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Perspective

Pinning synchronization of a complex network: Nodes, edges and higher-order edges

QiShu Jiang[1], Jin Zhou[1,2(a)], Bo Li[1], HaoRan Liu[1] and Jun-an Lu[1]

[1] *School of Mathematics and Statistics, Wuhan University - Wuhan, Hubei 430072, China*
[2] *Hubei Key Laboratory of Computational Science, Wuhan University - Wuhan, Hubei 430072, China*

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Abstract – In recent years, the interdisciplinary study of complex networks has become increasingly important in fields ranging from biology and physics to sociology and mathematics. This paper focuses on pinning control, an approach essential for achieving coordinated behavior in dynamic networks. We explore recent advancements in pinning control strategies, explaining theoretical frameworks and simulation techniques. Additionally, we discuss the significance of certain structures within networks across different orders. Finally, we conclude with a summary of key insights and propose our outlook on future research.

Introduction. – Complex networks are present across diverse domains like social interactions, biological systems, mathematical structures, scientific fields and engineering applications [1–4]. They have become indispensable to our everyday existence. Researchers from mathematics, physics, biology, engineering and social sciences are increasingly interested in understanding and managing the intricate dynamics exhibited by networks composed of numerous interacting nodes [5–11].

In complex dynamical networks, there is a fascinating phenomenon where the motion states of nodes in the network can synchronize under certain conditions. If a given network fails to synchronize, or if the synchronized state is not the desired state, control becomes necessary to guide the desired synchronization in the network [12–16]. However, in practical applications, directly controlling every part of a dynamic network with numerous nodes or (and) edges may be impossible and unnecessary. Therefore, achieving synchronization by directly adding control inputs to a fraction of nodes, edges, or higher-order edges in the network is very important. Many scholars have conducted significant explorations in this area [17–19].

This review is organized as follows: some mathematical preliminaries are given in the next section. In the third section, significant pinning control methods on nodes (0-order edges) are discussed. In the fourth section, approaches for adding control on traditional edges (1-order edges) are provided. In the fifth section, we introduce approaches to achieve synchronization by adding control inputs to both nodes and edges (0-order and 1-order edges). Then, exploration from traditional networks to higher-order networks by controlling 2-order or higher-order edges is discussed in the sixth section. Finally, conclusions and outlook are summarized in the last section.

Preliminaries. – In general, a traditional complex dynamical network consisting of N identical nonlinear oscillators with uncertain inputs is described as

$$\dot{x}_i = f(x_i, t) + \sum_{j=1}^{N} c_{ij} a_{ij} g(x_j) + u_i,\ 1 \le i \le N, \qquad (1)$$

whose node set and edge set are, respectively, $\mathcal{V}$ and $\mathcal{E}$. Let c_{ij} be the coupling strength and $A = (a_{ij})_{N\times N} \in R^{N\times N}$ be the coupling configuration matrix. If there is a link e_{ij} from node i to node j ($j \ne i$), then $a_{ij} = 1$; otherwise, $a_{ij} = 0$. Furthermore, in an undirected network, A is symmetric because $a_{ij} = a_{ji}$ always holds. $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in R^n$ is the n-dimensional state variable of the i-th node, $\dot{x} = f(x,t)$ denotes the dynamics of each isolated (uncoupled) node, $u_i \in R^n$ are the control inputs.

Network synchronization: If the coupled nodes satisfy $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for all $i, j = 1, \ldots, N$, then the network is said to be synchronized.

(a) E-mail: jzhou@whu.edu.cn (corresponding author)

In specific studies, researchers often consider a particular network synchronizing to a desired state $s(t)$ or an average state $\bar{x}(t)$. Here, we give definitions as well.

Network is said to synchronize all the states of the nodes in the dynamical network to a desired solution $s(t)$ if $\lim_{t\to\infty} \|x_i(t) - s(t)\| = 0$ for all $i = 1, \ldots, N$. Network achieves synchronization upon $\bar{x}$ if nodes in the network satisfy $\lim_{t\to\infty} \|x_i(t) - \bar{x}(t)\| = 0$ for all $i = 1, \ldots, N$.

Network synchronizability: The *Laplacian matrix* [20] is defined as $L = D - A$, where $A = (a_{ij})_{N\times N}$. D is a diagonal matrix with elements representing the sum of the corresponding rows of A.

For graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the second smallest Laplacian eigenvalue is referred to as the Fiedler value [20], also known as the algebraic connectivity which satisfies

$$\lambda_2(L) = \min_{\substack{x\perp\mathbf{1},\\ \|x\|=1}} x^T L x = \min_{\substack{x\perp\mathbf{1},\\ \|x\|=1}} \sum_{\mathcal{E}_{mn}\in\mathcal{G}} (x_m - x_n)^2, \qquad (2)$$

where $\mathbf{1}$ is a column vector with all components being 1, and x_m denotes the m-th component of vector x. Synchronizability of a network is mostly measured by the λ_2. The larger the value, the stronger the synchronizability.

Nonlinear restriction: In network pinning control, the movement state of each isolated node is often restricted, including but not limited to:

H1: The function $f(\cdot)$ is said to satisfy $f(\cdot) \in \text{QUAD}(P, \Delta)$, if there exist two positive definite diagonal matrices P and Δ, such that $(x-y)^T P(f(x) - f(y) - \Delta x + \Delta y) \le 0$ [21].

H2: The function $f(\cdot)$ is said to satisfy the Lipschitz condition. That is, there exists a positive constant δ satisfying $\|g(x,t) - g(y,t)\| \le \delta\|x - y\|$, where δ is the Lipschitz constant [21].

H3: Suppose that $\|Df(s)\|$ is bounded, where $Df(s)$ is the Jacobian of f evaluated at $x = s$. That is, there exists a non-negative constant α satisfying $\|Df(s)\| \le \alpha$ [22].

In fact, the three hypotheses give very similar nonlinear restriction of node dynamics $f(\cdot)$.

0-order edges (nodes). – In complex networks, node control is a crucial strategy aiming at influencing the dynamic behavior of the entire network by altering the state or inputs of nodes in a given network. In practice, directly controlling every node in a dynamical network with a huge number of nodes might be impossible or unnecessary. Therefore, pinning control strategy, that is, achieving the goal of control by directly adding control inputs to a fraction of nodes selected from the network, is very important. People may raise such questions: Which kinds of controllers can be applied to the controlled nodes? Which nodes should be controlled? In this section, we review some important findings and provide some answers.

Li *et al.* [23] defined stable conditions and investigated both the local and global stabilization of complex dynamical networks via the pinning control strategy. The authors pinned l nodes i_k, $k = 1, 2, \ldots, l$, in the network by introducing fewer negative feedback controllers

$$u_{i_k} = -c_{i_k i_k} d_{i_k} g(x_{i_k} - \bar{x}). \qquad (3)$$

They found that if $f(\cdot)$ is under certain conditions like the Lipschitz condition, gradually increasing the coupling strength $c = c_{i_k i_k}$ always leads to synchronization. Subsequently, numerous researchers conducted more refined and specific studies based on the stability conditions they provided.

Chen *et al.* [24] presented sufficient conditions to guarantee synchronization by pinning only one node with a simple linear input. Without any prior knowledge of the network topology, they found that a single controller can pin a complex network to a homogenous solution whether the function $g(\cdot)$ is linear or nonlinear. In light of their thorough and comprehensive argumentation, people can always pin a coupled complex network by adding a single controller if the coupling strength is large enough.

In addition to linear feedback controllers [23–27], many other control strategies are introduced for pinning control, including adaptive control [28,29], intermittent control [30, 31], impulsive control [32–34], finite-time control [35–37] and time-delay control [38]. Here we give several typical examples.

In refs. [22,39] the authors considered a complex dynamical network with linear diffusion coupling. They supposed $\|Df(s)\| \le \alpha$, $g(x_j) = \Gamma(x_j)$, where Γ is a irreducible diffusive matrix with $\|\Gamma\| = \gamma$ and $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_N$ are the eigenvalues of the matrix $\frac{\hat{B}+\hat{B}^T}{2}$, where $\hat{B}$ is a modified matrix of $B = (b_{ij})_{N\times N} = (c_{ij} \cdot a_{ij})_{N\times N}$. If there exists a natural number $1 \le l < N$ such that $\lambda_{l+1} < -\frac{\alpha}{\gamma}$, then the synchronous solution $S(t)$ of controlled network (1) is asymptotically stable under the pinning adaptive controllers

$$\begin{cases} u_{i_k} = -p_{i_k} e_{i_k}, \quad \dot{p}_{i_k} = q_{i_k} \|e_{i_k}\|_2, & 1 \le k \le l, \\ u_i = 0, & \text{otherwise}, \end{cases} \qquad (4)$$

where q_i are positive constants and $e_{i_k} = x_{i_k} - s$ is defined as error vector. Further, letting $c_{ij} = c(i \ne j)$, $\bar{B} = \frac{B}{c}$ be symmetric and $\bar{\lambda}_{l+1} < -\frac{\alpha}{c\gamma}$, the network can realize synchronization. On the one hand, for networks with fixed topological structure and coupling strength, based on $\bar{\lambda}_{l+1} < -\frac{\alpha}{c\gamma}$, the nodes which should be controlled are determined. On the other, after selecting the controlled nodes, from $c > -\frac{\alpha}{\bar{\lambda}_{l+1}\gamma}$, one can choose appropriate coupling strength to achieve network synchronization.

In ref. [38], authors discussed pinning synchronization of delayed neural networks. Due to the prevalent circumstance, neurons often receive delayed evolutionary information from their neighbors, the controlled network is a little bit different from (1). That is,

$$\dot{x}_i(t) = f(x_i, t) + \sum_{j=1}^{N} c_{ij} a_{ij} g(x_j) + \sum_{j=1}^{N} b_{ij} h(x_j(t - \tau(t))) + u_i,\ 1 \le i \le N, \quad (5)$$

u_i are the adaptive controllers given by

$$\begin{cases} u_{i_k}(t) = -\alpha_{i_k}(t)(x_{i_k}(t) - s(t)), & 1 \le k \le l, \\ \dot{\alpha}_{i_k}(t) = \beta_{i_k} \|x_{i_k}(t) - s(t)\|^2, & 1 \le k \le l, \\ u_{i_k}(t) = 0, & \text{otherwise}, \end{cases} \quad (6)$$

they proposed that by employing the adaptive controller above when the growth rates of $\tau(\cdot)$, $h(\cdot)$ and $g(\cdot)$ are constrained, the network can achieve synchronization if $A = (a_{ij})_{N\times N}$ and $B = (b_{ij})_{N\times N}$ are under certain conditions.

In many works, external controls are needed all the time, which may be non-economic. Liu and Chen [31] considered the linearly coupled network model with aperiodically intermittent control. For any time span $[t_i, t_{i+1})$, $[t_i, s_i]$ is the work time (control time), and $s_i - t_i$ is called the i-th control width (control duration); while (s_i, t_{i+1}) is the rest time, and $t_{i+1} - s_i$ is called the i-th rest width. Suppose all the rest widths $t_{i+1} - s_i$, $i = 0, 1, 2, \dots$, are bounded and function $f(\cdot) \in \mathrm{QUAD}(P, \Delta)$ then the coupled network with single linear pinning control

$$\begin{cases} u_1(t) = b \cdot g(s(t) - x_1(t)), & t \in [t_i, s_i], \\ u_i(t) = 0, & \text{otherwise}, \end{cases} \quad (7)$$

can realize global synchronization when the weight b is sufficiently large.

Zhou *et al.* [39] used ControlRank index to select the pinned nodes. Here a possible sequencing for node importance can be determined in descending order of $\lambda_q(L^s)$ $(1 \le q \le N)$, where $\lambda_q(L^s)$ is the minimum eigenvalue of the minor matrix of $\frac{L+L^T}{2}$ obtained by removing the q-th row and column pair. The larger the $\lambda_q(L^s)$, the more effective the pinning control. They gave an example of a network coupled with 11 Chua circuits. The topology of the network is a bi-star depicted by fig. 1. They calculated the ControlRank index of each node as $\lambda_1(L^s) = 0.17 > \lambda_2(L^s) = \lambda_7(L^s) = 0.09 > \lambda_3(L^s) = \lambda_4(L^s) = \lambda_5(L^s) = \lambda_5(L^s) = \lambda_8(L^s) = \lambda_9(L^s) = \lambda_{10}(L^s) = \lambda_{11}(L^s) = 0.05$. $\lambda_q(L^s)$ represents the minimum eigenvalue of the minor matrix by removing the q-th row-column pair of Laplacian matrix. They came out with the result that node 1 is the most efficient for control. However, in terms of degree centrality [40], nodes 2 and 7 emerge as the most significant. Node 1 acts as a pivotal "bridge" connecting two prominent clusters, facilitating extensive information exchange. Following nodes 2 and 7, which hold equal importance in the network, the remaining nodes also contribute significantly, maintaining network coherence. To verify their results, they controlled the 1st, the 2nd, the 7th and the 3rd

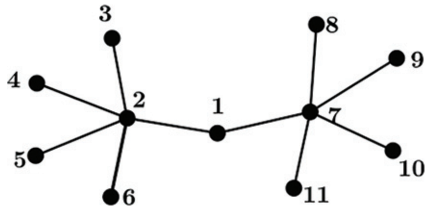

Fig. 1: Topology of the bi-star network. Source: ref. [39].

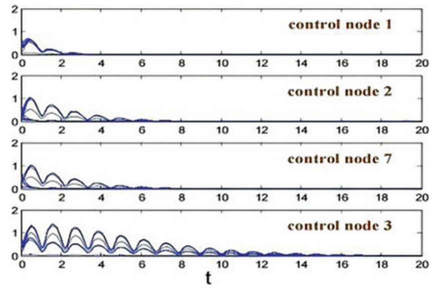

Fig. 2: Norm of error $\|e_j\|_2$ $(j = 1, 2, \dots, 11)$ *vs.* time t. Controlling node j_1 $(j_1 \in 1, 2, 7, 3)$. Source: ref. [39].

nodes, respectively. It is shown in fig. 2 that the time sequence of achieving synchronization is in agreement with the sequence of node importance. Similarly, the importance sequencing of l nodes in a network is determined in descending order of $\lambda_q(L^s)$ $(1 \le q \le C_N^l)$, where $\lambda_q(L^s)$ is the minimum eigenvalue of the minor matrix of $\frac{L+L^T}{2}$ by removing the corresponding l row-column pairs. Their study thoroughly considers the network structure, making significant advancements.

Based on the properties of $\lambda_q(L^s)$, Liu *et al.* [41] found and proved an amazing phenomenon. It is better to pin the nodes with large degrees when the proportion of pinned nodes is relatively small, while it is better to pin nodes with small degrees when the proportion of pinned nodes is large.

Besides, researches on multi-layer networks [42–44] and large-scale [45] networks synchronization are constantly advancing. In addition, the optimization approaches [46,47] related to pinning control are also continuously developing.

1-order edges (edges). – In the theory of complex networks, edge control [48–51] refers to the manipulation of specific edge structures or weights within a network to influence the overall behavior or properties of the network. Specifically, edge pinning control can be understood as controlling the inner coupling matrix in the network.

In ref. [50] authors studied distributed adaptive control of synchronization in complex networks. An effective distributed adaptive strategy to tune a small fraction of coupling weights of a network was designed based on local information of node dynamics. Here the authors used Laplacian matrix and proposed an adaptive strategy to control edges in a subgraph $\widetilde{\mathcal{G}} \subseteq \mathcal{G}$ with a set of l selected undirected $\widetilde{\mathcal{E}} \subseteq \mathcal{E}$,

$$\dot{L}_{ij}(t) = \dot{L}_{ji}(t) = -\alpha_{ij}(x_i - x_j)^T g(x_i - x_j), (i,j) \in \widetilde{\mathcal{E}}, \quad (8)$$

where $\alpha_{ij} = \alpha_{ji}$ are positive constants.

In terms of assessing the importance of edges, researches in this area are not yet fully sufficient. The existing indicators almost just rank the importance of the present edges in a network. Zhou *et al.* [52] noted that the edge centrality in a network includes two aspects, namely, the importance of the edges (abbreviation for the present edges) and that of the absent edges. A new edge centrality measure was proposed based on the effect of an edge on the network algebraic connectivity, which connects the topology and the dynamical properties. They denoted the variation of the second smallest Laplacian eigenvalues as $\Delta\lambda_{ij}^- = \lambda_2(L^-) - \lambda_2(L)$ and $\Delta\lambda_{ij}^+ = \lambda_2(L^+) - \lambda_2(L)$, respectively, where $\Delta\lambda_{ij}^-$ ($\Delta\lambda_{ij}^+$) represents the decrease (increase) of the second smallest Laplacian eigenvalue caused by deleting (adding) edge e_{ij} in the network. They defined connectivity rank index (CRI) and proposed that the closer the absolute value of the CRI of an edge (absent edge) is to 1, the more important it is.

CRI of an edge: For any $e_{ij} \in G$, the CRI of edge e_{ij} is

$$I_{ij}^- = \frac{\Delta\lambda_{ij}^-}{|\min_{e_{mn}\in G} \Delta\lambda_{mn}^-|},$$

where $I_{ij}^- = 0$ if $\min_{e_{mn}\in G} \Delta\lambda_{mn}^- = 0$.

CRI of an absent edge: For any $e_{ij} \notin G$, the CRI of absent edge e_{ij} is

$$I_{ij}^+ = \frac{\Delta\lambda_{ij}^+}{|\max_{e_{mn}\notin G} \Delta\lambda_{mn}^+|},$$

where $I_{ij}^+ = 0$ if $\min_{e_{mn}\notin G} \Delta\lambda_{mn}^+ = 0$.

In directed network, Jiang *et al.* [53] found that the dynamical contribution value of directed edge e_{ij} from node i to node j can be determined by Fiedler vectors: $I_{e_{ij}}^d = \mathrm{Re}[y_j(x_j - x_i)]$, where x_p and y_p are the p-th components of x and y, x is the right eigenvector and y^T is the left eigenvector, respectively. The right Fiedler vector provides information about the starting and ending nodes, whereas the left Fiedler vector only provides information about the end node. Consequently, comprehensive information about the edge's contribution can be provided by the combination of the left and right Fiedler vectors in a network.

0-order and 1-order edges (nodes and edges). – To achieve network synchronization, people can apply controllers to regulate both nodes and edges within the network simultaneously.

Liu *et al.* [31] considered synchronization with adaptive intermittent control on both nodes and edges. Adaptive intermittent control expressions below

$$\begin{cases} u_i(t) = c_{ij} \cdot c \cdot g(s(t) - x_i(t)), & i = 1, \\ \dot{c}_{ij}(t) = h \sum_{j=1}^{N} \|x_j(t) - s(t)\|^2, & h > 0, \end{cases} \quad (9)$$

are simple and put on single node and every edge only in work time.

Su *et al.* [54] partitioned the entire network into different clusters $[G_1, \ldots, G_d]$ and thereby defined cluster synchronization and global synchronization. They applied the same adaptive controller to at least one node x_i and applied adaptive controllers to edges within the same cluster,

$$\begin{cases} u_i(t) = h_i c_i(t)(\bar{x}_i(t) - x_i(t)), \\ \dot{c}_i(t) = h_i k_i (x_i(t) - \bar{x}_i(t))^T P(x_i(t) - \bar{x}_i(t)), \\ \dot{c}_{ij}(t) = h_{ij} a_{ij} k_{ij} (x_i(t) - x_j(t))^T P(x_i(t) - x_j(t)), \end{cases} \quad (10)$$

where $\bar{x}_l(t)$ is the desired state of the l-th cluster G_l and $P = \mathrm{diag}\{p_1, \ldots, p_n\}$ is a positive-definite diagonal matrix. The positive constants $k_{ij} = k_{ji}$ and k_i are the weights of the adaptive laws for parameters $c_{ij}(t)$ and $c_i(t)$, respectively. If node i is selected to be pinned, then $h_i = 1$; otherwise, $h_i = 0$. If nodes i and j are in the same cluster, then $h_{ij} = 1$; otherwise, $h_{ij} = 0$. Then, all clusters asymptotically synchronize to their given states, namely

$$\lim_{t\to\infty} \sum_{l=1}^{d} \sum_{i\in G_l} \|x_i(t) - \bar{x}_l(t)\| = 0, \quad (11)$$

their approach is to use a decentralized method, *i.e.*, each node only needs the state information of its neighbors and only selects those few nodes that have the information of their desired states.

Exploration from lower order to higher. – Researchers have continuously studied the pinning control of higher-order edges in networks, excluding 0-order and 1-order edges. In this section, by introducing cycles we will discuss higher-order edges, such as simplexes.

A simplex of order $(k-1)$ is formed by the interaction of k nodes $i_1, i_2, \ldots, i_k$, denoted as $[i_1, i_2, \ldots, i_k]$. A simplicial complex is formed by a collection of simplexes. For example, the set of n simplexes $\xi_1, \xi_2, \ldots, \xi_n$ forms a simplicial complex K. The highest order of these simplices $\xi_1, \xi_2, \ldots, \xi_n$ is defined as the order of the simplicial complex.

A cycle in a network is simply defined as a closed (non-repeating) path with the same starting and ending node. It is necessary to form higher-order edges. Studying the important ranking of cycles will not only have a significant impact on the control of edges, but also benefits the study of controlling higher-order edges. Therefore, it is an important "bridge" that we explore from lower to higher orders.

To study the impact of cycles on λ_2 in an undirected network, Jiang *et al.* [55] introduced the edge adding operation $\mathcal{G}_e = \mathcal{G} + e$ and node hanging operation $\mathcal{G}_{ev} = \mathcal{G} + ev$, which can be abbreviated as the adding and hanging operations, respectively. An edge is included in an adding operation, whereas the hanging operation adds a node and connects it to an existing one. By studying the impact of adding operation and hanging operation on algebraic connectivity they found that the hanging operation decreases

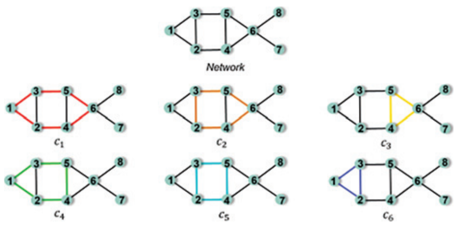

Fig. 3: A sample network with 6 cycles. Cycles are identified as $c_1, c_2, \ldots, c_6$ which are depicted by distinct colors. Source: ref. [55].

the λ_2, while adding operation increases the λ_2. Every time an edge is added in a connected graph, at least one cycle is introduced into the graph. So they proposed that the more edges are added, the more cycles appear in a network, and the larger the λ_2.

Also, they searched for the key cycles in a complex network. Let $C = \{c_i \mid i = 1, 2, \ldots, k\}$ be the set of all cycles in a network, where k is the number of cycles and c_i represents one of the cycles. They proposed a new cycle ranking index $I_{c_i} = \sum_{(p,q)\in\mathcal{E}_i}(x_p - x_q)^2$ to measure the importance of a cycle, where $\mathcal{E}_i$ is the set of all edges of cycle c_i, x_p is the p-th component of x in eq. (2). The larger the I_{c_i} is, the more key the cycle is. They gave an example considering a network coupled with Chua's circuits in fig. 3. They found $I_{c_1} = 0.3741 > I_{c_2} = 0.3490 > I_{c_3} = 0.1917 > I_{c_4} = 0.1824 > I_{c_5} = 0.1573 > I_{c_6} = 0.0251$, satisfying the experimental results: controlling c_1 achieves synchronization the fastest.

To study the pinning control of simplex in an undirected, weighted and connected network, Zhou *et al.* [56] considered a simplicial complex of order d, $\mathcal{G}^h = (\mathcal{V}, \mathcal{E})$, where the vertex set and simplex set are denoted as $\mathcal{V}$ and $\mathcal{E}$ respectively, and the dynamics of the i-th node in this higher-order network is described as

$$\begin{aligned}\dot{x}_i = {} & f(x_i) + c_1 \sum_{j_1=1}^{N} \sigma_{ij_1}^{(1)}(t) A_{ij_1}^{(1)} h^{(1)}(x_i, x_{j_1}) \\ & + c_2 \sum_{j_1=1}^{N}\sum_{j_2=1}^{N} \sigma_{ij_1j_2}^{(2)}(t) A_{ij_1j_2}^{(2)} h^{(2)}(x_i, x_{j_1}, x_{j_2}) \\ & \vdots \\ & + c_D \sum_{j_1=1}^{N} \cdots \sum_{j_D=1}^{N} \sigma_{ij_1\ldots j_D}^{(D)}(t) A_{ij_1\ldots j_D}^{(D)} h^{(D)}(x_i, \ldots, x_{j_D}) \\ & i = 1, \ldots, N, \end{aligned} \tag{12}$$

where $f(\cdot)$ represents the dynamics of a single node, c_d represents the d-th order coupling strength of a higher-order network and $h^{(d)}(x_i, x_{j_1}, \ldots, x_{j_d})\colon R^{(d+1)\times n} \to R^n$ represents the d-th order inner-coupling function. $h^{(d)}(x_i, x_{j_1}, \ldots, x_{j_d})$ usually satisfies the natural coupling condition $h^{(d)}(x, x, \ldots, x) = \cdots = h^{(2)}(x, x) = h^{(1)}(x)$, $d = 1, \ldots, D$. $\sigma_{ij_1\cdots j_d}^{(d)}(t)$ represents the weight of the d-order interaction constituted by the nodes $i, j_1, \ldots, j_d$ at time t. If nodes i and j interact, then $A_{ij}^{(1)} = 1$ in the tensor matrix; otherwise, $A_{ij}^{(1)} = 0$. Similarly, if the nodes $i, j_1, \ldots, j_d$ form a higher-order interaction, then $A_{ij_1\cdots j_d}^{(d)} = 1$ in the tensor matrix; otherwise, $A_{ij_1\cdots j_d}^{(d)} = 0$. Let $\widetilde{\mathcal{G}^h} = (\widetilde{\mathcal{V}}^{(d)}, \widetilde{\mathcal{E}}^{(d)})$ be a subgraph of $\mathcal{G}^h$, where $\widetilde{\mathcal{V}}^{(d)}$ is the set of nodes corresponding to $\widetilde{\mathcal{E}}^{(d)}$, with $d = 1, 2$. They proposed if $\|Df(s)\|$ is bounded, the 2nd-order simplicial complex corresponding to $D = 2$ in model (12) synchronizes under the adaptive control law

$$\begin{cases} \dot{\sigma}_{ij}^{(1)}(t) = -\alpha_{ij}(x_i - x_j)^T(x_i - x_j), \ [i^{2*}, j^{2*}] \in \widetilde{\mathcal{E}}^{(1)}, \\ \dot{\sigma}_{ijk}^{(2)}(t) = -[\beta_{ij}(x_i - x_j)^T(x_i - x_j) \\ +\beta_{ik}(x_i - x_k)^T(x_i - x_k) + \beta_{kj}(x_k - x_j)^T(x_k - x_j)], \\ [i^{2*}, j^{2*}, k^{2*}] \in \widetilde{\mathcal{E}}^{(2)}. \end{cases} \tag{13}$$

Also, they studied a method to rank the simplicial importance in networks. They considered a d-order simplicial complex $\mathcal{G}^h = (\mathcal{V}, \mathcal{E})$ which is completely determined by l_d simplexes of order d $[i_1^{d_1}, \ldots, i_{d+1}^{d_1}]$, $[i_1^{d_2}, \ldots, i_{d+1}^{d_2}], \ldots, [i_1^{d_{l_d}}, \ldots, i_{d+1}^{d_{l_d}}]$. The $(d+1)$-tuple $[i_1^{d_j}, \ldots, i_{d+1}^{d_j}]$ is a simple form of G^h of order d with label d_j, where $d = 1, \ldots, D$, $i_1^{d_j}, \ldots, i_{d+1}^{d_j} \in \mathcal{V}$ and $j = 1, \ldots, l_d$. The set of a d-order simplex $[i_1^{d_p}, \ldots, i_{d+1}^{d_p}]$ is denoted with label d_p in G^h and all its corresponding non-zero lower-order simplices are denoted with D^{d_p}. They increased the coupling strength of D^{d_p} by ϵ and denoted the modified simplices by $G_{d_p^\epsilon}^h$. The variation of the second smallest eigenvalue of the original network is denoted by $\Delta_{d_p^\epsilon}^s = \lambda_2(G_{d_p^\epsilon}^h) - \lambda_2$. They provided an approximate expression for $\Delta_{d_p^\epsilon}^s$. Regarding the problem of how to choose appropriate simplex pinning control to facilitate faster adaptive synchronization in higher-order networks, they proposed an approach. If the coupling strength of the D^{d_p} is increased by ϵ, the greater the influence on $\Delta_{d_p^\epsilon}^s$ is, the more the D^{d_p} should be controlled.

Conclusions. – In this review, we introduce various methods for integrating controllers into the nodes, edges, and higher-order edges of a complex network. In general, when considering node (0-order edge)-pinning synchronization, the minimum eigenvalue of the minor matrix of $\frac{L+L^T}{2}$ by removing the corresponding row-column pairs is considered. In the case of edge-pinning or higher-order edge-pinning synchronization, the second smallest Laplacian eigenvalue is investigated. Additionally, the study of higher-order networks is an emerging field, promising further breakthroughs.

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