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Asymptotic stability of wave patterns to compressible viscous and heat-conducting gases in the half-space

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Abstract

We study the large-time behavior of solutions to the compressible Navier–Stokes equations for a viscous and heat-conducting ideal polytropic gas in the one-dimensional half-space. A rarefaction wave and its superposition with a non-degenerate stationary solution are shown to be asymptotically stable for the outflow problem with large initial perturbation and general adiabatic exponent.

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1. Introduction

The one-dimensional motion of a compressible viscous and heat-conducting gas in the half-space $\mathbb{R}_+ := (0, \infty)$ can be formulated by the compressible Navier–Stokes equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P)_x = (\mu u_x)_x, \\ (\rho E)_t + (\rho u E + uP)_x = (\kappa\theta_x + \mu u u_x)_x, \end{cases} \tag{1.1}$$

where $t > 0$ and $x \in \mathbb{R}_+$ stand for the time variable and the spatial variable, respectively, and the primary dependent variables are the density ρ, the velocity u and the temperature θ. The specific total energy $E = e + \frac{1}{2}u^2$ with e being the specific internal energy. It is known from thermodynamics that only two of the thermodynamic variables ρ, θ, P (pressure), e and s (specific entropy) are independent. We focus on the ideal polytropic gas, which is expressed in normalized units by the following constitutive relations

$$P = R\rho\theta, \quad e = c_v\theta, \quad s = c_v \ln(\rho^{1-\gamma}\theta), \tag{1.2}$$

where $R > 0$ is the gas constant, $\gamma > 1$ the adiabatic exponent and $c_v = R/(\gamma - 1)$ the specific heat at constant volume. Positive constants μ and κ are the viscosity and the heat conductivity, respectively.

The system (1.1)–(1.2) is supplemented with the initial condition

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0), \tag{1.3}$$

which is assumed to satisfy the far-field condition

$$\lim_{x\to\infty} (\rho_0, u_0, \theta_0)(x) = (\rho_+, u_+, \theta_+), \tag{1.4}$$

where $\rho_+ > 0$, u_+ and $\theta_+ > 0$ are constants. For boundary conditions, we take

$$(u, \theta)(t, 0) = (u_-, \theta_-), \tag{1.5}$$

where u_- and $\theta_- > 0$ are constants. The initial data (1.3) is assumed to satisfy certain compatibility conditions as usual.

The boundary condition $u(t, 0) = u_- < 0$ means that the fluid blows out from the boundary, and hence the initial boundary value problem (1.1)–(1.5) with $u_- < 0$ is called the outflow problem. The problem (1.1)–(1.5) with $u_- = 0$ is called the impermeable wall problem, which has been studied in [6,7,20,21,31] and so on. According to the theory of well-posedness for initial boundary value problem, one has to impose one extra boundary condition $\rho(t, 0) = \rho_-$ on $\{x = 0\}$ for the case when $u_- > 0$. This case is called the inflow problem and has been investigated by Matsumura et al. [4,6,9,22,27,28]. We refer to Matsumura [19] for a complete classification about the large-time behaviors of solutions to initial boundary value problems of the isentropic compressible Navier–Stokes equations in the half-space $\mathbb{R}_+$.

The main purpose of this article is to study the large-time behavior of solutions to the outflow problem (1.1)–(1.5). The nonlinear stability of the stationary solution, the rarefaction wave and their composition has been addressed in [15,26] under small initial perturbation. For large perturbation case, Qin [26] proved that the non-degenerate stationary solution is asymptotically stable under the technical assumption that the adiabatic exponent γ is close to 1. Recently, Wan et al. [30] establish the asymptotic stability of the non-degenerate stationary solution with large initial perturbation and general adiabatic exponent γ. In this article we are going to study the case when the corresponding time-asymptotic state is a rarefaction wave or its superposition with a non-degenerate stationary solution under large initial perturbation.

We first investigate the large-time behavior of solutions toward the rarefaction wave for the outflow problem or the impermeable wall problem (1.1)–(1.5). To this end, we assume that positive constants ρ_+, $u_\pm$ and $\theta_\pm$ satisfy

$$u_- = u_+ + \int_{\rho_+}^{(\theta_-/\theta_+)^{\frac{1}{\gamma-1}}\rho_+} \sqrt{R\gamma\rho_+^{1-\gamma}\theta_+ z^{\frac{\gamma-3}{2}}}\,\mathrm{d}z < u_+, \tag{1.6}$$

so that $(\rho_+, u_+, \theta_+) \in R_3(\rho_-, u_-, \theta_-)$ for

$$\rho_- = (\theta_-/\theta_+)^{\frac{1}{\gamma-1}}\rho_+. \tag{1.7}$$

Here $R_3(\rho_-, u_-, \theta_-)$ is the 3-rarefaction wave curve through (ρ_-, u_-, θ_-) given by

$$R_3(\rho_-, u_-, \theta_-) := \left\{ (\rho, u, \theta) \,\middle|\, \begin{array}{l} \rho > \rho_-, \ \rho^{1-\gamma}\theta = \rho_-^{1-\gamma}\theta_-, \\ u = u_- + \displaystyle\int_{\rho_-}^{\rho} \sqrt{R\gamma\rho_-^{1-\gamma}\theta_- z^{\frac{\gamma-3}{2}}}\,\mathrm{d}z \end{array} \right\}.$$

We assume further that

$$u_- + \sqrt{R\gamma\theta_-} \geq 0. \tag{1.8}$$

Then as time t goes to infinity, the solution of the problem (1.1)–(1.5) is expected to converge to the 3-rarefaction wave $(\rho^R, u^R, \theta^R)(t, x)$ connecting (ρ_-, u_-, θ_-) and (ρ_+, u_+, θ_+), which is the unique entropy solution of the Riemann problem for the corresponding hyperbolic system of (1.1)–(1.2) (i.e. the compressible Euler system)

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P)_x = 0, \\ (\rho E)_t + (\rho u E + uP)_x = 0 \end{cases} \tag{1.9}$$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ with initial data

$$(\rho, u, \theta)(0, x) = \begin{cases} (\rho_-, u_-, \theta_-) & \text{for } x < 0, \\ (\rho_+, u_+, \theta_+) & \text{for } x > 0. \end{cases} \tag{1.10}$$

Note that $(\rho^R, u^R, \theta^R)(t, x) \equiv (\rho_-, u_-, \theta_-)$ for each $(t, x) \in [0, \infty) \times (-\infty, 0]$ due to (1.8).

We construct a smooth approximation $(\bar\rho, \bar u, \bar\theta)$ of (ρ^R, u^R, θ^R) for deriving the stability of the rarefaction wave. As in [9], we consider the Cauchy problem for the Burgers equation

$$
\begin{cases} w_t + w w_x = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ w(0,x) = w_- + \tilde{w} k_q \displaystyle\int_0^{x_+} z^q \mathrm{e}^{-z} \mathrm{d}z & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.11}
$$

where $\tilde{w} := w_+ - w_-$, $q \geq 16$ is some fixed constant, $x_+ := \max\{x, 0\}$ is the positive part of x, and the constant k_q satisfies

$$
k_q \int_0^\infty z^q \mathrm{e}^{-z} \mathrm{d}z = 1.
$$

The smoothed rarefaction wave $(\bar{\rho}, \bar{u}, \bar{\theta})$ connecting (ρ_-, u_-, θ_-) and (ρ_+, u_+, θ_+) is defined by

$$
\begin{cases} \lambda_3(\bar{\rho}, \bar{u}, \bar{\theta})(t,x) = w(1+t, x), \\ (\bar{\rho}^{1-\gamma}\bar{\theta})(t,x) = \rho_+^{1-\gamma}\theta_+, \\ \bar{u}(t,x) = u_+ + \displaystyle\int_{\rho_+}^{\bar{\rho}(t,x)} \sqrt{R\gamma\rho_+^{1-\gamma}\theta_+ z^{\frac{\gamma-3}{2}}}\, \mathrm{d}z, \end{cases} \tag{1.12}
$$

where ρ_- is given by (1.7) and $w(t,x)$ is the unique solution of (1.11) with $w_\pm = \lambda_3(\rho_\pm, u_\pm, \theta_\pm)$.

Now we state our stability result of the rarefaction wave $(\rho^R, u^R, \theta^R)(t,x)$ to the outflow problem or the impermeable wall problem with large initial perturbation.

Theorem 1. *Assume that $(\rho_+, u_\pm, \theta_\pm)$ and the initial data (ρ_0, u_0, θ_0) satisfy (1.6), (1.8) and*

$$
\inf_{x\in\mathbb{R}_+} \{\rho_0(x), \theta_0(x)\} > 0, \quad (\rho_0 - \bar{\rho}, u_0 - \bar{u}, \theta_0 - \bar{\theta}) \in H^1(\mathbb{R}_+). \tag{1.13}
$$

Then there exists a positive constant ϵ_1 such that if $u_- \leq 0$ and the boundary strength $\delta := |(u_+ - u_-, \theta_+ - \theta_-)| \leq \epsilon_1$, the initial boundary value problem (1.1)–(1.5) admits a unique solution (ρ, u, θ) satisfying

$$
\begin{gathered} (\rho - \bar{\rho}, u - \bar{u}, \theta - \bar{\theta}) \in C([0,\infty); H^1(\mathbb{R}_+)), \\ \rho_x - \bar{\rho}_x \in L^2(0,\infty; L^2(\mathbb{R}_+)), \quad (u_x - \bar{u}_x, \theta_x - \bar{\theta}_x) \in L^2(0,\infty; H^1(\mathbb{R}_+)). \end{gathered} \tag{1.14}
$$

Furthermore, the solution (ρ, u, θ) converges to the rarefaction wave (ρ^R, u^R, θ^R) uniformly as time tends to infinity:

$$
\lim_{t\to\infty} \sup_{x\in\mathbb{R}_+} \left| (\rho - \rho^R, u - u^R, \theta - \theta^R)(t,x) \right| = 0. \tag{1.15}
$$

Remark 1.1. For the Cauchy problem to the compressible Navier–Stokes equations (1.1)–(1.2) with generic adiabatic exponent γ in the whole space $\mathbb{R}$, we can employ the methodology developed in this paper to obtain the time-asymptotic stability of the rarefaction waves under large initial perturbation, which extends the corresponding stability results in [14,25] for the case with small initial perturbation or the case when γ is close to 1. We refer to a recent work [11] for the stability of superposition of viscous contact wave and rarefaction waves to the Cauchy problem for the compressible Navier–Stokes equations in Lagrangian coordinate.

Next we intend to study the time-asymptotic stability of the superposition of a non-degenerate stationary solution and a 3-rarefaction wave. For this purpose, we let $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ be the stationary solution of (1.1)–(1.5) connecting (u_-, θ_-) and (ρ_m, u_m, θ_m), namely $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ depends solely on the variable x and satisfies

$$\begin{cases} (\tilde{\rho}\tilde{u})' = 0, \\ (\tilde{\rho}\tilde{u}^2 + \tilde{P})' = \mu\tilde{u}'', \\ (\tilde{\rho}\tilde{u}\tilde{E} + \tilde{u}\tilde{P})' = \kappa\tilde{\theta}'' + \mu(\tilde{u}\tilde{u}')' \end{cases} \tag{1.16}$$

for $x \in \mathbb{R}_+$ and

$$(\tilde{u}, \tilde{\theta})(0) = (u_-, \theta_-), \qquad \lim_{x\to\infty} (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) = (\rho_m, u_m, \theta_m), \tag{1.17}$$

where $\tilde{P} := R\tilde{\rho}\tilde{\theta}$ and $\tilde{E} := c_v\tilde{\theta} + \frac{1}{2}\tilde{u}^2$. A stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ is called to be non-degenerate if for each $n \in \mathbb{N}$,

$$|\partial_x^n(\tilde{\rho} - \rho_m, \tilde{u} - u_m, \tilde{\theta} - \theta_m)(x)| \le C\tilde{\delta}\mathrm{e}^{-cx}, \tag{1.18}$$

where C, c are positive constants and $\tilde{\delta} := |(u_m - u_-, \theta_m - \theta_-)|$ is the boundary strength of the stationary solution. The existence of (non-degenerate) stationary solutions has been shown by Kawashima et al. [15] and will be restated in section 2.

We assume that

$$(\rho_+, u_+, \theta_+) \in R_3(\rho_m, u_m, \theta_m), \quad \sqrt{R\gamma\theta_m} \ge -u_m > 0, \tag{1.19}$$

so that there exist a 3-rarefaction wave (ρ^R, u^R, θ^R) connecting (ρ_m, u_m, θ_m) and (ρ_+, u_+, θ_+). It is expected that the large-time behavior of solutions to the outflow problem (1.1)–(1.5) is determined by the composition $(\check{\rho}, \check{u}, \check{\theta})$ of the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ and the 3-rarefaction wave (ρ^R, u^R, θ^R):

$$(\check{\rho}, \check{u}, \check{\theta})(t, x) = (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) + (\rho^R, u^R, \theta^R)(t, x) - (\rho_m, u_m, \theta_m). \tag{1.20}$$

In order to derive the stability result, we introduce the smoothed asymptotic state

$$(\hat{\rho}, \hat{u}, \hat{\theta})(t, x) = (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) + (\bar{\rho}, \bar{u}, \bar{\theta})(t, x) - (\rho_m, u_m, \theta_m), \tag{1.21}$$

where $(\bar{\rho}, \bar{u}, \bar{\theta})$ is the smooth rarefaction wave connecting (ρ_m, u_m, θ_m) and (ρ_+, u_+, θ_+). We define

$$\bar{\delta} := |(u_m - u_+, \theta_m - \theta_+)|.$$

Under the above preparation, we have the following stability result on the superposition $(\check{\rho}, \check{u}, \check{\theta})$ with large initial perturbation.

Theorem 2. *Assume that there exists a non-degenerate stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ connecting (u_-, θ_-) and (ρ_m, u_m, θ_m). Assume further that (ρ_+, u_+, θ_+) and the initial data (ρ_0, u_0, θ_0) satisfy (1.19) and*

$$\inf_{x\in\mathbb{R}_+} \{\rho_0(x), \theta_0(x)\} > 0, \quad (\rho_0 - \hat{\rho}, u_0 - \hat{u}, \theta_0 - \hat{\theta}) \in H^1(\mathbb{R}_+). \tag{1.22}$$

Then a positive constant ϵ_2 exists such that the outflow problem (1.1)–(1.5) with $\bar{\delta} + \tilde{\delta} \le \epsilon_2$ has a unique solution (ρ, u, θ) satisfying

$$\begin{gathered}(\rho - \hat{\rho}, u - \hat{u}, \theta - \hat{\theta}) \in C([0, \infty); H^1(\mathbb{R}_+)),\\ \rho_x - \hat{\rho}_x \in L^2(0, \infty; L^2(\mathbb{R}_+)), \quad (u_x - \hat{u}_x, \theta_x - \hat{\theta}_x) \in L^2(0, \infty; H^1(\mathbb{R}_+)).\end{gathered} \tag{1.23}$$

Furthermore, the solution (ρ, u, θ) converges to the composition $(\check{\rho}, \check{u}, \check{\theta})$ of the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ and the rarefaction wave (ρ^R, u^R, θ^R) uniformly as time tends to infinity:

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}_+} |(\rho - \check{\rho}, u - \check{u}, \theta - \check{\theta})(t, x)| = 0. \tag{1.24}$$

To derive the large-time behavior of solutions to the compressible Navier–Stokes equations (1.1), it suffices to deduce certain uniformly-in-time *a priori* estimates on the perturbations toward the asymptotic state and the essential step is to obtain the positive lower and upper bounds on the density $\rho(t, x)$ and the temperature $\theta(t, x)$ uniformly in time t and space x. In the case of small perturbation, one can use the smallness of the *a priori* H^1-norm of the perturbation to get the uniform bounds of the density ρ and the temperature θ. Owing to such uniform bounds and the smallness of the boundary strength δ, one can derive certain uniform a priori energy-type estimates as shown in [15,26]. In the case that the adiabatic exponent γ is close to 1, by observing that $\theta = \rho^{\gamma-1}e^{(\gamma-1)s/R}$ for ideal polytropic gases (1.1)–(1.2), one can deduce that $\|\theta - 1\|_{L^\infty([0,T]\times\mathbb{R})}$ can be sufficiently small. Thus the desired energy-type a priori estimates can be performed as in [25,26] based on the smallness of δ and the a priori assumption

$$\tfrac{1}{2} \le \theta(t, x) \le 2 \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}.$$

However, these arguments are no longer valid for the case with large initial perturbation and general adiabatic exponent. We note that, even for the asymptotic stability of constant state to the Cauchy problem for the system (1.1), the uniform positive lower and upper bounds on $\theta(t, x)$ are given only very recently by Li and Liang [18], although the corresponding uniform bounds on $\rho(t, x)$ were addressed in [12,13] thirteen years ago. In their work [18], Li–Liang considered the fixed-domain problems to the compressible Navier–Stokes equations in the Lagrangian coordinate and obtained the uniform positive lower and upper bounds on the temperature $\theta(t, x)$ through a time-asymptotically nonlinear stability analysis. However, the outflow problem (1.1)–(1.5) will be transformed into a free boundary problem in the Lagrangian coordinate, which makes the treatment of boundary more difficult. To overcome this difficulty, we shall make use of a direct energy method to the reformulated problem for the compressible Navier–Stokes equations (1.1) in the Eulerian coordinate and take account of the dissipative effect of the boundary terms.

The main point for deriving our main results, the stability of the rarefaction wave and its superposition with a non-degenerate stationary solution to the initial boundary value problem (1.1)–(1.5), is to employ the smallness of the boundary strength δ to control the possible growth of the perturbation suitably. Specifically, we first deduce the basic energy estimate with the aid of the decay properties of the smoothed rarefaction wave and the non-degenerate stationary solution provided that the boundary strength δ multiplied with a certain function of m_1 (the *a priori* lower bound of density ρ), m_2 (the *a priori* lower bound of temperature θ) and N (the *a priori* bound of the $L^\infty(0,T;H^1(\mathbb{R}_+))$-norm of perturbation) is suitably small (see Lemma 3.1 for detailed statement). Next, to get uniform pointwise bounds of the density $\rho(t,x)$, we transform the outflow problem (1.1)–(1.5) into a free boundary problem in the Lagrangian coordinate and modify Jiang's argument for fixed domains in [12,13]. Especially, we will use a cut-off function with parameter to localize the free boundary problem, and then we will deduce a local representation of the specific volume $v=1/\rho$ to establish the uniform bounds of v. With such uniform bounds of the density ρ in hand, we can derive the H^1-norm (in the spatial variable x) estimate of the perturbation uniformly in the time t in the Eulerian coordinate. And the maximum principle enables us to get the positive lower bound of the temperature $\theta(t,x)$ locally in time t. In view of the *a priori* assumption (3.8), we have to obtain the uniform positive lower bound of the temperature $\theta(t,x)$, which will be achieved by combining the locally-in-time lower bound of $\theta(t,x)$ and a well-designed continuation argument.

The layout of this paper is as follows. After stating the notations, we summarize the existence of the stationary solution and some properties of the smoothed rarefaction wave in Section 2. The basic energy estimate, the uniform bounds of the density ρ, the uniform H^1-norm estimate and the locally-in-time lower bound of the temperature θ will be obtained in subsections 3.1, 3.2, 3.3 and 3.4, respectively. The last part of this manuscript, subsection 3.5, is devoted to showing the proof of our main results by applying a well-designed continuation argument.

Notations Throughout this paper, $L^q(\mathbb{R}_+)$ $(1\le q\le\infty)$ stands for the usual Lebesgue space on $\mathbb{R}_+$ equipped with the norm $\|\cdot\|_{L^q}$ and $H^k(\mathbb{R}_+)$ $(k\in\mathbb{N})$ the usual Sobolev space in the L^2 sense with norm $\|\cdot\|_k$. We introduce $\|\cdot\|=\|\cdot\|_{L^2(\mathbb{R}_+)}$ for simplicity. The space of continuous functions on the interval I with values in $H^k(\mathbb{R}_+)$ is denoted by $C(I;H^k(\mathbb{R}_+))$ or simply by $C(I;H^k)$ while the space of L^2-functions on I with values in $H^k(\mathbb{R}_+)$ is denoted by $L^2(I;H^k(\mathbb{R}_+))$ or simply by $L^2(I;H^k)$. The Gaussian bracket $[x]$ means the largest integer not greater than x, and $x_+:=\max\{x,0\}$ is the positive part of x.

2. Preliminaries

It is well-known (see [2,29] for example) that for each $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$(\rho^R,u^R,\theta^R)(t,x)\in R_3(\rho_-,u_-,\theta_-),\quad \lambda_3(\rho^R,u^R,\theta^R)(t,x)=w^R(t,x),$$

where

$$\lambda_3(\rho,u,\theta):=u+\sqrt{R\gamma\theta}$$

is the 3-characteristic speed of the system (1.9) and $w^R(t,x)$ is the continuous weak solution of the Riemann problem on Burgers equation

$$\begin{cases} w_t^R + w^R w_x^R = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ w^R(0,x) = w_\pm & \text{for } \pm x > 0 \end{cases}$$

with $w_\pm = \lambda_3(\rho_\pm, u_\pm, \theta_\pm)$. Moreover, $w^R(t,x)$ takes the form of

$$w^R(t,x) = \begin{cases} w_- & \text{for } x \le w_- t, \\ x/t & \text{for } w_- t < x < w_+ t, \\ w_+ & \text{for } x \ge w_+ t. \end{cases}$$

The main idea in [8] is to approximate $w^R(t,x)$ by the solution $w(t,x)$ of the Cauchy problem (1.11). The following lemma can be deduced by virtue of the method of characteristics (see [8,23]).

Lemma 2.1. *Let $w_- < w_+$. Then the Burgers equation (1.11) has a unique smooth solution $w(t,x)$ satisfying*

(i) $w_x(t,x) \ge 0$, $w_- \le w(t,x) < w_+$ *for* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$;
(ii) *when* $x \le w_- t$, $w(t,x) - w_- = w_x(t,x) = w_{xx}(t,x) = 0$;
(iii) *for each* $p \in [1,\infty]$, *there exists a constant* $C_{p,q}$ *such that*

$$\|w_x(t)\|_{L^p(\mathbb{R})} \le C_{p,q} \min\{\tilde{w}, \tilde{w}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\},$$

$$\|w_{xx}(t)\|_{L^p(\mathbb{R})} \le C_{p,q} \min\{\tilde{w}, t^{-1+\frac{1}{q}(1-\frac{1}{p})}\};$$

(iv) $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |w(t,x) - w^R(t,x)| = 0$.

Having obtained $w(t,x)$, we can define the smoothed rarefaction wave $(\bar{\rho}, \bar{u}, \bar{\theta})$ according to (1.12). Then one can check from a direct calculation that $(\bar{\rho}, \bar{u}, \bar{\theta})$ solves the compressible Euler system (1.9) and

$$(\bar{\rho}, \bar{u}, \bar{\theta})(0,t) = (\rho_-, u_-, \theta_-), \qquad \lim_{x\to\infty} (\bar{\rho}, \bar{u}, \bar{\theta})(t,x) = (\rho_+, u_+, \theta_+) \quad \text{for each } t \ge 0. \tag{2.1}$$

In view of (1.12) and Lemma 2.1, we have the following properties for the smoothed rarefaction wave $(\bar{\rho}, \bar{u}, \bar{\theta})$.

Lemma 2.2. *The smooth approximation $(\bar{\rho}, \bar{u}, \bar{\theta})$ connecting (ρ_-, u_-, θ_-) and (ρ_+, u_+, θ_+) satisfies*

(i) $\rho_- \le \bar{\rho} \le \rho_+$, $\theta_- \le \bar{\theta} \le \theta_+$, $\bar{\theta}_x(t,x) \ge 0$, *and*

$$\bar{\rho}_x = \frac{1}{\gamma - 1} \bar{\rho} \bar{\theta}^{-1} \bar{\theta}_x, \quad \bar{u}_x = \frac{\sqrt{R\gamma}}{\gamma - 1} \bar{\theta}^{-\frac{1}{2}} \bar{\theta}_x; \tag{2.2}$$

(ii) *if* $x \le (u_- + \sqrt{R\gamma\theta_-})(1+t)$, *then* $(\bar\rho, \bar u, \bar\theta)(t,x) = (\rho_-, u_-, \theta_-)$;
(iii) *for each* $p \in [1,\infty]$, *there exists a constant* $C_{p,q}$ *such that*

$$\left\|(\bar\rho_x, \bar u_x, \bar\theta_x)(t)\right\|_{L^p(\mathbb{R}_+)} \le C_{p,q} \min\{\delta, \delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\}, \tag{2.3}$$

$$\left\|(\bar\rho_{xx}, \bar u_{xx}, \bar\theta_{xx})(t)\right\|_{L^p(\mathbb{R}_+)} \le C_{p,q} \min\{\delta, (1+t)^{-1+\frac{1}{q}(1-\frac{1}{p})}\}, \tag{2.4}$$

where $\delta := |(u_+ - u_-, \theta_+ - \theta_-)|$ *is the boundary strength;*
(iv) $\lim_{t\to\infty} \sup_{x\in\mathbb{R}_+} |(\bar\rho, \bar u, \bar\theta)(t,x) - (\rho^R, u^R, \theta^R)(1+t,x)| = 0$.

Next we state the existence and the properties of the stationary solution $(\tilde\rho, \tilde u, \tilde\theta)$ satisfying (1.16) and (1.17), which has been derived in [15]. To this end, we introduce the Mach number at infinity as

$$M_m := \frac{|u_m|}{c_m},$$

where $c_m := \sqrt{R\gamma\theta_m}$ is the sound speed.

Lemma 2.3 *([16]). Suppose that* (u_-, θ_-) *satisfies*

$$(u_-, \theta_-) \in \mathcal{M}^m := \left\{(u,\theta) \in \mathbb{R}^2 : |(u - u_m, \theta - \theta_m)| < \delta_0\right\}$$

for a certain positive constant δ_0.

(i) *For the case* $M_m > 1$, *there exists a unique smooth solution* $(\tilde\rho, \tilde u, \tilde\theta)$ *to the problem* (1.16)–(1.17) *satisfying the decay estimate* (1.18).
(ii) *For the case* $M_m = 1$, *there exists a certain region* $\mathcal{M}^0 \subset \mathcal{M}^m$ *such that if* $(u_-, \theta_-) \in \mathcal{M}^0$, *then there exists a unique smooth solution* $(\tilde\rho, \tilde u, \tilde\theta)$ *to* (1.16)–(1.17) *satisfying*

$$|\partial_x^n(\tilde\rho - \rho_m, \tilde u - u_m, \tilde\theta - \theta_m)(x)| \le \frac{C\tilde\delta^{n+1}}{(1+\tilde\delta x)^{k+1}} + C\tilde\delta e^{-cx} \quad \textit{for all } n \in \mathbb{N}.$$

(iii) *For the case* $M_m < 1$, *there exists a curve* $\mathcal{M}^- \subset \mathcal{M}^m$ *such that if* $(u_-, \theta_-) \in \mathcal{M}^m$, *then there exists a unique smooth solution* $(\tilde\rho, \tilde u, \tilde\theta)$ *to the problem* (1.16)–(1.17) *satisfying* (1.18).

Since the stationary solution $(\tilde\rho, \tilde u, \tilde\theta)$ and the smoothed rarefaction wave $(\bar\rho, \bar u, \bar\theta)$ are well-defined, one can deduce that $(\hat\rho, \hat u, \hat\theta)$ satisfies

$$\begin{cases} \hat\rho_t + \hat u \hat\rho_x + \hat\rho \hat u_x = \hat f_1, \\ \hat\rho(\hat u_t + \hat u \hat u_x) + \hat P_x = \mu \tilde u_{xx} + \hat f_2, \\ c_v \hat\rho(\hat\theta_t + \hat u \hat\theta_x) + \hat P \hat u_x = \kappa \tilde\theta_{xx} + \mu \tilde u_x^2 + \hat f_3 \end{cases} \tag{2.5}$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and the condition

$$(\hat\rho, \hat u, \hat\theta)(t,0) = (\rho_-, u_-, \theta_-), \quad \lim_{x\to\infty} (\hat\rho, \hat u, \hat\theta)(t,x) = (\rho_+, u_+, \theta_+) \quad \text{for each } t \ge 0. \tag{2.6}$$

Here $\hat{P} := P(\hat{\rho}, \hat{\theta}) = R\hat{\rho}\hat{\theta}$ and

$$\hat{f}_1 = \tilde{u}_x(\bar{\rho} - \rho_m) + \bar{u}_x(\tilde{\rho} - \rho_m) + \tilde{\rho}_x(\bar{u} - u_m) + \bar{\rho}_x(\tilde{u} - u_m), \tag{2.7}$$

$$\begin{aligned} \hat{f}_2 = \; & \hat{\rho}[\tilde{u}_x(\bar{u} - u_m) + \bar{u}_x(\tilde{u} - u_m)] + \tilde{u}\tilde{u}_x(\bar{\rho} - \rho_m) \\ & + (\hat{P} - \tilde{P} - \bar{P})_x - \bar{\rho}^{-1}(\tilde{\rho} - \rho_m)\bar{P}_x, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \hat{f}_3 = \; & c_v\hat{\rho}[\tilde{\theta}_x(\bar{u} - u_m) + \bar{\theta}_x(\tilde{u} - u_m)] + c_v(\bar{\rho} - \rho_m)\tilde{u}\tilde{\theta}_x \\ & + \tilde{u}_x(\hat{P} - \tilde{P}) + (\hat{P} - \bar{P})\bar{u}_x - R\bar{\theta}(\tilde{\rho} - \rho_m)\bar{u}_x. \end{aligned} \tag{2.9}$$

3. Stability analysis

This section is devoted to proving our main results: Theorem 1 and Theorem 2. We will concentrate on the proof of Theorem 2, that is, the stability of the composition of a rarefaction wave and a non-degenerate stationary solution. The proof of Theorem 1 is similar to and simpler than that of Theorem 2. We therefore omit it here for brevity.

First we introduce the perturbation (ϕ, ψ, ϑ) toward the superposition wave $(\hat{\rho}, \hat{u}, \hat{\theta})$ as

$$(\phi, \psi, \vartheta)(t, x) := (\rho, u, \theta)(t, x) - (\hat{\rho}, \hat{u}, \hat{\theta})(t, x),$$

where $(\hat{\rho}, \hat{u}, \hat{\theta})$ is given by (1.21). Then we subtract (2.5)–(2.6) from (1.1)–(1.5) to have the initial boundary value problem:

$$\begin{cases} \phi_t + u\phi_x + \rho\psi_x = f_1, \\ \rho(\psi_t + u\psi_x) + (P - \hat{P})_x = \mu\psi_{xx} + f_2, \\ c_v\rho(\vartheta_t + u\vartheta_x) + P\psi_x = \kappa\vartheta_{xx} + \mu\psi_x^2 + f_3 \end{cases} \tag{3.1}$$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$ with the initial and boundary conditions

$$(\phi, \psi, \vartheta)|_{t=0} = (\phi_0, \psi_0, \vartheta_0), \quad (\psi, \vartheta)|_{x=0} = (0, 0). \tag{3.2}$$

Here the initial condition $(\phi_0, \psi_0, \vartheta_0) := (\rho_0, u_0, \theta_0) - (\hat{\rho}, \hat{u}, \hat{\theta})|_{t=0}$ satisfies

$$\lim_{x\to\infty} (\phi_0, \psi_0, \vartheta_0)(x) = (0, 0, 0), \tag{3.3}$$

and

$$f_1 = -\hat{u}_x\phi - \hat{\rho}_x\psi - \hat{f}_1, \tag{3.4}$$

$$f_2 = \mu\bar{u}_{xx} - \mu\hat{\rho}^{-1}\phi\tilde{u}_{xx} + \hat{\rho}^{-1}\phi\hat{P}_x - \rho\psi\hat{u}_x - \hat{\rho}^{-1}\rho\hat{f}_2, \tag{3.5}$$

$$\begin{aligned} f_3 = \; & \kappa\bar{\theta}_{xx} - \hat{\rho}^{-1}\phi(\kappa\tilde{\theta}_{xx} + \mu\tilde{u}_x^2) + \mu\bar{u}_x^2 \\ & + 2\mu\psi_x\hat{u}_x + 2\mu\tilde{u}_x\bar{u}_x - R\rho\vartheta\hat{u}_x - c_v\rho\hat{\theta}_x\psi - \hat{\rho}^{-1}\rho\hat{f}_3, \end{aligned} \tag{3.6}$$

where $\hat{f}_i$ $(i = 1, 2, 3)$ are defined by (2.7)–(2.9), respectively.

We turn to deduce some desired a priori estimates for the perturbation (ϕ, ψ, ϑ) in the Sobolev space H^1. Before doing so, for some non-negative constants N, s, t and m_i $(i = 1, 2)$ with

$t \geq s$, we introduce the set in which we seek the solution of the initial boundary value problem (3.1)–(3.2) as follows

$$X(s,t;m_1,m_2,N) := \Big\{(\phi,\psi,\vartheta) \in C([s,t];H^1) : (\psi_x,\vartheta_x) \in L^2(s,t;H^1), \phi_x \in L^2(s,t;L^2),$$
$$\|(\phi,\psi,\vartheta)(t)\|_1 \leq N,\ (\phi+\hat\rho)(t,x) \geq m_1,\ (\vartheta+\hat\theta)(t,x) \geq m_2\ \forall (t,x) \in [s,t]\times\mathbb{R}_+\Big\}.$$

The letter C or C_i $(i \in \mathbb{N})$ will be employed to denote some positive constant which depends only on $\inf_{x\in\mathbb{R}_+}\{\rho_0(x),\theta_0(x)\}$ and $\|(\phi_0,\psi_0,\vartheta_0)\|_1$. The exact value denoted by C or C_i may therefore vary from line to line. For notational simplicity, we introduce $A \lesssim B$ if $A \leq CB$ holds uniformly for some constant C. The notation $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$. Besides, we will use the notation $(\rho,\theta) = (\phi+\hat\rho, \vartheta+\hat\theta)$.

To make the presentation clearly, we divide this section into five parts. The first four parts concern the a priori estimates for the solution $(\phi,\psi,\vartheta) \in X(0,T;m_1,m_2,N)$ to the problem (3.1)–(3.2), where $T > 0$ and it will be assumed that $m_i \leq 1 \leq N$ $(i=1,2)$ so that

$$\|(\phi,\psi,\vartheta)(t)\|_1 \leq N,\ \ m_1 \leq \rho(t,x) \lesssim N,\ \ m_2 \leq \theta(t,x) \lesssim N \text{ for all } (t,x) \in [0,T]\times\mathbb{R}_+. \tag{3.7}$$

In Subsection 3.5, the last part of this section, we will combine the energy estimates with a well-designed continuation argument to prove Theorem 2.

3.1. Basic energy estimate

In this part, we will show the following basic energy estimate.

Lemma 3.1. *Suppose that the conditions listed in Theorem 2 hold. Then there exists a sufficiently small $\epsilon_0 > 0$ such that if*

$$\Xi(m_1,m_2,N)(\bar\delta+\tilde\delta) \leq \epsilon_0 \tag{3.8}$$

with $\Xi(m_1,m_2,N) := m_1^{-50} m_2^{-50} N^{50}$, then

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}_+} \rho\mathcal{E}\mathrm{d}x + \int_0^T \rho\Phi\left(\frac{\hat\rho}{\rho}\right)(t,0)\mathrm{d}t + \int_0^T\int_{\mathbb{R}_+}\left[\frac{\psi_x^2}{\theta}+\frac{\vartheta_x^2}{\theta^2}\right]\mathrm{d}x\mathrm{d}t \lesssim 1, \tag{3.9}$$

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}_+} \frac{\phi_x^2}{\rho^3}\mathrm{d}x + \int_0^T \frac{\phi_x^2}{\rho^3}(t,0)\mathrm{d}t + \int_0^T\int_{\mathbb{R}_+}\frac{\theta\phi_x^2}{\rho^2}\mathrm{d}x\mathrm{d}t \lesssim N, \tag{3.10}$$

where

$$\mathcal{E} := R\hat\theta\Phi\left(\frac{\hat\rho}{\rho}\right) + \frac{1}{2}\psi^2 + c_v\hat\theta\Phi\left(\frac{\theta}{\hat\theta}\right), \quad \Phi(z) := z - \ln z - 1. \tag{3.11}$$

Proof. Step 1. After a straightforward calculation, one can derive

$$(\rho\mathcal{E})_t+\left[\rho u\mathcal{E}+\psi(P-\hat{P})-\mu\psi\psi_x-\kappa\frac{\vartheta\vartheta_x}{\theta}\right]_x+\mu\frac{\hat{\theta}\psi_x^2}{\theta}+\kappa\frac{\hat{\theta}\vartheta_x^2}{\theta^2}=\sum_{q=1}^{5}\mathcal{R}_q, \tag{3.12}$$

from which we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\rho\mathcal{E}\mathrm{d}x-u_-\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(t,0)+\int_{\mathbb{R}_+}\left[\mu\frac{\hat{\theta}\psi_x^2}{\theta}+\kappa\frac{\hat{\theta}\vartheta_x^2}{\theta^2}\right]\mathrm{d}x=\sum_{q=1}^{5}\int_{\mathbb{R}_+}\mathcal{R}_q\mathrm{d}x, \tag{3.13}$$

where each term $\mathcal{R}_q$ on the right-hand side of (3.12) will be defined below. Before defining and estimating all the terms on the right-hand side of (3.13), we set

$$\bar{U}:=(\bar{\rho},\bar{u},\bar{\theta}),\quad \tilde{U}:=(\tilde{\rho},\tilde{u},\tilde{\theta}),\quad U_m:=(\rho_m,u_m,\theta_m),\quad \Psi:=(\phi,\psi,\vartheta).$$

First we consider

$$\mathcal{R}_1:=\frac{\vartheta}{\theta}\left[\kappa\frac{\bar{\theta}_x\vartheta_x}{\theta}+2\mu\bar{u}_x\psi_x\right],$$

which is trivially estimated by Sobolev's inequality as

$$|\mathcal{R}_1|\lesssim m_2^{-2}|\bar{U}_x||\Psi||\Psi_x|\lesssim m_2^{-2}\|\Psi\|^{\frac12}\|\Psi_x\|^{\frac12}|\bar{U}_x||\Psi_x|.$$

In light of (2.3) and (3.7), we apply Hölder's and Young's inequalities to deduce

$$\begin{aligned}\int_{\mathbb{R}_+}|\mathcal{R}_1|\mathrm{d}x&\lesssim m_2^{-2}\|\Psi\|^{\frac12}\|\bar{U}_x\|\|\Psi_x\|^{\frac32}\\&\lesssim m_2^{-2}N^{\frac12}\bar{\delta}^{\frac12}(1+t)^{-\frac12}\|\Psi_x\|^{\frac32}\\&\lesssim (1+t)^{-2}+m_2^{-\frac83}N^{\frac23}\bar{\delta}^{\frac23}\|\Psi_x\|^2.\end{aligned} \tag{3.14}$$

Next we consider the term

$$\mathcal{R}_2:=\frac{\vartheta}{\theta}(\kappa\bar{\theta}_{xx}+\mu\bar{u}_x^2)+\mu\psi\bar{u}_{xx}.$$

This term can be controlled by Sobolev's inequality as

$$|\mathcal{R}_2|\lesssim m_2^{-1}|\Psi|\left[|\bar{U}_{xx}|+|\bar{U}_x|^2\right]\lesssim m_2^{-1}\|\Psi\|^{\frac12}\|\Psi_x\|^{\frac12}\left[|\bar{U}_{xx}|+|\bar{U}_x|^2\right].$$

According to (2.3), (2.4) and (3.7), we infer

$$\begin{aligned}\int_{\mathbb{R}_+} |\mathcal{R}_2|\mathrm{d}x &\lesssim m_2^{-1}\|\Psi\|^{\frac{1}{2}}\|\Psi_x\|^{\frac{1}{2}}\left[\|\bar{U}_{xx}\|_{L^1}+\|\bar{U}_x\|^2\right] \\ &\lesssim m_2^{-1}N^{\frac{1}{2}}\|\Psi_x\|^{\frac{1}{2}}\bar{\delta}^{\frac{1}{8}}(1+t)^{-\frac{7}{8}} \\ &\lesssim (1+t)^{-\frac{7}{6}}+m_2^{-4}N^2\bar{\delta}^{\frac{1}{2}}\|\Psi_x\|^2.\end{aligned} \tag{3.15}$$

Let us now consider the term

$$\mathcal{R}_3 := -R\frac{\hat{\theta}\phi\hat{f}_1}{\hat{\rho}}+2\mu\frac{\vartheta\bar{u}_x\tilde{u}_x}{\theta}-\rho\frac{\vartheta\hat{f}_3+\hat{\theta}\psi\hat{f}_2}{\hat{\rho}\hat{\theta}}.$$

It is not difficult to derive from (2.7)–(2.9) that

$$|(\hat{f}_1,\hat{f}_2,\hat{f}_3)|\lesssim|\bar{U}_x||\tilde{U}-U_m|+|\tilde{U}_x||\bar{U}-U_m|. \tag{3.16}$$

In view of Lemma 2.2, we deduce that $\bar{U}(t,0)\equiv U_m$ and hence we have that for $q\geq 1$,

$$\begin{aligned}&\left\||\bar{U}_x||\tilde{U}-U_m|+|\tilde{U}_x||\bar{U}-U_m|+|\bar{U}_x||\tilde{U}_x|\right\|_{L^q} \\ &\lesssim \left\||\bar{U}_x||\tilde{U}-U_m|+|\tilde{U}_x|\int_0^x|\bar{U}_x|(\cdot,y)\mathrm{d}y+|\bar{U}_x||\tilde{U}_x|\right\|_{L^q} \\ &\lesssim \|\bar{U}_x\|_{L^\infty}\left\||\tilde{U}-U_m|+x|\tilde{U}_x|+|\tilde{U}_x|\right\|_{L^q},\end{aligned}$$

which combined with (1.18) implies

$$\left\||\bar{U}_x||\tilde{U}-U_m|+|\tilde{U}_x||\bar{U}-U_m|+|\bar{U}_x||\tilde{U}_x|\right\|_{L^q}\lesssim\tilde{\delta}\|\bar{U}_x\|_{L^\infty}. \tag{3.17}$$

It follows from (3.7), (3.16) and (3.17) with $q=1$ that

$$\int_{\mathbb{R}_+}|\mathcal{R}_3|\mathrm{d}x\lesssim Nm_2^{-1}\|\Psi\|_{L^\infty}\tilde{\delta}\|\bar{U}_x\|_{L^\infty}.$$

We use (2.3) and Young's inequality to have

$$\int_{\mathbb{R}_+}|\mathcal{R}_3|\mathrm{d}x\lesssim Nm_2^{-1}\tilde{\delta}(1+t)^{-1}\|\Psi\|^{\frac{1}{2}}\|\Psi_x\|^{\frac{1}{2}}\lesssim(1+t)^{-\frac{4}{3}}+m_2^{-4}\tilde{\delta}^4N^6\|\Psi_x\|^2. \tag{3.18}$$

We then estimate the term

$$\begin{aligned}\mathcal{R}_4 := {} & \psi\tilde{\theta}_x\left[-\frac{c_v}{\hat{\theta}}\rho\vartheta + R\phi + R\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right) + c_v\rho\Phi\left(\frac{\theta}{\hat{\theta}}\right)\right] \\ & - \rho\tilde{u}_x\left[\frac{R^2\hat{\theta}}{c_v}\Phi\left(\frac{\hat{\rho}}{\rho}\right) + \psi^2 + R\hat{\theta}\Phi\left(\frac{\theta}{\hat{\theta}}\right)\right] + \mu\psi\tilde{u}_{xx}\left[1 - \frac{\rho}{\hat{\rho}}\right] \\ & + \vartheta\left[\frac{1}{\theta} - \frac{\rho}{\hat{\rho}\hat{\theta}}\right]\left[\kappa\tilde{\theta}_{xx} + \mu\tilde{u}_x^2\right] + \frac{\vartheta}{\theta}\left[\kappa\frac{\tilde{\theta}_x\vartheta_x}{\theta} + 2\mu\tilde{u}_x\psi_x\right] \\ & + \frac{\rho}{\hat{\rho}}\left[\kappa\tilde{\theta}_{xx} + \mu\tilde{u}_x^2 + \hat{f}_3\right]\left[\frac{R}{c_v}\Phi\left(\frac{\hat{\rho}}{\rho}\right) + \Phi\left(\frac{\theta}{\hat{\theta}}\right)\right].\end{aligned}$$

To this end, we first obtain from the identity

$$\Phi(z) = \int_0^1\int_0^1 \theta_1\Phi''(1 + \theta_1\theta_2(z-1))\mathrm{d}\theta_2\mathrm{d}\theta_1(z-1)^2$$

that

$$(z+1)^{-2}(z-1)^2 \lesssim \Phi(z) \lesssim (z^{-1}+1)^2(z-1)^2. \tag{3.19}$$

This last inequality implies

$$\Phi\left(\frac{\hat{\rho}}{\rho}\right) \lesssim m_1^{-2}\phi^2 \lesssim m_1^{-2}N|\phi|, \quad \Phi\left(\frac{\theta}{\hat{\theta}}\right) \lesssim m_2^{-2}\vartheta^2 \lesssim m_2^{-2}N|\vartheta|. \tag{3.20}$$

In light of (3.16) and (3.20), we discover

$$\int_{\mathbb{R}_+} |\mathcal{R}_4|\mathrm{d}x \lesssim N^2 m_1^{-2}m_2^{-2}\sum_{\ell=0}^{2}\sum_{k=0}^{1}\int_{\mathbb{R}_+}\left|\partial_x^\ell(\tilde{U} - U_m)\right|\left|\partial_x^k\Psi\right|^2\mathrm{d}x. \tag{3.21}$$

To estimate the terms on the right-hand side of (3.21), we utilize an idea in Nikkuni–Kawashima [24], that is, the following Poincaré type inequality

$$|\varphi(t,x)| \le |\varphi(t,0)| + \sqrt{x}\|\varphi_x(t)\| \quad \text{for } x \in \mathbb{R}_+.$$

Applying this inequality to Ψ, we deduce from (1.18) and (3.19) that

$$\int_{\mathbb{R}_+} |\partial_x^\ell(\tilde{U} - U_m)||\partial_x^k\Psi|^2\mathrm{d}x \lesssim \tilde{\delta}\phi(t,0)^2 + \tilde{\delta}\|\Psi_x\|^2 \lesssim N^2m_1^{-1}\tilde{\delta}\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(t,0) + \tilde{\delta}\|\Psi_x\|^2 \tag{3.22}$$

for $k = 0, 1$ and $\ell \in \mathbb{N}$. Plug (3.22) into (3.21) to deduce

$$\int_{\mathbb{R}_+} |\mathcal{R}_4|\mathrm{d}x \lesssim N^4m_1^{-3}m_2^{-2}\tilde{\delta}\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(t,0) + N^2m_1^{-2}m_2^{-2}\tilde{\delta}\|\Psi_x\|^2. \tag{3.23}$$

We next estimate the term

$$\begin{aligned}\mathcal{R}_5 := \ &\psi\bar{\theta}_x\left[-\frac{c_v}{\hat{\theta}}\rho\vartheta + R\phi + R\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right) + c_v\rho\Phi\left(\frac{\theta}{\hat{\theta}}\right)\right] \\ &- \rho\bar{u}_x\left[\frac{R^2\hat{\theta}}{c_v}\Phi\left(\frac{\hat{\rho}}{\rho}\right) + \psi^2 + R\hat{\theta}\Phi\left(\frac{\theta}{\hat{\theta}}\right)\right].\end{aligned}$$

To this end, we introduce

$$a := \ln\left(\frac{\hat{\rho}}{\rho}\right) \quad \text{and} \quad b := \ln\left(\frac{\theta}{\hat{\theta}}\right). \tag{3.24}$$

In light of (2.2), one can find

$$\mathcal{R}_5 = -\rho\bar{\theta}_x F(\psi, a, b) \tag{3.25}$$

with

$$\begin{aligned}F(\psi, a, b) := \ &R\psi a + c_v\psi b + \sqrt{R\gamma}R\bar{\theta}^{-\frac{1}{2}}\hat{\theta}(\mathrm{e}^a - a - 1) \\ &+ \frac{\sqrt{R\gamma}}{\gamma-1}\bar{\theta}^{-\frac{1}{2}}\psi^2 + \sqrt{R\gamma}c_v\bar{\theta}^{-\frac{1}{2}}\hat{\theta}(\mathrm{e}^b - b - 1).\end{aligned}$$

One can easily deduce that $(F, \partial_\psi F, \partial_a F, \partial_b F)(0,0,0) = (0,0,0,0)$ and that the Hessian matrix of F is

$$H_F(\psi, a, b) = \begin{pmatrix} 2\frac{\sqrt{R\gamma}}{\gamma-1}\bar{\theta}^{-\frac{1}{2}} & R & c_v \\ R & \sqrt{R\gamma}R\bar{\theta}^{-\frac{1}{2}}\hat{\theta}\mathrm{e}^a & 0 \\ c_v & 0 & \sqrt{R\gamma}c_v\bar{\theta}^{-\frac{1}{2}}\hat{\theta}\mathrm{e}^b \end{pmatrix}.$$

Let D_k $(k = 1, 2, 3)$ be the $k \times k$ leading principal minor of $H_F(0, 0, 0)$. Then we have

$$D_1 = 2\frac{\sqrt{R\gamma}}{\gamma-1}\bar{\theta}^{-\frac{1}{2}}, \quad D_2 = R^2\frac{2\gamma\bar{\theta}^{-1}\hat{\theta} - \gamma + 1}{\gamma - 1}, \quad D_3 = c_v\sqrt{\gamma R}R^2\bar{\theta}^{-\frac{1}{2}}\hat{\theta}\frac{2\gamma\bar{\theta}^{-1}\hat{\theta} - \gamma}{\gamma - 1}.$$

Apply Sylvester's criterion (see [5, Theorem 7.2.5]) to deduce that there exists a positive constant ε_1 such that if $\tilde{\delta} \le \varepsilon_1$, then $H_F(0, 0, 0)$ is positive definite. Then it follows from the Taylor formula (see [1, Theorem VII.5.8]) that

$$
\begin{aligned}
F(\psi,a,b)= & \sum_{q=0}^{2}\frac{1}{q!}(\psi\partial_\psi+a\partial_a+b\partial_b)^q F(0,0,0)\\
& +\int_0^1(1-t)\big\{(\psi\partial_\psi+a\partial_a+b\partial_b)^2F(t\psi,ta,tb)\\
& -(\psi\partial_\psi+a\partial_a+b\partial_b)^2F(0,0,0)\big\}\mathrm{d}t\\
\le\ & \frac{1}{2}(\psi,a,b)H_F(0,0,0)(\psi,a,b)^T\\
& +\sqrt{R\gamma}\bar{\theta}^{-\frac{1}{2}}\hat{\theta}\int_0^1(1-t)\left\{Ra^2\left(\mathrm{e}^{ta}-1\right)+c_v b^2\left(\mathrm{e}^{tb}-1\right)\right\}\mathrm{d}t.
\end{aligned}
$$

Due to $\bar{\theta}_x\ge 0$, we derive that if $\tilde{\delta}\le\varepsilon_1$, then

$$
\mathcal{R}_5\lesssim\rho\bar{\theta}_x\left[a^2(\mathrm{e}^{|a|}-1)+b^2(\mathrm{e}^{|b|}-1)\right]. \tag{3.26}
$$

We note that the identity

$$
\ln z=\int_0^1\frac{(z-1)\mathrm{d}\theta_1}{1+\theta_1(z-1)}
$$

implies

$$
|\ln z|^2\lesssim(z^{-1}+1)^2(z-1)^2. \tag{3.27}
$$

Then we apply (3.27) to a and b and use the estimate (3.26) to find

$$
\mathcal{R}_5\lesssim Nm_1^{-3}m_2^{-3}\bar{\theta}_x\,|\Psi|^3,
$$

which combined with (2.3) yields

$$
\begin{aligned}
\int_{\mathbb{R}_+}|\mathcal{R}_5|\mathrm{d}x\lesssim\ & Nm_1^{-3}m_2^{-3}\|\bar{\theta}_x\|_{L^\infty}\|\Psi\|_{L^\infty}\|\Psi\|^2\\
\lesssim\ & Nm_1^{-3}m_2^{-3}\bar{\delta}^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}\|\Psi\|^{\frac{5}{2}}\|\Psi_x\|^{\frac{1}{2}}\\
\lesssim\ & (1+t)^{-\frac{7}{6}}+m_1^{-12}m_2^{-12}\bar{\delta}^{\frac{1}{2}}N^{14}\|\Psi_x\|^2.
\end{aligned} \tag{3.28}
$$

Plug (3.14), (3.15), (3.18), (3.23) and (3.28) into (3.13) to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}_+}\rho\mathcal{E}\mathrm{d}x-u_-\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(t,0)+\int\limits_{\mathbb{R}_+}\left[\frac{\psi_x^2}{\theta}+\frac{\vartheta_x^2}{\theta^2}\right]\mathrm{d}x\\
&\quad\lesssim(1+t)^{-\frac{7}{6}}+m_1^{-12}m_2^{-12}N^{16}\left[\bar{\delta}^{\frac{1}{2}}+\tilde{\delta}\right]\left[\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(t,0)+\|\Psi_x\|^2\right].
\end{aligned}\tag{3.29}$$

Hence we can find a sufficiently small constant $\varepsilon_2>0$ such that if

$$m_1^{-12}m_2^{-12}N^{18}\left[\bar{\delta}^{\frac{1}{2}}+\tilde{\delta}\right]\le\varepsilon_2,\tag{3.30}$$

then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}_+}\rho\mathcal{E}\mathrm{d}x-\rho u_-\Phi\left(\frac{\hat{\rho}}{\rho}\right)(t,0)+\int\limits_{\mathbb{R}_+}\left[\frac{\psi_x^2}{\theta}+\frac{\vartheta_x^2}{\theta^2}\right]\mathrm{d}x\\
&\quad\lesssim(1+t)^{-\frac{7}{6}}+m_1^{-12}m_2^{-12}N^{16}\left[\bar{\delta}^{\frac{1}{2}}+\tilde{\delta}\right]\|\phi_x\|^2,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\sup_{0\le t\le T}\int\limits_{\mathbb{R}_+}\rho\mathcal{E}\mathrm{d}x+\int\limits_0^T\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(t,0)\mathrm{d}t+\int\limits_0^T\int\limits_{\mathbb{R}_+}\left[\frac{\psi_x^2}{\theta}+\frac{\vartheta_x^2}{\theta^2}\right]\\
&\quad\lesssim 1+m_1^{-12}m_2^{-12}N^{16}\left[\bar{\delta}^{\frac{1}{2}}+\tilde{\delta}\right]\int\limits_0^T\|\phi_x(t)\|^2\mathrm{d}t.
\end{aligned}\tag{3.31}$$

Step 2. We now make some estimates for the last term in (3.31). We first differentiate $(3.1)_1$ with respect to x and then multiply the resulting equation by ϕ_x/ρ^3 to find

$$\left(\frac{\phi_x^2}{2\rho^3}\right)_t+\left(\frac{u\phi_x^2}{2\rho^3}\right)_x-\hat{u}_x\frac{\phi_x^2}{\rho^3}+\hat{\rho}_x\frac{\phi_x\psi_x}{\rho^3}=f_{1x}\frac{\phi_x}{\rho^3}-\frac{\phi_x\psi_{xx}}{\rho^2}.\tag{3.32}$$

Multiply $(3.1)_2$ by ϕ_x/ρ^2 to have

$$\begin{aligned}
&\left(\frac{\phi_x\psi}{\rho}\right)_t-\left[\frac{\phi_t\psi}{\rho}+\frac{\hat{\rho}_x\psi^2}{\rho}\right]_x-\mu\frac{\phi_x\psi_{xx}}{\rho^2}\\
&=-\frac{\rho_x\hat{u}_x\phi\psi}{\rho^2}-\frac{\hat{\rho}_{xx}}{\rho}\psi^2-\frac{\hat{\rho}_x\psi\psi_x}{\rho}-\frac{\hat{\rho}_x\psi\hat{f}_1}{\rho^2}-\frac{\rho_x\psi\psi_x}{\rho}\\
&\quad+\frac{\hat{u}_x\psi_x\phi}{\rho}+\frac{\psi_x}{\rho}\hat{f}_1+\psi_x^2+\frac{u_x\phi_x\psi}{\rho}-(P-\hat{P})_x\frac{\phi_x}{\rho^2}+f_2\frac{\phi_x}{\rho^2}.
\end{aligned}\tag{3.33}$$

In light of (3.32) and (3.33), we have

$$\left[\frac{\mu\phi_x^2}{2\rho^3}+\frac{\phi_x\psi}{\rho}\right]_t+\left[\frac{\mu u\phi_x^2}{2\rho^3}-\frac{\phi_t\psi}{\rho}-\frac{\hat{\rho}_x\psi^2}{\rho}\right]_x+\frac{R\theta\phi_x^2}{\rho^2}=\sum_{q=1}^{5}\mathcal{Q}_q, \tag{3.34}$$

which combined with Cauchy's inequality implies

$$\int_{\mathbb{R}_+}\frac{\phi_x^2}{\rho^3}\mathrm{d}x+\int_0^t\frac{\phi_x^2}{\rho^3}(s,0)\mathrm{d}s+\int_0^t\int_{\mathbb{R}_+}\frac{\theta\phi_x^2}{\rho^2}\lesssim 1+\int_{\mathbb{R}_+}\rho\psi^2\mathrm{d}x+\sum_{q=1}^{5}\int_0^t\int_{\mathbb{R}_+}|\mathcal{Q}_q|, \tag{3.35}$$

where each term $\mathcal{Q}_q$ on the right-hand side of (3.34) will be defined below. First, let us define

$$\mathcal{Q}_1:=\psi_x^2-\frac{R\phi_x\vartheta_x}{\rho},$$

and by applying Cauchy's inequality, we have

$$\int_0^t\int_{\mathbb{R}_+}|\mathcal{Q}_1|\lesssim\epsilon\int_0^t\int_{\mathbb{R}_+}\frac{\theta\phi_x^2}{\rho^2}+\int_0^t\int_{\mathbb{R}_+}\left[\psi_x^2+C(\epsilon)\frac{\vartheta_x^2}{\theta}\right]. \tag{3.36}$$

Then we consider the term

$$\begin{aligned}\mathcal{Q}_2:=\ &-2\mu\frac{\hat{\rho}_x\phi_x\psi_x}{\rho^3}-\frac{\mu\phi_x}{\rho^3}(\tilde{u}_{xx}\phi+\tilde{\rho}_{xx}\psi)-\frac{\phi_x}{\rho^2\hat{\rho}}(\mu\tilde{u}_{xx}\phi-R\tilde{\rho}_x\hat{\theta}\phi-\tilde{u}_x\hat{\rho}\phi\psi)\\&-\frac{\phi\psi}{\rho^2}(\tilde{\rho}_x\hat{u}_x+\bar{\rho}_x\tilde{u}_x)-\frac{\tilde{\rho}_{xx}\psi^2}{\rho}-2\frac{\tilde{\rho}_x\psi\psi_x}{\rho}+\frac{\tilde{u}_x\phi\psi_x}{\rho}-\frac{R\tilde{\rho}_x\vartheta\phi_x}{\rho^2},\end{aligned}$$

which can be controlled as

$$|\mathcal{Q}_2|\lesssim m_1^{-3}(\bar{\delta}+\tilde{\delta})|\Psi_x|^2+m_1^{-3}|(\tilde{U}_x,\tilde{U}_{xx})||(\Psi,\Psi_x)|^2. \tag{3.37}$$

In light of (3.22), we have

$$\int_0^t\int_{\mathbb{R}_+}|\mathcal{Q}_2|\lesssim N^2m_1^{-4}\tilde{\delta}\int_0^t\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(s,0)\mathrm{d}s+m_1^{-3}(\bar{\delta}+\tilde{\delta})\int_0^t\|\Psi_x(s)\|^2\,\mathrm{d}s. \tag{3.38}$$

For the term

$$\begin{aligned}\mathcal{Q}_3:=&-\frac{\mu\phi_x}{\rho^3}(\bar{u}_{xx}\phi+\bar{\rho}_{xx}\psi)-\frac{\phi_x}{\rho^2\hat{\rho}}(-R\bar{\rho}_x\hat{\theta}\phi-\bar{u}_x\hat{\rho}\phi\psi)\\&-2\frac{\bar{\rho}_x\psi\psi_x}{\rho}+\frac{\bar{u}_x\phi\psi_x}{\rho}-\frac{R\bar{\rho}_x\vartheta\phi_x}{\rho^2},\end{aligned}$$

it follows that

$$|\mathcal{Q}_3| \lesssim m_1^{-3} \left|(\bar{U}_x, \bar{U}_{xx})\right| |\Psi| |\Psi_x| \lesssim m_1^{-3} \|\Psi\|^{\frac{1}{2}} \|\Psi_x\|^{\frac{1}{2}} \left|(\bar{U}_x, \bar{U}_{xx})\right| |\Psi_x|.$$

In view of (2.3), (2.4) and (3.7), we apply Hölder's and Young's inequalities to deduce

$$\begin{aligned}\int_0^t \int_{\mathbb{R}_+} |\mathcal{Q}_3| &\lesssim m_1^{-3} \int_0^t \|\Psi\|^{\frac{1}{2}} \|(\bar{U}_x, \bar{U}_{xx})\| \|\Psi_x\|^{\frac{3}{2}} \mathrm{d}s \\ &\lesssim m_1^{-3} N^{\frac{1}{2}} \bar{\delta}^{\frac{1}{2}} \int_0^t (1+s)^{-\frac{1}{2}+\frac{1}{4q}} \|\Psi_x(s)\|^{\frac{3}{2}} \mathrm{d}s \\ &\lesssim 1 + m_1^{-4} N^{\frac{2}{3}} \bar{\delta}^{\frac{2}{3}} \int_0^t \|\Psi_x(s)\|^2 \mathrm{d}s.\end{aligned} \tag{3.39}$$

For the term

$$\mathcal{Q}_4 := -\frac{\bar{\rho}_x \bar{u}_x \phi \psi}{\rho^2} - \frac{\bar{\rho}_{xx} \psi^2}{\rho},$$

we have

$$|\mathcal{Q}_4| \lesssim m_1^{-2} |\Psi|^2 \left[|\bar{U}_{xx}| + |\bar{U}_x|^2\right] \lesssim m_1^{-2} \|\Psi\| \|\Psi_x\| \left[|\bar{U}_{xx}| + |\bar{U}_x|^2\right],$$

which combined with (2.3)–(2.4) and (3.7) yields

$$\begin{aligned}\int_0^t \int_{\mathbb{R}_+} |\mathcal{Q}_4| &\lesssim m_1^{-2} \int_0^t \|\Psi\| \|\Psi_x\| \left[\|\bar{U}_{xx}\|_{L^1} + \|\bar{U}_x\|^2\right] \mathrm{d}s \\ &\lesssim m_1^{-2} N \int_0^t \|\Psi_x(s)\| \bar{\delta}^{\frac{1}{4}} (1+s)^{-\frac{3}{4}} \mathrm{d}s \\ &\lesssim 1 + m_1^{-4} N^2 \bar{\delta}^{\frac{1}{2}} \int_0^t \|\Psi_x(s)\|^2 \mathrm{d}s.\end{aligned} \tag{3.40}$$

Finally we consider the term

$$\mathcal{Q}_5 := -\frac{\mu \phi_x}{\rho^3} \hat{f}_{1x} - \frac{\phi_x}{\rho^2 \hat{\rho}} (-\mu \bar{u}_{xx} \hat{\rho} + \rho \hat{f}_2) - \frac{\hat{\rho}_x \psi \hat{f}_1}{\rho^2} + \frac{\psi_x \hat{f}_1}{\rho}.$$

Hölder's inequality gives

$$\int_0^t \int_{\mathbb{R}_+} |\mathcal{Q}_5| \lesssim m_1^{-3} \int_0^t \left[\|\Psi_x\| \|(\bar{u}_{xx}, \hat{f}_1, \hat{f}_2, \hat{f}_{1x})\| + \|\Psi\|_{L^\infty} \|\hat{f}_1\|_{L^1}\right] \mathrm{d}s. \tag{3.41}$$

We deduce from (2.7) that

$$|\hat{f}_{1x}| \lesssim |\bar{U}_x||\tilde{U}_x| + |\tilde{U}_{xx}||\bar{U} - U_m| + |\bar{U}_{xx}||\tilde{U} - U_m|. \tag{3.42}$$

Similar to the derivation of (3.17), we have that for $q \geq 1$,

$$\Big\| |\bar{U}_x||\tilde{U}_x| + |\tilde{U}_{xx}||\bar{U} - U_m| + |\bar{U}_{xx}||\tilde{U} - U_m| \Big\|_{L^q} \lesssim \tilde{\delta}\|(\bar{U}_x, \bar{U}_{xx})\|_{L^\infty}. \tag{3.43}$$

Combining the estimates (3.16), (3.17), (3.42) and (3.43) and utilizing Lemma 2.2, we deduce

$$\left\|\left(\bar{u}_{xx}, \hat{f}_1, \hat{f}_2, \hat{f}_{1x}\right)(s)\right\| \lesssim \|\bar{u}_{xx}(s)\| + \tilde{\delta}\left\|(\bar{U}_x, \bar{U}_{xx})(s)\right\|_{L^\infty} \lesssim \bar{\delta}^{\frac{3}{7}}(1+s)^{-\frac{15}{28}}.$$

Plug this last estimate into (3.41) and use (3.16)–(3.17) again to have

$$\begin{aligned}\int_0^t \int_{\mathbb{R}_+} |\mathcal{Q}_5| &\lesssim m_1^{-3} \int_0^t \left[\|\Psi_x\| \bar{\delta}^{\frac{3}{7}}(1+s)^{-\frac{15}{28}} + \tilde{\delta}(1+s)^{-1}\|\Psi\|^{\frac{1}{2}}\|\Psi_x\|^{\frac{1}{2}} \right] \mathrm{d}s \\ &\lesssim 1 + \left(m_1^{-6}\bar{\delta}^{\frac{6}{7}} + m_1^{-12}N^2\tilde{\delta}^4\right)\int_0^t \|\Psi_x(s)\|^2 \mathrm{d}s.\end{aligned} \tag{3.44}$$

Plugging (3.31), (3.36), (3.38), (3.39), (3.40) and (3.44) into (3.35), we take ϵ sufficiently small to find

$$\begin{aligned}&\int_{\mathbb{R}_+} \frac{\phi_x^2}{\rho^3}\mathrm{d}x + \int_0^t \frac{\phi_x^2}{\rho^3}(s,0)\mathrm{d}s + \int_0^t\int_{\mathbb{R}_+} \frac{\theta\phi_x^2}{\rho^2} \\ &\lesssim 1 + \int_0^t\int_{\mathbb{R}_+}\left[\psi_x^2 + \frac{\vartheta_x^2}{\theta}\right] + m_1^{-12}m_2^{-12}N^{16}\left[\bar{\delta}^{\frac{1}{2}} + \tilde{\delta}\right]\int_0^t \|\phi_x(s)\|^2\mathrm{d}s \\ &\quad + m_1^{-12}N^6\left[\bar{\delta}^{\frac{1}{2}} + \tilde{\delta}\right]\left[\int_0^t \rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(s,0)\mathrm{d}s + \int_0^t\int_{\mathbb{R}_+}\left(\frac{\psi_x^2}{\theta} + \frac{\vartheta_x^2}{\theta^2}\right)\right],\end{aligned}$$

which combined with (3.31) gives

$$\begin{aligned}&\int_{\mathbb{R}_+} \frac{\phi_x^2}{\rho^3}\mathrm{d}x + \int_0^t \frac{\phi_x^2}{\rho^3}(s,0)\mathrm{d}s + \int_0^t\int_{\mathbb{R}_+} \frac{\theta\phi_x^2}{\rho^2} \\ &\lesssim 1 + \int_0^t\int_{\mathbb{R}_+}\left[\psi_x^2 + \frac{\vartheta_x^2}{\theta}\right] + m_1^{-50}m_2^{-50}N^{50}\left[\bar{\delta}^{\frac{1}{2}} + \tilde{\delta}\right]\int_0^t\int_{\mathbb{R}_+} \frac{\theta\phi_x^2}{\rho^2}.\end{aligned}$$

We take $\epsilon_0 > 0$ small enough and use (3.8) to have

$$\int_{\mathbb{R}_+} \frac{\phi_x^2}{\rho^3}\mathrm{d}x + \int_0^t \frac{\phi_x^2}{\rho^3}(s,0)\mathrm{d}s + \int_0^t\int_{\mathbb{R}_+} \frac{\theta\phi_x^2}{\rho^2} \lesssim 1 + \int_0^t\int_{\mathbb{R}_+}\left[\psi_x^2 + \frac{\vartheta_x^2}{\theta}\right]. \tag{3.45}$$

The estimate (3.9) follows by plugging (3.45) into (3.31) and using the condition (3.8) for a sufficiently small $\epsilon_0 > 0$. Combine (3.9) and (3.45) to deduce (3.10). The proof of the lemma is completed. □

3.2. Uniform bounds on density

Having obtained the energy estimate (3.9), we can proceed to deduce the positive lower and upper bounds of the density $\rho(t,x)$ uniformly in time t and space x in this subsection. For this purpose, we transform the outflow problem into the corresponding problem in the Lagrangian coordinate by introducing the Lagrangian variable

$$y = -u_- \int_0^t \rho(s,0)\mathrm{d}s + \int_0^x \rho(t,z)\mathrm{d}z. \tag{3.46}$$

By the coordinate change $(t,x) \mapsto (t,y)$, the domain $[0,T]\times\mathbb{R}_+$ is mapped into

$$\Omega_T := \{(t,y) : 0 \le t \le T, y > Y(t)\} \quad \text{with} \quad Y(t) := -u_- \int_0^t \rho(s,0)\mathrm{d}s,$$

and the outflow problem (1.1)–(1.5) is transformed into the following initial boundary value problem

$$\begin{cases} v_t - u_y = 0, \\ u_t + P_y = \left(\dfrac{\mu u_y}{v}\right)_y, \\ \left(c_v\theta + \dfrac{u^2}{2}\right)_t + (Pu)_y = \left(\dfrac{\kappa\theta_y}{v} + \dfrac{\mu u u_y}{v}\right)_y & \text{for } y > Y(t), \\ (u,\theta)|_{y=Y(t)} = (u_-,\theta_-), \\ (v,u,\theta)|_{t=0} = (v_0,u_0,\theta_0). \end{cases} \tag{3.47}$$

Here $v = 1/\rho$ stands for the specific volume of the gas and $v_0 = 1/\rho_0$. The basic energy estimate (3.9) in Eulerian coordinate can be rewritten as a corresponding estimate in Lagrangian coordinate as a direct consequence of the transformation (3.46).

Corollary 3.2. *Suppose that the conditions listed in Lemma 3.1 hold. Then*

$$\sup_{0\le t\le T}\int_{Y(t)}^{\infty} \mathcal{E}\mathrm{d}y + \int_0^T\int_{Y(t)}^{\infty}\left[\frac{\psi_y^2}{v\theta} + \frac{\vartheta_y^2}{v\theta^2}\right]\mathrm{d}y\mathrm{d}t \lesssim 1. \tag{3.48}$$

Note that the function $Y(t)$ describing the boundary in the Lagrangian coordinate is part of the unknown, that is, the problem (3.47) is a free boundary problem. To obtain the uniform bounds of the specific volume v for the free boundary problem (3.47), we introduce the time-dependent domain $\Omega_i(t)$ with $i \in \mathbb{Z}$ and $t \in [0, T]$ as

$$\Omega_i(t) := \begin{cases} [Y(t), [Y(t)] + 2] & \text{if } i = [Y(t)] + 1, \\ [i, i+1] & \text{else}. \end{cases} \tag{3.49}$$

Based on the basic energy estimate (3.48), we have the following lemma.

Lemma 3.3. *Suppose that the conditions listed in Lemma 3.1 hold. Then there exists a positive constant C_0, depending solely on $\inf_{x\in\mathbb{R}_+}\{\rho_0(x), \theta_0(x)\}$ and $\|(\phi_0, \psi_0, \vartheta_0)\|_1$, such that for all pair (s, t) with $0 \le s \le t \le T$ and integer $i \ge [Y(t)] + 1$,*

$$C_0^{-1} \le \int_{\Omega_i(t)} v(s, y)\mathrm{d}y \le C_0, \quad C_0^{-1} \le \int_{\Omega_i(t)} \theta(s, y)\mathrm{d}y \le C_0, \tag{3.50}$$

and there are points $a_i(s, t), b_i(s, t) \in \Omega_i(t)$ satisfying

$$C_0^{-1} \le v(s, a_i(s, t)) \le C_0, \quad C_0^{-1} \le \theta(s, b_i(s, t)) \le C_0. \tag{3.51}$$

Proof. Let $0 \le s \le t \le T$ and $i \ge [Y(t)] + 1$. According to the definition of $Y(t)$ and the sign of u_-, we have $Y(s) \le Y(t)$ and $\Omega_i(t) \subset [Y(s), \infty)$. In view of (3.48), we get

$$\int_{\Omega_i(t)} \Phi\left(\frac{v}{\hat{v}}\right)(s, y)\mathrm{d}y + \int_{\Omega_i(t)} \Phi\left(\frac{\theta}{\hat{\theta}}\right)(s, y)\mathrm{d}y \lesssim 1.$$

Apply Jensen's inequality to the convex function Φ to obtain

$$\Phi\left(\frac{1}{|\Omega_i(t)|}\int_{\Omega_i(t)} \frac{v}{\hat{v}}(s, y)\mathrm{d}y\right) + \Phi\left(\frac{1}{|\Omega_i(t)|}\int_{\Omega_i(t)} \frac{\theta}{\hat{\theta}}(s, y)\mathrm{d}y\right) \le C.$$

Let α and β be the two positive roots of the equation $\Phi(z) = C$. Then we have

$$\alpha \le \frac{1}{|\Omega_i(t)|}\int_{\Omega_i(t)} \frac{v}{\hat{v}}(s, y)\mathrm{d}y \le \beta, \quad \alpha \le \frac{1}{|\Omega_i(t)|}\int_{\Omega_i(t)} \frac{\theta}{\hat{\theta}}(s, y)\mathrm{d}y \le \beta.$$

These estimates imply (3.50). Finally we employ the mean value theorem to (3.50) to find $a_i(s, t), b_i(s, t) \in \Omega_i(t)$ satisfying (3.51). The proof of the lemma is completed. □

We deduce a local representation of the solution v for the free boundary problem (3.47) in the next lemma by modifying Jiang's argument for fixed domains in [12,13]. To this end, we introduce the cutoff function $\varphi_z \in W^{1,\infty}(\mathbb{R})$ with parameter $z \in \mathbb{R}$ by

$$\varphi_z(y) = \begin{cases} 1, & y < [z]+4, \\ [z]+5-y, & [z]+4 \le y < [z]+5, \\ 0, & y \ge [z]+5. \end{cases} \tag{3.52}$$

Lemma 3.4. *Let $(\tau, z) \in \Omega_T$. Then*

$$v(t,y) = B_z(t,y)A_z(t) + \frac{R}{\mu}\int_0^t \frac{B_z(t,y)A_z(t)}{B_z(s,y)A_z(s)}\theta(s,y)\mathrm{d}s \tag{3.53}$$

for all $t \in [0,\tau]$ and $y \in I_z(\tau) := (Y(\tau), \infty) \cap ([z]-1, [z]+4)$, where

$$B_z(t,y) := v_0(y)\exp\left\{\frac{1}{\mu}\int_y^\infty (u_0(\xi) - u(t,\xi))\,\varphi_z(\xi)\mathrm{d}\xi\right\}, \tag{3.54}$$

$$A_z(t) := \exp\left\{\frac{1}{\mu}\int_0^t\int_{[z]+4}^{[z]+5}\left(\frac{\mu u_y}{v} - P\right)\mathrm{d}\xi\,\mathrm{d}s\right\}. \tag{3.55}$$

Proof. We multiply (3.47)$_2$ by φ_z to get

$$(\varphi_z u)_t = \left[\left(\mu\frac{u_y}{v} - P\right)\varphi_z\right]_y - \varphi_z'\left(\mu\frac{u_y}{v} - P\right). \tag{3.56}$$

Let $(t,y) \in [0,\tau] \times I_z(\tau)$. Since $y > Y(s)$ for each $s \in [0,\tau]$, we have $[0,\tau] \times [y,\infty) \subset \Omega_T$. In light of the identity $\varphi_z(y) = 1$ and (3.47)$_1$, we integrate (3.56) over $[0,t] \times [y,\infty)$ to get

$$-\int_y^\infty \varphi_z(\xi)(u(t,\xi) - u_0(\xi))\mathrm{d}\xi = \mu\ln\frac{v(t,y)}{v_0(y)} - R\int_0^t \frac{\theta(s,y)}{v(s,y)}\mathrm{d}s + \int_0^t\int_{[z]+4}^{[z]+5}\left(P - \mu\frac{u_y}{v}\right).$$

This implies that for each $t \in [0,\tau]$,

$$\frac{1}{v(t,y)}\exp\left\{\frac{R}{\mu}\int_0^t \frac{\theta(s,y)}{v(s,y)}\mathrm{d}s\right\} = \frac{1}{B_z(t,y)A_z(t)}. \tag{3.57}$$

Multiplying (3.57) by $R\theta(t,y)/\mu$ and integrating the resulting identity over $[0,t]$, we have

$$\exp\left\{\frac{R}{\mu}\int_0^t \frac{\theta(s,y)}{v(s,y)}\mathrm{d}s\right\} = 1 + \frac{R}{\mu}\int_0^t \frac{\theta(s,y)}{B_z(s,y)A_z(s)}\mathrm{d}s.$$

We then plug this identity into (3.57) to obtain (3.53) and complete the proof of the lemma. □

The following lemma is devoted to showing the bounds of the specific volume $v(\tau, z)$ uniformly in the time τ and the Lagrangian variable z.

Lemma 3.5. *Suppose that the conditions listed in Lemma 3.1 hold. Then*

$$C_1^{-1} \le v(\tau, z) \le C_1 \quad \text{for all } (\tau, z) \in \Omega_T. \tag{3.58}$$

Proof. Let $(\tau, z) \in \Omega_T$ be arbitrary but fixed. The proof is divided into three steps.

Step 1. It follows from Cauchy's inequality and (3.48) that

$$B_z(t, y) \sim 1 \quad \text{for all } (t, y) \in [0, \tau] \times I_z(\tau). \tag{3.59}$$

Let $0 \le s \le t \le \tau$. For each $0 \le t' \le t$, there exists $y(t') \in [[z] + 4, [z] + 5]$ such that

$$\theta(t', y(t')) = \inf_{([z]+4,[z]+5)} \theta(t', \cdot).$$

Apply Cauchy's inequality to have

$$\frac{\mu u_y}{v} - P \le \frac{C u_y^2}{v\theta} - \frac{R\theta}{2v} \le \frac{C\psi_y^2}{v\theta} + \frac{C\hat{u}_y^2}{v\theta} - \frac{R\theta}{2v}.$$

In view of (1.18), (2.3), (3.48), (3.8) and (3.50), we apply Jensen's inequality for the convex function $1/x$ to deduce

$$\begin{aligned}
&\int_s^t \int_{[z]+4}^{[z]+5} \left[\frac{\mu u_y}{v} - P\right] \\
&\le C + CNm_2^{-1}(\tilde{\delta} + \bar{\delta})(t - s) - \frac{R}{2}\int_s^t \theta(t', y(t')) \int_{[z]+4}^{[z]+5} v^{-1}(t', y)\mathrm{d}y\mathrm{d}t' \\
&\le C + CNm_2^{-1}(\tilde{\delta} + \bar{\delta})(t - s) - \frac{R}{2}\int_s^t \theta(t', y(t')) \left[\int_{[z]+4}^{[z]+5} v\mathrm{d}y\right]^{-1} \mathrm{d}t' \\
&\le C + C\epsilon_0(t - s) - C^{-1}\int_s^t \theta(t', y(t'))\mathrm{d}t'.
\end{aligned} \tag{3.60}$$

Since $[z] + 4 \ge [Y(t')] + 2$, we derive from (3.49) that $\Omega_{[z]+4}(t') = [[z] + 4, [z] + 5]$. We then apply Hölder's and Cauchy's inequalities to obtain from (1.18), Lemma 3.3 and (3.48) that

$$
\begin{aligned}
&\left|\int_s^t \int_{b_{[z]+4}(t',t')}^{y(t')} \frac{\theta_y}{\theta}(t',\xi)\mathrm{d}\xi\,\mathrm{d}t'\right| \\
&\le \int_s^t \int_{\Omega_{[z]+4}(t')} \left|\frac{\hat{\theta}_y}{\theta}(t',\xi)+\frac{\vartheta_y}{\theta}(t',\xi)\right|\mathrm{d}\xi\,\mathrm{d}t' \\
&\le m_2^{-1}(\tilde{\delta}+\bar{\delta})(t-s)+\int_s^t \left|\int_{Y(t')}^{\infty} \frac{\vartheta_y^2}{v\theta^2}(t',\xi)\mathrm{d}\xi\right|^{\frac{1}{2}} \left|\int_{\Omega_{[z]+4}(t')} v(t',\xi)\mathrm{d}\xi\right|^{\frac{1}{2}} \mathrm{d}t' \qquad (3.61)\\
&\le C(t-s)+C\int_s^t \int_{Y(t')}^{\infty} \frac{\vartheta_y^2}{v\theta^2}(t',\xi)\mathrm{d}\xi\,\mathrm{d}t' \\
&\le C(t-s)+C.
\end{aligned}
$$

Applying Jensen's inequality to the convex function e^x, we have from (3.51) and (3.61) that

$$
\begin{aligned}
&\int_s^t \theta(t',y(t'))\mathrm{d}t' = \int_s^t \exp\left(\ln\theta(t',y(t'))\right)\mathrm{d}t' \\
&\ge (t-s)\exp\left(\frac{1}{t-s}\int_s^t \ln\theta(t',y(t'))\mathrm{d}t'\right) \\
&\ge (t-s)\exp\left(\frac{1}{t-s}\int_s^t \left[\int_{b_{[z]+4}(t',t')}^{y(t')} \frac{\theta_y}{\theta}(t',\xi)\mathrm{d}\xi+\ln\theta(t',b_{[z]+4}(t',t'))\right]\mathrm{d}t'\right) \\
&\ge (t-s)\exp\left(-\ln C_0-\frac{1}{t-s}\left|\int_s^t \int_{b_{[z]+4}(t',t')}^{y(t')} \frac{\theta_y}{\theta}(t',\xi)\mathrm{d}\xi\,\mathrm{d}t'\right|\right) \\
&\ge \frac{t-s}{C}\exp\left(-\frac{C}{t-s}\right).
\end{aligned}
$$

This implies

$$
-\int_s^t \theta(t',y(t'))\mathrm{d}t' \le \begin{cases} 0 & \text{if } 0\le t-s\le 1, \\ -C^{-1}(t-s) & \text{if } t-s\ge 1. \end{cases} \qquad (3.62)
$$

Plugging (3.62) into (3.60) and taking $\epsilon_0>0$ small enough, we have for each $s\in[0,t]$ that

$$\int_s^t \int_{[z]+4}^{[z]+5} \left[\frac{\mu u_y}{v} - P\right] \leq C - C^{-1}(t-s).$$

According to the definition (3.55), we then obtain

$$0 \leq A_z(t) \leq C\mathrm{e}^{-t/C}, \quad \frac{A_z(t)}{A_z(s)} \leq C\mathrm{e}^{-(t-s)/C} \quad \text{for all } 0 \leq s \leq t \leq \tau. \tag{3.63}$$

Step 2. Plugging (3.59) and (3.63) into (3.53), we infer that for all $(t, y) \in [0, \tau] \times I_z(\tau)$,

$$\int_0^t \frac{A_z(t)}{A_z(s)}\theta(s, y)\mathrm{d}s \lesssim v(t, y) \lesssim 1 + \int_0^t \theta(s, y)\mathrm{e}^{-\frac{t-s}{C}}\mathrm{d}s. \tag{3.64}$$

In light of the fundamental theorem of calculus, we deduce from (1.18) and (3.50) that for $y \in I_z(\tau)$ and $0 \leq s \leq t \leq \tau$,

$$\begin{aligned}
&\left|\theta(s, y)^{\frac{1}{2}} - \theta(s, b_{[z]+2}(s, \tau))^{\frac{1}{2}}\right| \\
&\lesssim \int_{I_z(\tau)} \theta^{-\frac{1}{2}}\left|\hat{\theta}_y\right|(s, \xi)\mathrm{d}\xi + \int_{I_z(\tau)} \theta^{-\frac{1}{2}}|\vartheta_y|(s, \xi)\mathrm{d}\xi \\
&\lesssim m_2^{-\frac{1}{2}}(\tilde{\delta} + \bar{\delta}) + \left[\int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2}(s, \xi)\mathrm{d}\xi\right]^{\frac{1}{2}} \left[\int_{I_z(\tau)} v\theta(s, \xi)\mathrm{d}\xi\right]^{\frac{1}{2}} \\
&\lesssim m_2^{-\frac{1}{2}}(\tilde{\delta} + \bar{\delta}) + \sup_{I_z(\tau)} v^{\frac{1}{2}}(s, \cdot)\left[\int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2}(s, \xi)\mathrm{d}\xi\right]^{\frac{1}{2}},
\end{aligned} \tag{3.65}$$

where we have used $b_{[z]+2}(s, \tau) \in \Omega_{[z]+2}(\tau) \subset I_z(\tau)$. Combine (3.65) with (3.51) and (3.8) to give

$$1 - C\sup_{I_z(\tau)} v(s, \cdot)\int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2}(s, \xi)\mathrm{d}\xi \lesssim \theta(s, y) \lesssim 1 + \sup_{I_z(\tau)} v(s, \cdot)\int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2}(s, \xi)\mathrm{d}\xi. \tag{3.66}$$

We plug (3.66) into (3.64) to obtain

$$v(t, y) \lesssim 1 + \int_0^t \sup_{I_z(\tau)} v(s, \cdot)\int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2}(s, \xi)\mathrm{d}\xi\mathrm{d}s.$$

Taking the supremum over $I_z(\tau)$ with respect to y, we have

$$\sup_{I_z(\tau)} v(t,\cdot) \lesssim 1 + \int_0^t \sup_{I_z(\tau)} v(s,\cdot) \int_{\Omega_i(\tau)} \frac{\vartheta_y^2}{v\theta^2}(s,\xi)\mathrm{d}\xi\mathrm{d}s. \tag{3.67}$$

Applying Gronwall's inequality to (3.67), we can deduce from (3.48) that

$$\sup_{I_z(\tau)} v(t,\cdot) \le C_1 \quad \text{for all } t \in [0,\tau], \tag{3.68}$$

where $C_1 > 0$ is some constant independent of t, τ and z. Noting that $z \in I_z(\tau)$, we deduce from (3.68) that $v(\tau, z) \le C_1$. Since $(\tau, z) \in \Omega_T$ is arbitrary, we conclude

$$v(\tau,z) \le C_1 \quad \text{for all } (\tau,z) \in \Omega_T. \tag{3.69}$$

Step 3. On the other hand, in view of (3.50), (3.59) and (3.63), we integrate (3.53) on $I_z(\tau)$ with respect to y to find

$$1 \lesssim \int_{I_z(\tau)} v(t,y)\mathrm{d}y \lesssim \mathrm{e}^{-t/C} + \int_0^t \frac{A_z(t)}{A_z(s)}\mathrm{d}s.$$

Consequently, we have

$$\int_0^t \frac{A_z(t)}{A_z(s)}\mathrm{d}s \gtrsim 1 - C\mathrm{e}^{-t/C}. \tag{3.70}$$

Inserting (3.66), (3.69) and (3.70) into (3.64), we have

$$\begin{aligned}
v(t,y) &\gtrsim \int_0^t \frac{A_z(t)}{A_z(s)}\mathrm{d}s - C\int_0^t \frac{A_z(t)}{A_z(s)} \int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2}\mathrm{d}\xi\mathrm{d}s \\
&\gtrsim 1 - C\mathrm{e}^{-t/C} - C\left(\int_0^{t/2} + \int_{t/2}^t\right) \frac{A_z(t)}{A_z(s)} \int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2}\mathrm{d}\xi\mathrm{d}s \\
&\gtrsim 1 - C\mathrm{e}^{-t/C} - C\int_0^{t/2} \mathrm{e}^{-\frac{t-s}{C}} \int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2}\mathrm{d}\xi\mathrm{d}s - C\int_{t/2}^t \int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2} \\
&\gtrsim 1 - C\mathrm{e}^{-t/C} - C\mathrm{e}^{-\frac{t}{2C}} - C\int_{t/2}^t \int_{I_z(\tau)} \frac{\vartheta_y^2}{v\theta^2} \\
&\gtrsim 1 \quad \text{for all } (t,y) \in [T_0,\tau] \times I_z(\tau),
\end{aligned} \tag{3.71}$$

where T_0 is a positive constant independent of t. In particular, the estimate (3.71) implies

$$v(\tau, z) \gtrsim 1 \quad \text{for all } \tau \geq T_0,\ z > Y(\tau). \tag{3.72}$$

As in [16,17], we can derive a positive lower bound for v, that is,

$$v(\tau, z) \gtrsim \mathrm{e}^{-Ct} \quad \text{for } (\tau, z) \in \Omega_T. \tag{3.73}$$

Finally, we combine (3.73), (3.69) and (3.72) to get (3.58). This completes the proof. □

As a corollary of Lemma 3.5, we obtain the bounds for the density $\rho(t, x)$ uniformly in time t and space x.

Corollary 3.6. *Suppose that the conditions listed in Lemma 3.1 hold. Then*

$$C_1^{-1} \leq \rho(t, x) \leq C_1 \quad \textit{for all } (t, x) \in [0, T] \times \mathbb{R}_+, \tag{3.74}$$

where the positive constant C_1 depends solely on $\inf_{x\in\mathbb{R}_+}\{\rho_0(x), \theta_0(x)\}$ and $\|(\phi_0, \psi_0, \vartheta_0)\|_1$.

3.3. Uniform estimates for the perturbation

In this subsection, we will estimate the H_x^1-norm of the perturbation $(\phi, \vartheta, \psi)(t, x)$ uniformly in time t. First we can get the following uniform L^2-norm estimate.

Lemma 3.7. *Suppose that the conditions listed in Lemma 3.1 hold. Then*

$$\sup_{0\leq t\leq T} \|(\phi, \vartheta, \psi)(t)\|^2 + \int_0^T \int_{\mathbb{R}_+} \left[(1 + \theta + \psi^2)\psi_x^2 + \vartheta_x^2\right] \mathrm{d}x\mathrm{d}t \lesssim 1. \tag{3.75}$$

Proof. We divide the proof into five steps.

Step 1. First, for each $t \geq 0$ and $a > 0$, we denote

$$\Omega_a'(t) := \{x \in \mathbb{R}_+ : \vartheta(t, x) > a\}.$$

Then it follows from (3.9) and (3.74) that

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left[\int_{\mathbb{R}_+} \phi^2 \mathrm{d}x + \int_{\mathbb{R}_+\setminus\Omega_a'(t)} \vartheta^2 \mathrm{d}x + \int_0^t |\phi(s, 0)|^2 \mathrm{d}s\right] + \int_0^T \int_{\mathbb{R}_+\setminus\Omega_a'(t)} \left[\psi_x^2 + \vartheta_x^2\right] \\
&\leq C(a) \sup_{0\leq t\leq T}\left[\int_{\mathbb{R}_+} \rho\mathcal{E}\mathrm{d}x + \int_0^t \rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(s, 0)\mathrm{d}s\right] + C(a)\int_0^T\int_{\mathbb{R}_+}\left[\frac{\psi_x^2}{\theta} + \frac{\vartheta_x^2}{\theta^2}\right] \leq C(a).
\end{aligned} \tag{3.76}$$

Step 2. We now estimate the integral $\int_0^T \int_{\Omega_a'(t)} \vartheta_x^2$. To this end, we multiply $(3.1)_3$ by $(\vartheta - 2)_+ := \max\{\vartheta - 2, 0\}$ and integrate the resulting identity over $(0, t) \times \mathbb{R}_+$ to obtain

$$
\begin{aligned}
&\frac{c_v}{2}\int_{\mathbb{R}_+}\rho(\vartheta-2)_+^2\mathrm{d}x+\kappa\int_0^t\int_{\Omega_2'(s)}\vartheta_x^2+R\int_0^t\int_{\mathbb{R}_+}\rho\theta\psi_x(\vartheta-2)_+\\
&\quad=\frac{c_v}{2}\int_{\mathbb{R}_+}\rho_0(\vartheta_0-2)_+^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}f_3(\vartheta-2)_++\mu\int_0^t\int_{\mathbb{R}_+}\psi_x^2(\vartheta-2)_+.
\end{aligned}
\tag{3.77}
$$

To estimate the last term of (3.77), we multiply $(3.1)_2$ by $2\psi(\vartheta-2)_+$ and integrate the resulting identity over $(0,t)\times\mathbb{R}_+$ to find

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\psi^2\rho(\vartheta-2)_+\mathrm{d}x+2\mu\int_0^t\int_{\mathbb{R}_+}\psi_x^2(\vartheta-2)_+-\int_0^t\int_{\Omega_2'(s)}\rho\psi^2(\vartheta_t+u\vartheta_x)\\
&\quad=\int_{\mathbb{R}_+}\psi_0^2\rho_0(\vartheta_0-2)_+\mathrm{d}x+2R\int_0^t\int_{\mathbb{R}_+}\rho\theta\psi_x(\vartheta-2)_++2R\int_0^t\int_{\Omega_2'(s)}\rho\theta\psi\vartheta_x\\
&\qquad+2R\int_0^t\int_{\mathbb{R}_+}\psi(\hat\rho\hat\theta)_x(\vartheta-2)_+-2\mu\int_0^t\int_{\Omega_2'(s)}\psi\psi_x\vartheta_x+2\int_0^t\int_{\mathbb{R}_+}f_2\psi(\vartheta-2)_+.
\end{aligned}
\tag{3.78}
$$

Combining (3.78) and (3.77), we have from $(3.1)_3$ that

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\left[\frac{c_v}{2}\rho(\vartheta-2)_+^2+\psi^2\rho(\vartheta-2)_+\right]\mathrm{d}x+\kappa\int_0^t\int_{\Omega_2'(s)}\vartheta_x^2+\mu\int_0^t\int_{\mathbb{R}_+}\psi_x^2(\vartheta-2)_+\\
&\quad=\int_{\mathbb{R}_+}\left[\frac{c_v}{2}\rho_0(\vartheta_0-2)_+^2+\psi_0^2\rho_0(\vartheta_0-2)_+\right]\mathrm{d}x+\sum_{p=1}^{6}\mathcal{J}_p,
\end{aligned}
\tag{3.79}
$$

where each term $\mathcal{J}_p$ in the decomposition will be defined below. We now define and estimate all the terms in the decomposition. We first consider

$$
\mathcal{J}_1:=R\int_0^t\int_{\mathbb{R}_+}\rho\theta\psi_x(\vartheta-2)_+\quad\text{and}\quad\mathcal{J}_2:=2R\int_0^t\int_{\Omega_2'(s)}\rho\theta\psi\vartheta_x.
$$

In light of (3.74) and (3.9), we have

$$
\int_{\mathbb{R}_+}\psi^2\mathrm{d}x+\int_{\Omega_1'(s)}\theta\mathrm{d}x\lesssim\int_{\mathbb{R}_+}\rho\mathcal{E}\mathrm{d}x\lesssim 1.
\tag{3.80}
$$

From Cauchy's inequality and (3.74), we obtain

$$
\begin{aligned}
|\mathcal{J}_1| &\le \epsilon \int_0^t \int_{\mathbb{R}_+} \psi_x^2(\vartheta-2)_+ + C(\epsilon)\int_0^t \int_{\mathbb{R}_+} \theta^2(\vartheta-2)_+ \\
&\le \epsilon \int_0^t \int_{\mathbb{R}_+} \psi_x^2(\vartheta-2)_+ + C(\epsilon)\int_0^t \int_{\mathbb{R}_+} \theta(\vartheta-1)_+^2 \\
&\le \epsilon \int_0^t \int_{\mathbb{R}_+} \psi_x^2(\vartheta-2)_+ + C(\epsilon)\int_0^t \sup_{\mathbb{R}_+}(\vartheta-1)_+^2,
\end{aligned}
\tag{3.81}
$$

and

$$
\begin{aligned}
|\mathcal{J}_2| &\le \epsilon \int_0^t \int_{\Omega_2'(s)} \vartheta_x^2 + C(\epsilon)\int_0^t \int_{\Omega_2'(s)} \psi^2\theta^2 \\
&\le \epsilon \int_0^t \int_{\Omega_2'(s)} \vartheta_x^2 + C(\epsilon)\int_0^t \int_{\Omega_2'(s)} \psi^2(\vartheta-1)_+^2 \\
&\le \epsilon \int_0^t \int_{\Omega_2'(s)} \vartheta_x^2 + C(\epsilon)\int_0^t \sup_{\mathbb{R}_+}(\vartheta-1)_+^2.
\end{aligned}
\tag{3.82}
$$

Here we have used the fact that

$$\theta \le K(\vartheta-1)$$

with $K = 2 + \sup_{\mathbb{R}_+^2} \hat{\theta}$ provided that $\vartheta \ge 2$. Let us define

$$\mathcal{J}_3 := \int_0^t \int_{\Omega_2'(s)} \Big[f_3(\vartheta-2)_+ + c_v^{-1} f_3\psi^2 + 2 f_2\psi(\vartheta-2)_+ \Big].$$

According to (3.5) and (3.6), we use (3.16) and (3.74) to deduce

$$|(f_2, f_3)| \lesssim G + \big|(\Psi, \Psi_x)\big|\big|(\bar{U}_x, \tilde{U}_x, \tilde{U}_{xx})\big|, \tag{3.83}$$

with

$$G := \big|\bar{U}_{xx}\big| + \big|\bar{U}_x\big|^2 + |\bar{U}_x||\tilde{U} - U_m| + |\tilde{U}_x||\bar{U} - U_m| + |\bar{U}_x||\tilde{U}_x|. \tag{3.84}$$

Hence, we have

$$\mathcal{J}_3 \lesssim \int_0^t \int_{\Omega_2'(s)} \Big[G + \big|(\Psi, \Psi_x)\big|\big|(\bar{U}_x, \tilde{U}_x, \tilde{U}_{xx})\big| \Big] \Big[m_2^{-1}(\vartheta-2)_+^2 + \psi^2 \Big]. \tag{3.85}$$

It follows from Lemma 2.2 and (3.17) that

$$\|G(s)\|_{L^1} \lesssim \bar{\delta}^{\frac{1}{3}}(1+s)^{-\frac{2}{3}}, \tag{3.86}$$

from which we get

$$\begin{aligned}
&\int_0^t \int_{\Omega_2'(s)} G\left[m_2^{-1}(\vartheta-2)_+^2+\psi^2\right] \\
&\lesssim \bar{\delta}^{\frac{1}{3}} \int_0^t (1+s)^{-\frac{2}{3}}\|\psi\|\|\psi_x\| + m_2^{-1}\bar{\delta}^{\frac{1}{3}} \int_0^t \sup_{\mathbb{R}_+}(\vartheta-1)_+^2 \\
&\lesssim 1 + N^2\bar{\delta}^{\frac{2}{3}} \int_0^t \|\psi_x\|^2 + m_2^{-1}\bar{\delta}^{\frac{1}{3}} \int_0^t \sup_{\mathbb{R}_+}(\vartheta-1)_+^2.
\end{aligned} \tag{3.87}$$

Next we have from (3.7) and Lemma 2.2 that

$$\begin{aligned}
&m_2^{-1}\int_0^t \int_{\Omega_2'(s)} \left|(\Psi,\Psi_x)\right|\left|(\bar{U}_x,\tilde{U}_x,\tilde{U}_{xx})\right|(\vartheta-2)_+^2 \\
&\lesssim m_2^{-1}\int_0^t \int_{\Omega_2'(s)} \left|(\bar{U}_x,\tilde{U}_x,\tilde{U}_{xx})\right|\left[N^2(\vartheta-2)_+^2+\left|\Psi_x\right|^2\right] \\
&\lesssim m_2^{-1}N^2(\bar{\delta}+\tilde{\delta}) \int_0^t \left[\sup_{\mathbb{R}_+}(\vartheta-1)_+^2+\|\Psi_x\|^2\right].
\end{aligned} \tag{3.88}$$

Since

$$\left|(\Psi,\Psi_x)\right|\psi^2 \lesssim \left(1+\left|\Psi\right|\right)\left|\Psi\right|^3+\left|\Psi_x\right|^2 \lesssim N\left|\Psi\right|^3+\left|\Psi_x\right|^2, \tag{3.89}$$

we have

$$\begin{aligned}
&\int_0^t \int_{\Omega_2'(s)} \left|(\Psi,\Psi_x)\right|\left|(\bar{U}_x,\tilde{U}_x,\tilde{U}_{xx})\right|\psi^2 \\
&\lesssim \int_0^t \int_{\Omega_2'(s)} \left[N\left|\Psi\right|^3\left|\bar{U}_x\right|+N^2\left|\Psi\right|^2\left|(\tilde{U}_x,\tilde{U}_{xx})\right|\right]+(\bar{\delta}+\tilde{\delta})\int_0^t \|\Psi_x\|^2 \\
&\lesssim 1+\left[\bar{\delta}^{\frac{2}{9}}N^{\frac{26}{3}}+\tilde{\delta}N^2\right]\int_0^t \|\Psi_x\|^2+\tilde{\delta}m_1^{-1}N^4\int_0^t \rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(s,0)\mathrm{d}s.
\end{aligned} \tag{3.90}$$

To derive the last inequality in (3.88), we have used (3.22), Young's inequality and

$$\left|\bar{U}_x\right|\left|\Psi\right|^3 \lesssim \|\Psi\|_{L^\infty}^{\frac{3}{2}}\|\bar{U}_x\|_{L^4}\|\Psi\|^{\frac{3}{2}} \lesssim \bar{\delta}^{\frac{1}{12}}N^{\frac{9}{4}}(1+s)^{-\frac{11}{16}}\|\Psi_x\|^{\frac{3}{4}}.$$

Plugging (3.87), (3.88) and (3.90) into (3.85), we deduce from (3.8)–(3.10) that

$$|\mathcal{J}_3| \lesssim 1 + \int_0^t \sup_{\mathbb{R}_+}(\vartheta-1)_+^2. \tag{3.91}$$

Let us now consider the term

$$\mathcal{J}_4 := 2R\int_0^t\int_{\mathbb{R}_+} \psi(\hat{\rho}\hat{\theta})_x(\vartheta-2)_+,$$

which combined with (3.7) and (3.8) yields

$$|\mathcal{J}_4| \lesssim \int_0^t \sup_{\mathbb{R}_+}(\vartheta-2)_+\|\psi\|\|(\hat{\rho}_x,\hat{\theta}_x)\| \lesssim \int_0^t \sup_{\mathbb{R}_+}(\vartheta-1)_+^2. \tag{3.92}$$

For the term

$$\mathcal{J}_5 := \int_0^t\int_{\Omega_2'(s)} \left[\mu\psi_x^2c_v^{-1}\psi^2 - P\psi_xc_v^{-1}\psi^2 - 2\mu\psi\psi_x\vartheta_x\right],$$

we apply Cauchy's inequality and (3.80) to deduce

$$\begin{aligned}
|\mathcal{J}_5| &\lesssim \epsilon\int_0^t\int_{\Omega_2'(s)}\vartheta_x^2 + C(\epsilon)\int_0^t\int_{\Omega_2'(s)}\psi^2\psi_x^2 + \int_0^t\int_{\Omega_2'(s)}\psi^2\theta^2\\
&\lesssim \epsilon\int_0^t\int_{\Omega_2'(s)}\vartheta_x^2 + C(\epsilon)\int_0^t\int_{\Omega_2'(s)}\psi^2\psi_x^2 + \int_0^t\int_{\Omega_2'(s)}\psi^2(\vartheta-1)_+^2\\
&\lesssim \epsilon\int_0^t\int_{\Omega_2'(s)}\vartheta_x^2 + C(\epsilon)\int_0^t\int_{\Omega_2'(s)}\psi^2\psi_x^2 + \int_0^t\sup_{\mathbb{R}_+}(\vartheta-1)_+^2.
\end{aligned} \tag{3.93}$$

We finally consider

$$\mathcal{J}_6 := \int_0^t\int_{\Omega_2'(s)} c_v^{-1}\kappa\psi^2\vartheta_{xx}.$$

In order to estimate $\mathcal{J}_6$, we apply Lebesgue's dominated convergence theorem to find

$$\begin{aligned}\mathcal{J}_6 &= \frac{\kappa}{c_v}\lim_{\nu\to 0^+}\int_0^t\int_{\mathbb{R}_+}\varphi_\nu(\vartheta)\psi^2\vartheta_{xx}\\ &= \frac{\kappa}{c_v}\lim_{\nu\to 0^+}\int_0^t\int_{\mathbb{R}_+}\left[-2\varphi_\nu(\vartheta)\psi\psi_x\vartheta_x-\varphi_\nu'(\vartheta)\psi^2\vartheta_x^2\right]\\ &\le -\frac{\kappa}{c_v}\lim_{\nu\to 0^+}\int_0^t\int_{\mathbb{R}_+}2\varphi_\nu(\vartheta)\psi\psi_x\vartheta_x\\ &\le \epsilon\int_0^t\int_{\mathbb{R}_+}\vartheta_x^2+C(\epsilon)\int_0^T\int_{\mathbb{R}_+}\psi^2\psi_x^2,\end{aligned}\tag{3.94}$$

where the approximate scheme $\varphi_\nu(\vartheta)$ is defined by

$$\varphi_\nu(\vartheta)=\begin{cases}1, & \vartheta-2>\nu,\\ (\vartheta-2)/\nu, & 0<\vartheta-2\le\nu,\\ 0, & \vartheta-2\le 0.\end{cases}$$

Plugging (3.81)–(3.82), (3.91)–(3.94) into (3.79), we get from (3.74) that

$$\begin{aligned}&\int_{\mathbb{R}_+}(\vartheta-2)_+^2\mathrm{d}x+\int_0^t\int_{\Omega_2'(s)}\left[\vartheta_x^2+\psi_x^2(\vartheta-2)_+\right]\\ &\lesssim 1+\epsilon\int_0^t\int_{\mathbb{R}_+}\vartheta_x^2+C(\epsilon)\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x^2+C(\epsilon)\int_0^t\sup_{\mathbb{R}_+}(\vartheta-1)_+^2.\end{aligned}\tag{3.95}$$

Step 3. We obtain from (3.9) that

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}\left[\vartheta_x^2+\psi_x^2\theta\right] &\lesssim \int_0^t\int_{\Omega_3'(s)}\left[\vartheta_x^2+\psi_x^2(\vartheta-2)_+\right]+\int_0^t\int_{\mathbb{R}_+\backslash\Omega_3'(s)}\left[\frac{\vartheta_x^2}{\theta^2}+\frac{\psi_x^2}{\theta}\right]\\ &\lesssim \int_0^t\int_{\Omega_2'(s)}\left[\vartheta_x^2+\psi_x^2(\vartheta-2)_+\right]+1.\end{aligned}\tag{3.96}$$

Combining (3.96) and (3.95), and choosing ϵ sufficiently small, we have

$$\int_{\mathbb{R}_+}(\vartheta-2)_+^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}\left[\vartheta_x^2+\psi_x^2\theta\right]\lesssim 1+\int_0^t\sup_{\mathbb{R}_+}(\vartheta-1)_+^2+\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x^2.\tag{3.97}$$

Step 4. To estimate the last term of (3.97), we multiply $(3.1)_2$ by ψ^3 and then integrate the resulting identity over $(0,t)\times\mathbb{R}_+$ to have

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{R}_+}\rho\psi^4\mathrm{d}x+3\mu\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x^2-\frac{1}{4}\int_{\mathbb{R}_+}\rho_0\psi_0^4\mathrm{d}x\\&=3R\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x\hat{\theta}\phi+3R\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x\rho\vartheta+\int_0^t\int_{\mathbb{R}_+}f_2\psi^3.\end{aligned}\tag{3.98}$$

From (3.76) and (3.80), we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x\hat{\theta}\phi+\int_0^t\int_{\mathbb{R}_+\backslash\Omega_2'(s)}\psi^2\psi_x\rho\vartheta\\&\lesssim\int_0^t\|\psi\|_{L_x^\infty}^2\|\psi_x\|\left[\|\phi\|+\|\vartheta\|_{L^2(\mathbb{R}_+\backslash\Omega_2'(s))}\right]\lesssim\int_0^t\|\psi_x\|^2.\end{aligned}\tag{3.99}$$

We then apply Cauchy's inequality to derive

$$\begin{aligned}\int_0^t\int_{\Omega_2'(s)}\psi^2\psi_x\rho\vartheta&\le\epsilon\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x^2+C(\epsilon)\int_0^t\int_{\Omega_2'(s)}\psi^2\vartheta^2\\&\le\epsilon\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x^2+C(\epsilon)\int_0^t\sup_{\mathbb{R}_+}(\vartheta-1)_+^2.\end{aligned}\tag{3.100}$$

In view of (3.83), we utilize (3.86), (3.89), (3.22) and the Young's inequalities to have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}f_2\psi^3&\lesssim N\int_0^t\int_{\mathbb{R}_+}\left[G+\left|(\Psi,\Psi_x)\right|\left|(\bar{U}_x,\tilde{U}_x,\tilde{U}_{xx})\right|\right]\psi^2\\&\lesssim 1+\left[\bar{\delta}^{\frac{2}{9}}N^{\frac{34}{3}}+\tilde{\delta}N^3\right]\int_0^t\|\Psi_x\|^2+\tilde{\delta}m_1^{-1}N^5\int_0^t\rho\Phi\left(\frac{\hat{\rho}}{\rho}\right)(s,0)\mathrm{d}s.\end{aligned}\tag{3.101}$$

Plugging (3.99)–(3.101) into (3.98), and taking ϵ sufficiently small, we derive from (3.8)–(3.10) that

$$\int_{\mathbb{R}_+}\psi^4\mathrm{d}x+\int_0^t\int_{\mathbb{R}_+}\psi^2\psi_x^2\lesssim 1+\int_0^t\int_{\mathbb{R}_+}\psi_x^2+\int_0^t\sup_{\mathbb{R}_+}(\vartheta-1)_+^2.\tag{3.102}$$

It follows from (3.9) that

$$\int_0^t \int_{\mathbb{R}_+} \psi_x^2 \le \epsilon \int_0^t \int_{\mathbb{R}_+} \theta \psi_x^2 + C(\epsilon) \int_0^t \int_{\mathbb{R}_+} \frac{\psi_x^2}{\theta} \le \epsilon \int_0^t \int_{\mathbb{R}_+} \theta \psi_x^2 + C(\epsilon). \tag{3.103}$$

Combination of (3.103) and (3.102) yields

$$\int_{\mathbb{R}_+} \psi^4 \mathrm{d}x + \int_0^t \int_{\mathbb{R}_+} (1+\psi^2)\psi_x^2 \lesssim C(\epsilon) + \epsilon \int_0^t \int_{\mathbb{R}_+} \theta \psi_x^2 + \int_0^t \sup_{\mathbb{R}_+} (\vartheta - 1)_+^2 . \tag{3.104}$$

We plug (3.104) into (3.97) and choose ϵ suitable small to find

$$\int_{\mathbb{R}_+} \left[(\vartheta - 2)_+^2 + \psi^4 \right] \mathrm{d}x + \int_0^t \int_{\mathbb{R}_+} \left[\vartheta_x^2 + \psi_x^2 \left(1 + \theta + \psi^2 \right) \right] \lesssim 1 + \int_0^t \sup_{\mathbb{R}_+} (\vartheta - 1)_+^2 . \tag{3.105}$$

Step 5. It remains to estimate the last term of (3.105). According to the fundamental theorem of calculus, we have from (3.80) that

$$\begin{aligned}
\int_0^T \sup_{\mathbb{R}_+} (\vartheta - 1)_+^2 &\le \int_0^T \left[\int_{\Omega_1'(s)} |\vartheta_x| \right]^2 \\
&\le \int_0^T \left[\int_{\Omega_1'(s)} \frac{\vartheta_x^2}{\theta} \int_{\Omega_1'(s)} \theta \right] \\
&\le \epsilon \int_0^T \int_{\mathbb{R}_+} \vartheta_x^2 + C(\epsilon) \int_0^T \int_{\mathbb{R}_+} \frac{\vartheta_x^2}{\theta^2} \\
&\le \epsilon \int_0^T \int_{\mathbb{R}_+} \vartheta_x^2 + C(\epsilon).
\end{aligned} \tag{3.106}$$

Plug (3.106) into (3.105) and choose $\epsilon > 0$ suitable small to obtain (3.75). This completes the proof of the lemma. □

We obtain the uniform bound of the H_x^1-norm of $(\phi, \psi, \vartheta)(t, x)$ uniformly in time t in the next lemma.

Lemma 3.8. *Suppose that the conditions listed in Lemma 3.1 hold. Then*

$$\sup_{0\le t\le T}\|(\phi,\psi,\vartheta)(t)\|_1^2+\int_0^T\left[\|\sqrt{\theta}\phi_x(t)\|^2+\|(\psi_x,\vartheta_x)(t)\|_1^2\right]\mathrm{d}t\le C_2^2, \tag{3.107}$$

where the positive constant C_2 *depends only on* $\inf_{x\in\mathbb{R}_+}\{\rho_0(x),\theta_0(x)\}$ *and* $\|(\phi_0,\psi_0,\vartheta_0)\|_1$.

Proof. First, plugging (3.74) into (3.45), we deduce

$$\int_{\mathbb{R}_+}\phi_x^2+\int_0^t\int_{\mathbb{R}_+}\theta\phi_x^2\lesssim 1+\int_0^t\int_{\mathbb{R}_+}\left[\psi_x^2+\vartheta_x^2+\frac{\vartheta_x^2}{\theta^2}\right]\lesssim 1, \tag{3.108}$$

where we employed (3.9) and (3.75) in the last inequality.

Next, multiply $(3.1)_2$ by ψ_{xx}/ρ to derive

$$\left(\frac{\psi_x^2}{2}\right)_t-\left[\psi_t\psi_x+\frac{1}{2}u\psi_x^2\right]_x+\frac{1}{2}u_x\psi_x^2+\mu\frac{\psi_{xx}^2}{\rho}=\frac{(P-\hat{P})_x}{\rho}\psi_{xx}-\frac{f_2}{\rho}\psi_{xx}.$$

Integrating this last identity over $(0,t)\times\mathbb{R}_+$, we obtain from (3.74) and Cauchy's inequality that

$$\begin{aligned}&\int_{\mathbb{R}_+}\psi_x^2\mathrm{d}x-\int_0^t\psi_x^2(s,0)\mathrm{d}s+\int_0^t\int_{\mathbb{R}_+}\psi_{xx}^2\\&\lesssim 1+\int_0^t\int_{\mathbb{R}_+}\left[(P-\hat{P})_x^2+f_2^2+|\hat{u}_x|\psi_x^2+|\psi_x|^3\right].\end{aligned} \tag{3.109}$$

Apply Sobolev's inequality and (3.75) to obtain

$$\begin{aligned}\int_0^t\psi_x^2(s,0)\mathrm{d}s+\int_0^t\int_{\mathbb{R}_+}|\psi_x|^3&\lesssim\int_0^t\|\psi_x\|\|\psi_{xx}\|+\int_0^t\|\psi_x\|^{\frac{5}{2}}\|\psi_{xx}\|^{\frac{1}{2}}\\&\lesssim\epsilon\int_0^t\int_{\mathbb{R}_+}\psi_{xx}^2+C(\epsilon)\int_0^t\left[\|\psi_x\|^2+\|\psi_x\|^{\frac{10}{3}}\right]\\&\lesssim C(\epsilon)\left[1+\sup_{0\le s\le t}\|\psi_x(s)\|^{\frac{4}{3}}\right]+\epsilon\int_0^t\int_{\mathbb{R}_+}\psi_{xx}^2.\end{aligned} \tag{3.110}$$

Using $(P-\hat{P})_x=R(\theta\phi_x+\rho\vartheta_x+\phi\hat{\theta}_x+\vartheta\hat{\rho}_x)$, we derive from (3.83), (3.108), (3.75) that

$$\int_0^T\int_{\mathbb{R}_+}\left[(P-\hat{P})_x^2+|\hat{u}_x|\psi_x^2\right]\lesssim\int_0^T\int_{\mathbb{R}_+}\left[|(\theta\phi_x,\psi_x,\vartheta_x)|^2+|(\hat{\rho}_x,\hat{u}_x,\hat{\theta}_x)|^2|\Psi|^2\right]$$
$$\lesssim 1+\|\theta\|_{L^\infty([0,T]\times\mathbb{R}_+)}. \tag{3.111}$$

According to (3.83) and (3.84), we have from Lemma 2.2, (3.22), (3.8), (3.9) and (3.108) that

$$\int_0^T\int_{\mathbb{R}_+}|(f_2,f_3)|^2\lesssim 1. \tag{3.112}$$

We plug (3.110)–(3.112) into (3.109) to get

$$\sup_{0\le t\le T}\|\psi_x(t)\|^2+\int_0^T\int_{\mathbb{R}_+}\psi_{xx}^2\lesssim 1+\|\theta\|_{L^\infty([0,T]\times\mathbb{R}_+)}+\sup_{0\le t\le T}\|\psi_x(t)\|^{\frac{4}{3}}.$$

Then Young's inequality yields the estimate

$$\sup_{0\le t\le T}\|\psi_x(t)\|^2+\int_0^T\int_{\mathbb{R}_+}\psi_{xx}^2\lesssim 1+\|\theta\|_{L^\infty([0,T]\times\mathbb{R}_+)}. \tag{3.113}$$

Next, multiply $(3.1)_3$ by ϑ_{xx}/ρ and integrate the resulting identity over $(0,T)\times\mathbb{R}_+$ to have

$$\frac{c_v}{2}\int_{\mathbb{R}_+}\vartheta_x^2\mathrm{d}x+\kappa\int_0^T\int_{\mathbb{R}_+}\frac{\vartheta_{xx}^2}{\rho}=\frac{c_v}{2}\int_{\mathbb{R}_+}\vartheta_{0x}^2\mathrm{d}x+\int_0^T\int_{\mathbb{R}_+}\left[c_vu\vartheta_x-\mu\frac{\psi_x^2}{\rho}+R\theta\psi_x-\frac{f_3}{\rho}\right]\vartheta_{xx},$$

which combined with (3.74) implies

$$\int_{\mathbb{R}_+}\vartheta_x^2\mathrm{d}x+\int_0^T\int_{\mathbb{R}_+}\vartheta_{xx}^2\lesssim 1+\int_0^T\int_{\mathbb{R}_+}\left[u^2\vartheta_x^2+\|\psi_x\|_{L^\infty}^2\psi_x^2+\theta^2\psi_x^2+h^2\right]$$
$$\lesssim 1+\int_0^T(1+\|\psi\|\|\psi_x\|)\|\vartheta_x\|^2+\int_0^T\|\psi_x\|^3\|\psi_{xx}\| \tag{3.114}$$
$$+\|\theta\|_{L^\infty([0,T]\times\mathbb{R}_+)}\int_0^T\int_{\mathbb{R}_+}\theta\psi_x^2+\int_0^T\int_{\mathbb{R}_+}f_3^2.$$

From (3.113), we have

$$\int_0^T \|\psi_x\|^3\|\psi_{xx}\| \lesssim \sup_{0\le t\le T}\|\psi_x\|^2 \int_0^T \left(\|\psi_x\|^2+\|\psi_{xx}\|^2\right) \lesssim 1+\|\theta\|^2_{L^\infty([0,T]\times\mathbb{R}_+)}.$$

In light of (3.108), (3.113) and (3.75), we then obtain

$$\sup_{0\le t\le T}\int_{\mathbb{R}_+}\vartheta_x^2\mathrm{d}x+\int_0^T\int_{\mathbb{R}_+}\vartheta_{xx}^2 \lesssim 1+\|\theta\|^2_{L^\infty([0,T]\times\mathbb{R}_+)}. \tag{3.115}$$

Finally, it follows from (3.75) and (3.115) that

$$\|\theta-\hat\theta\|^2_{L^\infty([0,T]\times\mathbb{R}_+)} \lesssim \sup_{0\le t\le T}\|\vartheta(t)\|\|\vartheta_x(t)\| \lesssim 1+\|\theta\|_{L^\infty([0,T]\times\mathbb{R}_+)},$$

from which we have

$$\theta(t,x)\lesssim 1 \quad \text{for all } (t,x)\in[0,T]\times\mathbb{R}_+. \tag{3.116}$$

Combine (3.109), (3.113) and (3.115) to give

$$\sup_{0\le t\le T}\int_{\mathbb{R}_+}[\phi_x^2+\psi_x^2+\vartheta_x^2]\mathrm{d}x+\int_0^T\int_{\mathbb{R}_+}[\theta\phi_x^2+\psi_{xx}^2+\vartheta_{xx}^2]\lesssim 1,$$

which together with (3.75) yields (3.107). This completes the proof of the lemma. □

3.4. Local lower bound of temperature

In this subsection, we employ the maximum principle to get the lower bound for the temperature, which does depend on the time t.

Lemma 3.9. *Suppose that the conditions listed in Lemma 3.1 hold. Then*

$$\inf_{\mathbb{R}_+}\theta(t,\cdot)\ge\frac{\inf_{\mathbb{R}_+}\theta(s,\cdot)}{C_3\inf_{\mathbb{R}_+}\theta(s,\cdot)(t-s)+1} \quad \textit{for } 0\le s\le t\le T, \tag{3.117}$$

where the positive constant C_3 depends solely on $\inf_{x\in\mathbb{R}_+}\{\rho_0(x),\theta_0(x)\}$ and $\|(\phi_0,\psi_0,\vartheta_0)\|_1$.

Proof. It follows from $(1.1)_3$ that θ satisfies

$$\theta_t+u\theta_x-\frac{\kappa}{c_v\rho}\theta_{xx}=\frac{\mu}{c_v\rho}\left[u_x^2-\frac{P}{\mu}u_x\right]\ge-\frac{P^2}{4\mu c_v\rho}=-\frac{R^2\rho}{4\mu c_v}\theta^2.$$

Hence we deduce from (3.74) that

$$\theta_t+u\theta_x-\frac{\kappa}{c_v\rho}\theta_{xx}+C_3\theta^2\ge 0.$$

Let $\Theta := \theta - \underline{\theta}$ with $\underline{\theta} := \frac{\inf_{\mathbb{R}_+}\theta(s,\cdot)}{C_3 \inf_{\mathbb{R}_+}\theta(s,\cdot)(t-s)+1}$. We observe

$$\Theta|_{x=0,\infty} \geq 0, \quad \Theta|_{t=s} \geq 0,$$

and

$$\Theta_t + u\Theta_x - \frac{\kappa}{c_v\rho}\Theta_{xx} + C_2(\theta+\underline{\theta})\Theta = \theta_t + u\theta_x - \frac{\kappa}{c_v\rho}\theta_{xx} + C_2\theta^2 \geq 0.$$

Applying the weak maximum principle (see [3, Section 7.1]), we have that $\Theta(t,x) \geq 0$ for $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}_+$. This completes the proof of the lemma. □

3.5. Proof of Theorem 2

This subsection is devoted to proving the stability of the superposition of a rarefaction wave and a non-degenerate stationary solution, i.e. Theorem 2. To this end, we first give the local existence of solutions to the problem (3.1)–(3.2) in the following proposition. It can be proved by the standard iteration method (see [10] for example) and hence we omit the proof for brevity.

Proposition 3.10 *(Local existence). Suppose that the conditions in Theorem 2 hold. Let M, λ_1 and λ_2 be some positive constants such that $\|(\phi_0, \psi_0, \vartheta_0)\|_1 \leq M$, $\phi_0(x) + \hat{\rho}(x) \geq \lambda_1$ and $\vartheta_0(x) + \hat{\theta}(x) \geq \lambda_2$ for all $x \in \mathbb{R}_+$. Then there exists a positive constant $T_0 = T_0(\lambda_1, \lambda_2, M)$, depending only on λ_1, λ_2 and M, such that the problem (3.1)–(3.2) admits a unique solution $(\phi, \psi, \vartheta) \in X\left(0, T_0; \frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, 2M\right)$.*

Next we will give the proof of Theorem 2 in six steps by employing the continuation argument.

Step 1. Let Π and λ_i $(i = 1, 2, 3)$ be some positive constants such that $\|(\phi_0, \psi_0, \vartheta_0)\|_1 \leq \Pi$ and

$$\rho_0(x) \geq \lambda_1, \quad \theta_0(x) \geq \lambda_2, \quad \hat{\theta}(t,x) \geq \lambda_3 \quad \text{for all } t, x \geq 0.$$

Set $T_1 = 128\lambda_3^{-4}C_2^4$, where C_2 is exactly the same constant as in (3.107). Applying Proposition 3.10, we infer that the problem (3.1)–(3.2) has a unique solution $(\phi, \psi, \vartheta) \in X(0, t_1; \frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, 2\Pi)$ for some positive constant

$$t_1 = \min\{T_1, T_0(\lambda_1, \lambda_2, \Pi)\}.$$

Let $0 < \delta \leq \delta_1$ with

$$\Xi\left(\tfrac{1}{2}\lambda_1, \tfrac{1}{2}\lambda_2, 2\Pi\right)\delta_1 = \epsilon_0.$$

Then we can apply Lemmas 3.5, 3.8 and 3.9 with $T = t_1$ to obtain that for each $t \in [0, t_1]$, the local solution (ϕ, ψ, ϑ) constructed above satisfies that

$$\theta(t,x) \geq \frac{\lambda_2}{C_3\lambda_2 T_1 + 1} =: C_4 \quad \text{for all } x \in \mathbb{R}_+, \tag{3.118}$$

and

$$\begin{aligned}&C_1^{-1} \le \rho(t,x) \le C_1 \quad \text{for all } x \in \mathbb{R}_+,\\ &\|(\phi,\psi,\vartheta)(t)\|_1^2 + \int_0^t \left[\|\sqrt{\theta}\phi_x(s)\|^2 + \|(\psi_x,\vartheta_x)(s)\|_1^2\right] \mathrm{d}s \le C_2^2.\end{aligned} \tag{3.119}$$

Step 2. If we take $(\phi,\psi,\vartheta)(t_1,\cdot)$ as the initial data, we can apply Proposition 3.10 and extend the local solution (ϕ,ψ,ϑ) to the time interval $[0, t_1+t_2]$ with

$$t_2 = \min\{T_1 - t_1, T_0(C_1^{-1}, C_4, C_2)\}.$$

Moreover, for all $(t,x) \in [t_1, t_1+t_2] \times \mathbb{R}_+$,

$$\rho(t,x) \ge \tfrac{1}{2}C_1^{-1}, \quad \theta(t,x) \ge \tfrac{1}{2}C_4, \quad \|(\phi,\psi,\vartheta)(t)\|_1 \le 2C_2.$$

Take $0 < \delta \le \min\{\delta_1, \delta_2\}$ with

$$\Xi\left(\tfrac{1}{2}C_1^{-1}, \tfrac{1}{2}C_4, 2C_2\right)\delta_2 = \epsilon_0.$$

Then we can employ Lemmas 3.5, 3.8 and 3.9 with $T = t_1 + t_2$ to deduce that the local solution (ϕ,ψ,ϑ) satisfies (3.118) and (3.119) for each $t \in [0, t_1+t_2]$.

Step 3. We repeat the argument in Step 2, to extend our solution (ϕ,ψ,ϑ) to the time interval $[0, t_1+t_2+t_3]$ with

$$t_3 = \min\{T_1 - (t_1+t_2), T_0(C_1^{-1}, C_4, C_2)\}.$$

Assume that $0 < \delta \le \min\{\delta_1, \delta_2\}$. Continuing, after finitely many steps we construct the unique solution (ϕ,ψ,ϑ) existing on $[0, T_1]$ and satisfying (3.118) and (3.119) for each $t \in [0, T_1]$.

Step 4. Since $T_1 \ge 128\lambda_3^{-4}C_3^4$ and

$$\sup_{0\le t\le T_1} \|\vartheta(t)\|_1^2 + \int_{T_1/2}^{T_1} \|\vartheta_x(t)\|_1^2 \mathrm{d}t \le C_2^2,$$

we can find a $t_0' \in [T_1/2, T_1]$ such that

$$\|\vartheta(t_0')\| \le C_2, \quad \|\vartheta_x(t_0')\| \le \tfrac{1}{8}C_2^{-1}\lambda_3^2.$$

Sobolev's inequality yields

$$\|\vartheta(t_0')\|_{L^\infty} \le \sqrt{2}\|\vartheta(t_0')\|^{\frac{1}{2}}\|\vartheta_x(t_0')\|^{\frac{1}{2}} \le \tfrac{1}{2}\lambda_3,$$

and so

$$\theta(t_0', x) \geq \hat{\theta}(t_0', x) - \|\vartheta(t_0')\|_{L^\infty} \geq \tfrac{1}{2}\lambda_3 \quad \text{for all } x \in \mathbb{R}_+.$$

We note here that

$$\|(\phi, \psi, \vartheta)(t_0')\|_1 \leq C_2, \quad \rho(t_0', x) \geq C_1^{-1} \quad \text{for all } x \in \mathbb{R}_+.$$

Applying Proposition 3.10 again by taking $(\phi, \psi, \vartheta)(t_0', \cdot)$ as the initial data, we see that the problem (3.1)–(3.2) admits a unique solution $(\phi, \psi, \vartheta) \in X(t_0', t_0' + t_1'; \frac{1}{2}C_1^{-1}, \frac{1}{4}\lambda_3, 2C_2)$ with

$$t_1' = \min\{T_1, T_0(C_1^{-1}, \tfrac{1}{2}\lambda_3, C_2)\}.$$

If we take $0 < \delta \leq \min\{\delta_1, \delta_2, \delta_3\}$ with

$$\Xi\left(\tfrac{1}{2}C_1^{-1}, \tfrac{1}{4}\lambda_3, 2C_2\right)\delta_3 = \epsilon_0,$$

then we can apply Lemmas 3.5, 3.8 and 3.9 with $T = t_0' + t_1'$ to obtain that for each time $t \in [t_0', t_0' + t_1']$, the local solution (ϕ, ψ, ϑ) satisfies (3.119) and

$$\theta(t, x) \geq \frac{\inf_{\mathbb{R}_+} \theta(t_0', \cdot)}{C_3 \inf_{\mathbb{R}_+} \theta(t_0', \cdot) T_1 + 1} \geq \frac{\lambda_3}{C_3\lambda_3 T_1 + 2} =: C_5 \quad \text{for all } x \in \mathbb{R}_+. \tag{3.120}$$

Step 5. Next if we take $(\phi, \psi, \vartheta)(t_0' + t_1', \cdot)$ as the initial data, we apply Proposition 3.10 and construct the solution (ϕ, ψ, ϑ) existing on the time interval $[0, t_0' + t_1' + t_2']$ with

$$t_2' = \min\{T_1 - t_1', T_0(C_1^{-1}, C_5, C_2)\}$$

and satisfying

$$\rho(t, x) \geq \tfrac{1}{2}C_1^{-1}, \quad \theta(t, x) \geq \tfrac{1}{2}C_5, \quad \|(\phi, \psi, \vartheta)(t)\|_1 \leq 2C_2$$

for all $(t, x) \in [t_0' + t_1', t_0' + t_1' + t_2'] \times \mathbb{R}_+$. Let $0 < \delta \leq \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$ with

$$\Xi\left(\tfrac{1}{2}C_1^{-1}, \tfrac{1}{2}C_5, 2C_2\right)\delta_4 = \epsilon_0.$$

Then we can infer from Lemmas 3.5, 3.8 and 3.9 with $T = t_0' + t_1' + t_2'$ that the local solution (ϕ, ψ, ϑ) satisfies (3.120) and (3.119) for each $t \in [t_0', t_0' + t_1' + t_2']$. By assuming $0 < \delta \leq \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, we can repeatedly apply the argument above to extend the local solution to the time interval $[0, t_0' + T_1]$. Furthermore, we deduce that (3.120) and (3.119) hold for each $t \in [t_0', t_0' + T_1]$. In view of $t_0' + T_1 \geq \frac{3}{2}T_1$, we have shown that the problem (3.1)–(3.2) admits a unique solution (ϕ, ψ, ϑ) on $[0, \frac{3}{2}T_1]$.

Step 6. We take $0 < \delta \leq \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. As in Steps 4 and 5, we can find $t_0'' \in [t_0' + T_1/2, t_0' + T_1]$ such that the problem (3.1)–(3.2) admits a unique solution (ϕ, ψ, ϑ) on $[0, t_0'' + T_1]$, which satisfies (3.120) and (3.119) for each $t \in [t_0', t_0'' + T_1]$. Since $t_0'' + T_1 \geq t_0' + \frac{3}{2}T_1 \geq 2T_1$, we have extended the local solution (ϕ, ψ, ϑ) to $[0, 2T_1]$. Repeating the above procedure, we can then extend the solution (ϕ, ψ, ϑ) step by step to a global one provided that $\delta \leq \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Choosing $\epsilon_2 = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, we derive that the problem (3.1) has a unique solution $(\phi, \psi, \vartheta) \in X(0, \infty; C_1^{-1}, \min\{C_4, C_5\}, C_2)$ satisfying (3.119) for each $t \in [0, \infty)$.

Therefore, we can find constant C_6 depending only on $\inf_{x\in\mathbb{R}_+}\{\rho_0(x), \theta_0(x)\}$ and $\|(\phi_0, \psi_0, \vartheta_0)\|_1$ such that

$$\sup_{0\le t<\infty} \|(\phi, \psi, \vartheta)(t)\|_1^2 + \int_0^\infty \left[\|\phi_x(t)\|^2 + \|(\psi_x, \vartheta_x)(t)\|_1^2\right] \mathrm{d}t \le C_6^2,$$

from which the large-time behavior (1.24) follows in a standard argument (cf. [23]). This completes the proof of Theorem 2.

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