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Well-posedness for the free-boundary ideal compressible magnetohydrodynamic equations with surface tension

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Abstract

We establish the local existence and uniqueness of solutions to the free-boundary ideal compressible magnetohydrodynamic equations with surface tension in three spatial dimensions by a suitable modification of the Nash–Moser iteration scheme. The main ingredients in proving the convergence of the scheme are the tame estimates and unique solvability of the linearized problem in the anisotropic Sobolev spaces H^m_* for m large enough. In order to derive the tame estimates, we make full use of the boundary regularity enhanced from the surface tension. The unique solution of the linearized problem is constructed by designing some suitable ε–regularization and passing to the limit $\varepsilon \to 0$.

Keywords Free boundary problem · Ideal compressible magnetohydrodynamics · Surface tension · Well-posedness · Nash–Moser iteration

Mathematics Subject Classification 35L65 · 35R35 · 76N10 · 76W05

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1 Introduction

We consider the free-boundary ideal compressible magnetohydrodynamic (MHD) equations with surface tension governing the dynamics of inviscid, compressible, and electrically conducting fluids in three spatial dimensions. Let $\Omega(t) := \{x \in \mathbb{R}^3 : x_1 > \varphi(t, x')\}$ be the changing volume occupied by the conducting fluid at time t, where $x' := (x_2, x_3)$ is the tangential coordinate. The free boundary problem reads

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho v) = 0 && \text{in } \Omega(t), && (1.1\text{a})\\
&\partial_t(\rho v) + \nabla \cdot (\rho v \otimes v - H \otimes H) + \nabla q = 0 && \text{in } \Omega(t), && (1.1\text{b})\\
&\partial_t H - \nabla \times (v \times H) = 0 && \text{in } \Omega(t), && (1.1\text{c})\\
&\partial_t(\rho E + \tfrac{1}{2}|H|^2) + \nabla \cdot (v(\rho E + p) + H \times (v \times H)) = 0 && \text{in } \Omega(t), && (1.1\text{d})\\
&\partial_t \varphi = v \cdot N && \text{on } \Sigma(t), && (1.1\text{e})\\
&q = \mathfrak{s}\mathcal{H}(\varphi) && \text{on } \Sigma(t), && (1.1\text{f})\\
&(\rho, v, H, S, \varphi)|_{t=0} = (\rho_0, v_0, H_0, S_0, \varphi_0), && && (1.1\text{g})
\end{aligned}
$$

supplemented with the constraints

$$\nabla \cdot H = 0 \quad \text{in } \Omega(t), \tag{1.2}$$
$$H \cdot N = 0 \quad \text{on } \Sigma(t), \tag{1.3}$$

for $\partial_t := \frac{\partial}{\partial t}$ and $\nabla := (\partial_1, \partial_2, \partial_3)^\mathsf{T}$ with $\partial_i := \frac{\partial}{\partial x_i}$, where the density ρ, velocity $v \in \mathbb{R}^3$, magnetic field $H \in \mathbb{R}^3$, specific entropy S, and interface function φ are to be determined. Symbols $q = p + \frac{1}{2}|H|^2$ and $E = e + \frac{1}{2}|v|^2$ stand for the total pressure and specific total energy, respectively, where p is the pressure and e is the specific internal energy. The thermodynamic variables ρ and e are given smooth functions of p and S satisfying the Gibbs relation

$$\vartheta \, \mathrm{d}S = \mathrm{d}e + p \, \mathrm{d}\left(\frac{1}{\rho}\right), \tag{1.4}$$

where $\vartheta > 0$ is the absolute temperature. We denote by $\Sigma(t) := \{x \in \mathbb{R}^3 : x_1 = \varphi(t, x')\}$ the moving vacuum boundary, by $N := (1, -\partial_2\varphi, -\partial_3\varphi)^\mathsf{T}$ the normal vector to $\Sigma(t)$, by $\mathfrak{s} > 0$ the constant coefficient of surface tension, and by $\mathcal{H}(\varphi)$ twice the mean curvature of the boundary, that is,

$$\mathcal{H}(\varphi) := \mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'}\varphi}{\sqrt{1 + |\mathrm{D}_{x'}\varphi|^2}}\right) \quad \text{with } \mathrm{D}_{x'} := \begin{pmatrix} \partial_2 \\ \partial_3 \end{pmatrix}. \tag{1.5}$$

The boundary condition (1.1e) states that the free boundary moves with the velocity of the conducting fluid and makes the free surface $\Sigma(t)$ *characteristic*. The boundary condition (1.1f) results from surface tension and zero vacuum magnetic field. We refer to LANDAU–LIFSHITZ [18, §65] and DELHAYE [11] for the derivation of the MHD equations (1.1a)–(1.1d) and the boundary condition (1.1f). It should be noted that the effect of surface tension is especially important for modelling MHD flows in liquid metals (see, e.g., [25,37] and references therein). Even for MHD modelling of large scales phenomena like those in astrophysical plasmas, where the effect of surface tension and diffusion is usually neglected, it is still useful to keep surface tension as a stabilizing mechanism in numerical simulations of magnetic Rayleigh–Taylor instability [31,32].

The absence of the magnetic field, *i.e.*, $H \equiv 0$, reduces the problem (1.1) to the moving vacuum boundary problem for the compressible Euler equations, which has been studied extensively in the recent decades. In the case of zero surface tension, $\mathfrak{s} = 0$, an ill-posedness result have been shown by EBIN [12] for the incompressible Euler equations when the Rayleigh–Taylor sign condition is violated, while the local well-posedness has been established in [8,20,36,38] and [21,34] respectively for incompressible and compressible liquids under the Rayleigh–Taylor sign condition, and in [9,17] for compressible gases under the physical vacuum condition. In the case of positive surface tension, $\mathfrak{s} > 0$, the local well-posedness has been proved independently by COUTAND–SHKOLLER [8] and SHATAH–ZENG [29,30] for the incompressible Euler equations, and by COUTAND ET AL. [10] for compressible isentropic liquids, without imposing the Rayleigh–Taylor sign condition. The results of [8,10,29,30] indicate that the surface tension provides a regularizing effect on the vacuum boundary.

On the other hand, there have been only few results on the free boundary problem (1.1) for the ideal compressible MHD, due to the difficulties caused by the complex interplay between the velocity and magnetic fields. For the free-boundary ideal incompressible MHD without surface tension, the *a priori* estimates and local well-posedness have been respectively proved by HAO–LUO [14] and GU–WANG [13] under the generalized Rayleigh–Taylor sign condition on the total pressure, while a counterexample to well-posedness has been provided by HAO–LUO [15] when the generalized sign condition fails. Regarding the ideal compressible MHD equations (1.1), the authors [35] have recently established the local well-posedness for the case of zero surface tension $\mathfrak{s} = 0$ under the generalized sign condition. It would expect that the surface tension can have a stabilization effect on the evolution as for the cases without magnetic fields. The goal of the present paper is to confirm this effect rigorously, or, more precisely, to construct the unique solution of the problem (1.1) with surface tension replacing the generalized sign condition.

Suppose that the sound speed $a := a(\rho, S)$ is smooth and satisfies

$$a(\rho, S) := \sqrt{\frac{\partial p}{\partial \rho}(\rho, S)} > 0 \quad \text{for all } \rho \in (\rho_*, \rho^*),$$

where ρ_* and ρ^* are positive constants with $\rho_* < \rho^*$. In this paper we consider the liquids for which the density ρ belongs to (ρ_*, ρ^*). Consequently, it follows from the constraint (1.2) and the Gibbs relation (1.4) that the MHD equations (1.1a)–(1.1d) are equivalent to the symmetric hyperbolic system

$$A_0(U)\partial_t U + \sum_{i=1}^{3} A_i(U)\partial_i U = 0 \qquad \text{in } \varOmega(t), \tag{1.6}$$

where we choose $U := (q, v, H, S)^{\mathsf{T}}$ as the primary unknowns, and

$$A_0(U) := \begin{pmatrix} \frac{1}{\rho a^2} & 0 & -\frac{1}{\rho a^2}H^{\mathsf{T}} & 0 \\ 0 & \rho I_3 & O_3 & 0 \\ -\frac{1}{\rho a^2}H & O_3 & I_3 + \frac{1}{\rho a^2}H \otimes H & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.7}$$

$$A_i(U) := \begin{pmatrix} \frac{v_i}{\rho a^2} & e_i^{\mathsf{T}} & -\frac{v_i}{\rho a^2}H^{\mathsf{T}} & 0 \\ e_i & \rho v_i I_3 & -H_i I_3 & 0 \\ -\frac{v_i}{\rho a^2}H & -H_i I_3 & v_i I_3 + \frac{v_i}{\rho a^2}H \otimes H & 0 \\ 0 & 0 & 0 & v_i \end{pmatrix} \quad \text{for } i = 1, 2, 3. \tag{1.8}$$

Here and in what follows, O_m and I_m denote the zero and identity matrices of order m, respectively, and $\{e_1 := (1, 0, 0)^{\mathsf{T}}, e_2 := (0, 1, 0)^{\mathsf{T}}, e_3 := (0, 0, 1)^{\mathsf{T}}\}$ is the standard basis of $\mathbb{R}^3$.

We reduce the free boundary problem (1.1) into a fixed domain by straightening the unknown surface $\Sigma(t)$. More precisely, we introduce

$$U_\sharp(t,x) := U(t,\Phi(t,x),x'), \tag{1.9}$$

where Φ takes the following form that is suggested by MÉTIVIER [22, p. 70]:

$$\Phi(t,x) := x_1 + \kappa_\sharp \chi(x_1)\varphi(t,x'), \tag{1.10}$$

with the constant $\kappa_\sharp > 0$ and function $\chi \in C_0^\infty(\mathbb{R})$ satisfying

$$4\kappa_\sharp \|\varphi_0\|_{L^\infty(\mathbb{R}^2)} \le 1, \quad \|\chi'\|_{L^\infty(\mathbb{R})} < 1, \quad \chi \equiv 1 \text{ on } [0,1]. \tag{1.11}$$

This change of variables is admissible on the time interval $[0,T]$, provided $T > 0$ is suitably small. Without loss of generality we set $\kappa_\sharp = 1$. As in [22,33–35], we employ the cut-off function χ to avoid assumptions about compact support of the initial data (shifted to an equilibrium state).

Then the moving boundary problem (1.1) is reformulated as the following fixed-boundary problem:

$$\mathbb{L}(U,\Phi) := L(U,\Phi)U = 0 \qquad \text{in } [0,T]\times\Omega, \tag{1.12a}$$
$$\mathbb{B}(U,\varphi) := \begin{pmatrix} \partial_t\varphi - v\cdot N \\ q - \mathfrak{s}\mathcal{H}(\varphi) \end{pmatrix} = 0 \qquad \text{on } [0,T]\times\Sigma, \tag{1.12b}$$
$$(U,\varphi)|_{t=0} = (U_0,\varphi_0), \tag{1.12c}$$

where we drop for convenience the subscript "♯" in $U_\sharp$, and define the fixed domain $\Omega := \{x\in\mathbb{R}^3 : x_1 > 0\}$, the boundary $\Sigma := \{x\in\mathbb{R}^3 : x_1 = 0\}$, and the operator

$$L(U,\Phi) := A_0(U)\partial_t + \widetilde{A}_1(U,\Phi)\partial_1 + A_2(U)\partial_2 + A_3(U)\partial_3, \tag{1.13}$$

with

$$\widetilde{A}_1(U,\Phi) := \frac{1}{\partial_1\Phi}\big(A_1(U) - \partial_t\Phi A_0(U) - \partial_2\Phi A_2(U) - \partial_3\Phi A_3(U)\big). \tag{1.14}$$

In the new variables, the identities (1.2)–(1.3) become

$$\partial_1^\Phi H_1 + \partial_2^\Phi H_2 + \partial_3^\Phi H_3 = 0 \qquad \text{if } x_1 > 0, \tag{1.15}$$
$$H\cdot N = 0 \qquad \text{if } x_1 = 0, \tag{1.16}$$

where

$$\partial_t^\Phi := \partial_t - \frac{\partial_t\Phi}{\partial_1\Phi}\partial_1, \quad \partial_1^\Phi := \frac{1}{\partial_1\Phi}\partial_1, \quad \partial_i^\Phi := \partial_i - \frac{\partial_i\Phi}{\partial_1\Phi}\partial_1 \quad \text{for } i=2,3. \tag{1.17}$$

The identities (1.15)–(1.16) can be taken as initial constraints; we refer the reader to [33, Appendix A] for the proof.

Let us denote by $\lfloor s \rfloor$ the floor function mapping $s \in \mathbb{R}$ to the greatest integer less than or equal to s. Now we are in a position to state the local well-posedness theorem for the problem (1.12), which clearly implies a corresponding theorem for the original free boundary problem (1.1).

Theorem 1.1 *Let $m \in \mathbb{N}$ with $m \geq 20$. Suppose that the initial data (U_0, φ_0) satisfy the hyperbolicity condition $\rho_* < \inf_\Omega \rho(U_0) \leq \sup_\Omega \rho(U_0) < \rho^*$, the constraints (1.15)–(1.16), and the compatibility conditions up to order m (see Definition 3.1). Suppose further that $(U_0 - \overline{U}, \varphi_0)$ belongs to $H^{m+3/2}(\Omega) \times H^{m+2}(\mathbb{R}^2)$ for some constant equilibrium $\overline{U}$ with $\rho(\overline{U}) \in (\rho_*, \rho^*)$. Then there are a constant $T > 0$ and a unique solution (U, φ) of the problem (1.12) on the time interval $[0, T]$, such that*

$$U - \overline{U} \in H^{\lfloor (m-9)/2 \rfloor}([0, T] \times \Omega), \quad (\varphi, \mathrm{D}_{x'}\varphi) \in H^{m-9}([0, T] \times \mathbb{R}^2).$$

We will construct smooth solutions to the nonlinear characteristic problem (1.12) through a suitable modification of the Nash–Moser iteration scheme developed by Hörmander [16] and Coulombel–Secchi [7]; see [2,27] for a general description of the Nash–Moser method. The main ingredients in proving the convergence of the Nash–Moser iteration scheme are the *tame estimates* and *unique solvability* of the linearized problem in certain function spaces (*cf.* [27, Assumptions 2.1–2.2]).

For the linearized problem around a basic state, if the generalized Rayleigh–Taylor sign condition on the total pressure were imposed as in our previous work [35] for the case of zero surface tension, then we could deduce a uniform-in-$\mathfrak{s}$ L^2 energy estimate with no loss of derivatives from the source term to the solution (*cf.* (2.22)), allowing us to construct the unique solution by a direct application of the classical duality argument in Lax–Phillips [19] and Chazarain–Piriou [3, Chapter 7].

However, in this paper we aim to release the Rayleigh–Taylor sign condition and consider the general case for which the L^2 energy estimate is not closed. To deal with this situation, we propose here to build the basic *a priori* estimate in the anisotropic Sobolev space H^1_* rather than in L^2 for the linearized problem by taking advantage of the boundary regularity gained from the surface tension. More precisely, we first derive the L^2 energy estimates for the solutions and their *tangential spatial derivatives*, where the surface tension can provide good terms for the first and second order spatial derivatives of the interface function, respectively (*cf.* (2.24) and (2.32)). However, it is difficult to deduce the estimate of the *time derivative*, since the mean curvature operator (*cf.* (1.5)) does not involve the time derivative of the interface function. To overcome this difficulty, we reduce the tough boundary term $\mathcal{J}_{1a}$ defined in (2.34) to the volume integral containing the normal derivative of the noncharacteristic variable W_2, which can be controlled by virtue of the interior equations (*cf.* (2.40)–(2.41)). After that, using the estimate of the time derivative of the interface function, we achieve the H^1_* *a priori* estimate for the linearized problem (*cf.* (2.46)). To obtain high-order energy estimates for the linearized problem, we make full use of the improved boundary regularity and combine some estimates in [35] that are still available for our case $\mathfrak{s} > 0$ (see Sects. § 2.4–2.5 for the complete derivation).

The other main ingredient in our proof is to construct the unique solution of the linearized problem. For this purpose, we design for the linearized problem some suitable ε–regularization, for which we can deduce an L^2 *a priori* estimate with a constant $C(\varepsilon)$ depending on the small parameter $\varepsilon > 0$. Furthermore, an L^2 *a priori* estimate can be also shown for the corresponding dual problem. Then for any fixed and small parameter $\varepsilon > 0$, the existence and uniqueness of solutions in L^2 can be established by the duality argument. However, our constant $C(\varepsilon)$ tends to infinity as $\varepsilon \to 0$, and hence we are not able to use the L^2 estimate obtained for the regularized problem to take the limit $\varepsilon \to 0$. To overcome this difficulty, we derive a uniform-in-ε estimate in H^1_* for the regularized problem, which enables us to solve the linearized problem by passing to the limit $\varepsilon \to 0$.

It is worth mentioning that the anisotropic Sobolev spaces, introduced first by Chen [6], have been shown to be appropriate and effective for studying general symmetric hyperbolic problems with characteristic boundary; see Secchi [26] for a general theory and [4,28,33,35] for other results on characteristic problems in ideal compressible MHD.

We emphasize that our energy estimate (2.13) for the linearized problem exhibits a *fixed* loss of derivatives from the basic state to the solution and hence is a so-called *tame estimate*. To compensate this loss of derivatives and solve the nonlinear problem, we employ the modified Nash–Moser iteration technique, which has been also applied to the study of characteristic discontinuities [4,5,7,24,33] and vacuum free-boundary problems [28,34,35]. Nevertheless, in view of the aforementioned works [8,10,29,30], it is expected to avoid the loss of regularity and show the well-posedness for the nonlinear problem (1.1) without resorting to the Nash–Moser method. The proof of this expectation is an interesting open problem for future research.

The rest of this paper is organized as follows. Sect. 2 is devoted to the proof of Theorem 2.1, that is, the well-posedness of the linearized problem in the anisotropic Sobolev spaces H^m_* for any integer m large enough. To be more precise, for the effective linear problem (2.7), we deduce the H^1_* *a priori* estimate in Sect. 2.2, construct the unique solution in Sect. 2.3 by passing to the limit $\varepsilon \to 0$ in H^1_* from some certain ε–regularization, and complete the proof of Theorem 2.1 in Sect. 2.5 with the aid of the high-order energy estimates obtained in Sect. 2.4. For convenience, we collect a list of notation before the statement of Theorem 2.1 in Sect. 2.1. In Sect. 3, the existence part of Theorem 1.1 is proved by using a modified Nash–Moser iteration scheme (*cf.* Sects. 3.1–3.4), while the uniqueness part follows from the H^1_* energy estimate for the difference of solutions (*cf.* Sect. 3.5).

2 Well-posedness of the linearized problem

This section is devoted to showing the tame estimates and unique solvability for the linearized problem of (1.12) in anisotropic Sobolev spaces H^m_* with integer m large enough.

2.1 Main theorem for the linearized problem

For $T>0$, we denote $\Omega_T:=(-\infty,T)\times\Omega$ and $\Sigma_T:=(-\infty,T)\times\Sigma$. Let the basic state $(\mathring{U},\mathring{\varphi})$ with $\mathring{U}:=(\mathring{q},\mathring{v},\mathring{H},\mathring{S})^{\mathsf{T}}$ be sufficiently smooth and satisfy

$$\rho_*<\inf_{\Omega}\rho(\mathring{U})\le\sup_{\Omega}\rho(\mathring{U})<\rho^* \qquad \text{in } \Omega_T, \tag{2.1}$$

$$\partial_t\mathring{\varphi}=\mathring{v}\cdot\mathring{N},\quad \mathring{H}\cdot\mathring{N}=0 \qquad \text{on } \Sigma_T, \tag{2.2}$$

where

$$\mathring{N}:=(1,-\partial_2\mathring{\varphi},-\partial_3\mathring{\varphi})^{\mathsf{T}}=(1,-\mathrm{D}_{x'}\mathring{\varphi})^{\mathsf{T}}.$$

We also denote $\mathring{\Psi}:=\chi(x_1)\mathring{\varphi}(t,x')$ and $\mathring{\Phi}:=x_1+\mathring{\Psi}$ with $\chi\in C_0^\infty(\mathbb{R})$ satisfying (1.11). Then $\partial_1\mathring{\Phi}\ge 1/2$ on Ω_T, provided we without loss of generality assume that $\|\mathring{\varphi}\|_{L^\infty(\Sigma_T)}\le 1/2$. Furthermore, we suppose that

$$\|\mathring{U}\|_{W^{3,\infty}(\Omega_T)}+\|\mathring{\varphi}\|_{W^{4,\infty}(\Sigma_T)}\le K, \tag{2.3}$$

for some constant $K>0$.

The linearized operator around the basic state $(\mathring{U},\mathring{\varphi})$ for (1.12a) reads

$$\begin{aligned}\mathbb{L}'(\mathring{U},\mathring{\Phi})(V,\Psi)&:=\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{L}\big(\mathring{U}+\theta V,\ \mathring{\Phi}+\theta\Psi\big)\Big|_{\theta=0}\\&=L(\mathring{U},\mathring{\Phi})V+\mathcal{C}(\mathring{U},\mathring{\Phi})V-L(\mathring{U},\mathring{\Phi})\Psi\frac{\partial_1\mathring{U}}{\partial_1\mathring{\Phi}},\end{aligned}$$

where $\Psi:=\chi(x_1)\psi(t,x')$, the operator L is given in (1.13), and $\mathcal{C}$ is defined by

$$\mathcal{C}(U,\Phi)V:=\sum_{k=1}^{8}V_k\left(\frac{\partial A_0}{\partial U_k}(U)\partial_t U+\frac{\partial\widetilde{A}_1}{\partial U_k}(U,\Phi)\partial_1 U+\sum_{i=2,3}\frac{\partial A_i}{\partial U_k}(U)\partial_i U\right).$$

Introducing the good unknown of Alinhac [1]:

$$\dot{V}:=V-\frac{\partial_1\mathring{U}}{\partial_1\mathring{\Phi}}\Psi, \tag{2.4}$$

we have (*cf.* [22, Proposition 1.3.1])

$$\mathbb{L}'(\mathring{U},\mathring{\Phi})(V,\Psi)=L(\mathring{U},\mathring{\Phi})\dot{V}+\mathcal{C}(\mathring{U},\mathring{\Phi})\dot{V}+\frac{\Psi}{\partial_1\mathring{\Phi}}\partial_1\big(L(\mathring{U},\mathring{\Phi})\mathring{U}\big). \tag{2.5}$$

Regarding the linearized operator for (1.12b), we compute from (1.5) that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{H}(\mathring{\varphi}+\theta\psi)\Big|_{\theta=0} = \mathrm{D}_{x'}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{D}_{x'}(\mathring{\varphi}+\theta\psi)}{\sqrt{1+|\mathrm{D}_{x'}(\mathring{\varphi}+\theta\psi)|^2}}\right)\Big|_{\theta=0}$$
$$= \mathrm{D}_{x'}\cdot\left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|}-\frac{\mathrm{D}_{x'}\mathring{\varphi}\cdot\mathrm{D}_{x'}\psi}{|\mathring{N}|^3}\mathrm{D}_{x'}\mathring{\varphi}\right),$$

and hence

$$\mathbb{B}'(\mathring{U},\mathring{\varphi})(V,\psi) := \frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{B}\big(\mathring{U}+\theta V,\ \mathring{\varphi}+\theta\psi\big)\Big|_{\theta=0}$$
$$= \begin{pmatrix} (\partial_t+\mathring{v}_2\partial_2+\mathring{v}_3\partial_3)\psi - v\cdot\mathring{N} \\ q-\mathfrak{s}\mathrm{D}_{x'}\cdot\left(\dfrac{\mathrm{D}_{x'}\psi}{|\mathring{N}|}-\dfrac{\mathrm{D}_{x'}\mathring{\varphi}\cdot\mathrm{D}_{x'}\psi}{|\mathring{N}|^3}\mathrm{D}_{x'}\mathring{\varphi}\right)\end{pmatrix}. \tag{2.6}$$

Neglecting the last term of (2.5) as in [7,33–35], we write down the *effective linear problem*

$$\mathbb{L}_e'(\mathring{U},\mathring{\Phi})\dot{V} := L(\mathring{U},\mathring{\Phi})\dot{V}+\mathcal{C}(\mathring{U},\mathring{\Phi})\dot{V} = f \qquad \text{if } x_1>0, \tag{2.7a}$$
$$\mathbb{B}_e'(\mathring{U},\mathring{\varphi})(\dot{V},\psi) = g \qquad \text{if } x_1=0, \tag{2.7b}$$
$$(\dot{V},\psi) = 0 \qquad \text{if } t<0, \tag{2.7c}$$

where the identity $\mathbb{B}_e'(\mathring{U},\mathring{\varphi})(\dot{V},\psi)=\mathbb{B}'(\mathring{U},\mathring{\varphi})(V,\psi)$ results in the exact form

$$\mathbb{B}_e'(\mathring{U},\mathring{\varphi})(\dot{V},\psi) := \begin{pmatrix} (\partial_t+\mathring{v}_2\partial_2+\mathring{v}_3\partial_3)\psi-\partial_1\mathring{v}\cdot\mathring{N}\psi-\dot{v}\cdot\mathring{N} \\ \dot{q}+\partial_1\mathring{q}\psi-\mathfrak{s}\mathrm{D}_{x'}\cdot\left(\dfrac{\mathrm{D}_{x'}\psi}{|\mathring{N}|}-\dfrac{\mathrm{D}_{x'}\mathring{\varphi}\cdot\mathrm{D}_{x'}\psi}{|\mathring{N}|^3}\mathrm{D}_{x'}\mathring{\varphi}\right)\end{pmatrix}.$$

As in our previous work [35] for the case of zero surface tension, we will study the linearized problem (2.7) in anisotropic Sobolev spaces H_*^m that are defined below.

• *Notation.* Throughout this paper we adopt the following notation.

(i) We use letter C to denote any universal positive constant. Symbol $C(\cdot)$ denotes any generic positive constant depending on the quantities listed in the parenthesis. We employ $X\lesssim Y$ or $Y\gtrsim X$ to denote the statement that $X\le CY$ for some universal constant $C>0$.

(ii) Recall that $\nabla := (\partial_1,\partial_2,\partial_3)^{\mathsf{T}}$ and $\mathrm{D}_{x'} := (\partial_2,\partial_3)^{\mathsf{T}}$ with $\partial_i := \frac{\partial}{\partial x_i}$ for $i=1,2,3$. The time derivative $\partial_t := \frac{\partial}{\partial t}$ will be denoted in many cases by $\partial_0 := \frac{\partial}{\partial t}$. For any multi-index $\alpha := (\alpha_2,\alpha_3)\in\mathbb{N}^2$ and $m\in\mathbb{N}$, we define

$$\mathrm{D}_{x'}^{\alpha} := \partial_2^{\alpha_2}\partial_3^{\alpha_3},\quad |\alpha| := \alpha_2+\alpha_3,\quad \mathrm{D}_{x'}^{m} := (\partial_2^m,\partial_2^{m-1}\partial_3,\dots,\partial_3^m)^{\mathsf{T}}. \tag{2.8}$$

(iii) Symbol D will be employed to denote the space-time gradient

$$\mathrm{D} := (\partial_t,\partial_1,\partial_2,\partial_3)^{\mathsf{T}}.$$

For $m \in \mathbb{N}$, we denote by $\mathring{\mathrm{c}}_m$ a generic and smooth matrix-valued function of $\{(\mathrm{D}^\alpha \mathring{V}, \mathrm{D}^\alpha \mathring{\Psi}) : |\alpha| \le m\}$, where $\mathrm{D}^\alpha := \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ and $\alpha := (\alpha_0, \dots, \alpha_3) \in \mathbb{N}^4$ with the convention $|\alpha| := \alpha_0 + \cdots + \alpha_3$. The exact form of $\mathring{\mathrm{c}}_m$ may vary at different places.

(iv) We employ symbol D_*^α to mean that

$$\mathrm{D}_*^\alpha := \partial_t^{\alpha_0}(\sigma\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}\partial_1^{\alpha_4}, \quad \alpha := (\alpha_0, \dots, \alpha_4) \in \mathbb{N}^5, \tag{2.9}$$

with $\langle\alpha\rangle := |\alpha| + \alpha_4$ and $|\alpha| := \alpha_0 + \cdots + \alpha_4$, where $\sigma = \sigma(x_1)$ is an increasing smooth function on $[0, +\infty)$ such that $\sigma(x_1) = x_1$ for $0 \le x_1 \le 1/2$ and $\sigma(x_1) = 1$ for $x_1 \ge 1$.

(v) For $m \in \mathbb{N}$ and $I \subset \mathbb{R}$, we define the function space $H_*^m(I \times \Omega)$ as

$$H_*^m(I \times \Omega) := \{u \in L^2(I \times \Omega) : \mathrm{D}_*^\alpha u \in L^2(I \times \Omega) \text{ for } \langle\alpha\rangle \le m\},$$

equipped with the norm $\|\cdot\|_{H_*^m(I\times\Omega)}$, where

$$\|u\|^2_{H_*^m(I\times\Omega)} := \sum_{\langle\alpha\rangle\le m} \|\mathrm{D}_*^\alpha u\|^2_{L^2(I\times\Omega)}. \tag{2.10}$$

We will abbreviate

$$\|u\|_{m,*,t} := \|u\|_{H_*^m(\Omega_t)}, \quad \mathring{\mathrm{C}}_m := 1 + \|(\mathring{V}, \mathring{\Psi})\|^2_{m,*,T}. \tag{2.11}$$

Clearly, $H^m(I \times \Omega) \hookrightarrow H_*^m(I \times \Omega) \hookrightarrow H^{\lfloor m/2 \rfloor}(I \times \Omega)$ for all $m \in \mathbb{N}$ and $I \subset \mathbb{R}$.

The well-posedness for the effective linear problem (2.7) is provided in the following theorem.

Theorem 2.1 *Let $K_0 > 0$ and $m \in \mathbb{N}$ with $m \ge 6$. Then there exist constants $T_0 > 0$ and $C(K_0) > 0$ such that if for some $0 < T \le T_0$, the basic state $(\mathring{U}, \mathring{\varphi})$ satisfies (2.1)–(2.3), the source terms $f \in H_*^m(\Omega_T)$, $g \in H^{m+1}(\Sigma_T)$ vanish in the past, and $\mathring{V} := \mathring{U} - \overline{U} \in H_*^{m+4}(\Omega_T)$, $\mathring{\varphi} \in H^{m+4}(\Sigma_T)$ satisfy*

$$\|\mathring{V}\|_{10,*,T} + \|\mathring{\varphi}\|_{H^{10}(\Sigma_T)} \le K_0, \tag{2.12}$$

then the problem (2.7) has a unique solution $(\dot{V}, \psi) \in H_*^m(\Omega_T) \times H^m(\Sigma_T)$ satisfying the tame estimate

$$\begin{aligned} &\|(\dot{V}, \Psi)\|_{m,*,T} + \|(\psi, \mathrm{D}_{x'}\psi)\|_{H^m(\Sigma_T)} \\ &\quad \le C(K_0)\Big\{\|f\|_{m,*,T} + \|g\|_{H^{m+1}(\Sigma_T)} \\ &\qquad + \|(\mathring{V}, \mathring{\Psi})\|_{m+4,*,T}\left(\|f\|_{6,*,T} + \|g\|_{H^7(\Sigma_T)}\right)\Big\}. \end{aligned} \tag{2.13}$$

The above assumption that the source terms f and g vanish in the past corresponds to the nonlinear problem with zero initial data. The case of general initial data will be reduced in the subsequent nonlinear analysis.

2.2 H^1_* *a priori* estimate

This subsection is devoted to obtaining the *a priori* estimate in H^1_* for solutions of the linearized problem (2.7) by exploiting the stabilization effect of the surface tension on the evolution of the interface. For clear presentation, we divide this subsection into five parts.

2.2.1 Partial homogenization

It is convenient to reduce the problem (2.7) to homogeneous boundary conditions. As in [35, Sect. 3.3], there exists a function $V_\natural \in H^{m+2}_*(\Omega_T)$ vanishing in the past such that

$$\mathbb{B}'_e(\mathring{U}, \mathring{\varphi})(V_\natural, 0)\big|_{\Sigma_T} = g, \quad \|V_\natural\|_{s+2,*,T} \lesssim \|g\|_{H^{s+1}(\Sigma_T)} \text{ for } s = 0, \dots, m. \tag{2.14}$$

Consequently, the vector $V_\flat := \dot{V} - V_\natural$ solves

$$\begin{aligned} &\mathbb{L}'_e(\mathring{U}, \mathring{\Phi})V = \tilde{f} := f - \mathbb{L}'_e(\mathring{U}, \mathring{\Phi})V_\natural && \text{if } x_1 > 0, && (2.15a)\\ &\mathbb{B}'_e(\mathring{U}, \mathring{\varphi})(V, \psi) = 0 && \text{if } x_1 = 0, && (2.15b)\\ &(V, \psi) = 0 && \text{if } t < 0, && (2.15c) \end{aligned}$$

where we have dropped subscript "♭" for simplicity of notation.

To separate the noncharacteristic variables from others for the problem (2.15), we introduce the new unknown

$$W := (q, v_1 - \partial_2\mathring{\Phi} v_2 - \partial_3\mathring{\Phi} v_3, v_2, v_3, H_1, H_2, H_3, S)^{\mathsf{T}} = J(\mathring{\Phi})^{-1}V,$$

with

$$J(\Phi) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & \partial_2\Phi & \partial_3\Phi & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.16}$$

For $i = 0, 2, 3$, we denote

$$\boldsymbol{A}_1 := J(\mathring{\Phi})^{\mathsf{T}}\widetilde{A}_1(\mathring{U}, \mathring{\Phi})J(\mathring{\Phi}), \ \boldsymbol{A}_4 := J(\mathring{\Phi})^{\mathsf{T}}\mathbb{L}'_e(\mathring{U}, \mathring{\Phi})J(\mathring{\Phi}), \ \boldsymbol{A}_i := J(\mathring{\Phi})^{\mathsf{T}}A_i(\mathring{U})J(\mathring{\Phi}).$$

Then we reformulate the problem (2.15) into

$$LW := \sum_{i=0}^{3} A_i \partial_i W + A_4 W = f \quad \text{in } \Omega_T, \tag{2.17a}$$

$$W_2 = (\partial_t + \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3)\psi - \partial_1 \mathring{v} \cdot \mathring{N}\psi =: \mathrm{B}\psi \quad \text{on } \Sigma_T, \tag{2.17b}$$

$$W_1 = -\partial_1 \mathring{q}\psi + \mathfrak{s}\mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) \quad \text{on } \Sigma_T, \tag{2.17c}$$

$$(W, \psi) = 0 \quad \text{if } t < 0, \tag{2.17d}$$

with $\partial_0 := \partial/\partial t$ and $f := J(\mathring{\Phi})^{\mathsf{T}} \tilde{f}$. In view of (2.2), we calculate

$$A_1\big|_{x_1=0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & O_6 \end{pmatrix} =: A_1^{(1)}. \tag{2.18}$$

Set $A_1^{(0)} := A - A_1^{(1)}$ so that $A_1^{(0)}|_{x_1=0} = 0$. According to the kernel of $A_1|_{x_1=0}$, we denote by $W_{\mathrm{nc}} := (W_1, W_2)^{\mathsf{T}}$ the noncharacteristic part of the unknown W.

2.2.2 L^2 estimate of W

We derive the L^2 energy estimate for solutions of (2.17) as follows. Taking the scalar product of the problem (2.17a) with W leads to

$$\int_{\Omega} A_0 W \cdot W(t,x)\,\mathrm{d}x - \int_{\Sigma_t} A_1 W \cdot W \le C(K) \|(f, W)\|^2_{L^2(\Omega_t)}. \tag{2.19}$$

From (2.18) and (2.17b)–(2.17c), we have

$$\begin{aligned} &- A_1 W \cdot W = -2W_1 W_2 = -2W_1 \mathrm{B}\psi \\ &= \partial_t \left\{ \partial_1 \mathring{q}\psi^2 + \mathfrak{s}\left(\frac{|\mathrm{D}_{x'}\psi|^2}{|\mathring{N}|} - \frac{|\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} \right) \right\} + \mathfrak{s}\mathring{\mathrm{c}}_2 \mathrm{D}_{x'}\psi \cdot \begin{pmatrix} \psi \\ \mathrm{D}_{x'}\psi \end{pmatrix} \\ &\quad + \sum_{i=2,3} \partial_i \left\{ \partial_1 \mathring{q}\mathring{v}_i \psi^2 + \mathfrak{s}\mathring{v}_i \left(\frac{|\mathrm{D}_{x'}\psi|^2}{|\mathring{N}|} - \frac{|\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} \right) \right\} + \mathring{\mathrm{c}}_2 \psi^2 \\ &\quad - 2\mathfrak{s}\mathrm{D}_{x'} \cdot \left\{ \left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) \mathrm{B}\psi \right\} \quad \text{on } \Sigma_T, \end{aligned} \tag{2.20}$$

where $\mathring{\mathrm{c}}_s$ denotes a generic and smooth matrix-valued function of $\{(\mathrm{D}^\alpha \mathring{V}, \mathrm{D}^\alpha \mathring{\Psi}) : |\alpha| \le s\}$ for any $s \in \mathbb{N}$ whose exact form may change from line to line. We substitute the above identities into (2.19) to get

$$\int_{\Omega} A_0 W \cdot W(t,x)\,\mathrm{d}x + \int_{\Sigma} \left(\partial_1 \mathring{q} \psi^2 + \mathfrak{s} \frac{|\mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} \right) \mathrm{d}x' \\ \le C(K)\Big(\|(f,W)\|^2_{L^2(\Omega_t)} + \|(\psi, \sqrt{\mathfrak{s}}\mathrm{D}_{x'}\psi)\|^2_{L^2(\Sigma_t)} \Big). \tag{2.21}$$

If the generalized Rayleigh–Taylor sign condition $\partial_1 \mathring{q} \gtrsim 1$ were assumed, then we could apply Grönwall's inequality to (2.21) and obtain the *uniform-in-*$\mathfrak{s}$ *estimate*

$$\|W(t)\|^2_{L^2(\Omega)} + \|(\psi, \sqrt{\mathfrak{s}}\mathrm{D}_{x'}\psi)(t)\|^2_{L^2(\Sigma)} \le C(K)\|f\|^2_{L^2(\Omega_t)}. \tag{2.22}$$

But in this paper, we focus on the case of positive surface tension and aim to release the sign condition. For this purpose, we fix the coefficient $\mathfrak{s} > 0$ and use integration by parts to get

$$\|\psi(t)\|^2_{L^2(\Sigma)} = 2\int_{\Sigma_t} \psi \partial_t \psi \lesssim \|(\psi, \partial_t\psi)\|^2_{L^2(\Sigma_t)}. \tag{2.23}$$

Combining (2.21) and (2.23) gives

$$\|W(t)\|^2_{L^2(\Omega)} + \|(\psi, \mathrm{D}_{x'}\psi)(t)\|^2_{L^2(\Sigma)} \\ \le C(K)\Big(\|(f,W)\|^2_{L^2(\Omega_t)} + \|(\psi, \mathrm{D}_{x'}\psi, \partial_t\psi)\|^2_{L^2(\Sigma_t)} \Big),$$

from which we deduce

$$\|W(t)\|^2_{L^2(\Omega)} + \|(\psi, \mathrm{D}_{x'}\psi)(t)\|^2_{L^2(\Sigma)} \le C(K)\Big(\|f\|^2_{L^2(\Omega_t)} + \|\partial_t\psi\|^2_{L^2(\Sigma_t)} \Big). \tag{2.24}$$

2.2.3 L^2 estimate of $\mathrm{D}_{x'} W$

Let us proceed to close the energy estimate in H^1_*. To this end, we set $k \in \{0, 2, 3\}$. Applying the operator ∂_k to the interior equations (2.17a) yields

$$\sum_{i=0}^{3} A_i \partial_i \partial_k W = \partial_k f - \partial_k (A_4 W) - \sum_{i=0}^{3} \partial_k A_i \partial_i W. \tag{2.25}$$

It follows from (2.18) that $\partial_k A_1 = 0$ on Σ_T, and hence

$$\|\partial_k A_1(x_1)\|_{L^\infty((-\infty,T)\times\mathbb{R}^2)} \le C(K)\sigma(x_1) \ \text{ for } x_1 \ge 0 \text{ and } k \in \{0,2,3\}. \tag{2.26}$$

In view of (2.18) and (2.26), we take the scalar product of (2.25) with $\partial_k W$ to get

$$\int_{\Omega} A_0 \partial_k W \cdot \partial_k W(t,x)\,\mathrm{d}x - 2\int_{\Sigma_t} \partial_k W_1 \partial_k W_2 \le C(K)\|(f,W)\|^2_{1,*,t}, \tag{2.27}$$

where the norm $\|\cdot\|_{1,*,t} := \|\cdot\|_{H^1_*(\Omega_t)}$ is defined by (2.9)–(2.11). We utilize (2.17b)–(2.17c) to obtain that on Σ_T,

$$-2\partial_k W_1 \partial_k W_2 = -2\partial_k W_1 \partial_k \mathrm{B}\psi, \tag{2.28}$$

and

$$\begin{aligned}
-2\partial_k W_1 \partial_k \mathrm{B}\psi = {} & \partial_t \left\{ \partial_1 \mathring{q} (\partial_k \psi)^2 \right\} + \sum_{i=2,3} \partial_i \left\{ \partial_1 \mathring{q} \mathring{v}_i (\partial_k \psi)^2 \right\} \\
& + 2\partial_k \left(\partial_k \partial_1 \mathring{q} \psi \mathrm{B}\psi \right) + \sum_{|\alpha|\le 1} \mathring{c}_3 \begin{pmatrix} \psi \\ \partial_k \psi \end{pmatrix} \cdot \begin{pmatrix} \mathrm{D}^\alpha_{x'} \psi \\ \partial_t \psi \end{pmatrix} \\
& - 2\mathfrak{s} \mathrm{D}_{x'} \cdot \left\{ \partial_k \left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) \partial_k \mathrm{B}\psi \right\} \\
& + \underbrace{2\mathfrak{s} \partial_k \left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) \cdot \mathrm{D}_{x'} \partial_k \mathrm{B}\psi}_{\mathcal{T}_k},
\end{aligned} \tag{2.29}$$

where $\mathrm{D}^\alpha_{x'} := \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ for $\alpha = (\alpha_2, \alpha_3) \in \mathbb{N}^2$. A lengthy but straightforward calculation gives

$$\begin{aligned}
\mathcal{T}_k = {} & \mathfrak{s} \partial_t \left(\frac{|\mathrm{D}_{x'}\partial_k \psi|^2}{|\mathring{N}|} - \frac{|\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\partial_k \psi|^2}{|\mathring{N}|^3} + \mathring{c}_2 \mathrm{D}_{x'}\psi \cdot \mathrm{D}_{x'}\partial_k \psi \right) \\
& + \sum_{i=2,3} \partial_i \left\{ \mathfrak{s} \mathring{v}_i \left(\frac{|\mathrm{D}_{x'}\partial_k \psi|^2}{|\mathring{N}|} - \frac{|\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\partial_k \psi|^2}{|\mathring{N}|^3} + \mathring{c}_2 \mathrm{D}_{x'}\psi \cdot \mathrm{D}_{x'}\partial_k \psi \right) \right\} \\
& + \sum_{|\alpha|\le 2} \mathring{c}_3 \begin{pmatrix} \mathrm{D}_{x'}\psi \\ \partial_k \mathrm{D}_{x'}\psi \end{pmatrix} \cdot \begin{pmatrix} \partial_k \psi \\ \mathrm{D}^\alpha_{x'}\psi \\ \partial_t \mathrm{D}_{x'}\psi \end{pmatrix} \quad \text{for } k = 0, 2, 3.
\end{aligned} \tag{2.30}$$

Plug (2.28)–(2.29) into (2.27) with $k = 2, 3$, and use (2.30) to get

$$\begin{aligned}
& \sum_{k=2,3} \left(\int_\Omega A_0\, \partial_k W \cdot \partial_k W(t,x)\, \mathrm{d}x + \mathfrak{s} \int_\Sigma \frac{|\mathrm{D}_{x'}\partial_k \psi|^2}{|\mathring{N}|^3} \mathrm{d}x' \right) \\
& \le C(K) \Bigg(\|(f, W)\|^2_{1,*,t} + \sum_{k=2,3} \int_\Sigma \left(|\partial_k \psi|^2 + |\mathrm{D}_{x'}\psi||\mathrm{D}_{x'}\partial_k \psi| \right) \mathrm{d}x' \\
& \quad + \epsilon \|\partial_t \psi\|^2_{L^2(\Sigma_t)} + C(\epsilon) \sum_{|\alpha|\le 2} \|(\mathrm{D}^\alpha_{x'}\psi, \mathrm{D}_{x'}\partial_t \psi)\|^2_{L^2(\Sigma_t)} \Bigg)
\end{aligned}$$

for all $\epsilon > 0$. Here we have applied Young's inequality with a constant ϵ. Below we will always use the letter ϵ to denote such temporary constants in analogous situations.

Using Cauchy's inequality and the basic estimate

$$\|\mathrm{D}_{x'}\psi(t)\|_{L^2(\Sigma)} \lesssim \|(\mathrm{D}_{x'}\psi, \mathrm{D}_{x'}\partial_t\psi)\|_{L^2(\Sigma_t)}, \tag{2.31}$$

we have

$$\begin{aligned}&\|\mathrm{D}_{x'}W(t)\|^2_{L^2(\Omega)} + \|\mathrm{D}^2_{x'}\psi(t)\|^2_{L^2(\Sigma)}\\ &\quad \le C(K)\bigg(\|(f, W)\|^2_{1,*,t} + \sum_{|\alpha|\le 2}\|(\mathrm{D}^{\alpha}_{x'}\psi, \partial_t\psi, \mathrm{D}_{x'}\partial_t\psi)\|^2_{L^2(\Sigma_t)}\bigg).\end{aligned} \tag{2.32}$$

2.2.4 L^2 estimate of $\partial_t W$

For $k = 0$, we use (2.17b)–(2.17c) to find

$$-2\int_{\Sigma_t}\partial_t W_1\partial_t W_2 = \underbrace{2\int_{\Sigma_t}\partial_1\mathring{q}\,\partial_t\psi\,\partial_t W_2}_{\mathcal{J}_1} + \underbrace{2\int_{\Sigma_t}\partial_t\partial_1\mathring{q}\,\psi\,\partial_t W_2}_{\mathcal{J}_2} + \int_{\Sigma_t}\mathcal{T}_0, \tag{2.33}$$

where $\mathcal{T}_0$ is defined in (2.29). By virtue of (2.17b), we infer

$$\mathcal{J}_1 = \underbrace{2\int_{\Sigma_t}\partial_1\mathring{q}\,W_2\partial_t W_2}_{\mathcal{J}_{1a}} + \underbrace{2\int_{\Sigma_t}\partial_1\mathring{q}\,\partial_t W_2(-\mathring{v}_2\partial_2\psi - \mathring{v}_3\partial_3\psi + \partial_1\mathring{v}\cdot\mathring{N}\psi)}_{\mathcal{J}_{1b}}. \tag{2.34}$$

Passing to the volume integral yields

$$\mathcal{J}_{1a} = -2\int_{\Omega_t}\partial_1(\partial_1\mathring{q}\,W_2\partial_t W_2).$$

Consequently, we have

$$\begin{aligned}\mathcal{J}_{1a} &= -2\int_{\Omega}\partial_1\mathring{q}\,W_2\partial_1 W_2\,\mathrm{d}x - 2\int_{\Omega_t}\Big(\partial_1^2\mathring{q}\,W_2\partial_t W_2 - \partial_1\partial_t\mathring{q}\,W_2\partial_1 W_2\Big)\\ &\ge -\epsilon\|\partial_1 W_2(t)\|^2_{L^2(\Omega)} - C(K)C(\epsilon)\|(W_2, \partial_t W_2, \partial_1 W_2)\|^2_{L^2(\Omega_t)}\end{aligned} \tag{2.35}$$

for all $\epsilon > 0$. It follows from integration by parts that

$$
\begin{aligned}
\mathcal{J}_{1b} + \mathcal{J}_2 = 2 \int_{\Sigma} W_2 & \left\{ \partial_1 \mathring{q} (-\mathring{v}_2 \partial_2 \psi - \mathring{v}_3 \partial_3 \psi + \partial_1 \mathring{v} \cdot \mathring{N} \psi) + \partial_t \partial_1 \mathring{q} \psi \right\} \mathrm{d}x' \\
& - 2 \int_{\Sigma_t} W_2 \partial_t \left\{ \partial_1 \mathring{q} (-\mathring{v}_2 \partial_2 \psi - \mathring{v}_3 \partial_3 \psi + \partial_1 \mathring{v} \cdot \mathring{N} \psi) + \partial_t \partial_1 \mathring{q} \psi \right\} \\
& \geq - \| W_2(t) \|^2_{L^2(\Sigma)} - \| W_2 \|^2_{L^2(\Sigma_t)} \\
& - C \sum_{i=0,1} \| \partial_t^i \left\{ \partial_1 \mathring{q} (-\mathring{v}_2 \partial_2 \psi - \mathring{v}_3 \partial_3 \psi + \partial_1 \mathring{v} \cdot \mathring{N} \psi) + \partial_t \partial_1 \mathring{q} \psi \right\} \|^2_{L^2(\Sigma_t)} \\
& \geq - \| W_2(t) \|^2_{L^2(\Sigma)} - C(K) \sum_{|\alpha| \leq 2} \| (W_2, \mathrm{D}^{\alpha}_{x'} \psi, \partial_t \psi, \mathrm{D}_{x'} \partial_t \psi) \|^2_{L^2(\Sigma_t)}.
\end{aligned}
\tag{2.36}
$$

By virtue of (2.30) and (2.31), we get

$$
\int_{\Sigma_t} \mathcal{T}_0 \geq \frac{\mathfrak{s}}{2} \int_{\Sigma} \frac{|\mathrm{D}_{x'} \partial_t \psi|^2}{|\mathring{N}|^3} \mathrm{d}x' - C(K) \sum_{|\alpha| \leq 2} \| (\mathrm{D}^{\alpha}_{x'} \psi, \partial_t \psi, \mathrm{D}_{x'} \partial_t \psi) \|^2_{L^2(\Sigma_t)}. \tag{2.37}
$$

Substitute (2.33) into (2.27) and use (2.34)–(2.37) to obtain

$$
\begin{aligned}
& \| \partial_t W(t) \|^2_{L^2(\Omega)} + \| \mathrm{D}_{x'} \partial_t \psi(t) \|^2_{L^2(\Sigma)} \\
& \quad \leq C(K) \Big(\| W_2(t) \|^2_{L^2(\Sigma)} + \epsilon \| \partial_1 W_2(t) \|^2_{L^2(\Omega)} + C(\epsilon) \| \partial_1 W_2 \|^2_{L^2(\Omega_t)} \\
& \qquad + C(\epsilon) \| (f, W) \|^2_{1,*,t} + \sum_{|\alpha| \leq 2} \| (W_2, \mathrm{D}^{\alpha}_{x'} \psi, \partial_t \psi, \mathrm{D}_{x'} \partial_t \psi) \|^2_{L^2(\Sigma_t)} \Big).
\end{aligned}
\tag{2.38}
$$

For the first term on the right-hand side, a direct computation gives

$$
\| W_2(t) \|^2_{L^2(\Sigma)} = -2 \int_{\Omega} W_2 \partial_1 W_2 \lesssim \epsilon \| \partial_1 W_2(t) \|^2_{L^2(\Omega)} + \epsilon^{-1} \| W(t) \|^2_{L^2(\Omega)} \tag{2.39}
$$

for all $\epsilon > 0$. The $L^2(\Omega)$–norm of $\partial_1 W_2$ can be controlled by virtue of (2.17a) and (2.18). More precisely, from (2.17a) and (2.18), we have

$$
\begin{pmatrix} \partial_1 W_2 \\ \partial_1 W_1 \\ 0 \end{pmatrix} = f - A_4 W - \sum_{i=0,2,3} A_i \partial_i W - A_1^{(0)} \partial_1 W, \tag{2.40}
$$

which implies the estimate

$$
\| \partial_1 W_{\mathrm{nc}}(t) \|^2_{L^2(\Omega)} \leq C(K) \sum_{\langle \beta \rangle \leq 1} \| \mathrm{D}^{\beta}_{*} W(t) \|^2_{L^2(\Omega)} + C(K) \| f(t) \|^2_{L^2(\Omega)} \tag{2.41}
$$

for the noncharacteristic variables $W_{\rm nc} := (W_1, W_2)^{\mathsf{T}}$. Here we recall that the operator D_*^β is defined by (2.9). Substituting (2.41) into (2.39), we deduce

$$\|W_2(t)\|^2_{L^2(\Sigma)} \le C(K)\bigg(\epsilon \sum_{\langle\beta\rangle\le 1} \|\mathrm{D}_*^\beta W(t)\|^2_{L^2(\Omega)} + C(\epsilon)\|(f, W)\|^2_{1,*,t}\bigg). \tag{2.42}$$

Then we combine (2.38) and (2.41)–(2.42) to obtain

$$\begin{aligned}&\|\partial_t W(t)\|^2_{L^2(\Omega)} + \epsilon\|\partial_1 W_2(t)\|^2_{L^2(\Omega)} + \|(W_2, \mathrm{D}_{x'}\partial_t\psi)(t)\|^2_{L^2(\Sigma)}\\ &\quad\le C(K)\bigg(C(\epsilon)\|(f, W)\|^2_{1,*,t} + C(\epsilon)\|\partial_1 W_2\|^2_{L^2(\Omega_t)} + \epsilon \sum_{\langle\beta\rangle\le 1}\|\mathrm{D}_*^\beta W(t)\|^2_{L^2(\Omega)}\\ &\qquad + \sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}_{x'}^\alpha\psi, \partial_t\psi, \mathrm{D}_{x'}\partial_t\psi)\|^2_{L^2(\Sigma_t)}\bigg) \text{ for all } \epsilon > 0.\end{aligned} \tag{2.43}$$

2.2.5 H_*^1 Estimate of W

In view of the boundary condition (2.17b), we infer

$$\|\partial_t\psi\|^2_{L^2(\Sigma_t)} \le C(K)\|(W_2, \psi, \mathrm{D}_{x'}\psi)\|^2_{L^2(\Sigma_t)}. \tag{2.44}$$

Apply operator $\sigma\partial_1$ to (2.17a) and take the resulting equation with $\sigma\partial_1 W$ to get

$$\|\sigma\partial_1 W(t)\|^2_{L^2(\Omega)} \le C(K)\|(f, W)\|^2_{1,*,t}.$$

Combining the last estimate with (2.24), (2.32), and (2.43)–(2.44), we choose $\epsilon > 0$ small enough to obtain

$$\begin{aligned}&\sum_{\langle\beta\rangle\le 1}\|\mathrm{D}_*^\beta W(t)\|^2_{L^2(\Omega)} + \|\partial_1 W_2(t)\|^2_{L^2(\Omega)} + \sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}_{x'}^\alpha\psi, \mathrm{D}_{x'}\partial_t\psi)(t)\|^2_{L^2(\Sigma)}\\ &\quad\le C(K)\bigg(\|(f, W)\|^2_{1,*,t} + \|\partial_1 W_2\|^2_{L^2(\Omega_t)} + \sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}_{x'}^\alpha\psi, \mathrm{D}_{x'}\partial_t\psi)\|^2_{L^2(\Sigma_t)}\bigg).\end{aligned}$$

By virtue of Grönwall's inequality, we infer

$$\sum_{\langle\beta\rangle\le 1}\|\mathrm{D}_*^\beta W(t)\|^2_{L^2(\Omega)} + \sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}_{x'}^\alpha\psi, \mathrm{D}_{x'}\partial_t\psi)(t)\|^2_{L^2(\Sigma)} \le C(K)\|f\|^2_{1,*,t}, \tag{2.45}$$

which together with (2.41), (2.44), and (2.17c) yields

$$\begin{aligned}&\|W\|_{1,*,T} + \|\partial_1 W_{\rm nc}\|_{L^2(\Omega_T)} + \|W_{\rm nc}\|_{L^2(\Sigma_T)}\\ &\quad+\|(\psi, \mathrm{D}_{x'}\psi)\|_{H^1(\Sigma_T)} \le C(K)\|f\|_{1,*,T}.\end{aligned} \tag{2.46}$$

This is the desired H^1_* *a priori* estimate for solutions W of the linearized problem (2.17), which also contains the L^2 estimate for the traces of the noncharacteristic variables $W_{\rm nc} := (W_1, W_2)^{\mathsf{T}}$ on the boundary $\{x_1 = 0\}$.

2.3 Well-posedness in H^1_*

The main purpose of this subsection is to construct the unique solution of the linearized problem (2.17). Since the L^2 *a priori* estimate (2.21) is not closed without assuming the generalized Rayleigh–Taylor sign condition, the duality argument cannot be applied directly for the solvability of the problem (2.17). To overcome this difficulty, we design for the linearized problem (2.17) some suitable ε–regularization, for which we can deduce an L^2 *a priori* estimate with a constant $C(\varepsilon)$ depending on the small parameter $\varepsilon \in (0,1)$. It will turn out that an L^2 *a priori* estimate can be also achieved for the corresponding dual problem. Then we prove the existence and uniqueness of solutions in L^2 for any fixed and small parameter $\varepsilon \in (0,1)$ by the duality argument. However, our constant $C(\varepsilon)$ tends to infinity as $\varepsilon \to 0$, and hence we are not able to use the L^2 estimate obtained for the regularized problem to take the limit $\varepsilon \to 0$. To deal with this situation, we then derive a uniform-in-ε estimate in H^1_* for the regularized problem, which enables us to solve the linearized problem (2.17) by passing to the limit $\varepsilon \to 0$.

More precisely, we define the following regularized problem:

$$L_\varepsilon W := \sum_{i=0}^{3} A_i \partial_i W - \varepsilon J \partial_1 W + A_4 W = f \qquad \text{in } \Omega_T, \tag{2.47a}$$

$$W_2 = (\partial_t + \mathring{v}_2 \partial_2 + \mathring{v}_3 \partial_3)\psi - \partial_1 \mathring{v} \cdot \mathring{N} \psi + \varepsilon(\partial_2^4 + \partial_3^4)\psi \qquad \text{on } \Sigma_T, \tag{2.47b}$$

$$W_1 = -\partial_1 \mathring{q} \psi + \mathfrak{s} \mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) \qquad \text{on } \Sigma_T, \tag{2.47c}$$

$$(W, \psi) = 0 \qquad \text{if } t < 0, \tag{2.47d}$$

where $J := \operatorname{diag}(0, 1, 0, \dots, 0)$. The term $-\varepsilon J \partial_1 W$ containing in the interior equations (2.47a) helps us to deduce the L^2 energy estimate for the problem (2.47), while the term $\varepsilon(\partial_2^4 + \partial_3^4)\psi$ added in the boundary condition (2.47b) is particularly useful in the derivation of the L^2 energy estimate for the dual problem of (2.47).

This subsection is divided into three parts as follows.

2.3.1 L^2 estimate for the regularized problem

Let us first show the L^2 *a priori* estimate for the problem (2.47). Taking the scalar product of (2.47a) with W implies

$$\int_{\Omega} A_0 W \cdot W(t,x)\,\mathrm{d}x + \int_{\Sigma_t} (\varepsilon J - A_1) W \cdot W \le C(K) \|(f, W)\|^2_{L^2(\Omega_t)}. \tag{2.48}$$

By virtue of (2.18) and (2.47b), we have

$$(\varepsilon J - A_1)W \cdot W = \varepsilon W_2^2 - 2W_1 \mathrm{B}\psi - 2\varepsilon \sum_{i=2,3} W_1 \partial_i^4 \psi \quad \text{on } \Sigma_T, \tag{2.49}$$

where the operator B is defined in (2.17b). For $i = 2, 3$, we get from (2.47c) that

$$\begin{aligned} -\int_{\Sigma_t} W_1 \partial_i^4 \psi &= \mathfrak{s} \int_{\Sigma_t} \partial_i^2 \left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) \cdot \partial_i^2 \mathrm{D}_{x'}\psi \\ &\quad + \int_{\Sigma_t} \partial_i^2 (\partial_1 \mathring{q} \psi) \partial_i^2 \psi \geq \mathfrak{s} \int_{\Sigma_t} \frac{|\partial_i^2 \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} \\ &\quad - \int_{\Sigma_t} \left| [\partial_i^2, \mathring{\mathfrak{c}}_1] \mathrm{D}_{x'}\psi \cdot \partial_i^2 \mathrm{D}_{x'}\psi + \partial_i^2 (\partial_1 \mathring{q} \psi) \partial_i^2 \psi \right| \\ &\geq \frac{\mathfrak{s}}{2} \int_{\Sigma_t} \frac{|\partial_i^2 \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} - C(K) \sum_{|\alpha| \leq 2} \|\mathrm{D}_{x'}^\alpha \psi\|_{L^2(\Sigma_t)}^2. \end{aligned} \tag{2.50}$$

Substituting (2.49) into (2.48), we utilize (2.50) and the last identity in (2.20) to infer

$$\begin{aligned} &\|W(t)\|_{L^2(\Omega)}^2 + \|\mathrm{D}_{x'}\psi(t)\|_{L^2(\Sigma)}^2 + \varepsilon \|(W_2, \partial_2^2 \mathrm{D}_{x'}\psi, \partial_3^2 \mathrm{D}_{x'}\psi)\|_{L^2(\Sigma_t)}^2 \\ &\quad \leq C(K) \Big(\|(f, W)\|_{L^2(\Omega_t)}^2 + \sum_{|\alpha| \leq 2} \|(\psi, \mathrm{D}_{x'}\psi, \sqrt{\varepsilon} \mathrm{D}_{x'}^\alpha \psi)\|_{L^2(\Sigma_t)}^2 + \|\psi(t)\|_{L^2(\Sigma)}^2 \Big). \end{aligned} \tag{2.51}$$

To estimate the last term in (2.51), we multiply (2.47b) with ψ and obtain

$$\begin{aligned} &\|\psi(t)\|_{L^2(\Sigma)}^2 + 2\varepsilon \|(\partial_2^2 \psi, \partial_3^2 \psi)\|_{L^2(\Sigma_t)}^2 \\ &\quad \leq C(K) \|\psi\|_{L^2(\Sigma_t)}^2 + 2 \int_{\Sigma_t} |\psi W_2| \\ &\quad \leq \epsilon \varepsilon \|W_2\|_{L^2(\Sigma_t)}^2 + \left(C(K) + \epsilon^{-1} \varepsilon^{-1} \right) \|\psi\|_{L^2(\Sigma_t)}^2 \quad \text{for all } \epsilon > 0. \end{aligned} \tag{2.52}$$

Since

$$\|\partial_2 \partial_3 \mathrm{D}_{x'}^\alpha \psi\|_{L^2(\Sigma)}^2 = \int_{\Sigma} \partial_2^2 \mathrm{D}_{x'}^\alpha \psi \, \partial_3^2 \mathrm{D}_{x'}^\alpha \psi \leq \|(\partial_2^2 \mathrm{D}_{x'}^\alpha \psi, \partial_3^2 \mathrm{D}_{x'}^\alpha \psi)\|_{L^2(\Sigma)}^2 \tag{2.53}$$

for any $\alpha \in \mathbb{N}^2$, it follows from (2.52) that

$$\|\psi(t)\|_{L^2(\Sigma)}^2 + \varepsilon \|\mathrm{D}_{x'}^2 \psi\|_{L^2(\Sigma_t)}^2 \leq \epsilon \varepsilon \|W_2\|_{L^2(\Sigma_t)}^2 + C(K) C(\epsilon \varepsilon) \|\psi\|_{L^2(\Sigma_t)}^2 \tag{2.54}$$

for all $\epsilon > 0$. Here we have formally that $C(\epsilon\varepsilon) \to +\infty$ as $\varepsilon \to 0$. The same is true for all constants depending on ε which appear below. This is however unimportant because the parameter ε is now fixed. Plug (2.54) into (2.51), take $\epsilon > 0$ sufficiently small, and use (2.53) with $|\alpha| = 1$ to deduce

$$\begin{aligned}&\|W(t)\|^2_{L^2(\Omega)} + \|(\psi, \mathrm{D}_{x'}\psi)(t)\|^2_{L^2(\Sigma)} + \varepsilon\|(W_2, \mathrm{D}^2_{x'}\psi, \mathrm{D}^3_{x'}\psi)\|^2_{L^2(\Sigma_t)}\\ &\quad\le C(K)\left(\|(f, W)\|^2_{L^2(\Omega_t)} + C(\varepsilon)\|(\psi, \mathrm{D}_{x'}\psi)\|^2_{L^2(\Sigma_t)}\right).\end{aligned}$$

Then we apply Grönwall's inequality and utilize (2.47c) to get

$$\begin{aligned}&\|W(t)\|^2_{L^2(\Omega)} + \|(\psi, \mathrm{D}_{x'}\psi)(t)\|^2_{L^2(\Sigma)}\\ &\quad+\|(W_{\mathrm{nc}}, \mathrm{D}^2_{x'}\psi, \mathrm{D}^3_{x'}\psi)\|^2_{L^2(\Sigma_t)} \le C(K, \varepsilon)\|f\|^2_{L^2(\Omega_t)}.\end{aligned} \tag{2.55}$$

Moreover, we differentiate (2.47b) two times with respect to x_i and multiply the resulting identity with $\partial_i^2\psi$, for $i = 2, 3$, to deduce

$$\begin{aligned}&\sum_{i=2,3}\|\partial_i^2\psi(t)\|^2_{L^2(\Sigma)} + 2\varepsilon\sum_{i=2,3}\|(\partial_2^2\partial_i^2\psi, \partial_3^2\partial_i^2\psi)\|^2_{L^2(\Sigma_t)}\\ &\quad\le C(K)\sum_{|\alpha|\le 2}\|\mathrm{D}^\alpha_{x'}\psi\|^2_{L^2(\Sigma_t)} + 2\sum_{i=2,3}\int_{\Sigma_t}\left|\partial_i^4\psi\, W_2\right|\\ &\quad\le \epsilon\varepsilon\sum_{i=2,3}\|\partial_i^4\psi\|^2_{L^2(\Sigma_t)} + \left(C(K) + \frac{1}{\epsilon\varepsilon}\right)\sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}^\alpha_{x'}\psi)\|^2_{L^2(\Sigma_t)}\end{aligned}$$

for all $\epsilon > 0$. Taking $\epsilon > 0$ sufficiently small in the last estimate, we use (2.53) and (2.55) to infer

$$\begin{aligned}&\|W(t)\|^2_{L^2(\Omega)} + \sum_{|\alpha|\le 2}\|\mathrm{D}^\alpha_{x'}\psi(t)\|^2_{L^2(\Sigma)}\\ &\quad+\|(W_{\mathrm{nc}}, \mathrm{D}^2_{x'}\psi, \mathrm{D}^3_{x'}\psi, \mathrm{D}^4_{x'}\psi)\|^2_{L^2(\Sigma_t)} \le C(K, \varepsilon)\|f\|^2_{L^2(\Omega_t)},\end{aligned}$$

which combined with (2.47b) implies

$$\|W\|_{L^2(\Omega_T)} + \sum_{|\alpha|\le 4}\|(W_{\mathrm{nc}}, \mathrm{D}^\alpha_{x'}\psi, \partial_t\psi)\|_{L^2(\Sigma_T)} \le C(K, \varepsilon)\|f\|_{L^2(\Omega_T)}. \tag{2.56}$$

The *a priori* L^2 estimate (2.56) (together with the L^2 estimate (2.62) below for the dual problem) is suitable to prove the existence of solutions of the regularized problem (2.47) for any *fixed* and small $\varepsilon > 0$. However, since $C(K, \varepsilon) \to +\infty$ as $\varepsilon \to 0$, for passing to the limit $\varepsilon \to 0$ we will have to deduce an additional uniform-in-ε estimate.

2.3.2 Existence of solutions for the regularized problem

We solve the regularized problem (2.47) by means of the duality argument. To this end, it suffices to derive an L^2 *a priori* estimate without loss of derivatives for the following dual problem of (2.47):

$$L_\varepsilon^* W^* = f^* \qquad \text{if } x_1 > 0, \tag{2.57a}$$

$$\begin{aligned}&\partial_t w^* + \partial_2(\mathring{v}_2 w^*) + \partial_3(\mathring{v}_3 w^*) - \varepsilon(\partial_2^4 + \partial_3^4) w^* + \partial_1 \mathring{v} \cdot \mathring{N} w^* \\ &\quad + \partial_1 \mathring{q} W_2^* - \mathfrak{s} \mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'} W_2^*}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'} W_2^*}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) = 0 \qquad \text{if } x_1 = 0,\end{aligned} \tag{2.57b}$$

$$W^*|_{t>T} = 0, \tag{2.57c}$$

where $w^* := W_1^* - \varepsilon W_2^*$, and L_ε^* is the formal adjoint of L_ε (*cf.* (2.47a)):

$$L_\varepsilon^* := -\sum_{i=0}^{3} A_i \partial_i + \varepsilon J \partial_1 + A_4^\mathsf{T} - \sum_{i=0}^{3} \partial_i A_i.$$

The boundary condition (2.57b) is imposed to guarantee that

$$\begin{aligned}&\int_{\Omega_T} \left(L_\varepsilon W \cdot W^* - W \cdot L_\varepsilon^* W^* \right) \\ &\quad = \int_{\Sigma_T} (\varepsilon J - A_1)\, W \cdot W^* = -\int_{\Sigma_T} \left(W_2 w^* + W_1 W_2^* \right) = 0,\end{aligned}$$

thanks to (2.47b)–(2.47d) and (2.57c). To derive the energy estimate for the dual problem (2.57), we let $\tilde{t} := T - t$ and define

$$\widetilde{W}^*(\tilde{t}, x) := W^*(t, x), \quad \widetilde{f}^*(\tilde{t}, x) := f^*(t, x).$$

Then

$$\left(A_0 \partial_{\tilde{t}} - \sum_{i=1}^{3} A_i \partial_i + \varepsilon J \partial_1 + A_4^\mathsf{T} - \sum_{i=0}^{3} \partial_i A_i \right) \widetilde{W}^* = \widetilde{f}^* \qquad \text{if } x_1 > 0, \tag{2.58a}$$

$$\begin{aligned}&\partial_{\tilde{t}} \widetilde{w}^* - \partial_2(\mathring{v}_2 \widetilde{w}^*) - \partial_3(\mathring{v}_3 \widetilde{w}^*) + \varepsilon(\partial_2^4 + \partial_3^4) \widetilde{w}^* - \partial_1 \mathring{v} \cdot \mathring{N} \widetilde{w}^* \\ &\quad - \partial_1 \mathring{q} \widetilde{W}_2^* + \mathfrak{s} \mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'} \widetilde{W}_2^*}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'} \widetilde{W}_2^*}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) = 0 \qquad \text{if } x_1 = 0,\end{aligned} \tag{2.58b}$$

$$\widetilde{W}^*|_{\tilde{t}<0} = 0, \tag{2.58c}$$

where $\widetilde{w}^* := \widetilde{W}_1^* - \varepsilon \widetilde{W}_2^*$. Taking the scalar product of (2.58a) with $\widetilde{W}^*$ yields

$$\int_\Omega A_0 \widetilde{W}^* \cdot \widetilde{W}^*(\tilde{t}, x)\,\mathrm{d}x + \int_{\Sigma_{\tilde{t}}} (A_1 - \varepsilon J)\widetilde{W}^* \cdot \widetilde{W}^* \leq C(K)\|(\widetilde{f}^*, \widetilde{W}^*)\|^2_{L^2(\Omega_{\tilde{t}})}.$$

Since

$$(A_1 - \varepsilon J)\widetilde{W}^* \cdot \widetilde{W}^* = 2\widetilde{W}_1^* \widetilde{W}_2^* - \varepsilon(\widetilde{W}_2^*)^2 = 2\widetilde{w}^* \widetilde{W}_2^* + \varepsilon(\widetilde{W}_2^*)^2 \ \text{ if } x_1 = 0,$$

we use Cauchy's inequality to infer

$$\begin{aligned} &\|\widetilde{W}^*(\tilde{t})\|^2_{L^2(\Omega)} + \varepsilon\|\widetilde{W}_2^*\|^2_{L^2(\Sigma_{\tilde{t}})} \\ &\quad \leq C(K)\left(\|(\widetilde{f}^*, \widetilde{W}^*)\|^2_{L^2(\Omega_{\tilde{t}})} + \varepsilon^{-1}\|\widetilde{w}^*\|^2_{L^2(\Sigma_{\tilde{t}})}\right). \end{aligned} \tag{2.59}$$

Multiply the boundary condition (2.58b) by $\widetilde{w}^*$ to deduce

$$\begin{aligned} &\|\widetilde{w}^*(\tilde{t})\|^2_{L^2(\Sigma)} + 2\varepsilon\|(\partial_2^2\widetilde{w}^*, \partial_3^2\widetilde{w}^*)\|^2_{L^2(\Sigma_{\tilde{t}})} \\ &\quad \leq C(K)\|(\widetilde{w}^*, \widetilde{W}_2^*)\|^2_{L^2(\Sigma_{\tilde{t}})} + 2\mathfrak{s}\left|\int_{\Sigma_{\tilde{t}}} \widetilde{W}_2^* \mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'}\widetilde{w}^*}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\widetilde{w}^*}{|\mathring{N}|^3}\mathrm{D}_{x'}\mathring{\varphi}\right)\right| \\ &\quad \leq \epsilon\varepsilon \sum_{|\alpha|\leq 2} \|\mathrm{D}^\alpha_{x'}\widetilde{w}^*\|^2_{L^2(\Sigma_{\tilde{t}})} + C(K, \epsilon\varepsilon)\|(\widetilde{w}^*, \widetilde{W}_2^*)\|^2_{L^2(\Sigma_{\tilde{t}})} \ \text{ for all } \epsilon > 0. \end{aligned} \tag{2.60}$$

It follows from integration by parts that

$$\begin{aligned} \|\partial_2\partial_3\widetilde{w}^*\|^2_{L^2(\Sigma_{\tilde{t}})} &= \int_{\Sigma_{\tilde{t}}} \partial_2^2\widetilde{w}^* \partial_3^2\widetilde{w}^* \leq \|(\partial_2^2\widetilde{w}^*, \partial_3^2\widetilde{w}^*)\|^2_{L^2(\Sigma_{\tilde{t}})}, \\ \|\mathrm{D}_{x'}\widetilde{w}^*\|^2_{L^2(\Sigma_{\tilde{t}})} &= -\int_{\Sigma_{\tilde{t}}} \widetilde{w}^* \mathrm{D}_{x'} \cdot \mathrm{D}_{x'}\widetilde{w}^* \leq \|(\widetilde{w}^*, \mathrm{D}^2_{x'}\widetilde{w}^*)\|^2_{L^2(\Sigma_{\tilde{t}})}. \end{aligned}$$

Plugging the above estimates into (2.60) and taking $\epsilon > 0$ suitably small, we get

$$\|\widetilde{w}^*(\tilde{t})\|^2_{L^2(\Sigma)} + \varepsilon \sum_{|\alpha|\leq 2} \|\mathrm{D}^\alpha_{x'}\widetilde{w}^*\|^2_{L^2(\Sigma_{\tilde{t}})} \leq C(K, \varepsilon)\|(\widetilde{w}^*, \widetilde{W}_2^*)\|^2_{L^2(\Sigma_{\tilde{t}})}. \tag{2.61}$$

Combine (2.59) and (2.61) to derive

$$\begin{aligned} &\|\widetilde{W}^*(\tilde{t})\|^2_{L^2(\Omega)} + \|\widetilde{w}^*(\tilde{t})\|^2_{L^2(\Sigma)} + \sum_{|\alpha|\leq 2} \|(\widetilde{W}_2^*, \mathrm{D}^\alpha_{x'}\widetilde{w}^*)\|^2_{L^2(\Sigma_{\tilde{t}})} \\ &\quad \leq C(K, \varepsilon)\left(\|(\widetilde{f}^*, \widetilde{W}^*)\|^2_{L^2(\Omega_{\tilde{t}})} + \|\widetilde{w}^*\|^2_{L^2(\Sigma_{\tilde{t}})}\right). \end{aligned}$$

We apply Grönwall's inequality to the last estimate and obtain

$$\|\widetilde{W}^*(\tilde{t})\|^2_{L^2(\Omega)} + \|\widetilde{w}^*(\tilde{t})\|^2_{L^2(\Sigma)} + \sum_{|\alpha|\le 2} \|(\widetilde{W}_2^*, \mathrm{D}^\alpha_{x'}\widetilde{w}^*)\|^2_{L^2(\Sigma_{\tilde{t}})} \le C(K,\varepsilon)\|\widetilde{f}^*\|^2_{L^2(\Omega_{\tilde{t}})}.$$

As a result, for the dual problem (2.57) we obtain the following estimate:

$$\|W^*\|_{L^2(\Omega_T)} + \sum_{|\alpha|\le 2} \|(W_2^*, \mathrm{D}^\alpha_{x'} w^*)\|_{L^2(\Sigma_T)} \le C(K,\varepsilon)\|f^*\|_{L^2(\Omega_T)}. \tag{2.62}$$

Here, as in (2.55), $C(K,\varepsilon) \to +\infty$ as $\varepsilon \to 0$, but this is not important if we consider a fixed parameter $\varepsilon > 0$.

With the L^2 estimates (2.56) and (2.62) in hand, we can prove the existence and uniqueness of a weak solution $W \in L^2(\Omega_T)$ of the regularized problem (2.47) with the traces $W_{\mathrm{nc}}|_{x_1=0}$ belonging to $L^2(\Sigma_T)$ for any fixed and small parameter $\varepsilon \in (0,1)$ by the classical duality argument in [19].

We then consider (2.47b) as a fourth-order parabolic equation for ψ with the given source term $W_2|_{x_1=0} \in L^2(\Sigma_T)$ and zero initial data $\psi|_{t=0} = 0$ (*cf.* (2.47d)). Referring to [3, Theorem 5.2], we conclude that the Cauchy problem for this parabolic equation has a unique solution $\psi \in C([0,T], H^4(\mathbb{R}^2)) \bigcap C^1([0,T], L^2(\mathbb{R}^2))$ (implying that $\psi \in L^2((-\infty,T], H^4(\mathbb{R}^2))$ and $\partial_t\psi \in L^2(\Sigma_T)$). In fact, we have already obtained the *a priori* estimate for solutions ψ of this Cauchy problem in (2.56).

Therefore, we have proved the existence of a unique solution $(W,\psi) \in L^2(\Omega_T) \times L^2((-\infty,T], H^4(\mathbb{R}^2))$ of the regularized problem (2.47) for any fixed and small parameter $\varepsilon > 0$ with $W_{\mathrm{nc}}|_{x_1=0} \in L^2(\Sigma_T)$ and $\partial_t\psi \in L^2(\Sigma_T)$. Then, tangential differentiation gives the existence of a unique solution $(W,\psi) \in H^1_*(\Omega_T) \times H^1((-\infty,T], H^4(\mathbb{R}^2))$, again for any fixed and small parameter $\varepsilon > 0$. Moreover, in next subsection we will prove a uniform-in-ε *a priori* estimate for this solution.

2.3.3 Uniform estimate and passing to the limit

We are going to show the *uniform-in-ε* energy estimate for the regularized problem (2.47) in H^1_*.

• L^2 estimate of W. Substitute (2.23) into (2.51) and use (2.53) to get

$$\begin{aligned} &\|W(t)\|^2_{L^2(\Omega)} + \|(\psi, \mathrm{D}_{x'}\psi)(t)\|^2_{L^2(\Sigma)} + \varepsilon\|(W_2, \mathrm{D}^3_{x'}\psi)\|^2_{L^2(\Sigma_t)} \\ &\quad \le C(K)\Big(\|(f,W)\|^2_{L^2(\Omega_t)} + \sum_{|\alpha|\le 2} \|(\psi, \mathrm{D}_{x'}\psi, \partial_t\psi, \sqrt{\varepsilon}\mathrm{D}^\alpha_{x'}\psi)\|^2_{L^2(\Sigma_t)}\Big). \end{aligned} \tag{2.63}$$

• L^2 estimate of $\mathrm{D}_{x'}W$. Let $k \in \{0,2,3\}$. Apply the operator ∂_k to (2.47a), take the scalar product of the resulting equation with $\partial_k W$, and use (2.26) to discover

$$\int_\Omega A_0\partial_k W \cdot \partial_k W \,\mathrm{d}x + \int_{\Sigma_t} (\varepsilon J - A_1)\partial_k W \cdot \partial_k W \le C(K)\|(f,W)\|^2_{1,*,t}. \tag{2.64}$$

It follows from (2.18) that

$$
\begin{aligned}
(\varepsilon J - A_1)\partial_k W \cdot \partial_k W &= \varepsilon(\partial_k W_2)^2 - 2\partial_k W_1 \partial_k W_2 \\
&= \varepsilon(\partial_k W_2)^2 - 2\partial_k W_1 \partial_k \mathrm{B}\psi - 2\varepsilon \sum_{i=2,3} \partial_k W_1 \partial_k \partial_i^4 \psi \quad \text{on } \Sigma_T,
\end{aligned} \tag{2.65}
$$

with B being defined in (2.17b). In view of (2.47c), we have

$$
-2\varepsilon \sum_{i=2,3} \int_{\Sigma_t} \partial_k W_1 \partial_k \partial_i^4 \psi = \underbrace{-2\varepsilon \sum_{i=2,3} \int_{\Sigma_t} \partial_i \partial_k (\partial_1 \mathring{q} \psi) \partial_k \partial_i^3 \psi}_{\mathcal{J}_3^{(k)}} + \mathcal{J}_4^{(k)}, \tag{2.66}
$$

where

$$
\mathcal{J}_4^{(k)} := 2\varepsilon\mathfrak{s} \sum_{i=2,3} \int_{\Sigma_t} \partial_k \partial_i^2 \left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) \cdot \partial_k \partial_i^2 \mathrm{D}_{x'}\psi. \tag{2.67}
$$

Applying Cauchy's inequality yields

$$
\begin{aligned}
\mathcal{J}_4^{(k)} &= 2\varepsilon\mathfrak{s} \sum_{i=2,3} \int_{\Sigma_t} \left(\frac{|\partial_k \partial_i^2 \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|} - \frac{|\mathrm{D}_{x'}\mathring{\varphi} \cdot \partial_k \partial_i^2 \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} \right) \\
&\quad + 2\varepsilon\mathfrak{s} \sum_{i=2,3} \int_{\Sigma_t} [\partial_k \partial_i^2,\ h(\mathrm{D}_{x'}\mathring{\varphi})] \mathrm{D}_{x'}\psi \cdot \partial_k \partial_i^2 \mathrm{D}_{x'}\psi \\
&\geq \varepsilon\mathfrak{s} \sum_{i=2,3} \int_{\Sigma_t} \frac{|\partial_k \partial_i^2 \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} - \varepsilon C(K) \sum_{|\alpha|\leq 2} \|(\mathrm{D}_{x'}^\alpha \partial_k \psi, \mathrm{D}_{x'}^\alpha \mathrm{D}_{x'}\psi)\|_{L^2(\Sigma_t)}^2,
\end{aligned} \tag{2.68}
$$

$$
\mathcal{J}_3^{(k)} \geq -\frac{\varepsilon\mathfrak{s}}{2} \sum_{i=2,3} \int_{\Sigma_t} \frac{|\partial_k \partial_i^2 \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} - \varepsilon C(K) \sum_{|\alpha|\leq 1} \|(\mathrm{D}_{x'}^\alpha \psi, \mathrm{D}_{x'}^\alpha \partial_k \psi)\|_{L^2(\Sigma_t)}^2, \tag{2.69}
$$

where $h(\mathrm{D}_{x'}\mathring{\varphi})$ is some smooth matrix-valued function of $\mathrm{D}_{x'}\mathring{\varphi}$. Plugging (2.65) into (2.64), we use (2.29)–(2.31), (2.68)–(2.69), and (2.53) to infer

$$
\begin{aligned}
&\|\mathrm{D}_{x'} W(t)\|_{L^2(\Omega)}^2 + \|\mathrm{D}_{x'}^2 \psi(t)\|_{L^2(\Sigma)}^2 + \varepsilon \|(\mathrm{D}_{x'} W_2, \mathrm{D}_{x'}^4 \psi)\|_{L^2(\Sigma_t)}^2 \\
&\quad \leq C(K) \Bigg(\|(f, W)\|_{1,*,t}^2 + \varepsilon \sum_{|\alpha|\leq 2} \|\mathrm{D}_{x'}^\alpha \mathrm{D}_{x'}\psi\|_{L^2(\Sigma_t)}^2 + \epsilon \|\partial_t \psi\|_{L^2(\Sigma_t)}^2 \\
&\qquad + C(\epsilon) \sum_{|\alpha|\leq 2} \|(\mathrm{D}_{x'}^\alpha \psi, \mathrm{D}_{x'} \partial_t \psi)\|_{L^2(\Sigma_t)}^2 \Bigg) \quad \text{for all } \epsilon > 0.
\end{aligned} \tag{2.70}
$$

In view of (2.47b), we obtain

$$\|\partial_t\psi\|^2_{L^2(\Sigma_t)} \le C(K)\|(W_2, \psi, \mathrm{D}_{x'}\psi, \varepsilon\partial_2^4\psi, \varepsilon\partial_3^4\psi)\|^2_{L^2(\Sigma_t)}.$$

We plug the last inequality into (2.70) and take $\epsilon > 0$ sufficiently small to get

$$\begin{aligned}&\|\mathrm{D}_{x'}W(t)\|^2_{L^2(\Omega)} + \|\mathrm{D}^2_{x'}\psi(t)\|^2_{L^2(\Sigma)} + \|(\partial_t\psi, \sqrt{\varepsilon}\mathrm{D}_{x'}W_2, \sqrt{\varepsilon}\mathrm{D}^4_{x'}\psi)\|^2_{L^2(\Sigma_t)}\\ &\quad\le C(K)\Big(\|(f, W)\|^2_{1,*,t} + \sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}^\alpha_{x'}\psi, \mathrm{D}_{x'}\partial_t\psi, \sqrt{\varepsilon}\mathrm{D}^\alpha_{x'}\mathrm{D}_{x'}\psi)\|^2_{L^2(\Sigma_t)}\Big).\end{aligned} \tag{2.71}$$

• L^2 estimate of $\partial_t W$. For $k = 0$, in view of (2.18) and (2.47c), we have

$$\begin{aligned}(\varepsilon J - A_1)\partial_t W\cdot\partial_t W &= \varepsilon(\partial_t W_2)^2 + 2\partial_1\mathring{q}\,\partial_t\psi\,\partial_t W_2 + 2\partial_t\partial_1\mathring{q}\,\psi\,\partial_t W_2\\ &\quad - 2\mathfrak{s}\partial_t\mathrm{D}_{x'}\cdot\left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi}\cdot\mathrm{D}_{x'}\psi}{|\mathring{N}|^3}\mathrm{D}_{x'}\mathring{\varphi}\right)\partial_t W_2 \quad \text{on } \Sigma_T.\end{aligned}$$

Then it follows from (2.47b) that

$$\int_{\Sigma_t}(\varepsilon J - A_1)\partial_t W\cdot\partial_t W = \varepsilon\int_{\Sigma_t}(\partial_t W_2)^2 + \mathcal{J}_1 + \mathcal{J}_2 + \int_{\Sigma_t}\mathcal{T}_0 + \mathcal{J}_4^{(0)}, \tag{2.72}$$

where the terms $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{T}_0$, and $\mathcal{J}_4^{(0)}$ are defined in (2.33), (2.29), and (2.67). Using the boundary condition (2.47b) again, we get

$$\mathcal{J}_1 = \mathcal{J}_{1a} + \mathcal{J}_{1b} \underbrace{-2\varepsilon\sum_{i=2,3}\int_{\Sigma_t}\partial_1\mathring{q}\,\partial_t W_2\,\partial_i^4\psi}_{\mathcal{J}_{1c}}, \tag{2.73}$$

where the terms $\mathcal{J}_{1a}$ and $\mathcal{J}_{1b}$ are defined in (2.34). Clearly, we have

$$|\mathcal{J}_{1c}| \le \frac{\varepsilon}{2}\int_{\Sigma_t}(\partial_t W_2)^2 + \varepsilon C(K)\|(\partial_2^4\psi, \partial_3^4\psi)\|^2_{L^2(\Sigma_t)}. \tag{2.74}$$

Pugging (2.72)–(2.73) into (2.64) with $k = 0$, and utilizing (2.35)–(2.36), (2.74), (2.30)–(2.31), (2.68), (2.39), and (2.53) imply

$$\begin{aligned} &\|\partial_t W(t)\|^2_{L^2(\Omega)} + \|(W_2, \partial_t \mathrm{D}_{x'}\psi)(t)\|^2_{L^2(\Sigma)} + \varepsilon\|(\partial_t W_2, \mathrm{D}^3_{x'}\partial_t\psi)\|^2_{L^2(\Sigma_t)} \\ &\le C(K)\bigg(\epsilon\|\partial_1 W_2(t)\|^2_{L^2(\Omega)} + C(\epsilon)\|\partial_1 W_2\|^2_{L^2(\Omega_t)} \\ &\quad + C(\epsilon)\|(f, W)\|^2_{1,*,t} + \sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}^\alpha_{x'}\psi, \mathrm{D}_{x'}\partial_t\psi)\|^2_{L^2(\Sigma_t)} \\ &\quad + \|\partial_t\psi\|^2_{L^2(\Sigma_t)} + \varepsilon\sum_{|\alpha|\le 2}\|(\mathrm{D}^\alpha_{x'}\mathrm{D}_{x'}\psi, \mathrm{D}^\alpha_{x'}\partial_t\psi, \partial_2^4\psi, \partial_3^4\psi)\|^2_{L^2(\Sigma_t)}\bigg). \end{aligned} \tag{2.75}$$

• H^1_* estimate of W. To estimate the first term on the right-hand side, we use (2.47a) and (2.18) to get

$$\begin{pmatrix} \partial_1 W_2 \\ \partial_1 W_1 - \varepsilon\partial_1 W_2 \\ 0 \end{pmatrix} = f - A_4 W - \sum_{i=0,2,3} A_i\partial_i W - A_1^{(0)}\partial_1 W,$$

from which we obtain

$$\|\partial_1 W_{\mathrm{nc}}(t)\|^2_{L^2(\Omega)} \le C(K)\sum_{\langle\beta\rangle\le 1}\|\mathrm{D}^\beta_* W(t)\|^2_{L^2(\Omega)} + C(K)\|f\|^2_{1,*,t}. \tag{2.76}$$

Applying the operator $\sigma\partial_1$ to (2.47a) implies

$$\|\sigma\partial_1 W(t)\|^2_{L^2(\Omega)} \le C(K)\|(f, W)\|^2_{1,*,t}.$$

Combining the last estimate with (2.63), (2.71), (2.75)–(2.76), and

$$\|\partial_k \mathrm{D}^2_{x'}\psi\|^2_{L^2(\Sigma_t)} \le \epsilon\|\partial_k\mathrm{D}^3_{x'}\psi\|^2_{L^2(\Sigma_t)} + C(\epsilon)\|\partial_k\mathrm{D}_{x'}\psi\|^2_{L^2(\Sigma_t)}$$

for $k = 0, 2, 3$, we take $\epsilon > 0$ sufficiently small to obtain

$$\begin{aligned} &\sum_{\langle\beta\rangle\le 1}\|\mathrm{D}^\beta_* W(t)\|^2_{L^2(\Omega)} + \|\partial_1 W_2(t)\|^2_{L^2(\Omega)} + \sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}^\alpha_{x'}\psi, \mathrm{D}_{x'}\partial_t\psi)(t)\|^2_{L^2(\Sigma)} \\ &+ \|\partial_t\psi\|^2_{L^2(\Sigma_t)} + \varepsilon\sum_{2\le|\alpha|\le 3}\|(W_2, \mathrm{D}_{x'}W_2, \partial_t W_2, \mathrm{D}^\alpha_{x'}\mathrm{D}_{x'}\psi, \mathrm{D}^\alpha_{x'}\partial_t\psi)\|^2_{L^2(\Sigma_t)} \\ &\le C(K)\bigg(\|(f, W)\|^2_{1,*,t} + \|\partial_1 W_2\|^2_{L^2(\Omega_t)} + \sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}^\alpha_{x'}\psi, \mathrm{D}_{x'}\partial_t\psi)\|^2_{L^2(\Sigma_t)}\bigg). \end{aligned}$$

Then it follows from Grönwall's inequality that

$$\sum_{\langle\beta\rangle\leq 1}\|\mathrm{D}_*^{\beta}W(t)\|^2_{L^2(\Omega)}+\sum_{|\alpha|\leq 2}\|(W_2,\mathrm{D}_{x'}^{\alpha}\psi,\mathrm{D}_{x'}\partial_t\psi)(t)\|^2_{L^2(\Sigma)}+\|\partial_t\psi\|^2_{L^2(\Sigma_t)}$$
$$+\varepsilon\sum_{2\leq|\alpha|\leq 3}\|(W_2,\mathrm{D}_{x'}W_2,\partial_t W_2,\mathrm{D}_{x'}\mathrm{D}_{x'}^{\alpha}\psi,\mathrm{D}_{x'}^{\alpha}\partial_t\psi)\|^2_{L^2(\Sigma_t)}\leq C(K)\|f\|^2_{1,*,t}. \tag{2.77}$$

Combining (2.77) with (2.47d) and (2.76), we derive the *uniform-in-ε* estimate

$$\begin{aligned}&\|W\|_{1,*,T}+\|\partial_1 W_{\mathrm{nc}}\|_{L^2(\Omega_T)}+\|W_{\mathrm{nc}}\|_{L^2(\Sigma_T)}\\&+\|(\psi,\mathrm{D}_{x'}\psi)\|_{H^1(\Sigma_T)}+\sqrt{\varepsilon}\|\mathrm{D}_{x'}^4\psi\|_{L^2(\Sigma_T)}\leq C(K)\|f\|_{1,*,T},\end{aligned} \tag{2.78}$$

which allows us to construct the unique solution of the linearized problem (2.17) by passing to the limit $\varepsilon\to 0$. Indeed, due to the last estimate, we can extract a subsequence weakly convergent to $(W,\psi)\in H_*^1(\Omega_T)\times H^1((-\infty,T],H^2(\mathbb{R}^2))$ with $\partial_1 W_{\mathrm{nc}}\in L^2(\Omega_T)$ and $W_{\mathrm{nc}}|_{x_1=0}\in L^2(\Sigma_T)$. Since $\partial_1 W_2$ and $\sqrt{\varepsilon}(\partial_2^4+\partial_3^4)\psi$ are uniformly bounded in $L^2(\Omega_T)$ and $L^2(\Sigma_T)$ respectively (*cf.* (2.78)), the passage to the limit $\varepsilon\to 0$ in (2.47a)–(2.47c) verifies that (W,ψ) is a solution of the problem (2.17). Moreover, the uniqueness follows from the *a priori* estimate (2.46).

2.4 High-order energy estimates

Let us derive the high-order energy estimates for solutions of the problem (2.17). Let $m\in\mathbb{N}_+$ and $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\mathbb{N}^5$ with $\langle\alpha\rangle:=\sum_{i=0}^4\alpha_i+\alpha_4\leq m$. Apply the operator $\mathrm{D}_*^{\alpha}:=\partial_t^{\alpha_0}(\sigma\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}\partial_1^{\alpha_4}$ to (2.17a) and take the scalar product of the resulting equations with $\mathrm{D}_*^{\alpha}W$ to obtain

$$\mathcal{R}_\alpha(t)+\mathcal{Q}_\alpha(t)=\int_{\Omega}A_0\mathrm{D}_*^{\alpha}W\cdot\mathrm{D}_*^{\alpha}W(t,x)\,\mathrm{d}x\gtrsim\|\mathrm{D}_*^{\alpha}W(t)\|^2_{L^2(\Omega)}, \tag{2.79}$$

where

$$\mathcal{Q}_\alpha(t):=2\int_{\Sigma_t}\mathrm{D}_*^{\alpha}W_1\mathrm{D}_*^{\alpha}W_2, \tag{2.80}$$

$$\mathcal{R}_\alpha(t):=\int_{\Omega_t}\mathrm{D}_*^{\alpha}W\cdot\Big(2R_\alpha+\sum_{i=0}^{3}\partial_i A_i\mathrm{D}_*^{\alpha}W\Big), \tag{2.81}$$

with $R_\alpha:=\mathrm{D}_*^{\alpha}f-\mathrm{D}_*^{\alpha}(A_4W)-\sum_{i=0}^{3}[\mathrm{D}_*^{\alpha},A_i\partial_i]W$. Since the estimate of $\mathcal{R}_\alpha(t)$ does not involve any boundary condition, from [35, Lemma 3.5], we obtain

$$\sum_{\langle\alpha\rangle\leq m}\mathcal{R}_\alpha(t)\leq C(K)\mathcal{M}_1(t), \tag{2.82}$$

where

$$\mathcal{M}_1(t) := \|(f, W)\|^2_{m,*,t} + \mathring{C}_{m+4}\|(f, W)\|^2_{W^{2,\infty}_*(\Omega_t)}, \tag{2.83}$$

with $\mathring{C}_m$ being defined by (2.11) and

$$\|u\|_{W^{2,\infty}_*(\Omega_t)} := \sum_{\langle\alpha\rangle\le 1} \|D^\alpha_* u\|_{W^{1,\infty}(\Omega_t)}.$$

• *Case* $\alpha_1 > 0$. For $\alpha_1 > 0$, we have $\mathcal{Q}_\alpha(t) = 0$. Hence it follows directly from (2.79) and (2.82) that

$$\sum_{\langle\alpha\rangle\le m,\, \alpha_1>0} \|D^\alpha_* W(t)\|^2_{L^2(\Omega)} \le C(K)\mathcal{M}_1(t), \tag{2.84}$$

where $\mathcal{M}_1(t)$ is given in (2.83).
• *Case* $\alpha_1 = 0$ *and* $\alpha_4 > 0$. Next let us consider the case with $\alpha_1 = 0$ and $\alpha_4 > 0$. From (2.40), we have

$$\mathcal{Q}_\alpha(t) \lesssim \sum_{i=0,2,3} \|D^{\alpha-e}_*(f, A_4 W, A_i\partial_i W, A_1^{(0)}\partial_1 W)\|^2_{L^2(\Sigma_t)},$$

for the multi-index $e := (0, 0, 0, 0, 1)$. Then we can employ the proof of [35, Lemma 3.7] to deduce

$$\begin{aligned}&\sum_{\langle\alpha\rangle\le m,\, \alpha_1=0,\, \alpha_4>0} \|D^\alpha_* W(t)\|^2_{L^2(\Omega)}\\ &\quad\lesssim \epsilon \sum_{\langle\beta\rangle\le m} \|D^\beta_* W(t)\|^2_{L^2(\Omega)} + C(\epsilon, K)\mathcal{M}_1(t) \quad \text{for } \epsilon \in (0, 1).\end{aligned} \tag{2.85}$$

• *Case* $\alpha_1 = \alpha_4 = 0$. Now we let $\alpha_1 = \alpha_4 = 0$ and $\langle\alpha\rangle \le m$. Then $D^\alpha_* = \partial_t^{\alpha_0}\partial_2^{\alpha_2}\partial_2^{\alpha_3}$ and $\alpha_0 + \alpha_2 + \alpha_3 \le m$. Using (2.17b)–(2.17c), we calculate

$$\mathcal{Q}_\alpha(t) = \sum_{i=1}^{4} \mathcal{Q}^{(i)}_\alpha(t), \tag{2.86}$$

where

$$\mathcal{Q}_\alpha^{(1)}(t) := -2\int_{\Sigma_t} \Big\{\partial_1\mathring{q}\mathrm{D}_*^\alpha\psi + [\mathrm{D}_*^\alpha, \partial_1\mathring{q}]\psi\Big\}(\partial_t + \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3)\mathrm{D}_*^\alpha\psi,$$

$$\mathcal{Q}_\alpha^{(2)}(t) := -2\mathfrak{s}\int_{\Sigma_t} \mathrm{D}_*^\alpha\bigg(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi}\cdot\mathrm{D}_{x'}\psi}{|\mathring{N}|^3}\mathrm{D}_{x'}\mathring{\varphi}\bigg)\cdot(\partial_t + \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3)\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi,$$

$$\mathcal{Q}_\alpha^{(3)}(t) := -2\int_{\Sigma_t} \mathrm{D}_*^\alpha(\partial_1\mathring{q}\psi)\Big\{[\mathrm{D}_*^\alpha, \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3]\psi - \mathrm{D}_*^\alpha\big(\partial_1\mathring{v}\cdot\mathring{N}\psi\big)\Big\},$$

$$\mathcal{Q}_\alpha^{(4)}(t) := -2\mathfrak{s}\int_{\Sigma_t} \mathrm{D}_*^\alpha\bigg(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi}\cdot\mathrm{D}_{x'}\psi}{|\mathring{N}|^3}\mathrm{D}_{x'}\mathring{\varphi}\bigg)$$
$$\cdot\Big\{[\mathrm{D}_*^\alpha\mathrm{D}_{x'}, \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3]\psi - \mathrm{D}_*^\alpha\mathrm{D}_{x'}\big(\partial_1\mathring{v}\cdot\mathring{N}\psi\big)\Big\}.$$

Apply integration by parts, Cauchy's and Moser-type calculus inequalities to infer

$$\begin{aligned}
\mathcal{Q}_\alpha^{(1)}(t) &\le -\int_{\Sigma} \partial_1\mathring{q}(\mathrm{D}_*^\alpha\psi)^2\mathrm{d}x' - 2\int_{\Sigma}[\mathrm{D}_*^\alpha, \partial_1\mathring{q}]\psi\mathrm{D}_*^\alpha\psi + C(K)\|\mathrm{D}_*^\alpha\psi\|_{L^2(\Sigma_t)}^2\\
&\quad + \|\partial_t[\mathrm{D}_*^\alpha, \partial_1\mathring{q}]\psi\|_{L^2(\Sigma_t)}^2 + \sum_{i=2,3}\|\partial_i(\mathring{v}_i[\mathrm{D}_*^\alpha, \partial_1\mathring{q}]\psi)\|_{L^2(\Sigma_t)}^2\\
&\le C(K)\Big(\|\mathrm{D}_*^\alpha\psi(t)\|_{L^2(\Sigma)}^2 + \|\mathrm{D}_*^\alpha\psi\|_{L^2(\Sigma_t)}^2 + \|[\mathrm{D}_*^\alpha, \partial_1\mathring{q}]\psi\|_{H^1(\Sigma_t)}^2\Big)\\
&\le C(K)\Big(\|\mathrm{D}_*^\alpha\psi(t)\|_{L^2(\Sigma)}^2 + \|\psi\|_{H^m(\Sigma_t)}^2 + \mathring{\mathrm{C}}_{m+4}\|\psi\|_{L^\infty(\Sigma_t)}^2\Big).
\end{aligned}$$

It follows from the Moser-type calculus inequalities that

$$\begin{aligned}
\mathcal{Q}_\alpha^{(3)}(t) &\lesssim \|(\partial_1\mathring{q}\psi, \partial_1\mathring{v}\cdot\mathring{N}\psi)\|_{H^m(\Sigma_t)}^2 + \|[\mathrm{D}_*^\alpha, \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3]\psi\|_{L^2(\Sigma_t)}^2\\
&\le C(K)\Big(\|\psi\|_{H^m(\Sigma_t)}^2 + \mathring{\mathrm{C}}_{m+4}\|\psi\|_{L^\infty(\Sigma_t)}^2\Big),\\
\mathcal{Q}_\alpha^{(4)}(t) &\lesssim \|(\mathring{\mathrm{c}}_1\mathrm{D}_{x'}\psi, \mathrm{D}_{x'}\mathring{\mathrm{c}}_1\psi)\|_{H^m(\Sigma_t)}^2 + \|[\mathrm{D}_*^\alpha\mathrm{D}_{x'}, \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3]\psi\|_{L^2(\Sigma_t)}^2\\
&\le C(K)\Big(\|(\psi, \mathrm{D}_{x'}\psi)\|_{H^m(\Sigma_t)}^2 + \mathring{\mathrm{C}}_{m+4}\|(\psi, \mathrm{D}_{x'}\psi)\|_{L^\infty(\Sigma_t)}^2\Big).
\end{aligned}$$

For the term $\mathcal{Q}_\alpha^{(2)}(t)$, we calculate

$$\begin{aligned}
&-2\mathrm{D}_*^\alpha\bigg(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi}\cdot\mathrm{D}_{x'}\psi}{|\mathring{N}|^3}\mathrm{D}_{x'}\mathring{\varphi}\bigg)\cdot(\partial_t + \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3)\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi\\
&\quad = \partial_t\bigg\{-\frac{|\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi|^2}{|\mathring{N}|} + \frac{|\mathrm{D}_{x'}\mathring{\varphi}\cdot\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} + [\mathrm{D}_*^\alpha, \mathring{\mathrm{c}}_1]\mathrm{D}_{x'}\psi\cdot\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi\bigg\}\\
&\qquad + \sum_{i=2,3}\partial_i\bigg\{\mathring{v}_i\bigg(-\frac{|\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi|^2}{|\mathring{N}|} + \frac{|\mathrm{D}_{x'}\mathring{\varphi}\cdot\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3} + [\mathrm{D}_*^\alpha, \mathring{\mathrm{c}}_1]\mathrm{D}_{x'}\psi\cdot\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi\bigg)\bigg\}\\
&\qquad + \mathring{\mathrm{c}}_2\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi\cdot\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi - \bigg\{\partial_t[\mathrm{D}_*^\alpha, \mathring{\mathrm{c}}_1]\mathrm{D}_{x'}\psi + \sum_{i=2,3}\partial_i(\mathring{v}_i[\mathrm{D}_*^\alpha, \mathring{\mathrm{c}}_1]\mathrm{D}_{x'}\psi)\bigg\}\cdot\mathrm{D}_*^\alpha\mathrm{D}_{x'}\psi.
\end{aligned}$$

Then it follows from Cauchy's inequality, integration by parts, and Moser-type calculus inequalities that

$$\begin{aligned}\mathcal{Q}_\alpha^{(2)}(t) &\le -\mathfrak{s}\int_\Sigma \frac{|\mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3}\mathrm{d}x' + \int_\Sigma |[\mathrm{D}_*^\alpha, \mathring{\mathfrak{c}}_1]\mathrm{D}_{x'}\psi||\mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi|\mathrm{d}x' \\ &\quad + C(K)\sum_{i=2,3}\|(\partial_t[\mathrm{D}_*^\alpha, \mathring{\mathfrak{c}}_1]\mathrm{D}_{x'}\psi, \partial_i(\mathring{v}_i[\mathrm{D}_*^\alpha, \mathring{\mathfrak{c}}_1]\mathrm{D}_{x'}\psi), \mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi)\|^2_{L^2(\Sigma_t)} \\ &\le -\frac{\mathfrak{s}}{2}\int_\Sigma \frac{|\mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3}\mathrm{d}x' + C(K)\left(\|\mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi\|^2_{L^2(\Sigma_t)} + \|[\mathrm{D}_*^\alpha, \mathring{\mathfrak{c}}_1]\mathrm{D}_{x'}\psi\|^2_{H^1(\Sigma_t)}\right) \\ &\le -\frac{\mathfrak{s}}{2}\int_\Sigma \frac{|\mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3}\mathrm{d}x' + C(K)\left(\|\mathrm{D}_{x'}\psi\|^2_{H^m(\Sigma_t)} + \mathring{\mathrm{C}}_{m+4}\|\mathrm{D}_{x'}\psi\|^2_{L^\infty(\Sigma_t)}\right).\end{aligned}$$

Substituting the above estimates of $\mathcal{Q}_\alpha^{(i)}(t)$, for $i = 1, 2, 3, 4$, into (2.86), we get

$$\mathcal{Q}_\alpha(t) + \frac{\mathfrak{s}}{2}\int_\Sigma \frac{|\mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3}\mathrm{d}x' \le C(K)\Big(\|\mathrm{D}_*^\alpha\psi(t)\|^2_{L^2(\Sigma)} + \mathcal{M}_2(t)\Big) \tag{2.87}$$

for all $\alpha = (\alpha_0, 0, \alpha_2, \alpha_3, 0) \in \mathbb{N}^5$ with $|\alpha| \le m$, where

$$\mathcal{M}_2(t) := \|(\psi, \mathrm{D}_{x'}\psi)\|^2_{H^m(\Sigma_t)} + \mathring{\mathrm{C}}_{m+4}\|(\psi, \mathrm{D}_{x'}\psi)\|^2_{L^\infty(\Sigma_t)}. \tag{2.88}$$

Let us estimate the first term on the right-hand side of (2.87). If $|\alpha| \le m - 1$ or $\alpha_2 + \alpha_3 \ge 1$, then

$$\|\mathrm{D}_*^\alpha\psi(t)\|^2_{L^2(\Sigma)} \lesssim \int_{\Sigma_t} |\mathrm{D}_*^\alpha\psi||\partial_t \mathrm{D}_*^\alpha\psi| \lesssim \|(\psi, \mathrm{D}_{x'}\psi)\|^2_{H^m(\Sigma_t)}. \tag{2.89}$$

If $\alpha_2 = \alpha_3 = 0$ and $\alpha_0 = m$, then it follows from (2.17b) that

$$\mathrm{D}_*^\alpha\psi = \partial_t^{m-1}(W_2 - \mathring{v}_2\partial_2\psi - \mathring{v}_3\partial_3\psi + \partial_1\mathring{v}\cdot\mathring{N}\psi),$$

which yields

$$\begin{aligned}\|\mathrm{D}_*^\alpha\psi(t)\|^2_{L^2(\Sigma)} &\lesssim \|\partial_t^{m-1}W_2(t)\|^2_{L^2(\Sigma)} + \|\mathring{v}_2\partial_2\psi + \mathring{v}_3\partial_3\psi - \partial_1\mathring{v}\cdot\mathring{N}\psi\|^2_{H^m(\Sigma_t)} \\ &\lesssim \|\partial_t^{m-1}W_2(t)\|^2_{L^2(\Sigma)} + \mathcal{M}_2(t),\end{aligned} \tag{2.90}$$

with $\mathcal{M}_2(t)$ being given in (2.88). For the first term on the right-hand side, we use integration by parts to obtain

$$\begin{aligned}\|\partial_t^{m-1}W_2(t)\|^2_{L^2(\Sigma)} &\lesssim \epsilon\|\partial_t^{m-1}\partial_1 W_2(t)\|^2_{L^2(\Omega)} + \epsilon^{-1}\|\partial_t^{m-1}W_2(t)\|^2_{L^2(\Omega)} \\ &\lesssim \epsilon\|\partial_t^{m-1}\partial_1 W_2(t)\|^2_{L^2(\Omega)} + \epsilon^{-1}\|W_2\|^2_{m,*,t}\end{aligned} \tag{2.91}$$

for $\epsilon \in (0,1)$. From (2.40) and (2.18), we infer

$$
\begin{aligned}
\|\partial_t^{m-1}\partial_1 W_2(t)\|^2_{L^2(\Omega)} &\lesssim \sum_{\langle\beta\rangle\le m}\|\mathrm{D}_*^\beta W(t)\|^2_{L^2(\Omega)} + \|(f, A_4 W)\|^2_{m,*,t} \\
&\quad + \sum_{i=0,2,3}\|([\partial_t^{m-1}, A_i\partial_i]W, [\partial_t^{m-1}, A_1^{(0)}\partial_1]W)(t)\|^2_{L^2(\Omega)} \\
&\lesssim \sum_{\langle\beta\rangle\le m}\|\mathrm{D}_*^\beta W(t)\|^2_{L^2(\Omega)} + \mathcal{M}_1(t).
\end{aligned}
\tag{2.92}
$$

Plug (2.91)–(2.92) into (2.90) and combine the resulting estimate with (2.87)–(2.89) to deduce

$$
\begin{aligned}
&\mathcal{Q}_\alpha(t) + \|\mathrm{D}_*^\alpha\psi(t)\|^2_{L^2(\Sigma)} + \frac{\mathfrak{s}}{2}\int_\Sigma \frac{|\mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi|^2}{|\mathring{N}|^3}\,\mathrm{d}x' \\
&\quad\le C(K)\Big(\epsilon\sum_{\langle\beta\rangle\le m}\|\mathrm{D}_*^\beta W(t)\|^2_{L^2(\Omega)} + C(\epsilon)\mathcal{M}_1(t) + C(\epsilon)\mathcal{M}_2(t)\Big)
\end{aligned}
\tag{2.93}
$$

for all $\alpha \in \mathbb{N}^5$ with $\alpha_1 = \alpha_4 = 0$ and $\langle\alpha\rangle \le m$, where $\mathcal{M}_1(t)$ and $\mathcal{M}_2(t)$ are defined by (2.83) and (2.88), respectively. Substituting (2.82) and (2.93) into (2.79) leads to

$$
\begin{aligned}
&\sum_{\langle\alpha\rangle\le m,\,\alpha_1=\alpha_4=0}\Big(\|\mathrm{D}_*^\alpha W(t)\|^2_{L^2(\Omega)} + \|(\mathrm{D}_*^\alpha\psi, \mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi)(t)\|^2_{L^2(\Sigma)}\Big) \\
&\quad\le C(K)\Big(\epsilon\sum_{\langle\beta\rangle\le m}\|\mathrm{D}_*^\beta W(t)\|^2_{L^2(\Omega)} + C(\epsilon)\mathcal{M}_1(t) + C(\epsilon)\mathcal{M}_2(t)\Big)
\end{aligned}
\tag{2.94}
$$

for $\epsilon \in (0,1)$.

2.5 Proof of Theorem 2.1

Combining (2.84)–(2.85) and (2.94), we take $\epsilon > 0$ sufficiently small to derive

$$
\mathcal{I}(t) \le C(K)\mathcal{M}_1(t) + C(K)\mathcal{M}_2(t),
\tag{2.95}
$$

where $\mathcal{M}_1(t)$ and $\mathcal{M}_2(t)$ are given in (2.83) and (2.88), respectively, and

$$
\mathcal{I}(t) := \sum_{\langle\alpha\rangle\le m}\|\mathrm{D}_*^\alpha W(t)\|^2_{L^2(\Omega)} + \sum_{\langle\alpha\rangle\le m,\,\alpha_1=\alpha_4=0}\|(\mathrm{D}_*^\alpha\psi, \mathrm{D}_*^\alpha \mathrm{D}_{x'}\psi)(t)\|^2_{L^2(\Sigma)}.
$$

It follows from definition that, for $s \in \mathbb{N}$,

$$
\|\mathring{\varphi}\|_{H^s(\Sigma_t)} \le \|\mathring{\Psi}\|_{s,*,t} \lesssim \|\mathring{\varphi}\|_{H^s(\Sigma_t)},\quad \|\psi\|_{H^s(\Sigma_t)} \le \|\Psi\|_{s,*,t} \lesssim \|\psi\|_{H^s(\Sigma_t)}.
\tag{2.96}
$$

Since

$$\int_0^t \mathcal{I}(\tau)\,\mathrm{d}\tau = \|W\|^2_{m,*,t} + \|(\psi, \mathrm{D}_{x'}\psi)\|^2_{H^m(\Sigma_t)},$$

we apply Grönwall's inequality to (2.95) and infer

$$\mathcal{I}(t) \le C(K)\mathrm{e}^{C(K)T}\mathcal{N}(T) \quad \text{for } t \in [0, T], \tag{2.97}$$

where

$$\mathcal{N}(t) := \|\tilde{f}\|^2_{m,*,t} + \mathring{\mathrm{C}}_{m+4}\left(\|(\tilde{f}, W)\|^2_{W^{2,\infty}_*(\Omega_t)} + \|(\psi, \mathrm{D}_{x'}\psi)\|^2_{L^\infty(\Sigma_t)}\right).$$

It follows from the embedding inequalities (see, *e.g.*, [35, Lemma 3.3]) that

$$\mathcal{N}(T) \lesssim \|\tilde{f}\|^2_{m,*,T} + \mathring{\mathrm{C}}_{m+4}\left(\|(\tilde{f}, W)\|^2_{6,*,T} + \|\psi\|^2_{H^3(\Sigma_T)}\right). \tag{2.98}$$

Recalling the definition (2.11) for $\mathring{\mathrm{C}}_{m+4}$, we integrate (2.97) over $[0, T]$ and take $T > 0$ sufficiently small to obtain

$$\begin{aligned} &\|W\|^2_{m,*,T} + \|(\psi, \mathrm{D}_{x'}\psi)\|^2_{H^m(\Sigma_T)} \\ &\quad \le C(K)T\mathrm{e}^{C(K)T}\Big\{\|\tilde{f}\|^2_{m,*,T} \\ &\qquad + \|(\mathring{V}, \mathring{\Psi})\|^2_{m+4,*,T}\left(\|(\tilde{f}, W)\|^2_{6,*,T} + \|\psi\|^2_{H^3(\Sigma_T)}\right)\Big\} \quad \text{for } m \ge 6. \end{aligned} \tag{2.99}$$

Using (2.99) with $m = 6$, (2.12), (2.96), and the embedding $H^9_*(\Omega_T) \hookrightarrow W^{3,\infty}(\Omega_T)$, we can find a suitably small constant $T_0 > 0$, depending on K_0, such that if $0 < T \le T_0$, then

$$\|W\|^2_{6,*,T} + \|(\psi, \mathrm{D}_{x'}\psi)\|^2_{H^6(\Sigma_T)} \le C(K_0)\|\tilde{f}\|^2_{6,*,T}.$$

Substitute the last estimate into (2.99) to derive

$$\begin{aligned} &\|W\|^2_{m,*,T} + \|(\psi, \mathrm{D}_{x'}\psi)\|^2_{H^m(\Sigma_T)} \\ &\quad \le C(K_0)\left(\|\tilde{f}\|^2_{m,*,T} + \|(\mathring{V}, \mathring{\Psi})\|^2_{m+4,*,T}\|\tilde{f}\|^2_{6,*,T}\right) \quad \text{for } m \ge 6. \end{aligned} \tag{2.100}$$

In Sect. 2.3, we have proved that for $(f, g) \in H^1_*(\Omega_T) \times H^2(\Sigma_T)$ vanishing in the past, the problem (2.17) admits a unique solution $(W, \psi) \in H^1_*(\Omega_T) \times H^1(\Sigma_T)$. Applying the arguments in [3, Chapter 7] and using the energy estimate (2.100), one can establish the existence and uniqueness of solutions (W, ψ) of the problem (2.17) in $H^m_*(\Omega_T) \times H^m(\Sigma_T)$ for $m \ge 6$.

It remains to prove the tame estimates (2.13) of the problem (2.7). For this purpose, we use (2.14) to derive

$$
\begin{aligned}
\|\dot{V}\|_{m,*,T}^2 &\le C(K_0)\Big(\|W\|_{m,*,T}^2 + \|W\|_{W_*^{1,\infty}(\Omega_T)}^2 \mathring{\mathrm{C}}_{m+2} + \|g\|_{H^m(\Sigma_T)}^2\Big),\\
\|\tilde{f}\|_{m,*,T}^2 &\le C(K_0)\Big(\|f\|_{m,*,T}^2 + \|V_\natural\|_{W_*^{2,\infty}(\Omega_T)}^2 \mathring{\mathrm{C}}_{m+2} + \|g\|_{H^{m+1}(\Sigma_T)}^2\Big),
\end{aligned}
$$

which combined with (2.100), (2.96), and the embedding inequalities yield the tame estimates (2.13). The proof of Theorem 2.1 is complete.

3 Proof of Theorem 1.1

In this section, we prove the main theorem of this paper by using a Nash–Moser iteration to handle the loss of regularity from the coefficients (*i.e.*, the basic states) to the solutions in (2.13). We refer the reader to ALINHAC–GÉRARD [2, Chapter 3] and SECCHI [27] for a general description of the method.

3.1 Reducing the nonlinear problem

In order to employ Theorem 2.1, we will reformulate the nonlinear problem (1.12) into a problem with zero initial data by absorbing the initial data into the interior equations via the approximate solutions. Before this, we introduce the compatibility conditions on the initial data that are necessary in the construction of the approximate solutions.

Let $m \in \mathbb{N}$ with $m \ge 3$. Suppose that the initial data (1.12c) satisfy $\widetilde{U}_0 := U_0 - \overline{U} \in H^{m+3/2}(\Omega)$, $\varphi_0 \in H^{m+2}(\mathbb{R}^2)$, and $\|\varphi_0\|_{L^\infty(\mathbb{R}^2)} \le 1/4$. Then

$$
\partial_1 \Phi_0 \ge \frac{3}{4} \quad \text{in } \Omega, \quad \text{for } \Phi_0 := x_1 + \Psi_0, \tag{3.1}
$$

where $\Psi_0 := \chi(x_1)\varphi_0$ and $\chi \in C_0^\infty(\mathbb{R})$ satisfies (1.11). We denote the perturbation $U - \overline{U}$ by $\widetilde{U}$. Then we define $(\widetilde{U}_{(j)},\ \varphi_{(j)}) := (\partial_t^j \widetilde{U},\ \partial_t^j \varphi)|_{t=0}$ and $\Psi_{(j)} := \chi(x_1)\varphi_{(j)}$ for any integer j. Setting $\xi := \mathrm{D}_{x'}\varphi \in \mathbb{R}^2$ and $\mathcal{W} := (\widetilde{U}, \nabla\widetilde{U}, \mathrm{D}\Psi)^\mathsf{T} \in \mathbb{R}^{36}$, we can rewrite the second condition in (1.12b) and the equations (1.12a) as

$$
q = \mathfrak{s}\mathrm{D}_{x'} \cdot \mathfrak{f}(\xi), \quad \partial_t \widetilde{U} = G\big(\mathcal{W}\big), \tag{3.2}
$$

where

$$
\mathfrak{f}(\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}, \tag{3.3}
$$

and G is a suitable C^∞–function vanishing at the origin. Applying the generalized Faà di Bruno's formula (see [23, Theorem 2.1]) and the Leibniz's rule to (3.2) and the first condition in (1.12b), respectively, we compute

$$q_{(j)}|_{\Sigma} = \sum_{\substack{\alpha_k \in \mathbb{N}^2 \\ |\alpha_1|+\cdots+j|\alpha_j|=j}} \mathfrak{s}\mathrm{D}_{x'} \cdot \left(\mathrm{D}^{\alpha_1+\cdots+\alpha_j} \mathfrak{f}\big(\xi_{(0)}\big) \prod_{k=1}^{j} \frac{j!}{\alpha_k!} \left(\frac{\xi_{(k)}}{k!} \right)^{\alpha_k} \right), \tag{3.4}$$

$$\widetilde{U}_{(j+1)} = \sum_{\substack{\alpha_k \in \mathbb{N}^{35} \\ |\alpha_1|+\cdots+j|\alpha_j|=j}} \mathrm{D}^{\alpha_1+\cdots+\alpha_j} G\big(\mathcal{W}_{(0)}\big) \prod_{k=1}^{j} \frac{j!}{\alpha_k!} \left(\frac{\mathcal{W}_{(k)}}{k!} \right)^{\alpha_k}, \tag{3.5}$$

$$\varphi_{(j+1)} = v_{1(j)}\big|_{\Sigma} - \sum_{k=0}^{j} \sum_{i=2,3} \frac{j!}{(j-k)!k!} v_{i(k)}\big|_{\Sigma} \partial_i \varphi_{(j-k)}, \tag{3.6}$$

where $\xi_{(k)} := \mathrm{D}_{x'}\varphi_{(k)}$ and $\mathcal{W}_{(k)} := (\widetilde{U}_{(k)}, \nabla\widetilde{U}_{(k)}, \mathrm{D}\Psi_{(k)})^{\mathsf{T}}$. The identities (3.5)–(3.6) can determine $\widetilde{U}_{(j)}$ and $\varphi_{(j)}$ for integers j inductively. More precisely, we have the following lemma whose proof can be found from [22, Lemma 4.2.1].

Lemma 3.1 *Let $m \in \mathbb{N}$ with $m \geq 3$, $\widetilde{U}_0 := U_0 - \overline{U} \in H^{m+3/2}(\Omega)$, and $\varphi_0 \in H^{m+2}(\mathbb{R}^2)$. Then the equations (3.6) and (3.5) determine $\widetilde{U}_{(j)} \in H^{m+3/2-j}(\Omega)$ and $\varphi_{(j)} \in H^{m+2-j}(\mathbb{R}^2)$, for $j = 1, \ldots, m$, satisfying*

$$\sum_{j=0}^{m} \Big(\big\| \widetilde{U}_{(j)} \big\|_{H^{m+3/2-j}(\Omega)} + \big\| \varphi_{(j)} \big\|_{H^{m+2-j}(\mathbb{R}^2)} \Big) \leq C M_0,$$

where constant $C > 0$ depends only on m, $\|\widetilde{U}_0\|_{W^{1,\infty}(\Omega)}$, and $\|\varphi_0\|_{W^{1,\infty}(\mathbb{R}^2)}$, and

$$M_0 := \big\| \widetilde{U}_0 \big\|_{H^{m+3/2}(\Omega)} + \|\varphi_0\|_{H^{m+2}(\mathbb{R}^2)}. \tag{3.7}$$

We are now ready to introduce the compatibility conditions on the initial data.

Definition 3.1 Let $m \geq 3$ be an integer. Assume that $\widetilde{U}_0 := U_0 - \overline{U} \in H^{m+3/2}(\Omega)$ and $\varphi_0 \in H^{m+2}(\mathbb{R}^2)$ satisfy (3.1). If the functions $\widetilde{U}_{(j)}$ and $\varphi_{(j)}$ determined by (3.5)–(3.6) satisfy (3.4) for $j = 0, \ldots, m$, then we say that the initial data (U_0, φ_0) are compatible up to order m.

Imposing the above compatibility conditions on the initial data, we can construct the approximate solution in the next lemma. We omit the detailed proof since it is similar to that of [35, Lemma 4.2].

Lemma 3.2 *Let $m \geq 3$ be an integer. Assume that the initial data (U_0, φ_0) satisfy the constraint (1.16), $\widetilde{U}_0 := U_0 - \overline{U} \in H^{m+3/2}(\Omega)$, $\varphi_0 \in H^{m+2}(\mathbb{R}^2)$, and the compatibility conditions up to order m. Then we can find positive constants $T_1(M_0)$ and $C(M_0)$ (cf. (3.7)), such that if $0 < T \leq T_1(M_0)$, then there exist U^a and φ^a*

satisfying

$$\|\widetilde{U}^a\|_{H^{m+1}(\Omega_T)} + \|\varphi^a\|_{H^{m+5/2}(\Sigma_T)} \le C(M_0), \tag{3.8}$$

$$\rho(U^a) \in (\rho_*, \rho^*), \quad \partial_1 \Phi^a \ge \frac{5}{8} \quad \textit{in } \Omega_T, \tag{3.9}$$

where $\widetilde{U}^a := U^a - \overline{U}$ *and* $\Phi^a := x_1 + \Psi^a$ *with* $\Psi^a := \chi(x_1)\varphi^a$. *Moreover,*

$$\partial_t^k \mathbb{L}(U^a, \Phi^a)\big|_{t=0} = 0 \quad \textit{for } k = 0, \dots, m-1, \qquad \textit{in } \Omega, \tag{3.10}$$

$$\mathbb{B}(U^a, \varphi^a) = 0, \qquad H^a \cdot N^a = 0 \qquad \textit{on } \Sigma_T, \tag{3.11}$$

$$(U^a, \varphi^a)\big|_{t=0} = (U_0, \varphi_0), \tag{3.12}$$

$$\left(\partial_t^{\Phi^a} + v^a \cdot \nabla^{\Phi^a}\right) H^a - (H^a \cdot \nabla^{\Phi^a}) v^a + H^a \nabla^{\Phi^a} \cdot v^a = 0 \qquad \textit{in } \Omega_T,$$

where we denote $N^a := (1, -\partial_2\Psi^a, -\partial_3\Psi^a)^T$ *and* $\nabla^{\Phi} := (\partial_1^{\Phi}, \partial_2^{\Phi}, \partial_3^{\Phi})^T$ *with* ∂_t^{Φ} *and* ∂_i^{Φ} *defined by* (1.17).

The vector function (U^a, φ^a) constructed in the Lemma 3.2 is called the *approximate solution* to the problem (1.12). Let us define

$$f^a := \begin{cases} -\mathbb{L}(U^a, \Phi^a) & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{3.13}$$

Then it follows from (3.8) and (3.10) that $f^a \in H^m(\Omega_T)$ and

$$\|f^a\|_{H^m(\Omega_T)} \le \delta_0(T), \tag{3.14}$$

where $\delta_0(T) \to 0$ as $T \to 0$. By virtue of (3.10)–(3.13), we infer that $(U, \varphi) = (U^a, \varphi^a) + (V, \psi)$ solves the nonlinear problem (1.12) on $[0, T] \times \Omega$, provided V, ψ, and $\Psi := \chi(x_1)\psi$ are solutions of the problem

$$\begin{cases} \mathcal{L}(V, \Psi) := \mathbb{L}(U^a + V, \Phi^a + \Psi) - \mathbb{L}(U^a, \Phi^a) = f^a & \text{in } \Omega_T, \\ \mathcal{B}(V, \psi) := \mathbb{B}(U^a + V, \varphi^a + \psi) = 0 & \text{on } \Sigma_T, \\ (V, \psi) = 0, & \text{if } t < 0. \end{cases} \tag{3.15}$$

3.2 Nash–Moser iteration scheme

We first quote the following result from [33, Proposition 10].

Proposition 3.3 *Let* $T > 0$ *be a real number and* $m \ge 3$ *be an integer. Denote* $\mathscr{F}_*^s(\Omega_T) := \{u \in H_*^s(\Omega_T) : u = 0 \text{ for } t < 0\}$. *Then there is a family of smoothing* *operators* $\{\mathcal{S}_\theta\}_{\theta\geq 1}: \mathscr{F}_*^3(\Omega_T) \to \bigcap_{s\geq 3}\mathscr{F}_*^s(\Omega_T)$, *such that*

$$\|\mathcal{S}_\theta u\|_{k,*,T} \leq C\theta^{(k-j)_+}\|u\|_{j,*,T} \qquad \textit{for } k,\ j=1,\ldots,m, \tag{3.16a}$$
$$\|\mathcal{S}_\theta u - u\|_{k,*,T} \leq C\theta^{k-j}\|u\|_{j,*,T} \qquad \textit{for } 1\leq k\leq j\leq m, \tag{3.16b}$$
$$\left\|\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{S}_\theta u\right\|_{k,*,T} \leq C\theta^{k-j-1}\|u\|_{j,*,T} \qquad \textit{for } k,\ j=1,\ldots,m, \tag{3.16c}$$

where $k,\ j$ *are integers,* $(k-j)_+ := \max\{0,\ k-j\}$, *and the constant* C *depends only on* m*. Moreover, there exists another family of smoothing operators (still denoted by* $\mathcal{S}_\theta$*) acting on the functions defined on* Σ_T *and satisfying the properties in (3.16) with norms* $\|\cdot\|_{H^j(\Sigma_T)}$.

Let us follow [7,33,35] to describe the iteration scheme for (3.15).

Assumption (A-1) Set $(V_0, \psi_0) = 0$. Let (V_k, ψ_k) be given and vanish in the past, and set $\Psi_k := \chi(x_1)\psi_k$, for $k = 0,\ldots,n$.

We consider

$$V_{n+1} = V_n + \delta V_n, \quad \psi_{n+1} = \psi_n + \delta\psi_n, \quad \delta\Psi_n := \chi(x_1)\delta\psi_n. \tag{3.17}$$

The differences δV_n and $\delta\psi_n$ will be specified via

$$\begin{cases} \mathbb{L}_e'(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\delta\dot{V}_n = f_n & \text{in } \Omega_T,\\ \mathbb{B}_e'(U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})(\delta\dot{V}_n, \delta\psi_n) = g_n & \text{on } \Sigma_T,\\ (\delta\dot{V}_n, \delta\psi_n) = 0 & \text{for } t<0, \end{cases} \tag{3.18}$$

where $\Psi_{n+1/2} := \chi(x_1)\psi_{n+1/2}$, the good unknown $\delta\dot{V}_n$ is defined by (*cf.* (2.4))

$$\delta\dot{V}_n := \delta V_n - \frac{\partial_1(U^a + V_{n+1/2})}{\partial_1(\Phi^a + \Psi_{n+1/2})}\delta\Psi_n, \tag{3.19}$$

and $(V_{n+1/2}, \psi_{n+1/2})$ is a smooth modified state to be defined in Proposition 3.8 such that $(U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})$ satisfies (2.1)–(2.3). The source terms f_n and g_n will be chosen through the accumulated error terms at Step n later on.

Assumption (A-2) Set $f_0 := \mathcal{S}_{\theta_0} f^a$ and $(e_0, \tilde{e}_0, g_0) := 0$ for $\theta_0 \geq 1$ sufficiently large, and let $(f_k, g_k, e_k, \tilde{e}_k)$ be given and vanish in the past for $k = 1,\ldots,n-1$.

With Assumptions (A-1)–(A-2) in hand, we calculate the accumulated error terms at Step n for $n \geq 1$ by

$$E_n := \sum_{k=0}^{n-1} e_k, \quad \widetilde{E}_n := \sum_{k=0}^{n-1}\tilde{e}_k. \tag{3.20}$$

Then we compute f_n and g_n from

$$\sum_{k=0}^{n} f_k + \mathcal{S}_{\theta_n} E_n = \mathcal{S}_{\theta_n} f^a, \quad \sum_{k=0}^{n} g_k + \mathcal{S}_{\theta_n} \widetilde{E}_n = 0, \tag{3.21}$$

where $\mathcal{S}_{\theta_n}$ are the smoothing operators defined in Proposition 3.3 with $\theta_0 \geq 1$ and $\theta_n := \sqrt{\theta_0^2 + n}$. Once f_n and g_n are chosen, we can use Theorem 2.1 to obtain $(\delta \dot{V}_n, \delta\psi_n)$ from (3.18). Then we get δV_n and (V_{n+1}, ψ_{n+1}) from (3.19) and (3.17) respectively.

The error terms at Step n are defined as follows:

$$\begin{aligned}
&\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n)\\
&\quad = \mathbb{L}'(U^a + V_n, \Phi^a + \Psi_n)(\delta V_n, \delta\Psi_n) + e_n'\\
&\quad = \mathbb{L}'(U^a + \mathcal{S}_{\theta_n} V_n, \Phi^a + \mathcal{S}_{\theta_n}\Psi_n)(\delta V_n, \delta\Psi_n) + e_n' + e_n''\\
&\quad = \mathbb{L}'(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})(\delta V_n, \delta\Psi_n) + e_n' + e_n'' + e_n'''\\
&\quad = \mathbb{L}_e'(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})\delta\dot{V}_n + e_n' + e_n'' + e_n''' + D_{n+1/2}\delta\Psi_n, \qquad (3.22)\\
&\mathcal{B}(V_{n+1}, \psi_{n+1}) - \mathcal{B}(V_n, \psi_n)\\
&\quad = \mathbb{B}'(U^a + V_n, \varphi^a + \psi_n)(\delta V_n, \delta\psi_n) + \tilde{e}_n'\\
&\quad = \mathbb{B}'(U^a + \mathcal{S}_{\theta_n} V_n, \varphi^a + \mathcal{S}_{\theta_n}\psi_n)(\delta V_n, \delta\psi_n) + \tilde{e}_n' + \tilde{e}_n''\\
&\quad = \mathbb{B}_e'(U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})(\delta\dot{V}_n, \delta\psi_n) + \tilde{e}_n' + \tilde{e}_n'' + \tilde{e}_n''', \qquad (3.23)
\end{aligned}$$

where

$$D_{n+1/2} := \frac{1}{\partial_1(\Phi^a + \Psi_{n+1/2})}\partial_1\mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}) \tag{3.24}$$

is the remaining error term and we have used (2.5) to derive the last identity in (3.22). Setting

$$e_n := e_n' + e_n'' + e_n''' + D_{n+1/2}\delta\Psi_n, \quad \tilde{e}_n := \tilde{e}_n' + \tilde{e}_n'' + \tilde{e}_n''', \tag{3.25}$$

completes the description of the iterative scheme.

Let $m \geq 13$ be an integer and let $\widetilde{\alpha} := m - 5$. Assume that the initial data (U_0, φ_0) satisfy $\widetilde{U}_0 := U_0 - \overline{U} \in H^{m+3/2}(\Omega)$ and $\varphi_0 \in H^{m+2}(\mathbb{R}^2)$. By virtue of Lemma 3.2, we have

$$\|\widetilde{U}^a\|_{H^{\widetilde{\alpha}+6}(\Omega_T)} + \|\varphi^a\|_{H^{\widetilde{\alpha}+15/2}(\Sigma_T)} \leq C(M_0), \;\; \|f^a\|_{H^{\widetilde{\alpha}+5}(\Omega_T)} \leq \delta_0(T), \tag{3.26}$$

where M_0 is defined by (3.7) and $\delta_0(T) \to 0$ as $T \to 0$. Suppose further that Assumptions (A-1)–(A-2) are satisfied. Our inductive hypothesis reads

$$(\mathbf{H}_{n-1}) \begin{cases} \text{(a) } \|(\delta V_k, \delta\Psi_k)\|_{s,*,T} + \|(\delta\psi_k, \mathrm{D}_{x'}\delta\psi_k)\|_{H^s(\Sigma_T)} \le \epsilon\theta_k^{s-\alpha-1}\Delta_k \\ \qquad \text{for all } k = 0, \dots, n-1 \text{ and } s = 6, \dots, \widetilde{\alpha}; \\ \text{(b) } \|\mathcal{L}(V_k, \Psi_k) - f^a\|_{s,*,T} \le 2\epsilon\theta_k^{s-\alpha-1} \\ \qquad \text{for all } k = 0, \dots, n-1 \text{ and } s = 6, \dots, \widetilde{\alpha} - 2; \\ \text{(c) } \|\mathcal{B}(V_k, \psi_k)\|_{H^s(\Sigma_T)} \le \epsilon\theta_k^{s-\alpha-1} \\ \qquad \text{for all } k = 0, \dots, n-1 \text{ and } s = 7, \dots, \alpha, \end{cases}$$

for integer $\alpha \ge 7$, constant $\epsilon > 0$, and $\Delta_k := \theta_{k+1} - \theta_k$. We are going to show that hypothesis $(\mathbf{H}_{n-1})$ implies $(\mathbf{H}_n)$ and that $(\mathbf{H}_0)$ holds, provided $T > 0$ and $\epsilon > 0$ are small enough and $\theta_0 \ge 1$ is suitably large.

We first let hypothesis $(\mathbf{H}_{n-1})$ hold. Then we get the following lemma.

Lemma 3.4 ([33, Lemma 7]) *If θ_0 is sufficiently large, then*

$$\|(V_k, \Psi_k)\|_{s,*,T} + \|\psi_k\|_{H^s(\Sigma_T)} \le \begin{cases} \epsilon\theta_k^{(s-\alpha)_+} & \text{if } s \ne \alpha, \\ \epsilon \log\theta_k & \text{if } s = \alpha, \end{cases} \tag{3.27}$$

$$\|(I - \mathcal{S}_{\theta_k})(V_k, \Psi_k)\|_{s,*,T} + \|(I - \mathcal{S}_{\theta_k})\psi_k\|_{H^s(\Sigma_T)} \le C\epsilon\theta_k^{s-\alpha}, \tag{3.28}$$

for all $k = 0, \dots, n-1$ and $s = 6, \dots, \widetilde{\alpha}$. Moreover,

$$\|(\mathcal{S}_{\theta_k} V_k, \mathcal{S}_{\theta_k}\Psi_k)\|_{s,*,T} + \|\mathcal{S}_{\theta_k}\psi_k\|_{H^s(\Sigma_T)} \le \begin{cases} C\epsilon\theta_k^{(s-\alpha)_+} & \text{if } s \ne \alpha, \\ C\epsilon \log\theta_k & \text{if } s = \alpha, \end{cases} \tag{3.29}$$

for all $k = 0, \dots, n-1$ and $s = 6, \dots, \widetilde{\alpha} + 6$.

3.3 Error estimates

This subsection is devoted to the estimate of the quadratic error terms e_k' and $\tilde{e}_k'$, the first substitution error terms e_k'' and $\tilde{e}_k''$, the second substitution error terms e_k''' and $\tilde{e}_k'''$, and the last error term $D_{k+1/2}\delta\Psi_k$ (*cf.* (3.22)–(3.24)). First we find

$$e_k' = \int_0^1 \mathbb{L}''\big(U^a + V_k + \tau\delta V_k, \Phi^a + \Psi_k + \tau\delta\Psi_k\big)\big((\delta V_k, \delta\Psi_k), (\delta V_k, \delta\Psi_k)\big)(1-\tau)\,\mathrm{d}\tau,$$

$$\tilde{e}_k' = \int_0^1 \mathbb{B}''\big(U^a + V_k + \tau\delta V_k, \varphi^a + \psi_k + \tau\delta\psi_k\big)\big((\delta V_k, \delta\psi_k), (\delta V_k, \delta\psi_k)\big)(1-\tau)\,\mathrm{d}\tau,$$

where $\mathbb{L}''$ and $\mathbb{B}''$ are the second derivatives of the operators $\mathbb{L}$ and $\mathbb{B}$, respectively, that is,

$$\mathbb{L}''(\mathring{U}, \mathring{\Phi})\big((V, \Psi), (\widetilde{V}, \widetilde{\Psi})\big) := \frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{L}'\big(\mathring{U} + \theta \widetilde{V}, \mathring{\Phi} + \theta \widetilde{\Psi}\big)(V, \Psi)\Big|_{\theta=0},$$

$$\mathbb{B}''(\mathring{U}, \mathring{\varphi})\big((V, \psi), (\widetilde{V}, \tilde{\psi})\big) := \frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{B}'(\mathring{U} + \theta \widetilde{V}, \mathring{\varphi} + \theta \tilde{\psi})(V, \psi)\Big|_{\theta=0}.$$

For our problem (1.12), from (2.6), applying a direct calculation yields

$$\begin{aligned} &\mathbb{B}''(\mathring{U}, \mathring{\varphi})\big((V, \psi), (\widetilde{V}, \tilde{\psi})\big) \\ &= \begin{pmatrix} (\tilde{v}_2 \partial_2 + \tilde{v}_3 \partial_3)\psi + (v_2 \partial_2 + v_3 \partial_3)\tilde{\psi} \\ \mathfrak{s} \mathrm{D}_{x'} \cdot \left(\dfrac{\mathring{\zeta} \cdot \tilde{\zeta}}{|\mathring{N}|^3} \zeta - \dfrac{\tilde{\zeta} \cdot \zeta}{|\mathring{N}|^3} \mathring{\zeta} - \dfrac{\mathring{\zeta} \cdot \zeta}{|\mathring{N}|^3} \tilde{\zeta} + \dfrac{3(\mathring{\zeta} \cdot \zeta)(\mathring{\zeta} \cdot \tilde{\zeta})}{|\mathring{N}|^5} \mathring{\zeta} \right) \end{pmatrix}, \end{aligned} \tag{3.30}$$

where $\zeta := \mathrm{D}_{x'}\psi$, $\mathring{\zeta} := \mathrm{D}_{x'}\mathring{\varphi}$, $\tilde{\zeta} := \mathrm{D}_{x'}\tilde{\psi}$, and $\mathring{N} := (1, -\partial_2\mathring{\varphi}, -\partial_3\mathring{\varphi})^{\mathsf{T}}$. Using the Moser-type calculus and embedding inequalities, and omitting detailed calculations, we obtain the following estimates for the operators $\mathbb{L}''$ and $\mathbb{B}''$.

Proposition 3.5 *Let $T > 0$ be a real number and $s \geq 6$ be an integer. Suppose that $(\widetilde{V}, \widetilde{\Psi}) \in H^{s+2}_*(\Omega_T)$ and $\tilde{\varphi} \in H^{s+2}(\Sigma_T)$ satisfy*

$$\|(\widetilde{V}, \widetilde{\Psi})\|_{W^{2,\infty}_*(\Omega_T)} + \|\tilde{\varphi}\|_{W^{1,\infty}(\Sigma_T)} \leq \widetilde{K}$$

for some constant $\widetilde{K} > 0$. Then there is a constant $C(\widetilde{K}) > 0$, such that, if $(V_i, \Psi_i) \in H^{s+2}_*(\Omega_T)$ and $(W_i, \psi_i) \in H^s(\Sigma_T) \times H^{s+2}(\Sigma_T)$ for $i = 1, 2$, then

$$\begin{aligned} &\big\|\mathbb{L}''\big(\overline{U} + \widetilde{V}, x_1 + \widetilde{\Psi}\big)\big((V_1, \Psi_1), (V_2, \Psi_2)\big)\big\|_{s,*,T} \\ &\quad \leq C(\widetilde{K}) \sum_{i \neq j} \Big\{ \big\|(V_i, \Psi_i)\big\|_{6,*,T} \big\|(V_j, \Psi_j)\big\|_{s+2,*,T} \\ &\qquad + \big\|(V_1, \Psi_1)\big\|_{6,*,T} \big\|(V_2, \Psi_2)\big\|_{6,*,T} \big\|\big(\widetilde{V}, \widetilde{\Psi}\big)\big\|_{s+2,*,T} \Big\}, \end{aligned}$$

and

$$\begin{aligned} &\big\|\mathbb{B}''\big(\overline{U} + \widetilde{V}, \widetilde{\varphi}\big)\big((W_1, \psi_1), (W_2, \psi_2)\big)\big\|_{H^s(\Sigma_T)} \\ &\quad \leq C(\widetilde{K}) \sum_{i \neq j} \Big\{ \|\psi_i\|_{H^3(\Sigma_T)} \|\psi_j\|_{H^{s+2}(\Sigma_T)} + \|\psi_1\|_{H^3(\Sigma_T)} \|\psi_2\|_{H^3(\Sigma_T)} \|\tilde{\varphi}\|_{H^{s+2}(\Sigma_T)} \\ &\qquad + \|W_i\|_{H^s(\Sigma_T)} \|\psi_j\|_{H^3(\Sigma_T)} + \|W_i\|_{H^2(\Sigma_T)} \|\psi_j\|_{H^{s+1}(\Sigma_T)} \Big\}. \end{aligned}$$

We first estimate the quadratic error terms e'_k and $\tilde{e}'_k$.

Lemma 3.6 *Let $\alpha \geq 7$. If $\epsilon > 0$ is small enough and $\theta_0 \geq 1$ is sufficiently large, then*

$$\|e_k'\|_{s,*,T} + \|\tilde{e}_k'\|_{H^s(\Sigma_T)} \lesssim \epsilon^2 \theta_k^{\varsigma_1(s)-1} \Delta_k, \tag{3.31}$$

for $k = 0, \ldots, n-1$ and $s = 6, \ldots, \widetilde{\alpha} - 2$, where

$$\varsigma_1(s) := \max\{s + 6 - 2\alpha, (s + 2 - \alpha)_+ + 10 - 2\alpha\}.$$

Proof Using the embedding theorem, hypothesis ($\mathbf{H}_{n-1}$), and (3.26)–(3.27) yields

$$\|(\widetilde{U}^a, V_k, \delta V_k, \Psi^a, \Psi_k, \delta\Psi_k)\|_{W_*^{2,\infty}(\Omega_T)} + \|(\varphi^a, \psi_k, \delta\psi_k)\|_{W^{1,\infty}(\Sigma_T)} \lesssim 1.$$

So Proposition 3.5 can be applied for estimating e_k' and $\tilde{e}_k'$. Precisely, we use the trace theorem, hypothesis ($\mathbf{H}_{n-1}$), and (3.26) to obtain that for $s = 6, \ldots, \widetilde{\alpha} - 2$,

$$\begin{aligned}\|\tilde{e}_k'\|_{H^s(\Sigma_T)} &\lesssim \|\delta\psi_k\|_{H^6(\Sigma_T)} \|\delta\psi_k\|_{H^{s+2}(\Sigma_T)} + \|\delta\psi_k\|^2_{H^6(\Sigma_T)} \|(\varphi^a, \psi_k, \delta\psi_k)\|_{H^{s+2}(\Sigma_T)} \\ &\quad + \|\delta V_k\|_{s+1,*,T} \|\delta\psi_k\|_{H^6(\Sigma_T)} + \|\delta V_k\|_{6,*,T} \|\delta\psi_k\|_{H^{s+1}(\Sigma_T)} \\ &\lesssim \epsilon^2 \theta_k^{s+6-2\alpha} \Delta_k^2 + \epsilon^2 \theta_k^{10-2\alpha} \Delta_k^2 \big(1 + \|\psi_k\|_{H^{s+2}(\Sigma_T)}\big).\end{aligned}$$

If $s + 2 \neq \alpha$, then

$$\|\tilde{e}_k'\|_{H^s(\Sigma_T)} \lesssim \epsilon^2 \Delta_k^2 \big(\theta_k^{s+6-2\alpha} + \theta_k^{(s+2-\alpha)_+ + 10 - 2\alpha}\big) \lesssim \epsilon^2 \theta_k^{\varsigma_1(s)-1} \Delta_k,$$

due to (3.27) and $\Delta_k \lesssim \theta_k^{-1}$.

If $s + 2 = \alpha$, then it follows from (3.27) and $\alpha \geq 7$ that

$$\|\tilde{e}_k'\|_{H^s(\Sigma_T)} \lesssim \epsilon^2 \Delta_k^2 \big(\theta_k^{4-\alpha} + \theta_k^{11-2\alpha}\big) \lesssim \epsilon^2 \theta_k^{\varsigma_1(\alpha-2)-1} \Delta_k.$$

The estimate of e_k' can be obtained similarly, so we omit the details and finish the proof of the lemma. □

The following lemma concerns the estimate of e_k'' and $\tilde{e}_k''$ defined in (3.22)–(3.23).

Lemma 3.7 *Let $\alpha \geq 7$. If $\epsilon > 0$ is small enough and $\theta_0 \geq 1$ is sufficiently large, then*

$$\|e_k''\|_{s,*,T} + \|\tilde{e}_k''\|_{H^s(\Sigma_T)} \lesssim \epsilon^2 \theta_k^{\varsigma_2(s)-1} \Delta_k, \tag{3.32}$$

for $k = 0, \ldots, n-1$ and $s = 6, \ldots, \widetilde{\alpha} - 2$, where

$$\varsigma_2(s) := \max\{s + 8 - 2\alpha, (s + 2 - \alpha)_+ + 12 - 2\alpha\}. \tag{3.33}$$

Proof We can rewrite the term $\tilde{e}''_k$ as

$$\tilde{e}''_k = \int_0^1 \mathbb{B}''\Big(U^a + \mathcal{S}_{\theta_k} V_k + \tau(I - \mathcal{S}_{\theta_k})V_k,\ \varphi^a + \mathcal{S}_{\theta_k}\psi_k + \tau(I - \mathcal{S}_{\theta_k})\psi_k\Big)\Big(\big(\delta V_k, \delta\psi_k\big),\ \big((I - \mathcal{S}_{\theta_k})V_k, (I - \mathcal{S}_{\theta_k})\psi_k\big)\Big)\,\mathrm{d}\tau.$$

It follows from the embedding theorem, hypothesis ($\mathbf{H}_{n-1}$), and (3.26)–(3.29) that

$$\|(\widetilde{U}^a, \mathcal{S}_{\theta_k} V_k, V_k, \Psi^a, \mathcal{S}_{\theta_k}\Psi_k, \Psi_k)\|_{W^{2,\infty}_*(\Omega_T)} + \|(\varphi^a, \mathcal{S}_{\theta_k}\psi_k, \psi_k)\|_{W^{1,\infty}(\Sigma_T)} \lesssim 1.$$

Then we can employ Proposition 3.5 to infer that, for $s = 6, \dots, \widetilde{\alpha} - 2$,

$$\|\tilde{e}''_k\|_{H^s(\Sigma_T)} \lesssim \epsilon^2\theta_k^{s+7-2\alpha}\Delta_k + \epsilon^2\theta_k^{11-2\alpha}\Delta_k\big(1 + \|(\mathcal{S}_{\theta_k}\psi_k, \psi_k)\|_{H^{s+2}(\Sigma_T)}\big).$$

Analyzing the cases $s+2 \neq \alpha$ and $s+2 = \alpha$ separately as in the proof of Lemma 3.6, we utilize (3.27)–(3.29) to derive (3.32) and complete the proof. □

In order to solve (3.18), the smooth modified state $(V_{n+1/2}, \psi_{n+1/2})$ will be constructed to ensure that the constraints (2.1)–(2.3) hold for $(U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})$. For $T > 0$ sufficiently small, $(U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})$ will satisfy (2.1) and (2.3), because $(V_{n+1/2}, \psi_{n+1/2})$ will be specified to vanish in the past and (U^a, φ^a) satisfies (3.8)–(3.9) and (3.11). Hence it suffices to focus on the constraints (2.2). As a matter of fact, we can quote the following proposition in our previous paper [35], since the construction and estimate of the smooth modified state therein are independent of the second boundary condition in (1.12b).

Proposition 3.8 ([35, Proposition 4.8]) *Let $\alpha \geq 8$. Then there are $V_{n+1/2}$ and $\psi_{n+1/2}$ vanishing in the past, such that $(U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2})$ satisfies (2.2) for the approximate solution (U^a, φ^a) constructed in Lemma 3.2. Moreover,*

$$\psi_{n+1/2} = \mathcal{S}_{\theta_n}\psi_n, \quad v_{2,n+1/2} = \mathcal{S}_{\theta_n} v_{2,n}, \quad v_{3,n+1/2} = \mathcal{S}_{\theta_n} v_{3,n}, \tag{3.34}$$

$$\|\mathcal{S}_{\theta_n}\Psi_n - \Psi_{n+1/2}\|_{s,*,T} \lesssim \epsilon\theta_n^{s-\alpha} \quad \textit{for } s = 6, \dots, \widetilde{\alpha} + 6, \tag{3.35}$$

$$\|\mathcal{S}_{\theta_n} V_n - V_{n+1/2}\|_{s,*,T} \lesssim \epsilon\theta_n^{s+2-\alpha} \quad \textit{for } s = 6, \dots, \widetilde{\alpha} + 4. \tag{3.36}$$

The second substitution error term e'''_k given in (3.22) can be rewritten as

$$e'''_k = \int_0^1 \mathbb{L}''\Big(U^a + \tau(\mathcal{S}_{\theta_k} V_k - V_{k+1/2}) + V_{k+1/2},\ \Phi^a + \tau(\mathcal{S}_{\theta_k}\Psi_k - \Psi_{k+1/2}) + \Psi_{k+1/2}\Big)\Big((\delta V_k, \delta\Psi_k),\ (\mathcal{S}_{\theta_k} V_k - V_{k+1/2}, \mathcal{S}_{\theta_k}\Psi_k - \Psi_{k+1/2})\Big)\,\mathrm{d}\tau.$$

For $\tilde{e}_k'''$ defined in (3.23), we have from (3.34) and (3.30) that

$$\tilde{e}_k''' = \int_0^1 \mathbb{B}''\Big(U^a + \tau(\mathcal{S}_{\theta_k} V_k - V_{k+1/2}) + V_{k+1/2},\ \varphi^a + \psi_{k+1/2}\Big)\Big((\delta V_k, \delta\psi_k),\ (\mathcal{S}_{\theta_k} V_k - V_{k+1/2}, 0)\Big)\,\mathrm{d}\tau = 0.$$

Then we can obtain the following lemma by using Propositions 3.5 and 3.8.

Lemma 3.9 *Let $\alpha \geq 8$. If $\epsilon > 0$ is small enough and $\theta_0 \geq 1$ is sufficiently large, then*

$$\tilde{e}_k''' = 0, \quad \|e_k'''\|_{s,*,T} \lesssim \epsilon^2 \theta_k^{\varsigma_3(s)-1} \Delta_k, \tag{3.37}$$

for $k = 0, \dots, n-1$ and $s = 6, \dots, \widetilde{\alpha} - 2$, where

$$\varsigma_3(s) := \max\{s + 10 - 2\alpha, (s + 2 - \alpha)_+ + 14 - 2\alpha\}.$$

The next lemma provides the estimate of $D_{k+1/2}\delta\Psi_k$ defined by (3.24).

Lemma 3.10 ([35, Lemma 4.10]) *Let $\alpha \geq 8$ and $\widetilde{\alpha} \geq \alpha + 2$. If $\epsilon > 0$ is small enough and $\theta_0 \geq 1$ is sufficiently large, then*

$$\|D_{k+1/2}\delta\Psi_k\|_{s,*,T} \lesssim \epsilon^2 \theta_k^{\varsigma_4(s)-1} \Delta_k, \tag{3.38}$$

for $k = 0, \dots, n-1$ and $s = 6, \dots, \widetilde{\alpha} - 2$, where

$$\varsigma_4(s) := \max\{s + 12 - 2\alpha, (s - \alpha)_+ + 18 - 2\alpha\}. \tag{3.39}$$

With Lemmas 3.6–3.10 in hand, we can estimate the accumulated error terms E_n and $\widetilde{E}_n$ defined by (3.20) (also see (3.25)).

Lemma 3.11 ([35, Lemma 4.12]) *Let $\alpha \geq 12$ and $\widetilde{\alpha} = \alpha + 3$. If $\epsilon > 0$ is small enough and $\theta_0 \geq 1$ is sufficiently large, then*

$$\|E_n\|_{\alpha+1,*,T} \lesssim \epsilon^2 \theta_n, \quad \|\widetilde{E}_n\|_{H^{\alpha+1}(\Sigma_T)} \lesssim \epsilon^2. \tag{3.40}$$

3.4 Proof of existence

To derive hypothesis ($\mathbf{H}_n$) from ($\mathbf{H}_{n-1}$), we need the following estimates of f_n and g_n given in (3.21). The proof can be found in [35].

Lemma 3.12 ([35, Lemma 4.13]) *Let $\alpha \geq 12$ and $\widetilde{\alpha} = \alpha + 3$. If $\epsilon > 0$ is small enough and $\theta_0 \geq 1$ is sufficiently large, then*

$$\|f_n\|_{s,*,T} \lesssim \Delta_n\big(\theta_n^{s-\alpha-1}\|f^a\|_{\alpha,*,T} + \epsilon^2\theta_n^{s-\alpha-1} + \epsilon^2\theta_n^{\varsigma_4(s)-1}\big),$$
$$\|g_n\|_{H^{s+1}(\Sigma_T)} \lesssim \epsilon^2\Delta_n\big(\theta_n^{s-\alpha-1} + \theta_n^{\varsigma_2(s+1)-1}\big),$$

for $s = 6, \dots, \widetilde{\alpha}$, where $\varsigma_2(s)$ and $\varsigma_4(s)$ are given in (3.33) and (3.39) respectively.

The next lemma follows by using (3.35)–(3.36), the tame estimate (2.13), and Lemma 3.12. The proof is omitted here for brevity, since it is similar to that of [33, Lemma 15].

Lemma 3.13 *Let $\alpha \geq 12$ and $\widetilde{\alpha} = \alpha + 3$. If $\epsilon > 0$ and $\|f^a\|_{\alpha,*,T}/\epsilon$ are small enough, and if $\theta_0 \geq 1$ is sufficiently large, then*

$$\|(\delta V_n, \delta \Psi_n)\|_{s,*,T} + \|(\delta \psi_n, \mathrm{D}_{x'} \delta \psi_n)\|_{H^s(\Sigma_T)} \leq \epsilon \theta_n^{s-\alpha-1} \Delta_n. \tag{3.41}$$

for $s = 6, \dots, \widetilde{\alpha}$.

The above lemma provides the estimate (a) in hypothesis $(\mathbf{H}_n)$. The other estimates in $(\mathbf{H}_n)$ are given in the following lemma, whose proof is similar to that of [33, Lemma 16].

Lemma 3.14 *Let $\alpha \geq 12$ and $\widetilde{\alpha} = \alpha + 3$. If $\epsilon > 0$ and $\|f^a\|_{\alpha,*,T}/\epsilon$ are small enough, and if $\theta_0 \geq 1$ is sufficiently large, then*

$$\|\mathcal{L}(V_n, \Psi_n) - f^a\|_{s,*,T} \leq 2\epsilon \theta_n^{s-\alpha-1} \qquad \text{for } s = 6, \dots, \widetilde{\alpha} - 1, \tag{3.42}$$

$$\|\mathcal{B}(V_n, \psi_n)\|_{H^s(\Sigma_T)} \leq \epsilon \theta_n^{s-\alpha-1} \qquad \text{for } s = 7, \dots, \alpha. \tag{3.43}$$

If $\alpha \geq 12$, $\widetilde{\alpha} = \alpha + 3$, $\epsilon > 0$, and $\|f^a\|_{\alpha,*,T}/\epsilon$ are sufficiently small, and $\theta_0 \geq 1$ is large enough, then we can derive hypothesis $(\mathbf{H}_n)$ from $(\mathbf{H}_{n-1})$ in virtue of Lemmas 3.13–3.14. Fixing the constants $\alpha, \widetilde{\alpha}, \epsilon > 0$, and $\theta_0 \geq 1$, as in [33, Lemma 17], we can derive that $(\mathbf{H}_0)$ is true for a suitably small time.

Lemma 3.15 *If $T > 0$ is sufficiently small, then hypothesis $(\mathbf{H}_0)$ holds.*

Let us prove the existence of solutions to the nonlinear problem (1.12).

Proof of the existence part of Theorem 1.1 Suppose that we are given the initial data (U_0, φ_0) satisfying all the assumptions listed in Theorem 1.1. Let $\widetilde{\alpha} = m - 5$ and $\alpha = \widetilde{\alpha} - 3 \geq 12$. Then the initial data (U_0, φ_0) are compatible up to order $m = \widetilde{\alpha} + 5$. In view of (3.8) and (3.14), taking $\epsilon > 0$ and $T > 0$ sufficiently small, and $\theta_0 \geq 1$ large enough, we obtain all the requirements of Lemmas 3.13–3.15. Then, for suitably small time $T > 0$, hypothesis $(\mathbf{H}_n)$ holds for all $n \in \mathbb{N}$. In particular,

$$\sum_{n=0}^{\infty} \left(\|(\delta V_n, \delta \Psi_n)\|_{s,*,T} + \|(\delta \psi_n, \mathrm{D}_{x'} \delta \psi_n)\|_{H^s(\Sigma_T)} \right) \lesssim \sum_{n=0}^{\infty} \theta_n^{s-\alpha-2} < \infty$$

for $s = 6, \dots, \alpha - 1$. Consequently, the sequence (V_n, ψ_n) converges to some limit (V, ψ) in $H_*^{\alpha-1}(\Omega_T) \times H^{\alpha-1}(\Sigma_T)$, and also in $H^{\lfloor(\alpha-1)/2\rfloor}(\Omega_T) \times H^{\alpha-1}(\Sigma_T)$, owing to the embedding $H_*^s \hookrightarrow H^{\lfloor s/2 \rfloor}$. Moreover, we have $\mathrm{D}_{x'}\psi \in H^{\alpha-1}(\Sigma_T)$. Passing to the limit in (3.42)–(3.43) for $s = \alpha - 1 = m - 9$, we obtain (3.15). Hence $(U, \varphi) = (U^a + V, \varphi^a + \psi)$ solves the original nonlinear problem (1.12) on $[0, T]$. □

3.5 Proof of uniqueness

It remains to prove the uniqueness of solutions to the nonlinear problem (1.12). For this purpose, we assume that there exist two solutions (U, φ) and $(\mathring{U}, \mathring{\varphi})$ of the problem (1.12). Setting the differences $\widetilde{U} := U - \mathring{U}$ and $\psi := \varphi - \mathring{\varphi}$, we deduce

$$L(U, \Phi)\widetilde{U} - L(U, \Phi)\Psi \frac{\partial_1 \mathring{U}}{\partial_1 \mathring{\Phi}} = R_{\mathrm{int}} \qquad \text{in } [0, T] \times \Omega, \tag{3.44a}$$

$$(\partial_t + \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3)\psi - \tilde{v} \cdot N = 0 \qquad \text{on } [0, T] \times \Sigma, \tag{3.44b}$$

$$\tilde{q} - \mathfrak{s}\mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'}\varphi}{|N|} - \frac{\mathrm{D}_{x'}\mathring{\varphi}}{|\mathring{N}|} \right) = 0 \qquad \text{on } [0, T] \times \Sigma, \tag{3.44c}$$

and we can impose

$$(\widetilde{U}, \psi) = 0 \qquad \text{for } t < 0, \tag{3.45}$$

thanks to the trivial initial data $(\widetilde{U}, \psi)|_{t=0} = 0$. Here

$$\Phi(t, x) := x_1 + \chi(x_1)\varphi(t, x'), \qquad N := (1, -\partial_2\varphi, -\partial_3\varphi)^{\mathsf{T}},$$
$$\mathring{\Phi}(t, x) := x_1 + \chi(x_1)\mathring{\varphi}(t, x'), \qquad \mathring{N} := (1, -\partial_2\mathring{\varphi}, -\partial_3\mathring{\varphi})^{\mathsf{T}},$$

and $\Psi := \chi(x_1)\psi(t, x') = \Phi - \mathring{\Phi}$ with $C_0^\infty(\mathbb{R})$–function χ satisfying the requirements (1.11). Using the mean value theorem (see, *e.g.*, Zorich [39, Sect. 8.4.1]), we have

$$R_{\mathrm{int}} := (L(\mathring{U}, \mathring{\Phi}) - L(U, \mathring{\Phi}))\mathring{U} = \hat{a}_1\widetilde{U}, \tag{3.46}$$

where the matrix $\hat{a}_1$ depends on $\mathrm{D}\mathring{U}$, $\mathrm{D}\mathring{\Phi}$, and some 'mean value' U^* lying between U and $\mathring{U}$. Precisely, $U^* = \mathring{U} + \theta\widetilde{U}$ for some $\theta \in (0, 1)$, so that its norm can be controlled by the corresponding norms of U and $\mathring{U}$. Regarding the boundary condition (3.44c), we employ the Taylor's lemma (see [39, Sect. 8.4.4] for instance) to infer

$$\mathfrak{f}(\xi) - \mathfrak{f}(\mathring{\xi}) = (\zeta_1\partial_{\xi_1} + \zeta_2\partial_{\xi_2})\mathfrak{f}(\mathring{\xi}) + \frac{1}{2}(\zeta_1\partial_{\xi_1} + \zeta_2\partial_{\xi_2})^2\mathfrak{f}(\mathring{\xi} + \theta'\zeta) \ \text{ for } 0 < \theta' < 1,$$

where $\mathfrak{f}$ is defined by (3.3), $\xi := \mathrm{D}_{x'}\varphi$, $\mathring{\xi} := \mathrm{D}_{x'}\mathring{\varphi}$, and $\zeta := \xi - \mathring{\xi} = \mathrm{D}_{x'}\psi$. Moreover, we compute

$$\mathfrak{s}\mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'}\varphi}{|N|} - \frac{\mathrm{D}_{x'}\mathring{\varphi}}{|\mathring{N}|} \right) = \mathfrak{s}\mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi}{|\mathring{N}|^3}\mathrm{D}_{x'}\mathring{\varphi} \right) + R_{\mathrm{bdy}}, \tag{3.47}$$

where

$$R_{\mathrm{bdy}} = \sum_{i,j=2,3} \mathrm{D}_{x'} \cdot \big(\hat{b}_{ij}\psi_{x_i}\psi_{x_j}\big), \tag{3.48}$$

for generic vector-valued coefficients $\hat{b}_{ij}$ whose norms can be estimated through Sobolev's norms of the interface functions φ and $\mathring{\varphi}$.

As for the linearized problem in Sect. 2, we pass to the "good unknown" (*cf.* (2.4))

$$\dot{U} := \widetilde{U} - \frac{\partial_1 \mathring{U}}{\partial_1 \mathring{\Phi}} \Psi$$

for the difference $\widetilde{U}$ of solutions, and introduce the new unknown

$$W := J(\Phi)\dot{U},$$

where J is defined by (2.16). Taking into account (3.46)–(3.48) and omitting detailed computations, we reformulate the problem (3.44)–(3.45) into

$$\begin{aligned}
&\sum_{i=0}^{3} A_i \partial_i W + A_4 W = f && \text{in } \Omega_T, && (3.49\text{a})\\
&W_2 = (\partial_t + \mathring{v}_2\partial_2 + \mathring{v}_3\partial_3)\psi - \partial_1 \mathring{v} \cdot N\psi && \text{on } \Sigma_T, && (3.49\text{b})\\
&W_1 = -\partial_1 \mathring{q}\psi + \mathfrak{s}\mathrm{D}_{x'} \cdot \left(\frac{\mathrm{D}_{x'}\psi}{|\mathring{N}|} - \frac{\mathrm{D}_{x'}\mathring{\varphi} \cdot \mathrm{D}_{x'}\psi}{|\mathring{N}|^3} \mathrm{D}_{x'}\mathring{\varphi} \right) + R_{\mathrm{bdy}} && \text{on } \Sigma_T, && (3.49\text{c})\\
&(W, \psi) = 0 && \text{if } t < 0, && (3.49\text{d})
\end{aligned}$$

where

$$A_1 := J(\Phi)^{\mathsf{T}} \widetilde{A}_1(U, \Phi) J(\Phi), \;\; A_4 := -J(\Phi)^{\mathsf{T}} \hat{a}_1 J(\Phi), \;\; A_i := J(\Phi)^{\mathsf{T}} A_i(U) J(\Phi),$$

for $i = 0, 2, 3$, the term R_{bdy} is given in (3.48), and $f := J(\Phi)^{\mathsf{T}} \hat{a}_2 \Psi$ for some suitable matrix-valued function $\hat{a}_2$ depending on $\mathrm{D}\mathring{U}$, $\mathrm{D}\mathring{\Phi}$, and some 'mean value' U^* lying between U and $\mathring{U}$. Since $\partial_t \varphi = v \cdot N$ on the boundary Σ, we derive the identity

$$A_1\big|_{x_1=0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & O_6 \end{pmatrix},$$

which has been proved to be important in deriving the energy estimates for solutions of the linearized problem (2.17). Indeed, for the problem (3.49), we can also deduce the estimate (2.27) for $k = 0, 2, 3$. However, compared with the boundary condition (2.17c), there is one additional nonlinear (quadratic) term R_{bdy} in (3.49c). As a result, the second term in (2.27) will be decomposed into the integral of the right-hand side of (2.29) over Σ_t and the following integral:

$$\widetilde{\mathcal{J}}_k := -2 \int_{\Sigma_t} \partial_k R_{\mathrm{bdy}} \partial_k W_2 = -2 \sum_{i=2,3} \int_{\Sigma_t} \mathrm{D}_{x'} \cdot \partial_k \big(\hat{b}_{ij} \psi_{x_i} \psi_{x_j} \big) \partial_k W_2. \tag{3.50}$$

In order to close the energy estimate in H^1_*, we treat in

$$\sum_{i=2,3} \partial_k\big(\hat{b}_{ij}\psi_{x_i}\psi_{x_j}\big) = \sum_{i=2,3}(\partial_k\hat{b}_{ij}\psi_{x_i}\psi_{x_j} + \hat{b}_{ij}\partial_k\psi_{x_i}\psi_{x_j} + \hat{b}_{ij}\psi_{x_i}\partial_k\psi_{x_j}) = \hat{c}_k \mathrm{D}_{x'}\psi$$

the higher-order derivatives as coefficients whose norms can be bounded through Sobolev's norms of the solutions φ and $\mathring{\varphi}$. Then we have

$$\widetilde{\mathcal{J}}_k = 2\int_{\Sigma_t} \partial_k\left(\hat{c}_k \mathrm{D}_{x'}\psi \cdot \mathrm{D}_{x'}W_2\right) - 2\int_{\Sigma_t} \partial_k(\hat{c}_k \mathrm{D}_{x'}\psi)\cdot \mathrm{D}_{x'}W_2,$$

which combined with (3.49b) implies

$$\begin{aligned}\big|\widetilde{\mathcal{J}}_k\big| \lesssim &\sum_{|\alpha|\le 2} \|(\mathrm{D}^\alpha_{x'}\psi, \mathrm{D}_{x'}\partial_t\psi)\|^2_{L^2(\Sigma_t)} + \epsilon\|\mathrm{D}_{x'}\psi(t)\|^2_{L^2(\Sigma)}\\ &+ C(\epsilon)\sum_{|\alpha|\le 2}\|(\mathrm{D}^\alpha_{x'}\psi, \mathrm{D}_{x'}\partial_t\psi)(t)\|^2_{L^2(\Sigma)} \qquad \text{for } \epsilon > 0.\end{aligned}$$

Employing the entirely similar arguments as in Sect. 2.2, we finally derive the estimate (2.45) with $f = J(\Phi)^{\mathsf{T}}\hat{a}_2\chi(x_1)\psi$. Estimating f through ψ and using (2.44), we obtain

$$\begin{aligned}&\sum_{\langle\beta\rangle\le 1}\|\mathrm{D}^\beta_* W(t)\|^2_{L^2(\Omega)} + \sum_{|\alpha|\le 2}\|(W_2, \mathrm{D}^\alpha_{x'}\psi, \mathrm{D}_{x'}\partial_t\psi)(t)\|^2_{L^2(\Sigma)}\\ &\lesssim \|\psi\|^2_{H^1(\Sigma_t)} \lesssim \|(W_2, \psi, \mathrm{D}_{x'}\psi)\|^2_{L^2(\Sigma_t)}.\end{aligned}$$

Applying Grönwall's inequality to the last estimate leads to $W = 0$ and $\psi = 0$, which imply the uniqueness of solutions to the nonlinear problem (1.12), that is, $U = U'$ and $\varphi = \varphi'$. Therefore, the proof of Theorem 1.1 is complete.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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