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Perspective

Synchronization on higher-order networks

HAORAN LIU[1], JIN ZHOU[1,2(a)], BO LI[1], MENG HUANG[3], JUN-AN LU[1] and DINGHUA SHI[4]

[1] *School of Mathematics and Statistics, Wuhan University - Hubei 430072, China*
[2] *Hubei Key Laboratory of Computational Science, Wuhan University - Hubei 430072, China*
[3] *Hubei Key Laboratory of Power Equipment & System Security for Integrated Energy and School of Electrical Engineering and Automation, Wuhan University - Hubei 430072, China*
[4] *Department of Mathematics, College of Science, Shanghai University - Shanghai, China*

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Abstract – Network science has already been fruitful and confirmed effective on the description of real-world or abstract systems. An increasing number of researches and instances have successfully verified, however, that interactions in systems may occur among three, four, or even more components. The introduction of higher-order perspective brings a revolution on network science, and refreshes researchers' understanding of synchronization. Hence, an overview is presented here in regard of synchronization on higher-order networks. We start from an introduction of how the higher-order networks are represented using algebraic tools. Then a series of landmark researches on synchronization is reviewed under circumstances of whether or not the dynamics contains control. Finally, we summarize our conclusions and propose our outlooks on expectations of future works.

Introduction. – Researching on the large variety of complex systems existing in, for example, ecology [1,2], sociology [3–5] and neuroscience [6–8], necessitates an accurate and visualizable macroscopic-scale description of the system itself. Under the assumption that networks only capture pairwise connections, it is quite natural to abstract subjects of the system as nodes, and interactions between subjects as edges.

More and more real systems seem to imply, however, that interactions with pairwise form are not sufficient to describe the relationship between subjects of particular systems. Actually, there exist real-world systems whose interactions do not occur between only two nodes, but rather in a mesoscopic level with three, four, or even more nodes [1–8]. Interactions beyond pairwise, together with nodes as subjects of systems, form a generalized network structure, known as higher-order network. Simplicial complexes along with hypergraphs are two main models that are capable of describing higher-order systems. Similar to traditional networks, higher-order networks also have matrix (tensor) representation, and can define its own measures, dynamics, spreading progresses, and synchronization [9,10].

(a) E-mail: jzhou@whu.edu.cn (corresponding author)

The dynamics of a system is an intelligible model depicting the evolution of the system's components. As a fundamental dynamical phenomenon occurred in networked systems, synchronization of a system refers to a state where all nodes adjust their motion onto a common trajectory [11], or the process gradually reaching this state. Synchronization is widely discovered in real-world systems [12,13] as well as mathematical systems [14,15]. On the research of synchronization and its stability, impressive progress has been made concerning the criteria of local and global bounded synchronization [16], the influence of network topology on synchronization [17,18], and synchronization with time-varying delay [19]. Different types of synchronization has also been identified on networks with special structures, such as remote synchronization [20], chimera states [21] and generalized synchronization [22]. When considering the higher-order case, dynamics and synchronization phenomenon may exhibit a far more complicated behavior, including social contagions [23], diffusion [24] and random walks [25].

Due to the complexity and major significance of synchronization mentioned above, we devote ourselves to the synchronization of higher-order networks, hence this review is organized as follows. The next section describes the definition of matrix (tensor) representation of higher-order networks, the third section focuses on higher-order network synchronization in the absence of control, the fourth section briefly reviews synchronization with control, and finally conclusions and outlook are summarized in the last section.

Higher-order network representation. – Basic concepts of hypergraphs and simplicial complexes will be briefly introduced.

A hypergraph is expressed as a 2-element tuple $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ is a finite set implying nodes of this hypergraph, and $\mathcal{E} = \{\epsilon_1, \ldots, \epsilon_m\}$ is a family of sets with each set ϵ_i an arbitrary non-empty subset of $\mathcal{V}$ (usually under the limitation that the cardinality of the set should be no less than 2). Practically, each element of $\mathcal{E}$ is called *hyperedge*, which is naturally regarded as an extension of pairwise connections. If hyperedge ϵ_i contains n_i nodes, *i.e.*, $|\epsilon_i| = n_i$, the *order of a hyperedge* is a little counterintuitively defined as $n_i - 1$. In this framework, the set ϵ_i is unordered, that is, after any permutation of the elements of ϵ_i, it refers to the same hyperedge. Indeed, 1-order hyperedges are traditional edges, 2-order hyperedges are interactions occurred between 3 nodes, etc. For convenience, any node itself is designated as a 0-order hyperedge. The *order of a hypergraph* is defined as the maximal order of all hyperedges in the hypergraph.

The notions of simplex, along with simplicial complex (a collection of simplices obeying certain rules), are borrowed from algebraic topology. Given a simplicial complex with all nodes in a topological space, a k-order simplex (abbreviated as k-simplex) whose vertices are $\{v_0, \ldots, v_k\}$ is denoted as $\sigma = [v_0, \ldots, v_k]$. Assuming that simplex σ is extracted from a particular simplicial complex $\mathcal{K}$, the square bracket in the definition of simplex σ implies not only σ is a node set, but has a strong constraint that any sub-simplex of σ (*i.e.*, any simplex generated by a non-empty subset of σ) should also be contained in the simplicial complex $\mathcal{K}$. Similar to the definition of hyperedge, the set σ is unordered. Also, any node is designated as a 0-simplex. Hence, a tuple may represent a simplicial complex as $\mathcal{K} = (\mathcal{V}, \mathcal{S})$, where $\mathcal{V}$ is the node set and $\mathcal{S}$ is the set of all simplices contained therein.

The *full simplex* of a simplex $\sigma = [v_0, \ldots, v_k]$ is defined as the union of all sub-simplicies, except 0-order ones (*i.e.*, vertices), of σ. The full simplex of σ is here denoted as $\bar{\sigma}^\circ$ or $\overline{[v_0, \ldots, v_k]}^\circ$. Full simplex refers to interactions of all orders among a certain simplex, from every pairwise edges to the simplex itself as an interaction. Applications for full simplex appear in pinning control [26], and further details will be discussed in the fourth section.

For a clearer explanation over simplicial complexes and hypergraphs, a toy example up to 2-order (fig. 1) is designed. Given the interaction information of this higher-order network as $\{\{1,2\},\{2,3\},\{3,4,5\},\{2,5,6\}\}$, the corresponding simplicial complex and hypergraph can be easily developed. In this toy example, the node sets of simplicial complex and hypergraph are the same, and

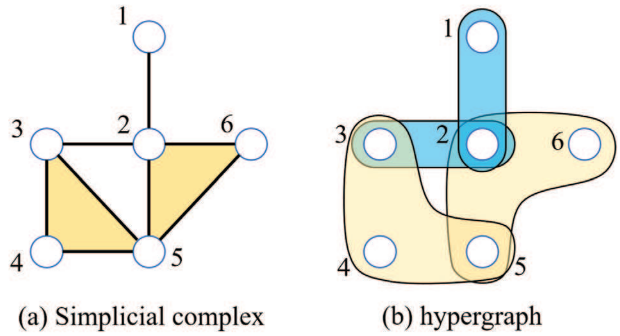


Fig. 1: Comparison between simplicial complex and hypergraph.

the difference lies in their interactions. Due to the restriction of simplicial complex, it is required that simplices $[3,4], [3,5], [4,5]$ and $[2,5], [2,6], [5,6]$ are all contained in the simplicial complex (fig. 1(a)). In the case of hypergraph (fig. 1(b)), however, there is no such limitation. For example, hyperedge $\{3,4,5\}$ does not necessarily require the existence of hyperedge $\{3,4\}$.

Simplicial complexes and hypergraphs possess matrix representations in a similar form. In the following, we focus our discussion on a D-order higher-order network with N nodes unless otherwise specified.

In a higher-order system, there should be an incidence matrix of every order. For any $d \in \{1, 2, \ldots, D\}$, the d-order incidence matrix is a $(0,1)$-matrix, describing whether every $(d-1)$-order edge is contained in a d-order edge. Take the simplicial complex of our toy model as an example (fig. 1(a)), there are 2 different incidence matrices written as

$$B^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$B^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},$$

where $B^{(1)}$ says whether 0-order simplices (nodes) are a vertex of 1-order simplicies (edges), and $B^{(2)}$ whether 1-order simplices are an edge in 2-order simplicies (triangles).

To describe the adjacency relationship of higher-order networks, an algebraic structure is introduced, namely tensor (or hyper-matrix [10]). A k-order tensor $\mathscr{A}$ $(k \geqslant 1)$ is here defined as matrix-like structure with $k+1$ indices. In analogy with the adjacency matrix of traditional networks, for any $d \in \{1, 2, \ldots, D\}$, the d-order adjacency tensor is denoted as $\mathscr{A}^{(d)}$. Its $(i_0,\dots,i_d)$-th entry $\mathscr{A}^{(d)}_{i_0\dots i_d}$ equals 1 if and only if $\{i_0,\dots,i_d\}$ is a d-order edge of this network, and 0 otherwise. Note that the higher-order edge regarded as a set is unordered, the adjacency tensor has the property of permutation symmetry, namely $A^{(d)}_{\pi(i_0,\dots,i_d)} = A^{(d)}_{i_0\dots i_d}$, where $\pi(i_0,\dots,i_d)$ is any permutation of $\{i_0,\dots,i_d\}$. Specifically, a 1-order adjacency tensor is an $N\times N$ matrix, a 2-order adjacency tensor is an $N\times N\times N$ tensor, and so on.

Laplacian matrix of a simple undirected graph provides abundant algebraic information of the graph itself. Laplacian eigenvalues are all non-negative, and the minimum eigenvalue of Laplacian is always zero. The second smallest Laplacian eigenvalue and its corresponding eigenvector are called Fiedler value and Fiedler vector [27], respectively. The eigenpairs of Laplacian matrix have been rediscovered for their hidden information on graph structure [28], cluster synchronization discrimination [29], important cycle identification [30] and so on.

For every order d ranging from 1 to D, there is a corresponding d-order Laplacian $L^{(d)}$ calculated from the d-order adjacency tensor $\mathscr{A}^{(d)}$. In refs. [31,32], d-order Laplacian $L^{(d)}$ is defined as

$$L^{(d)}_{ij} = \begin{cases} d!k^{(d)}_i, & i=j, \\ -(d-1)!k^{(d)}_{ij}, & i\neq j, \end{cases} \tag{1}$$

where d-order generalized degree of node i is $k^{(d)}_i = \frac{1}{d!}\sum_{j_1,\dots,j_d=1}^{N}\mathscr{A}^{(d)}_{ij_1\dots j_d}$, namely the number of d-hyperedges containing node i, and $k^{(d)}_{ij} = \frac{1}{(d-1)!}\sum_{k_2,\dots,k_d=1}^{N}\mathscr{A}^{(d)}_{ijk_2\dots k_d}$ means the total number of hyperedges containing the node pair (i,j). It can be easily verified that $L^{(d)}$ is symmetric and of zero row-sum, while in ref. [33] $L^{(d)}$ is defined as

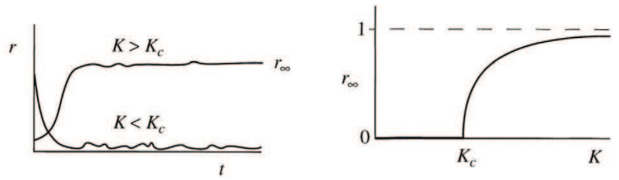
$$L^{(d)}_{ij} = dk^{(d)}_i\delta_{ij} - k^{(d)}_{ij}, \tag{2}$$

where δ_{ij} is the Kronecker delta. In fact, the Laplacian defined by (1) should be that of (2) multiplied by $(d-1)!$, if one notices that $A^{(d)}_{ijk_2\dots k_d}=0$ when $i=j$, therefore $k^{(d)}_{ij}=0$.

Using subscripts "SC" and "H" to distinguish the simplicial complex and the hypergraph, we give the 2-order Laplacian of our toy example as

$$L^{(2)}_{SC} = L^{(2)}_{H} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Synchronization in the absence of control. – Kuramoto, a pioneer on the description of oscillator synchronization, has thoroughly researched the synchronization of a finite number of fully coupled oscillators [34,35]. In this section, we will firstly review the Kuramoto model and its higher-order extensions, then general higher-order network collective dynamics is to be reported.



(a) r tendency over time (b) r_∞ tendency over K

Fig. 2: Kuramoto model abrupt synchronization. In our symbols, K_c is the coupling threshold K_1^*, r is the coherence parameter R_1, and r_∞ is the limit of r when $t\to+\infty$. Source: ref. [36].

Higher-order Kuramoto model. Kuramoto has proposed his well-known N-oscillator model [35] as follows:

$$\dot{\theta}_i = \omega_i + \frac{K_1}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i), \tag{3}$$

where θ_i is the phase of the i-th oscillator valued in the interval $[0,2\pi)$, ω_i is its natural frequency, and $K_1>0$ is the coupling constant of pairwise connections. In the absence of coupling terms, the phase of every oscillator i evolves linearly with the rate ω_i. When coupled with other nodes, the coupling drives the phase of node i towards synchronization. As a result, there exists a coupling threshold K_1^*. When the coupling strength K_1 is over K_1^*, the phases of this system can reach a state where all pairs of nodes' phases tend to differ by a constant value, which is called the *phase locking state* in physics. In order to better observe the relationship between synchronization process and coupling strength, a macroscopic parameter $Z_1(t) = R_1(t)e^{i\Phi_1(t)} = \frac{1}{N}\sum_{j=1}^{N}e^{i\theta_j}$ is defined, where i is the imaginary unit. The modulus of $Z_1(t)$ is $R_1(t)\in[0,1]$, which measures the degree of coherence; and $\Phi_1(t)$ is the average phase of all nodes, ranging in $[0,2\pi)$. Then the system in eq. (3) is rewritten in a mean-field manner as

$$\dot{\theta}_i = \omega_i + K_1R_1\sin(\Phi_1-\theta_i). \tag{4}$$

A remarkable discovery of the Kuramoto model is the abrupt synchronization phenomenon. When the coupling strength $K_1<K_1^*$, the coherence parameter $R_1\approx 0$ for all t, while in the $K_1>K_1^*$ case, R_1 abruptly increases to $O(1)$, see fig. 2 for intuitions of this phenomenon. A natural idea to extend the Kuramoto model is that nodes can be designed to couple each other in the form of a graph, instead of all-to-all coupling in the original Kuramoto model [37]:

$$\dot{\theta}_i = \omega_i + \frac{K_1}{N}\sum_{j=1}^{N}a_{ij}\sin(\theta_j-\theta_i), \tag{5}$$

where a_{ij} is the (i,j)-th entry of the adjacency matrix of the graph representing the connections of nodes.

For a further extension in the higher-order context, the first mode is to extend the "state" of a node from phase (a real number) to an arbitrarily dimensional real vector.

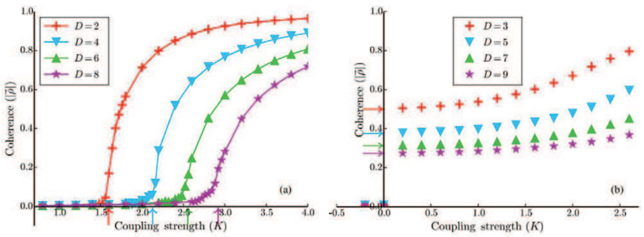


Fig. 3: D-dimensional Kuramoto model phase transition graphs. (a) When D is even, phase transition appear in a continuous form, and (b) when D is odd, $K^* = 0$ becomes the common threshold for abrupt coherence. Source: ref. [38].

Reference [38] provides a generalized Kuramoto model with the state of every node a D-dimensional vector. The traditional Kuramoto model in eq. (3) is rewritten in a 2D unit vector form as

$$\dot{\boldsymbol{\sigma}}_i = W_i \boldsymbol{\sigma}_i + \frac{K}{N} \sum_{j=1}^{N} [\boldsymbol{\sigma}_j - (\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_i)\boldsymbol{\sigma}_i], \tag{6}$$

where $\boldsymbol{\sigma}_i = (\cos\theta_i, \sin\theta_i)^T$ can be regarded as the coordinates of node i on unit circle, $W_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}$ is a anti-symmetric matrix indicating a rotating motion. In this way, a D-dimensional Kuramoto model is acquired as the same form of eq. (6), where $\boldsymbol{\sigma}_i$ here is a D-dimensional vector, and W_i is a $D \times D$ anti-symmetric matrix. Analogously to the traditional Kuramoto model, the authors defined a global order parameter $\boldsymbol{\rho} = N^{-1} \sum_{j=1}^{N} \boldsymbol{\sigma}_j$, and $|\boldsymbol{\rho}| = 1$ implies complete synchronization of this system. The phenomenon observed is also surprising (see fig. 3). Using a similar analytical framework of the traditional Kuramoto model, the authors obtained a good theoretical expression of $|\boldsymbol{\rho}|$ with regard to K, which approximates the numerical results quite well in the case of large N. A succesive work is carried out by Dai *et al.* [39], introducing a "positive feedback" factor into the coupling term:

$$\dot{\boldsymbol{\sigma}}_i = W_i \boldsymbol{\sigma}_i + \frac{K}{N} |\boldsymbol{\rho}| \sum_{j=1}^{N} [\boldsymbol{\sigma}_j - (\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_i)\boldsymbol{\sigma}_i], \tag{7}$$

where the modulus of order parameter $\boldsymbol{\rho}$ is multiplied by the coupling strength compared to eq. (6). When the dimension D is odd, an elusive phase oscillation phenomenon is discovered in this model.

The second mode, also a mode better fitting the higher-order context, is to extend the Kuramoto model onto a higher-order graph, and develop a coupling form on higher-order interactions.

Skardal *et al.* [40] gave a multilayer system:

$$\dot{\theta}_i = \omega_i + \frac{K_2}{N^2} \sum_{j,k=1}^{N} \sin(\theta_j + \theta_k - 2\theta_i), \tag{8}$$

$$\dot{\phi}_i = \nu_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_i) + d \sin(\theta_i - \phi_i), \tag{9}$$

which was abstracted from equations of power grids. Equation (8) focused on the situation that oscillators are coupled with pure 2-simplices, and considered a symmetric phase coupling mode. The authors also proposed generalized order parameters $z_q(t) = \frac{1}{N} \sum_{j=1}^{N} e^{qi\theta_j} = R_q e^{i\Phi_q}$, $q = 1, 2$. This system's dynamics appears an abrupt desynchronization transition when the 2-order coupling strength K_2 is decreasing, and a global multi-stability for 3-body interactions.

In their subsequent work [41], the authors considered a more general system containing simplices up to 3-order:

$$\begin{aligned} \dot{\theta}_i = \omega_i &+ \frac{K_1}{\langle k^{(1)} \rangle} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i) \\ &+ \frac{K_2}{2! \langle k^{(2)} \rangle} \sum_{j,l=1}^{N} B_{ijl} \sin(2\theta_j - \theta_l - \theta_i) \\ &+ \frac{K_3}{3! \langle k^{(3)} \rangle} \sum_{j,l,m=1}^{N} C_{ijlm} \sin(\theta_j + \theta_l - \theta_m - \theta_i), \end{aligned} \tag{10}$$

where $\langle k^{(1)} \rangle$, $\langle k^{(2)} \rangle$, $\langle k^{(3)} \rangle$ are the average q-order generalized degree among all nodes. The nodes of this system are coupled in a 3-order simplicial complex, whose adjacency tensors of all orders are respectively denoted as A_{ij}, B_{ijl}, and C_{ijlm}. It should be noted that the phase coupling adopts a non-symmetric mode in terms of order 2 and order 3, which can be treated using the dimensional reduction technique, as reported by the authors.

Lucas *et al.* [33] proposed a generalized Kuramoto model with all nodes coupled in an arbitrary order simplicial complex:

$$\begin{aligned} \dot{\theta}_i = \omega_i &+ \frac{K_1}{\langle k^{(1)} \rangle} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i) \\ &+ \frac{K_2}{2! \langle k^{(2)} \rangle} \sum_{j,k=1}^{N} B_{ijk} \sin(\theta_j + \theta_k - 2\theta_i) \\ &+ \cdots \\ &+ \frac{K_D}{D! \langle k^{(D)} \rangle} \sum_{j_1,\ldots,j_D=1}^{N} M_{ij_1 \ldots j_D} \sin\left(\sum_{m=1}^{N} \theta_{j_m} - \theta_i\right), \end{aligned} \tag{11}$$

and developed an analytical paradigm for the Lyapunov stability of this generalized system. Using the definition of Laplacian matrices of ref. [33], the linearized system of equations (11) is rearranged as

$$\delta\dot{\theta}_i = -\sum_{d=1}^{D} \frac{K_d}{\langle k^{(D)} \rangle} \sum_{j=1}^{N} L_{ij}^{(d)} \delta\theta_j = -\sum_{j=1}^{N} L_{ij}^{(mul)} \delta\theta_j, \tag{12}$$

where the multi-order Laplacian matrix is defined as $L^{(mul)} = \sum_{d=1}^{D} \frac{K_d}{\langle k^{(D)} \rangle} L^{(d)}$. Then the system's smallest non-zero Lyapunov exponent Λ_2 should be the opposite of the second smallest eigenvalue of multi-order Laplacian, *i.e.*, $\Lambda_2 = -\lambda_2^{(mul)}$. In particular, the more positive the $\lambda_2^{(mul)}$, the more stable the synchronized state.

General higher-order network collective dynamics. Based on recent progress made on Kuramoto model, general higher-order collective dynamical models have been proposed and researched in the last few years. Reference [32] gave a dynamics of undirected simplicial complexes and investigated the synchrozation stability of the proposed system, while ref. [31] gave a dynamics of directed hypergraphs and studied the impact of directionality on synchronization. In this section, the main results of [31,32] will be reviewed.

Gambuzza *et al.* [32] proposed their dynamics on simplicial complex as follows:

$$\begin{aligned}\dot{x_i} = f(x_i) &+ \sigma_1 \sum_{j_1=1}^{N} a_{ij_1}^{(1)} g^{(1)}(x_i, x_{j_1}) \\ &+ \sigma_2 \sum_{j_1,j_2=1}^{N} a_{ij_1j_2}^{(2)} g^{(2)}(x_i, x_{j_1}, x_{j_2}) + \cdots \\ &+ \sigma_D \sum_{j_1,\ldots,j_D=1}^{N} a_{ij_1\ldots j_D}^{(D)} g^{(D)}(x_i, x_{j_1}, \ldots, x_{j_D}),\end{aligned} \quad (13)$$

where x_i is the m-dimensional state vector of node i, $\{\sigma_i\}_{i=1}^{D}$ are coupling strengths, and for every order $d \in \{1,\ldots,D\}$, $a_{ij_1\ldots j_d}^{(d)}$ is the $(i,j_1,\ldots,j_d)$-th entry of d-order adjacency tensor. The coupling functions $g^{(d)}\colon \mathbb{R}^{(d+1)\times m} \to \mathbb{R}^m (d = 1,\ldots,D)$ are non-invasive, *i.e.*, $g^{(d)}(x, x, \ldots, x) \equiv 0, \forall d, \forall x \in \mathbb{R}^m$. Compared to the target trajectory x^s satisfying $\frac{\mathrm{d}}{\mathrm{d}t}x^s(t) = f(x^s(t))$, the variation ($\delta x_i = x_i - x^s$) of the above system in eq. (13) is organized as

$$\dot{\delta\boldsymbol{x}} = \left[I_N \otimes JF - \sum_{d=1}^{D} \sigma_d \mathcal{L}^{(d)} \otimes JG^{(d)} \right] \delta\boldsymbol{x}, \quad (14)$$

where $\delta\boldsymbol{x} = [\delta x_1^T, \ldots, \delta x_N^T]^T$ is the variation of all nodes' states. $JF, JG^{(1)}, \ldots, JG^{(D)}$ are the notations of Jacobian matrices at x^s. In order to separate the transverse system out of this coupling one, the authors chose a reference vector basis as the eigenvectors $v_1, \ldots, v_n$ of the classic Laplacian $\mathcal{L}^{(1)}$ ($V = [v_1, \ldots, v_N]$) and introduced new variables as $\eta = (V^{-1} \otimes I_m)\delta\boldsymbol{x}$. Then one gets

$$\begin{aligned}\dot{\eta} &= (V^{-1} \otimes I_m)[I_N \otimes JF - \sum_{d=1}^{D} \sigma_d \mathcal{L}^{(d)} \otimes JG^{(d)}](V \otimes I_m)\eta \\ &= [I_N \otimes JF - \sigma_1 \Lambda^{(1)} \otimes JG^{(1)} - \sum_{d=2}^{D} \sigma_d \tilde{\mathcal{L}}^{(d)} \otimes JG^{(d)}]\eta,\end{aligned} \quad (15)$$

where $\Lambda^{(1)}$ is a diagonal eigenvalue matrix of $\mathcal{L}^{(1)}$, and $\tilde{\mathcal{L}}^{(d)} = V^{-1}\mathcal{L}^{(d)}V, \forall d$, are also zero-row-sum matrices. The authors also proposed the *natural coupling assumption*, namely d-order coupling function $g^{(d)}(x_i, x_{j_1}, \ldots, x_{j_d})$ can be separated as $g^{(d)}(x_i, x_{j_1}, \ldots, x_{j_d}) = h^{(d)}(x_{j_1}, \ldots, x_{j_D}) - h^{(d)}(x_i, \ldots, x_i)$, where functions $h^{(d)}\colon \mathbb{R}^{md} \to \mathbb{R}^m$ satisfy $h^{(1)}(x) = h^{(2)}(x, x) = \cdots = h^{(D)}(x, x, \ldots, x)$, for all $x \in \mathbb{R}^m$. If we further assume that the coupling function satisfied the natural coupling condition, the system (14) can be simplified into an even more compact form:

$$\dot{\delta\boldsymbol{x}} = [I_N \otimes JF - \mathcal{L}^{(mul)} \otimes I_m]\delta\boldsymbol{x}, \quad (16)$$

where $\mathcal{L}^{(mul)} = \sum_{d=1}^{D} \sigma_d \mathcal{L}^{(d)}$ represents the multi-order Laplacian in this case. Then the classical MSF method can be imposed to analyze the proposed system.

Gallo *et al.* [31] proposed a dynamics on directed hypergraphs of arbitrary order which had a similar form as eq. (13). The difference between the proposed system and eq. (13) lies in the adjacency tensor. Reference [31] defined an m-directed d-hyperedge, which means that in this hyperedge, prechosen m nodes $\{i_1, \ldots, i_m\}$ are pointed by the remaining $s = d + 1 - m$ nodes $\{j_1, \ldots, j_s\}$, and the adjacency tensor has a locally permutation symmetry property:

$$A_{\pi(i_1,\ldots,i_m)\pi'(j_1,\ldots,j_s)}^{(d)} = 1,$$

where $\pi(i_1, \ldots, i_m)$ and $\pi'(j_1, \ldots, j_s)$ are any permutation of $\{i_1, \ldots, i_m\}$ and $\{j_1, \ldots, j_s\}$, respectively. Adopting the same analytical framework and assumptions of ref. [32], the authors finally obtained a master stability eq. (MSE) with complex parameter:

$$\dot{\eta} = [JF(x^s) - (\alpha + i\beta)JH(x^s)]\eta. \quad (17)$$

Because of the asymmetry property of the multi-order Laplacian, its eigenvalues may be complex number. Therefore, the maximum Lyapunov exponent of the original system should be regarded as a function over the real part α and the imaginary part β. Moreover, the authors discovered that the directionality largely influences (de)synchronization, by developing a parameter p moving from 0 to 1 indicating the strength of direction of every hyperedge.

Synchronization with pinning control. – How to drive the system state towards a target trajectory (sometimes in a limited time) has always been a fundamental question in the research of complex systems. Control theory has developed algebraic and analytical tools on linear systems [42,43] and non-linear systems [44–46], as well as a wide range of control schemes [46–49]. Among them is pinning control, a type of control scheme which needs only to control a small fraction of prechosen nodes (often called pinner) to accomplish a global control target. Basic theories of pinning control have been established in the last twenty years [50–53]. The general methodology of pinning control is to exert a signal to every pinner [54,55], and it also fits a time-delayed dynamical system [56]. Besides controllers located on nodes, Yu *et al.* developed an adaptive pinning scheme on edges using Lyapunov method to discriminate stability [57]. Node importance based on pinning control has also been developed [58]. Recent works on pinning control of higher-order networks [26,59,60] have mainly focused on the extension of traditional pinning control laws.

Li *et al.* discussed in ref. [26] the importance of the second smallest eigenvalue λ_2 of multi-order Laplacian, and how to choose pinning simplices based on λ_2 in order to achieve better control effects. The authors considered a dynamical system on simplicial complex with time-varying weights on each simplex, and assumed that the coupling function has a linear form, *i.e.*, $g^{(d)}(x_i, x_{j_1}, \ldots, x_{j_d}) = \frac{1}{d}(x_{j_1} + \cdots + x_{j_d} - dx_i), \forall d$. For simplification, the authors gave the error dynamics (compared to the average state of nodes $s = \frac{1}{N}\sum_{j=1}^{N} x_j$) and described adaptive control laws on 2-order undirected simplicial complex. Controllers were put on full simplices of a small fraction of 2-order simplices. The authors proposed a pinner selection method, relying on the improvement of λ_2 when adding a small weight on particular full simplex. The bigger the improvement is, the better the full simplex should perform in pinning control.

De Lellis *et al.* [60] provided a different perspective of pinning control, by decomposing a directed hypergraph into a signed graph. The authors also proposed two algorithms, respectively computing if the system can be asymptotically controlled onto the target trajectory, and how to add pinners in a sequence. Numerical simulation on Chua's circuits showed that the proposed adding-pinner method is far more effective than randomly choosing pinners.

Conclusions and outlook. – In this review, we have gone through the matrix (tensor) representation of higher-order systems in the second section, discussed the Kuramoto model and its variants in higher-order framework, then given general higher-order collective dynamical models in the third section, and finally mentioned some works of synchronization with pinning control in the fourth section. Due to the limited space, not all impressive researches on synchronization of higher-order networks have been reviewed, but one will gain an intuition of how the higher-order perspective revolutionarily changes the traditional network science.

However, as a very recently developed research field, there are still many blanks waiting to be filled in, and many open problems to be more deeply investigated.

– Simplicial complex itself is a topology concept. Current representations and dynamics with regard of simplicial complexes cannot, however, express the essence of its related topology structures.

– Current dynamics depicting higher-order systems are mostly the evolutionary equations of the states of nodes. Does there exist an even more general dynamics of higher-order edges, such as traditional edges or 3-body interactions?

– Research tools and methods on control of higher-order systems are mainly limited on the extension of traditional control theory. Is there any novel methods exclusively for higher-order analysis that can reveal deeper facts of higher-order structures?

Despite incompleteness in current researches, the field of higher-order systems is already fruitful and successful when combined with real-world engineering and biology problems. We are looking forward to the breakthrough of the pursuit of truth on higher-order systems.

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