*^{\alpha}\xi}_{\mathcal{I}_{1a}}+\underbrace{\int_{\Sigma_t}\mathring{c}_1\mathrm{D}_*^{\alpha'}(\mathring{c}_1\psi)\mathrm{D}_{x'}\mathrm{D}_*^{\alpha}\xi}_{\mathcal{I}_{1b}}+\mathcal{I}_{1c}.$$

As $\langle\alpha'\rangle\leqslant m-1$, the term $\mathcal{I}_{1b}$ can be controlled as $\mathcal{I}_3$, and the term $\mathcal{I}_{1c}$ is defined and estimated as

$$\mathcal{I}_{1c}:=\varepsilon\int_{\Sigma_t}\mathring{c}_1\mathrm{D}_*^{\alpha'}(\mathring{c}_0\Delta_{x'}\xi)\mathrm{D}_{x'}\mathrm{D}_*^{\alpha}\xi\lesssim_K\varepsilon\|\mathrm{D}_{x'}\xi\|^2_{H^m(\Sigma_t)}+\mathcal{M}_2(t).\tag{3.57}$$

Passing to the volume integral for $\mathcal{I}_{1a}$ yields

$$\begin{aligned}\mathcal{I}_{1a}=&\int_{\Omega_t^-}\mathring{c}_2\mathrm{D}_*^{\alpha'}(\mathring{c}_0\mathbf{u}_\sharp)\mathrm{D}_{x'}\mathrm{D}_*^{\alpha}\xi+\int_{\Omega_t^-}\mathring{c}_1\partial_1\mathrm{D}_*^{\alpha'}(\mathring{c}_0\mathbf{u}_\sharp)\mathrm{D}_{x'}\mathrm{D}_*^{\alpha}\xi\\&+\int_{\Omega_t^-}\mathring{c}_2\mathrm{D}_*^{\alpha'}(\mathring{c}_0\mathbf{u}_\sharp)\partial_1\mathrm{D}_*^{\alpha}\xi+\int_{\Omega_t^-}\mathring{c}_1\mathrm{D}_{x'}\mathrm{D}_*^{\alpha'}(\mathring{c}_0\mathbf{u}_\sharp)\partial_1\mathrm{D}_*^{\alpha}\xi,\end{aligned}$$

where $\mathbf{u}_\sharp(t,x_1,x'):=(W_2(t,-x_1,x'),W_5(t,-x_1,x'),(\mathring{A}\nabla\xi)_1(t,x_1,x'))^{\mathsf{T}}$. Using the identities (3.29), (3.33b), and

$$\begin{pmatrix}\partial_1W_2\\\partial_1W_1-\varepsilon\partial_1W_2\\0\end{pmatrix}=\boldsymbol{f}-\boldsymbol{A}_4W-\sum_{i=0,2,3}\boldsymbol{A}_i\partial_iW-\boldsymbol{A}_1^{(0)}\partial_1W,\tag{3.58}$$

we infer

$$\mathcal{I}_{1a}\lesssim_K\mathcal{M}_1(t)+\mathcal{M}_2(t),\tag{3.59}$$

where $\mathcal{M}_1(t)$ and $\mathcal{M}_2(t)$ are given in (3.44) and (3.48), respectively. Combine the above estimates to discover

$$\begin{aligned}
&\|D_*^\alpha W(t)\|_{L^2(\Omega^+)}^2 + \|D_*^\alpha \nabla\xi(t)\|_{L^2(\Omega^-)}^2 + \varepsilon\|D_*^\alpha W_2\|_{L^2(\Sigma_t)}^2 + \varepsilon\|D_*^\alpha D_{x'}\xi(t)\|_{L^2(\Sigma)}^2 \\
&\lesssim_K C(\epsilon)\mathcal{M}(t) + \varepsilon\|D_{x'}\xi\|_{H^m(\Sigma_t)}^2 + \epsilon \sum_{\langle\beta\rangle\leqslant m} \|D_*^\beta W(t)\|_{L^2(\Omega^+)}^2 + \|D_*^\alpha \psi(t)\|_{L^2(\Sigma)}^2
\end{aligned} \tag{3.60}$$

for $\alpha_1 = \alpha_4 = 0$, where $\mathcal{M}(t) := \mathcal{M}_1(t) + \mathcal{M}_2(t) + \mathcal{M}_3(t)$. If $\alpha_1 = \alpha_4 = 0$, then it follows from (3.56), (3.33b), (3.29), and (3.58) that

$$\begin{aligned}
\|D_*^\alpha \psi(t)\|_{L^2(\Sigma)}^2 \lesssim_K{}& C(\epsilon)\mathcal{M}(t) + \epsilon \sum_{\langle\beta\rangle\leqslant m} \|D_*^\beta W(t)\|_{L^2(\Omega^+)}^2 \\
&+ \epsilon \sum_{\langle\beta\rangle\leqslant m} \|D_*^\beta \nabla\xi(t)\|_{L^2(\Omega^-)}^2 + \epsilon^2 \sum_{\langle\gamma\rangle\leqslant m} \|D_*^\gamma D_{x'}\xi(t)\|_{L^2(\Sigma)}^2
\end{aligned} \tag{3.61}$$

for all $\epsilon > 0$, where $\gamma_1 = \gamma_4 = 0$. Thanks to (3.33b), by induction, we can infer

$$\sum_{|\beta|\leqslant m} \|D^\beta \nabla\xi(t)\|_{L^2(\Omega^-)}^2 \lesssim_K \mathcal{M}_2(t) + \sum_{\langle\alpha\rangle\leqslant m,\ \alpha_1=\alpha_4=0} \|D_*^\alpha \nabla\xi(t)\|_{L^2(\Omega^-)}^2. \tag{3.62}$$

Using (3.45) and (3.60)–(3.62), we take ϵ and ε small enough to get

$$\mathcal{E}(t) \lesssim_K \int_0^t \mathcal{E}(\tau)\mathrm{d}\tau \,+\, \mathcal{N}(t), \tag{3.63}$$

where

$$\begin{aligned}
\mathcal{E}(t) :={}& \sum_{\langle\alpha\rangle\leqslant m} \|D_*^\alpha W(t)\|_{L^2(\Omega^+)}^2 + \sum_{|\beta|\leqslant m} \|D^\beta \nabla\xi(t)\|_{L^2(\Omega^-)}^2 \\
&+ \sum_{\langle\alpha\rangle\leqslant m,\, \alpha_1=\alpha_4=0} \|D_*^\alpha \psi(t)\|_{L^2(\Sigma)}^2 + \varepsilon \sum_{\langle\alpha\rangle\leqslant m,\, \alpha_1=\alpha_4=0} \|D_*^\alpha D_{x'}\xi(t)\|_{L^2(\Sigma)}^2, \\
\mathcal{N}(t) :={}& \|\boldsymbol{f}\|_{H_*^m(\Omega_t^+)}^2 + \mathring{C}_{m+4}\Big(\|(\boldsymbol{f}, W)\|_{W_*^{2,\infty}(\Omega_t^+)}^2 + \|\nabla\xi\|_{L^\infty(\Omega_t^-)}^2 + \|\psi\|_{W^{2,\infty}(\Sigma_t)}^2 \Big).
\end{aligned}$$

Applying Grönwall's inequality to (3.63) and using Poincaré's inequality imply

$$\begin{aligned}
&\|W\|_{H_*^m(\Omega_T^+)}^2 + \|(\xi, \nabla\xi)\|_{H^m(\Omega_T^-)}^2 + \|\psi\|_{H^m(\Sigma_T)}^2 + \varepsilon\|D_{x'}\xi\|_{H^m(\Sigma_T)}^2 \\
&\lesssim_K \|\boldsymbol{f}\|_{H_*^m(\Omega_T^+)}^2 + \|\boldsymbol{f}\|_{H_*^6(\Omega_T^+)}^2 \Big(\|\mathring{U}\|_{H_*^{m+4}(\Omega_T^+)} + \|\mathring{h}\|_{H^{m+4}(\Omega_T^-)} + \|\mathring{\varphi}\|_{H^{m+4}(\Sigma_T)} \Big)
\end{aligned} \tag{3.64}$$

for $0 \leqslant T \leqslant T_0$ and $m \geqslant 6$, provided $\varepsilon > 0$ is sufficiently small.

The uniform-in-ε high-order estimate (3.64) allows us to obtain the solvability of the problem (3.28) by passing to the limit $\varepsilon \to 0$. Indeed, according to (3.64), we can extract a subsequence weakly convergent to $(W, \xi, \psi) \in H_*^m(\Omega_T^+) \times H^m(\Omega_T^-) \times H^m(\Sigma_T)$ satisfying the estimate (3.64) with $\varepsilon = 0$. As $\partial_1 W_2$ and $\sqrt{\varepsilon}\Delta_{x'}\xi$ are uniformly bounded in $H_*^{m-2}(\Omega_T^+)$ and $H^{m-2}(\Sigma_T)$, respectively, the passage to the limit $\varepsilon \to 0$ in (3.33) verifies that (W, ξ, ψ) solves the reduced problem (3.28). The uniqueness of solutions results from the estimate (3.64) with $\varepsilon = 0$. Then we can obtain for the effective linear problem (3.13) the existence and uniqueness of solutions $(\dot{V}, \dot{h}, \psi) \in H^m_*(\Omega^+_T) \times H^m(\Omega^-_T) \times H^m(\Sigma_T)$ satisfying the high-order estimate:

$$\begin{aligned} &\|\dot{V}\|_{H^m_*(\Omega^+_T)} + \|\dot{h}\|_{H^m(\Omega^-_T)} + \|\psi\|_{H^m(\Sigma_T)} \\ &\lesssim_K \left(\|\mathring{U}\|_{H^{m+4}_*(\Omega^+_T)} + \|\mathring{h}\|_{H^{m+4}(\Omega^-_T)} + \|\mathring{\varphi}\|_{H^{m+4}(\Sigma_T)}\right)\Big(\|f^+\|_{H^6_*(\Omega^+_T)} + \|f^-\|_{H^7(\Omega^-_T)} \\ &\qquad +\|g\|_{H^7\times H^8}\Big) + \|f^+\|_{H^m_*(\Omega^+_T)} + \|f^-\|_{H^{m+1}(\Omega^-_T)} + \|g\|_{H^{m+1}\times H^{m+2}}, \end{aligned} \tag{3.65}$$

where

$$\|g\|_{H^m\times H^{m+1}} := \|(g_1, g_2)\|_{H^m(\Sigma_T)} + \|g_4\|_{H^m(\Sigma^+_T)} + \|g_3\|_{H^{m+1}(\Sigma_T)} + \|g_5\|_{H^{m+1}(\Sigma^-_T)}.$$

Note that the inequality (3.65) is a tame estimate due to the *fixed* loss of regularity from the basic state to the solution, which allows us to apply a suitable Nash–Moser iteration scheme to solve the nonlinear problem (2.1); see [52, 63–65] for the part of the nonlinear analysis.

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