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# Global Solutions to the Compressible Navier-Stokes Equations for a Reacting Mixture with Temperature Dependent Transport Coefficients

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Dedicated to Professor Gui-Qiang G. Chen on the occasion of his 60th birthday, with admiration and affection.

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**Abstract.** We consider the compressible Navier-Stokes equations for a reacting ideal polytropic gas when the coefficients of viscosity, thermal conductivity, and species diffusion are general smooth functions of temperature. The choice of temperature-dependent transport coefficients is motivated by the kinetic theory and experimental results. We establish the existence, uniqueness, and time-asymptotic behavior of global solutions for one-dimensional, spherically symmetric, or cylindrically symmetric flows under certain assumptions on the $H^2$ norm of the initial data. This is a Nishida-Smoller type global solvability result, since the initial perturbations can be large if the adiabatic exponent is close to 1.

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**Key words**: Compressible Navier-Stokes equations, reacting mixture, global large solutions, temperature dependent transport coefficients, Nishida-Smoller type result.

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# 1 Introduction

We study global well-posedness of a mathematical model governing the dynamic combustion of viscous and exothermically reacting gases with large initial data and temperature dependent transport coefficients. The motion of the gas can be described by the following compressible Navier-Stokes equations for a reacting mixture in the Eulerian coordinates (see Williams [56]):

$$\begin{cases} \rho_t+\nabla\cdot(\rho\boldsymbol{u})=0, \\ (\rho\boldsymbol{u})_t+\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})+\nabla P=\nabla\cdot\mathbb{S}, \\ (\rho E)_t+\nabla\cdot(\rho E\boldsymbol{u}+P\boldsymbol{u})=\nabla\cdot(\kappa\nabla\theta+\mathbb{S}\boldsymbol{u}+qd\rho\nabla Z), \\ (\rho Z)_t+\nabla\cdot(\rho Z\boldsymbol{u})=\nabla\cdot(d\rho\nabla Z)-K\phi(\theta)\rho Z. \end{cases} \tag{1.1}$$

Here the density $\rho$, velocity $\boldsymbol{u}\in\mathbb{R}^n$, temperature $\theta$, and reactant mass fraction $Z\in[0,1]$ are the primary unknowns of time $t\geqslant 0$ and spatial variable $\boldsymbol{x}\in\mathbb{R}^n$ with space dimension $n\geqslant 1$. The specific total energy $E$ and viscous stress tensor $\mathbb{S}$ have the form

$$E=e+\frac{1}{2}|\boldsymbol{u}|^2+qZ, \quad \mathbb{S}=\mu\big(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathrm{T}}\big)+\lambda\nabla\cdot\boldsymbol{u}\mathbb{I}_n,$$

where the constant $q$ is the heat release, $\mu>0$ and $\lambda$ are the viscosity coefficients with $2\mu+n\lambda>0$, and $\mathbb{I}_n$ is the identity matrix of order $n$. The pressure $P$, the internal energy $e$, and the transport coefficients $\kappa$ (thermal conductivity), $d$ (species diffusion), $\mu$, and $\lambda$ are prescribed through constitutive relations as functions of $\rho$ and $\theta$. The constant $K$ is the rate of reactant, while $\phi(\theta)$ denotes the reaction rate function.

For ideal polytropic gases, the thermodynamic variables satisfy the equations of state

$$e=c_v\theta, \quad P=R\rho\theta, \quad c_v=\frac{R}{\gamma-1}, \tag{1.2}$$

where $R>0$ is the gas constant and $\gamma>1$ is the adiabatic exponent. We assume that $\phi(\theta)$ is nonnegative and smooth for $\theta>0$, which typically includes the modified Arrhenius equation $\phi(\theta)=\theta^\beta \mathrm{e}^{-A/\theta}$ with $\beta$ and $A>0$ being constants [29].

Gui-Qiang Chen [2] initiated the study of global solutions to the reactive Navier-Stokes equations (1.1) in one dimension. Since then, global well-posedness for (1.1) has become an active topic of research (cf. [3,4,8,9,11,13–18,20,31, 33–36,39,42–45,50,51,54,55,57,58,60]). More precisely, for ideal polytropic gases (1.2), Chen [2] first established the global existence of large-data solutions to (1.1) in one-dimensional bounded and unbounded domains. Also see Chen *et al.* [3,4] and Hoff [18] for multi-species models with large discontinuous initial data. The time-asymptotic behavior of global large solutions is proved in [2–4, 18, 31] for one-dimensional bounded and unbounded domains, respectively. One can refer to [14, 39, 42–45, 57] for asymptotic stability on various wave patterns. For a gas model that incorporates real gas effects, Wang [51] proved the global well-posedness of large solutions to the initial-boundary value problem of (1.1). For radiative gases ($P = R\rho\theta + a\theta^4/3$ and $e = c_v\theta + a\theta^4/\rho$ with constant $a > 0$), we refer the reader to [11, 17, 35, 36] for one-dimensional global large solutions and [15,16,33,58] for asymptotic stability of wave patterns.

For multi-dimensional reactive Navier-Stokes equations (1.1), the global existence of variational solutions was established by Donatelli and Trivisa [8, 9] for radiative gases in the spirit of Feireisl [12] and Lions [37]. However, the uniqueness of solutions in [8, 9] is still unknown. As for the classical solutions, the local existence was proved in [20] for sufficiently smooth initial data, while the global existence, uniqueness, optimal time decay, and pointwise estimates were obtained recently in [13, 54, 55] for small initial perturbations. Furthermore, the global solvability of (1.1) with large initial data was achieved in [34, 50, 60] for spherically and cylindrically symmetric flows.

Our main interest concerns the influence of temperature dependence of the transport coefficients $\mu, \lambda, \kappa$, and $d$ on the global large solutions for the Eqs. (1.1)-(1.2). For concreteness, we assume that $\mu, \lambda, \kappa$, and $d$ are general smooth functions of $\theta$ satisfying

$$\mu(\theta) > 0, \quad \kappa(\theta) > 0, \quad 2\mu(\theta) + n\lambda(\theta) > 0, \quad d(\theta) > 0 \quad \text{for} \quad \theta > 0. \tag{1.3}$$

This choice of temperature dependent transport coefficients is motivated by the kinetic theory and experimental results for gases (see Williams [56, Appendix E], Zel'dovich and Raizer [59]). It is worth pointing out that, while there is a significant literature for the well-posedness of global strong solutions of (1.1) (cf. [2–4, 11, 17, 18, 31, 34–36, 50, 51, 60]), a completely satisfactory theory with temperature dependent transport coefficients (1.3) and large initial data remains open.

In this paper, we establish the existence and large-time behavior of global solutions to the Eqs. (1.1)-(1.3) for one-dimensional, spherically symmetric, or cylindrically symmetric flows in the bounded domain $\Omega = \{\boldsymbol{x} = (x_1, \cdots, x_n) : a < r < b\}$, where

- $r = x_1$ and $(\rho, \boldsymbol{u}, \theta, Z)(t, \boldsymbol{x}) = (\hat{\rho}, \hat{u}, \hat{\theta}, \widehat{Z})(t, r)$ for one-dimensional flow ($n = 1$),
- $r = |\boldsymbol{x}|, (\rho, \theta, Z)(t, \boldsymbol{x}) = (\hat{\rho}, \hat{\theta}, \widehat{Z})(t, r)$, and $\boldsymbol{u}(t, \boldsymbol{x}) = \hat{u}(t, r)\boldsymbol{x}/r$ for spherically symmetric flow ($n \geqslant 2$),

- $r=\sqrt{x_1^2+x_2^2}$, $(\rho,\theta,Z)(t,\boldsymbol{x})=(\hat{\rho},\hat{\theta},\widehat{Z})(t,r)$, and
$$\boldsymbol{u}(t,\boldsymbol{x})=\frac{\hat{u}(t,r)}{r}\begin{pmatrix}x_1\\x_2\\0\end{pmatrix}+\frac{\hat{v}(t,r)}{r}\begin{pmatrix}-x_2\\x_1\\0\end{pmatrix}+\frac{\hat{w}(t,r)}{r}\begin{pmatrix}0\\0\\1\end{pmatrix}$$
for cylindrically symmetric flow ($n=3$).

The Eqs. (1.1)-(1.3) are supplemented with the initial data

$$(\rho,\boldsymbol{u},\theta,Z)|_{t=0}=(\rho_0,\boldsymbol{u}_0,\theta_0,Z_0)(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega, \tag{1.4}$$

and the boundary conditions

$$\boldsymbol{u}=0,\quad \frac{\partial\theta}{\partial\boldsymbol{n}}=\frac{\partial Z}{\partial\boldsymbol{n}}=0\quad \text{on}\ \ \partial\Omega, \tag{1.5}$$

where $a,b>0$ are constants, $0\leqslant Z_0(\boldsymbol{x})\leqslant 1$, and $\boldsymbol{n}$ is the unit exterior normal to $\partial\Omega$. The initial data are supposed to be compatible with the boundary conditions.

Set $m=0$ and $\hat{v}=\hat{w}\equiv 0$ for the one-dimensional case, $m=n-1\geqslant 1$ and $\hat{v}=\hat{w}\equiv 0$ for the spherically symmetric case, and $m=1$ for the cylindrically symmetric case. Then the scalar functions $\hat{\rho},\hat{u},\hat{v},\hat{w},\hat{\theta}$, and $\widehat{Z}$ satisfy

$$\hat{\rho}_t+\frac{(r^m\hat{\rho}\hat{u})_r}{r^m}=0, \tag{1.6a}$$

$$\hat{\rho}(\hat{u}_t+\hat{u}\hat{u}_r)-\frac{\hat{\rho}\hat{v}^2}{r}+\widehat{P}_r=\left[\frac{\hat{\nu}(r^m\hat{u})_r}{r^m}\right]_r-\frac{2m\hat{u}\hat{\mu}_r}{r}, \tag{1.6b}$$

$$\hat{\rho}(\hat{v}_t+\hat{u}\hat{v}_r)+\frac{\hat{\rho}\hat{u}\hat{v}}{r}=(\hat{\mu}\hat{v}_r)_r+\frac{2\hat{\mu}\hat{v}_r}{r^m}-\frac{m(\hat{\mu}r^{m-1}\hat{v})_r}{r^m}-\frac{\hat{\mu}\hat{v}}{r^{2m}}, \tag{1.6c}$$

$$\hat{\rho}(\hat{w}_t+\hat{u}\hat{w}_r)=(\hat{\mu}\hat{w}_r)_r+\frac{m\hat{\mu}\hat{w}_r}{r}, \tag{1.6d}$$

$$c_v\hat{\rho}(\hat{\theta}_t+\hat{u}\hat{\theta}_r)+\widehat{P}\frac{(r^m\hat{u})_r}{r^m}=\frac{\left(\hat{\kappa}r^m\hat{\theta}_r\right)_r}{r^m}+\widehat{\mathcal{Q}}+qK\phi(\hat{\theta})\hat{\rho}\widehat{Z}, \tag{1.6e}$$

$$\hat{\rho}(\widehat{Z}_t+\hat{u}\widehat{Z}_r)=\frac{\left(\hat{d}\hat{\rho}r^m\widehat{Z}_r\right)_r}{r^m}-K\phi(\hat{\theta})\hat{\rho}\widehat{Z} \tag{1.6f}$$

together with the initial and boundary conditions

$$(\hat{\rho},\hat{u},\hat{v},\hat{w},\hat{\theta},\widehat{Z})|_{t=0}=\left(\hat{\rho}_0,\hat{u}_0,\hat{v}_0,\hat{w}_0,\hat{\theta}_0,\widehat{Z}_0\right)(r),\quad a\leqslant r\leqslant b, \tag{1.7}$$

$$\left(\hat{u},\hat{v},\hat{w},\hat{\theta}_r,\widehat{Z}_r\right)|_{r=a,b}=0,\qquad t\geqslant 0, \tag{1.8}$$

where

$$\widehat{P}=R\hat{\rho}\hat{\theta},\quad (\hat{\mu},\hat{\kappa},\hat{d})=(\mu,\kappa,d)(\hat{\theta}),\quad \hat{\nu}=\nu(\hat{\theta})=2\mu(\hat{\theta})+\lambda(\hat{\theta}),$$
$$\widehat{Q}=\frac{\hat{\nu}(r^m\hat{u})_r^2}{r^{2m}}-\frac{2m\hat{\mu}(r^{m-1}\hat{u}^2)_r}{r^m}+\hat{\mu}\hat{w}_r^2+\hat{\mu}\left(\hat{v}_r-\frac{\hat{v}}{r^m}\right)^2.$$

For global solvability, it is convenient to reformulate the problem (1.6)-(1.8) in Lagrangian coordinates. To this end, we introduce

$$(\rho,u,v,w,\theta,Z)(t,x):=\big(\hat{\rho},\hat{u},\hat{v},\hat{w},\hat{\theta},\widehat{Z}\big)(t,r),$$

where $(t,x)$ are the Lagrangian variables defined by

$$r=r(t,x):=h^{-1}(x)+\int_0^t\hat{u}\big(s,r(s,x)\big)\mathrm{d}s,\quad h(r):=\int_a^r y^m\hat{\rho}_0(y)\mathrm{d}y. \tag{1.9}$$

Assume without loss of generality that $h(b)=1$. It follows from (1.6a) that

$$r_t(t,x)=u(t,x),\quad r_x(t,x)=r^{-m}\tau(t,x), \tag{1.10}$$

where $\tau=1/\rho$ is the specific volume. Then we can reformulate the problem (1.6)-(1.8) as

$$\tau_t=(r^mu)_x, \tag{1.11a}$$
$$u_t-\frac{v^2}{r}+r^mP_x=r^m\left(\frac{\nu(r^mu)_x}{\tau}\right)_x-2mr^{m-1}u\mu_x, \tag{1.11b}$$
$$v_t+\frac{uv}{r}=r\left(\frac{\mu rv_x}{\tau}\right)_x+2\mu v_x-(\mu v)_x-\frac{\mu\tau v}{r^2}, \tag{1.11c}$$
$$w_t=r\left(\frac{\mu rw_x}{\tau}\right)_x+\mu w_x, \tag{1.11d}$$
$$c_v\theta_t+P(r^mu)_x=\left(\frac{\kappa r^{2m}\theta_x}{\tau}\right)_x+qK\phi(\theta)Z+Q, \tag{1.11e}$$
$$Z_t=\left(\frac{dr^{2m}Z_x}{\tau^2}\right)_x-K\phi(\theta)Z \tag{1.11f}$$

for $t>0$ and $x\in I:=(0,1)$, subject to the initial and boundary conditions

$$(\tau,u,v,w,\theta,Z)|_{t=0}=(\tau_0,u_0,v_0,w_0,\theta_0,Z_0), \tag{1.12}$$
$$(u,v,w,\theta_x,Z_x)|_{x=0,1}=0, \tag{1.13}$$

where $\nu := 2\mu + \lambda$ and

$$\mathcal{Q} = \frac{\nu (r^m u)_x^2}{\tau} - 2m\mu (r^{m-1} u^2)_x + \mu\tau \left( \frac{r v_x}{\tau} - \frac{v}{r} \right)^2 + \frac{\mu r^2 w_x^2}{\tau}. \tag{1.14}$$

Let us state the main result of this paper in the following theorem, which is the global well-posedness for the problem (1.11)-(1.14) with temperature dependent transport coefficients (1.3) and large initial data. It is clear that this theorem can lead to an equivalent statement for the corresponding problem (1.6)-(1.8) in the Eulerian coordinates.

**Theorem 1.1.** *Suppose that the transport coefficients satisfy* (1.3). *Let the initial data* $(\tau_0, u_0, v_0, w_0, \theta_0, Z_0)$ *be compatible with the boundary conditions* (1.13) *and satisfy*

$$\begin{aligned} &\|(\tau_0 - 1, u_0, v_0, w_0, Z_0)\|_{H^2(I)} + \|Z_0\|_{L^1(I)} + \sqrt{c_v}\|\theta_0 - 1\|_{H^1(I)} + \|\theta_{0xx}\|_{L^2(I)} \leqslant \Pi_0, \\ &V_0^{-1} \leqslant \tau_0(x) \leqslant V_0, \quad \theta_0(x) \geqslant V_0^{-1}, \quad \forall x \in I, \end{aligned} \tag{1.15}$$

*where* $\Pi_0$ *and* $V_0$ *are positive constants independent of* $\gamma - 1$. *Then there exist positive constants* $\epsilon_0$ *and* $C_1$, *which depend only on* $\Pi_0$ *and* $V_0$ *such that if* $\gamma - 1 \leqslant \epsilon_0$, *then the problem* (1.11)-(1.14) *has a unique global solution* $(\tau, u, v, w, \theta, Z) \in C([0,\infty), H^2(I))$ *satisfying*

$$\frac{1}{2} \leqslant \theta(t,x) \leqslant 2, \quad C_1^{-1} \leqslant \tau(t,x) \leqslant C_1, \quad \forall (t,x) \in [0,\infty) \times I, \tag{1.16}$$

*and the exponential decay rate*

$$\left\| \left( \tau - \bar{\tau}, u, v, w, \theta - \bar{\theta}, Z - \bar{Z} \right)(t) \right\|_{H^1(I)} \leqslant C_\gamma \mathrm{e}^{-c_\gamma t}, \quad \forall t \in [0,\infty), \tag{1.17}$$

*where* $C_\gamma$ *and* $c_\gamma$ *are positive constants depending on* $\gamma$, *and*

$$\bar{\tau} = \int_I \tau_0 \mathrm{d}x, \quad c_v \bar{\theta} + q\bar{Z} = \int_I \left( c_v \theta_0 + \frac{1}{2}\left(u_0^2 + v_0^2 + w_0^2\right) + qZ_0 \right) \mathrm{d}x, \quad \phi(\bar{\theta})\bar{Z} = 0.$$

**Remark 1.1.** The initial perturbations can be large if the adiabatic exponent $\gamma$ is sufficiently close to 1. Hence, Theorem 1.1 provides a Nishida-Smoller type global solvability result for the Eqs. (1.11) with temperature dependent transport coefficients (1.3). See [41] for the corresponding original Nishida-Smoller type global existence result for one-dimensional isentropic compressible Euler equations.

**Remark 1.2.** We can apply the arguments in this paper to deduce corresponding well-posedness results for both the initial boundary value problem (1.11)-(1.12) with boundary conditions $(u,v,w,\theta,Z)|_{x=0,1}=(0,0,0,1,0)$ and the one-dimensional Cauchy problem (1.11)-(1.12) in the whole space $\mathbb{R}$ with the far-field condition

$$\lim_{x\to\pm\infty}(\tau_0,u_0,v_0,w_0,\theta_0,Z_0)(x)=(1,0,0,0,1,0).$$

However, global existence of large solutions to (1.11) with $m\geqslant 1$ (symmetric case) and (1.3) is still unknown for unbounded domains. See [50] for the result on the symmetric flows with constant transport coefficients in unbounded domains.

**Remark 1.3.** Only $H^2$ regularity of the initial perturbations is required in Theorem 1.1, and thus we improve the results in [38,48] for which the initial perturbations need to be in $H^3$.

We prove Theorem 1.1 in Section 2 by first establishing certain a priori energy estimates and then combining these estimates with the continuation argument to conclude the proof of the theorem. As shown in [2–4, 31, 50], the crucial step to construct global large solutions of the reactive Navier-Stokes equations (1.11) is to deduce the positive upper and lower bounds of the specific volume and the temperature. In the one-dimensional case ($m=v=w=0$) with constant viscosity, the pointwise bounds are achieved by employing an elaborate representation of the specific volume and the maximum principle in Kazhikhov *et al.* [1,27,28] for non-reacting flows and in Chen *et al.* [2–4] for reacting flows. See [22, 23, 30, 31, 36, 52] for the bounds uniformly in space and time over unbounded domains, and [21,32,34,40,49,50,60] for the spherically and cylindrically symmetric cases. In the one-dimensional case when the viscosity $\nu$ depends only on the specific volume $\tau$, one can employ the identity

$$\left(\frac{\nu(\tau)\tau_x}{\tau}-u\right)_t=P_x$$

observed by Kanel [25] to deduce global existence of large solutions of the compressible (reactive) Navier-Stokes equations for certain types of density-dependent viscosity, see, for instance, [5–7,26,35,47,51].

However, the above methodologies seem not valid for the case with temperature dependent viscosity, where the corresponding identity becomes

$$\left(\frac{\nu\tau_x}{\tau}-u\right)_t=P_x+\frac{\partial\nu}{\partial\theta}\frac{\tau_x\theta_t-\tau_t\theta_x}{\tau}. \tag{1.18}$$

The temperature dependence of the viscosity has turned out to have a strong influence on the solutions and lead to difficulty in mathematical analysis for global solvability with large data (cf. [19, 24]). One of the main difficulties in analysis arises from the last highly nonlinear term in (1.18). Nevertheless, some progress on the global existence has been made for ideal polytropic gases (1.2) recently.

Liu *et al.* [38] observed that $\|(\theta-1,\theta_t,\theta_x)\|_{L^\infty}$ can be small if the adiabatic exponent $\gamma$ is close to 1. Thanks to this observation, global large solutions to one-dimensional compressible Navier-Stokes equations with general temperature-dependent transport coefficients are constructed in [38] under certain assumptions on the $H^3$ norm of the initial perturbations. See Wan and Wang [48] for the corresponding result in the spherically and cylindrically symmetric cases. For the general adiabatic exponent case with the transport coefficients being proportional to $h(\tau)\theta^\alpha$, the global well-posedness of large solutions is obtained in [53] for certain non-degenerate function $h$ and in [46] for $h\equiv 0$, provided that the constant $|\alpha|$ is sufficiently small.

In this paper, we prove a Nishida-Smoller type global solvability result for the one-dimensional, spherically symmetric, or cylindrically symmetric reacting flows by a modification of the analysis in [38,48]. In particular, we show that the smallness of $\|\theta_t(t)\|_{L^2}$ and the $H^2$ regularity of the solutions is enough for deriving the estimate of $\|\tau_x/\tau(t)\|_{L^2}$ (cf. Lemma 2.3) and the pointwise bounds of the specific volume $\tau$ (cf. Lemma 2.4). Therefore, we refine the results in [38,48] that require the smallness of $\|\theta_t(t)\|_{L^\infty}$ and the $H^3$ regularity of the solutions. We refer to [10] for applying a similar argument to study the large-behavior for the one-dimensional compressible Navier-Stokes equations with temperature dependent transport coefficients.

## 2 Energy estimates

In this section, we shall prove Theorem 1.1 by establishing suitable a priori energy estimates of solutions $(\tau,u,v,w,\theta,Z)\in X(0,T;M,N)$ to the problem (1.11)-(1.13) for $T>0$ and $M,N>1$. The solution space is defined by

$$\begin{aligned}X(0,T;M,N):=\Big\{&(\tau-1,u,v,w,\theta-1,Z)\in C\big([0,T];H^2(I)\big):\\&\tau_x\in L^2\big(0,T;H^1(I)\big),(u_x,v_x,w_x,\theta_x,Z_x)\in L^2\big(0,T;H^2(I)\big),\\&\mathcal{E}(T)\leqslant N^2,\tau(t,x)\geqslant M^{-1},\forall (t,x)\in[0,T]\times I\Big\},\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}(T):=&\sup_{0\leqslant t\leqslant T}\big\|\big(\tau-1,u,v,w,\sqrt{c_v}(\theta-1)\big)(t)\big\|_{H^1}^2\\&+\sup_{0\leqslant t\leqslant T}\|(c_v\theta_t,\theta_{xx})(t)\|^2+\int_0^T\big\|\big(\sqrt{c_v}\theta_t,\tau_x\big)(s)\big\|_{H^1}^2\mathrm{d}s\leqslant N^2.\end{aligned}\tag{2.1}$$

Here and below, for notational simplicity, we use

$$\|\cdot\| := \|\cdot\|_{L^2(I)}, \quad \|\cdot\|_{H^k} := \|\cdot\|_{H^k(I)}, \quad \|\cdot\|_{L^q} := \|\cdot\|_{L^q(I)}.$$

From Sobolev's inequality, we have the a priori assumptions:

$$\|\theta_t(t)\| + \|(\theta-1)(t)\|_{H^1} + \|(\theta-1)(t)\|_{L^\infty} \lesssim (\gamma-1)^{\frac{1}{2}} N, \tag{2.2}$$

$$\|\theta_x(t)\|_{L^\infty} \lesssim (\gamma-1)^{\frac{1}{4}} N, \quad \int_0^t \|\theta_t(s)\|_{H^1}^2 \mathrm{d}s \lesssim (\gamma-1) N^2, \tag{2.3}$$

$$M^{-1} \leqslant \tau(t,x) \lesssim N, \quad \forall (t,x) \in [0,T] \times I. \tag{2.4}$$

For the (spherically or cylindrically) symmetric case that corresponds to $m \geqslant 1$, we will make repeated use of the bounds

$$a \leqslant r(t,x) \leqslant b, \quad \forall (t,x) \in [0,T] \times I, \tag{2.5}$$

which comes from (1.9)-(1.10).

We can deduce the following lemma by utilizing integration by parts and the maximum principle (cf. [2, Lemmas 1-2]).

**Lemma 2.1.** *Suppose that the conditions in Theorem* 1.1 *are satisfied. Then*

$$\|Z(t)\|_{L^1} + \int_0^t \int_I K\phi(\theta) Z = \|Z_0\|_{L^1}, \tag{2.6}$$

$$\|Z(t)\|^2 + 2\int_0^t \int_I \left( \frac{d r^{2m} Z_x^2}{\tau^2} + K\phi(\theta) Z^2 \right) = \|Z_0\|^2, \tag{2.7}$$

$$0 \leqslant Z(t,x) \leqslant 1, \quad \forall (t,x) \in [0,T] \times I. \tag{2.8}$$

Multiplying (1.11a)-(1.11e) by $1-\tau^{-1}, u, v, w$, and $1-\theta^{-1}$, respectively, and using the identity (2.6), we can follow the proof of [48, Lemma 2.1] to obtain the entropy-type energy estimate in the next lemma.

**Lemma 2.2.** *Under the conditions of Theorem* 1.1*, there is a positive constant $\epsilon_1 > 0$ depending only on $\Pi_0$ and $V_0$, such that if*

$$(\gamma-1)(MN)^{60} \leqslant \epsilon_1, \tag{2.9}$$

*then*

$$\frac{1}{2} \leqslant \theta(t,x) \leqslant 2, \quad \forall (t,x) \in [0,T] \times I, \tag{2.10}$$

$$\sup_{0 \leqslant t \leqslant T} \|\eta(t)\|_{L^1} + \int_0^T V(t) \mathrm{d}t \lesssim \left\| \left(\tau_0 - 1, u_0, v_0, w_0, \sqrt{c_v}(\theta_0 - 1)\right) \right\|^2 + \|Z_0\|_{L^1}, \tag{2.11}$$

*where*

$$\eta := R\psi(\tau) + \frac{u^2+v^2+w^2}{2} + c_v\psi(\theta), \quad \psi(z) := z - \ln z - 1,$$
$$V(t) := \int_I \left( m\tau u^2 + \tau v^2 + \frac{\tau_t^2 + u_x^2 + v_x^2 + w_x^2 + \theta_x^2}{\tau} \right) \mathrm{d}x.$$

To obtain pointwise bounds for the specific volume $\tau$, we make the estimate of $\|\tau_x/\tau\|$ in the following lemma.

**Lemma 2.3.** *Suppose that the conditions in Theorem* 1.1 *are satisfied. If* (2.9) *holds for a sufficiently small* $\epsilon_1 > 0$*, then*

$$\sup_{t\in[0,T]} \left\| \frac{\tau_x}{\tau}(t) \right\|^2 + \int_0^T \int_I \frac{\tau_x^2}{\tau^3} \lesssim 1 + \|\ln \tau\|_{L^\infty([0,T]\times I)}. \tag{2.12}$$

*Proof.* Using the chain rule and Eq. (1.11a) yields

$$\left( \frac{\nu \tau_x}{\tau} \right)_t = \left( \frac{\nu (r^m u)_x}{\tau} \right)_x + \frac{\nu'(\theta)}{\tau} (\tau_x \theta_t - \tau_t \theta_x). \tag{2.13}$$

Substitute the above identity into (1.11b) and multiply the resulting equation by $\nu\tau_x/\tau$ to discover

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_I \left( \frac{1}{2} \left( \frac{\nu\tau_x}{\tau} \right)^2 - \frac{u}{r^m} \frac{\nu\tau_x}{\tau} \right) + \int_I \frac{\nu\theta\tau_x^2}{\tau^3} = \sum_{q=1}^{7} \mathcal{K}_q, \tag{2.14}$$

where

$$\mathcal{K}_1 := \int_I \left( \frac{u}{r^m} \right)_x \frac{\nu\tau_t}{\tau}, \qquad \mathcal{K}_2 := \int_I \frac{\nu\tau_x\theta_x}{\tau^2}, \qquad \mathcal{K}_3 := \int_I \frac{\nu\tau_x}{\tau} \frac{mu^2 - v^2}{r^{m+1}},$$
$$\mathcal{K}_4 := \int_I \frac{\nu'(\theta)}{\tau} r^{-m} u \tau_t \theta_x, \quad \mathcal{K}_5 := \int_I \frac{\nu'(\theta)\nu}{\tau^2} \theta_t \tau_x^2,$$
$$\mathcal{K}_6 := \int_I \frac{\tau_x}{\tau} \theta_x \left( 2mr^{-1} u \mu'(\theta) \nu - \nu\nu'(\theta)\tau^{-1}\tau_t \right), \qquad \mathcal{K}_7 := -\int_I \frac{\nu'(\theta)}{\tau} r^{-m} u \tau_x \theta_t.$$

The idea in [38, 48] for controlling $\mathcal{K}_5$ is to use the smallness of $\|\theta_t\|_{L^\infty([0,T]\times I)}$, which requires the $H^3$ regularity of the solutions and initial data. In this paper, we estimate the term $\mathcal{K}_5$ in a different way, which requires only the smallness of $\|\theta_t(t)\|$ and the $H^2$ regularity of the solutions, see [10] for a similar argument.

More precisely, it follows from the a priori assumptions (2.1)-(2.4) and Young's inequality that

$$\begin{aligned}\int_0^T \mathcal{K}_5 \mathrm{d}t &\lesssim M^2 \int_0^T \|\tau_x(t)\|_{L^\infty} \|\theta_t(t)\| \|\tau_x(t)\| \mathrm{d}t \\ &\lesssim M^2 \sup_{0\leqslant t\leqslant T} \|\theta_t(t)\| \int_0^T \|\tau_x(t)\|^{\frac{3}{2}} \|\tau_{xx}(t)\|^{\frac{1}{2}} \mathrm{d}t \\ &\lesssim M^2 \sup_{0\leqslant t\leqslant T} \|\theta_t(t)\| \int_0^T \|\tau_x(t)\|_{H^1}^2 \mathrm{d}t \lesssim (\gamma-1)^{\frac{1}{2}} M^2 N^3.\end{aligned}$$

The other terms $\mathcal{K}_q$ ($q\neq 5$) in (2.14) can be estimated as in [48, Lemma 2.2] under the assumptions (2.1)-(2.4) and (2.9). Hence, we can conclude the proof of this lemma. □

Lemmas 2.2-2.3 enable us to apply the argument developed by Kanel [25,38] to establish the uniform positive bounds for the specific volume $\tau$. We refer to [48, Lemma 2.3] for the details of the proof.

**Lemma 2.4.** *Suppose that the conditions in Theorem* 1.1 *are satisfied. If* (2.9) *holds for a sufficiently small* $\epsilon_1 > 0$*, then*

$$\tau(t,x) \sim 1, \quad \forall (t,x) \in [0,T] \times I. \tag{2.15}$$

Let us make the estimates for the remaining first-order derivatives.

**Lemma 2.5.** *Suppose that the conditions in Theorem* 1.1 *are satisfied. If* (2.9) *holds for a sufficiently small* $\epsilon_1 > 0$*, then*

$$\sup_{0\leqslant t\leqslant T} \|(u_x, v_x, w_x, \sqrt{c_v}\theta_x, Z_x)(t)\|^2 + \int_0^T \|(u_{xx}, v_{xx}, w_{xx}, \theta_{xx}, Z_{xx})(t)\|^2 \mathrm{d}t \lesssim 1. \tag{2.16}$$

*Proof.* Multiplying (1.11b), (1.11c), (1.11d), and (1.11f) by $u_{xx}, v_{xx}, w_{xx}$, and $z_{xx}$, respectively, and following the proof of [48, Lemmas 2.5-2.6], we can use the assumptions (2.1)-(2.4) and Lemmas 2.1-2.4 to derive

$$\sup_{0\leqslant t\leqslant T} \|(u_x, v_x, w_x, Z_x)(t)\|^2 + \int_0^T \|(u_{xx}, v_{xx}, w_{xx}, Z_{xx})(t)\|^2 \mathrm{d}t \lesssim 1.$$

Multiply (1.11e) by $\theta_{xx}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\sqrt{c_v}\theta_x(t)\|^2 + 2\int_I \frac{\kappa r^{2m} \theta_{xx}^2}{\tau} = \sum_{q=13}^{16} \mathcal{K}_q,$$

where

$$\begin{aligned}
&\mathcal{K}_{13}:=\int_I 2\theta_{xx}\left(\frac{\theta(r^m u)_x}{\tau}-\frac{\nu(r^m u)_x^2}{\tau}+2m\mu(r^{m-1}u^2)_x\right),\\
&\mathcal{K}_{14}:=\int_I 2\theta_{xx}\left(-\frac{\mu r^{2m}w_x^2}{\tau}-\mu\tau\left(\frac{r^m v_x}{\tau}-\frac{v}{r^m}\right)^2\right),\\
&\mathcal{K}_{15}:=\int_I 2\theta_{xx}\left(\frac{\kappa r^{2m}\theta_{xx}}{\tau}-\left(\frac{\kappa r^{2m}\theta_x}{\tau}\right)_x\right),\\
&\mathcal{K}_{16}:=\int_I 2\theta_{xx}qK\phi(\theta)Z.
\end{aligned}$$

It follows from (2.7) that

$$\int_0^T \mathcal{K}_{16}\leqslant\epsilon\int_0^T\int_I\theta_{xx}^2+C(\epsilon)\int_0^T\int_I\phi(\theta)^2Z^2\leqslant\epsilon\int_0^T\int_I\theta_{xx}^2+C(\epsilon).$$

As in [48, Lemma 2.7], we can control the terms $\mathcal{K}_q$ $(q=13,14,15)$ and thus complete the proof of this lemma. □

The next lemma follows directly from the identities (1.10), Eqs. (1.11), and Lemmas 2.1-2.5 (cf. [48, Lemma 2.8]).

**Lemma 2.6.** *Suppose that the conditions in Theorem* 1.1 *are satisfied. If* (2.9) *holds for a sufficiently small* $\epsilon_1>0$, *then*

$$\sup_{0\leqslant t\leqslant T}\|\tau_t(t)\|^2+\int_0^T\|(\tau_{xt},u_t,v_t,w_t,c_v\theta_t,Z_t)(t)\|^2\mathrm{d}t\lesssim 1. \tag{2.17}$$

Let us derive the uniform $L^2$ bounds for the second-order derivatives. Under the assumptions (2.1)-(2.4), we can use the similar arguments as in [48, Lemmas 2.9-2.12] to establish the following result. We omit the proof for brevity.

**Lemma 2.7.** *Suppose that the conditions in Theorem* 1.1 *are satisfied. If* (2.9) *holds for a sufficiently small* $\epsilon_1>0$, *then*

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\|(u_t,v_t,w_t,c_v\theta_t,Z_t,\tau_{xt},u_{xx},v_{xx},w_{xx},\theta_{xx},Z_{xx})(t)\|^2\\
&\quad+\int_0^T\|(\tau_{tt},u_{xt},v_{xt},w_{xt},\sqrt{c_v}\theta_{xt},Z_{xt})(t)\|^2\mathrm{d}t\lesssim 1,
\end{aligned}\tag{2.18}$$

$$\int_0^T\left\|\left(u_{xxx},v_{xxx},w_{xxx},c_v^{-\frac{1}{2}}\theta_{xxx},Z_{xxx}\right)(t)\right\|^2\mathrm{d}t\lesssim 1+\int_0^T\|\tau_{xx}(t)\|^2\mathrm{d}t. \tag{2.19}$$

To control the last term in (2.19) and close the a priori estimates, we deduce the estimate for $\|\tau_{xx}\|$ in the following lemma.

**Lemma 2.8.** *Suppose that the conditions in Theorem* 1.1 *are satisfied. If* (2.9) *holds for a sufficiently small* $\epsilon_1>0$*, then*

$$\sup_{0\leqslant t\leqslant T}\|\tau_{xx}(t)\|^2+\int_0^T\left\|\left(\tau_{xx},u_{xxx},v_{xxx},w_{xxx},c_v^{-\frac{1}{2}}\theta_{xxx},Z_{xxx}\right)(t)\right\|^2\mathrm{d}t\lesssim 1. \tag{2.20}$$

*Proof.* Differentiating (1.11b) with respect to $x$ and multiplying the resulting identity by $(\nu\tau_x/\tau)_x$, we use (2.13) to obtain (cf. [48, Eq. (2.89)])

$$\left\|\left(\frac{\nu\tau_x}{\tau}\right)_x(t)\right\|^2+\int_0^t\left\|\left(\frac{\nu\tau_x}{\tau}\right)_x\right\|^2\lesssim 1+\sum_{i=1}^5\mathcal{J}_i, \tag{2.21}$$

where

$$\begin{aligned}
&\mathcal{J}_1:=\int_0^t\int_I|(\theta_{xx},\theta_x\tau_x,u_{xt},u_t,v_x,v,u_x\theta_x,\theta_{xx},\theta_x)|^2,\\
&\mathcal{J}_2:=\int_0^t\int_I|\theta_x(\tau_{tx},\tau_t\tau_x,\tau_t\theta_x)|^2,\quad \mathcal{J}_3:=\int_0^t\int_I\left|\left(\tau_x^2,\tau_x\theta_{tx},\tau_t\theta_{xx}\right)\right|^2,\\
&\mathcal{J}_4:=\int_0^t\int_I\tau_{xx}^2\theta_t^2,\qquad\qquad\qquad\quad \mathcal{J}_5:=\int_0^t\int_I|\tau_x\theta_t(\tau_x,\theta_x)|^2.
\end{aligned}$$

It follows from Lemmas 2.2-2.7 that $\mathcal{J}_1+\mathcal{J}_2\lesssim 1$. For the term $\mathcal{J}_3$, we have

$$\begin{aligned}
\mathcal{J}_3&\lesssim\sup_{0\leqslant s\leqslant t}\left(\|(\tau_x,\tau_t)(s)\|\|(\tau_{xx},\tau_{xt})(s)\|\right)\int_0^t\|(\tau_x,\theta_{xt},\theta_{xx})(s)\|^2\mathrm{d}s\\
&\lesssim\epsilon\sup_{0\leqslant s\leqslant t}\|\tau_{xx}(s)\|^2+C(\epsilon).
\end{aligned}$$

Regarding the terms $\mathcal{J}_4$ and $\mathcal{J}_5$, we infer from (2.2)-(2.3) that

$$\begin{aligned}
\mathcal{J}_4&\lesssim\int_0^t\|\tau_{xx}(s)\|^2\|\theta_t(s)\|\|\theta_{xt}(s)\|\mathrm{d}s\\
&\lesssim\sup_{0\leqslant s\leqslant t}\|\theta_t(s)\|\left(\int_0^t\|\theta_{xt}\|^2\right)^{\frac{1}{2}}\sup_{0\leqslant s\leqslant t}\|\tau_{xx}(s)\|\left(\int_0^t\|\tau_{xx}\|^2\right)^{\frac{1}{2}}\\
&\lesssim(\gamma-1)N^2\left(\sup_{0\leqslant s\leqslant t}\|\tau_{xx}(s)\|^2+\int_0^t\|\tau_{xx}\|^2\right),
\end{aligned}$$

$$
\begin{aligned}
\mathcal{J}_5 &\lesssim \int_0^t \|\theta_t\|^2 \|(\tau_x,\theta_x)\|_{L^\infty}^4 \\
&\lesssim \sup_{0\leqslant s\leqslant t} \|\theta_t(s)\|^2 \|\tau_x(s)\|^2 \int_0^t \|\tau_{xx}\|^2 + \int_0^t \|\theta_t\|^2 \\
&\lesssim (\gamma-1)N^2 \left(1+\int_0^t \|\tau_{xx}\|^2\right).
\end{aligned}
$$

Plugging the above estimates into (2.21) and using (2.19), we can obtain (2.20) by taking $\epsilon_1>0$ in (2.9) small enough. □

We combine the above a priori estimates to obtain the following corollary.

**Corollary 2.1.** *Suppose that the conditions in Theorem* 1.1 *are satisfied. If* (2.9) *holds for a sufficiently small* $\epsilon_1>0$, *then*

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\left(\|(\tau-1,u,v,w,Z)(t)\|_{H^2}^2 + \|(\sqrt{c_v}(\theta-1),\sqrt{c_v}\theta_x,\theta_{xx},c_v\theta_t)(t)\|^2\right) \\
&\quad + \int_0^T \left(\|\tau_x\|_{H^1}^2 + \|(u_x,v_x,w_x,Z_x)\|_{H^2}^2 + \left\|(\theta_x,\theta_{xx},c_v^{-\frac{1}{2}}\theta_{xxx},c_v\theta_t,\sqrt{c_v}\theta_{xt})\right\|^2\right) \lesssim 1.
\end{aligned}
$$

With the above uniform a priori estimates in hand, we can use the continuation argument and Poincaré's inequality to prove the existence, uniqueness, and large-time behavior of global solutions to the problem (1.11)-(1.13). We omit the details and refer to [48, Section 3] for brevity. The proof of Theorem 1.1 is then complete.

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