Uncover the Relation Between Consensus and Topology of Directed Network: The Minimum Node In-Degree of Directed Cycles

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***Abstract*—Directed networks widely exist in industrial, social, and natural networks, the consensus of which is characterized by the convergence rate. In this article, a quantitative analysis for the inhibitory effect of directed cycles on the consensus of directed network is provided. Specifically, directed cycle necessarily inhibits network consensus when there is at least one node with in-degree of 1 on the cycle, and the more the nodes with in-degree of 1, the greater the inhibition. The inhibition induced by multiple disjoint directed cycles is only determined by the directed cycle with the minimum node in-degree, independent of other directed cycles. Generally, the more intersected directed cycles, the smaller the convergence rate. Accordingly, to improve the consensus or synchronizability of a directed network, it is necessary to avoid the generation of multiple intersected directed cycles, especially those with a large number of in-degree 1.**

***Index Terms*—Complex network, consensus, directed cycle, directed network, synchronization.**

I. Introduction

Complex networks have played important roles in various fields such as bioscience [1], [2], disease transmission [3], [4], and engineering technology [5], [6]. The consensus of a network with a spanning tree is characterized by the minimal real part of the nonzero Laplacian eigenvalue, which is called the convergence rate [7], [8]. According to the directionality of edges in the network, complex networks can be divided into directed networks and undirected networks.

However, most existing results of the consensus performance are derived from undirected networks rather than directed networks [9], [10], [11], [12]. For instance, adding an edge to an undirected network increases the convergence rate [13], [14], and hanging a node decreases the convergence rate [15]. It is thus natural to consider the impact of edge-adding and hanging node operations on the consensus performance of directed networks. Directed acyclic network is a directed network that does not contain directed cycles [16], [17], [18], such as directed chain, directed star-chain, and directed binary tree. Recently, the dynamic analysis of directed acyclic network has been extensively explored via edge-adding operations, mainly involving synchronization or consensus. For example, Zhang et al. [19] investigated the effect of adding a reverse edge on the dominant convergence rate of directed chain and directed grid, and found that the reverse edge decreases the convergence rate. Subsequently, based on [19], Hao et al. [20] considered the impact of adding two reverse edges on the consensus performance of directed chain, and the effect of adding a reverse edge across a stem in a directed acyclic network is discussed in [21]. These existing results indicate that a reverse edge has a significant inhibitory effect on the convergence rate of directed acyclic network, since the new network becomes a directed cyclic network. Nevertheless, there is currently little literature exploring whether more complex edge-adding operations for general directed networks still possess this property, or what the underlying reason for this inhibitory effect might be. Directed cycles in directed networks can be seen as being formed by adding directed edges to directed acyclic networks. Accordingly, the main problem of this article is to investigate how directed cycles determine the consensus performance of a directed network from the perspective of adding edges to a directed acyclic network. It is found that the minimum in-degree of nodes on the directed cycles determines the convergence rate of the new networks, and the convergence rate decreases with the increase in the number of nodes having minimum in-degree on the cycle, whereas it increases with the number of other nodes. In addition, the impact of multiple disjoint directed cycles on the convergence rate is only determined by the directed cycle with the minimum node in-degree, and the convergence rate decreases as the number of intersected cycles increases. Furthermore, we apply our results to a class of directed star-chain networks.

The rest of this article is organized as follows. In Section II, some definitions of directed networks and preliminary lemma are given. In Section III, we investigate the effect of edge-adding operation on the convergence rate of directed acyclic network, and conclude that directed cycles suppress the consensus performance of directed network. In Section IV, several numerical examples are provided to verify the validity of the theoretical results. Finally, Section V concludes this article.

II. Preliminaries and Model Description

This article focuses on the effect of directed cycles on directed network consensus. For this purpose, we first provide the basic definitions of directed networks as below.

Definition 1: A node is called the *root node* (or *leader node*) of a directed network if it does not receive information from other nodes. For each directed edge of the network, the start node is called the *head node*, and the end node is called the *tail node*.

Definition 2: For a directed network, the in-degree of node i refers to the number of directed edges pointing toward node i from other nodes.

Definition 3: Adding a node and a directed edge pointing toward this node from other existing nodes is called a *forward hanging operation*.

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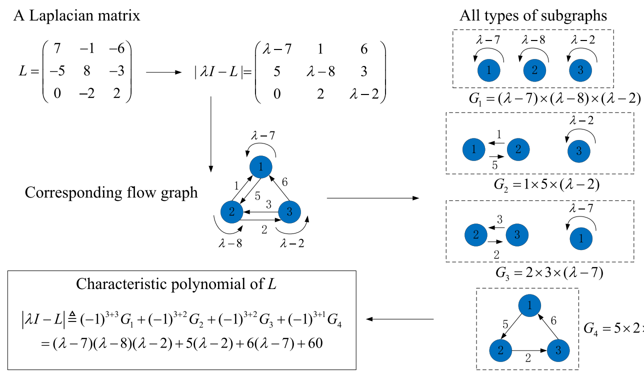

Fig. 1. Third-order Laplacian and its corresponding subgraphs.

Definition 4: A *directed cycle of length* p is defined as a directed closed loop containing p nodes $\{n_1, n_2, \ldots, n_p\}$ and p directed edges $\{(n_1, n_2), (n_2, n_3), \ldots, (n_p, n_1)\}$.

Definition 5: For a directed network with a spanning tree, the minimal real part of the nonzero Laplacian eigenvalues $\mathcal{R}e(\lambda_2)$ is called the *convergence rate*.

Definition 6: For l intersected directed cycles, if any two directed cycles have shared nodes, this pattern is called *full intersection*, otherwise it is called *nonfull intersection*.

Generally, to obtain the characteristic equation of the Laplacian for a directed network, an ordering of the nodes needs to be given in advance [19], [20], [21], [22]. However, due to the low fault tolerance, this traditional method is inconvenient for high-dimensional networks. To efficiently get the convergence rate of a directed network, Lemma 1 is employed.

Lemma 1 ([23]): For an n-order square matrix Q corresponding to a flow graph G, the determinant of Q is given as follows:

$$\det Q = \sum_{i=1}^{\rho} (-1)^{n+n_i} G_i$$

where ρ represents the number of subgraphs for G, each consisting of noncontact loops of n nodes. n_i and G_i are the number of directed loops and the product of the connection gains of each edge in the ith subgraph($i = 1, \ldots, \rho$).

Lemma 1 is actually a method for calculating the determinant of a square matrix, and its novelty lies in associating the determinant with the flow graph. An n-order matrix determines a graph of n nodes, and the elements in the matrix represent edges weights between the nodes in the graph. Therefore, Lemma 1 illustrates the transformation of the determinant of a square matrix into the calculation of the connection gains of noncontact loops corresponding to the flow graph. As long as we know the in-degree of all nodes and the distribution of directed cycles of the network, it is easy to obtain the Laplacian characteristic equation using Lemma 1. For example, Fig. 1 shows a third-order Laplacian and its corresponding flow graph, as well as all types of subgraphs.

Consider a dynamical directed acyclic network containing a spanning tree as follows:

$$\dot{x}_i = -\sum_{j=1}^{n} b_{ij}(x_i - x_j) \tag{1}$$

where $x_i \in \mathbb{R}$ represents the state of node i; $b_{ij} \in \mathbb{R}$ is the coupling strength between nodes i and j. If there is no directed edge from node j to node i, $b_{ij} = 0$; otherwise, $b_{ij} = 1$ for $i, j = 1, 2, \ldots, n$. Define the whole state of nodes by $x = [x_1, x_2, \ldots, x_n]$ and the Laplacian matrix by L, then (1) can be reformulated as follows:

$$\dot{x}(t) = -Lx(t). \tag{2}$$

Clearly, L has a simple eigenvalue 0 with the corresponding eigenvector $\mathbf{1}_n = (1, \ldots, 1)$, and the other $n-1$ eigenvalues have positive real parts, which is denoted as $\lambda_i(L)$ $(i = 1, 2, \ldots, n)$, and $0 = \mathcal{R}e(\lambda_1) < \mathcal{R}e(\lambda_2) \leq \cdots \leq \mathcal{R}e(\lambda_n)$.

III. Consensus in Directed Network

In this section, we first consider the dynamics of general directed acyclic networks, then further investigate the impact of directed cycles on the dynamics of directed cyclic networks by adding specific directed edges.

A. Directed Acyclic Network

Assume that the directed acyclic network contains a spanning tree and that there is at least one node with in-degree of 1. The general property on the convergence rate of a directed acyclic network is given by the following lemma.

Lemma 2: The convergence rate of a directed acyclic network is determined by the minimal nonzero in-degree of nodes, i.e., $\lambda_2 = 1$. Moreover, forward hanging operation maintains the convergence rate of the network.

Proof: For a directed acyclic network with n nodes, define the number of nodes with in-degree $\{k_1, k_2, \ldots, k_p\}$ as $\{j_1, j_2, \ldots, j_p\}$, where $1 = k_1 < k_2 < \cdots < k_p$ and $j_1 + j_2 + \cdots + j_p + 1 = n$. By Lemma 1, due to the absence of directed cycles in the directed acyclic network, its characteristic polynomial of Laplacian L only contains one term, given as follows:

$$|\lambda I - L| = \lambda \prod_{i=1}^{p} (\lambda - k_i)^{j_i}$$

and $\lambda_2 = k_1 = 1$, which shows that the convergence rate of the network is determined by the minimum in-degree of all nodes. Forward hanging operation on the directed acyclic network will not form a directed cycle, so the new network still has a directed acyclic graph, and the Laplacian characteristic polynomial is a single term, as shown below:

$$\left|\lambda I - L^{\text{forward}}\right| = \lambda(\lambda - k_1)^{j_1+1} \prod_{i=2}^{p} (\lambda - k_i)^{j_i}$$

and $\lambda_2^{\text{forward}} = k_1 = 1$, implying that the forward hanging operation does not change the convergence rate.

Lemma 3: For a directed acyclic network, if the newly added directed edges do not cause the generation of directed cycles, then the convergence rate of the network remains unchanged even if the number of edges is sufficiently large.

Proof: The new network is still a directed acyclic network containing a directed spanning tree and having one node with in-degree of 1, so adding directed edges without creating cycles will not affect the convergence rate of the network.

It is generally accepted that a reverse edge of directed chain can result in a directed cycle, so the characteristic polynomial becomes the sum of two terms, causing a decrease in the convergence rate. More generally, directed cycles may be formed by the addition of multiple edges (see Fig. 2). Therefore, what we are concerned with is the effect of added directed cycles on the convergence rate of the directed acyclic network; that is, the effect of directed cycles on the directed network dynamics.

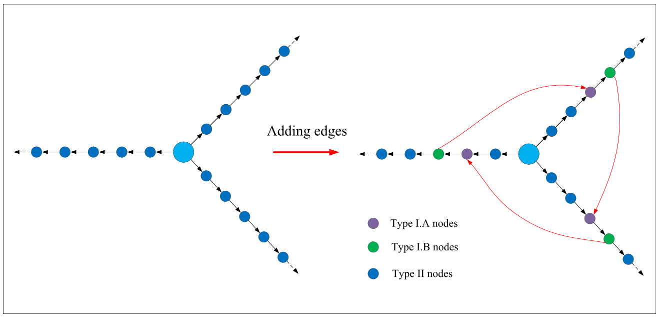

Fig. 2. Schematic diagram of adding edges to a directed star-chain network.

B. Directed Cyclic Network

In what follows, we first investigate the effect of a new directed cycle on the convergence rate of the directed cyclic network, and then extend it to multiple directed cycles.

Case I: A single directed cycle. For convenience, the following lemma on a polynomial equation is given in advance.

Lemma 4: For the polynomial equation

$$P(x) \triangleq x^{c_0} \prod_{i=1}^{q_1} (x + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (x + h_j)^{c_j} = 1 \tag{3}$$

with integers $c_0 \geq 1, q_1, q_2 \geq 1, w_i, c_j \geq 1, d_i, h_j \geq 1, i = 1, \ldots, q_1$, and $j = 1, \ldots, q_2$, one has

1) Equation (3) has a unique positive real root $x = x_1 \in (0, 1)$;
2) The real part of the other roots of (3) is smaller than the positive real root x_1;
3) The positive real root x_1 depends on c_0, d_i, w_i, h_j, and c_j, explicitly denoted as follows:

$$x_1(c_0, D, W, H, C)$$

with $D = [d_1, \ldots, d_{q_1}]$, $W = [w_1, \ldots, w_{q_1}]$, $H = [h_1, \ldots, h_{q_2}]$, and $C = [c_1, \ldots, c_{q_2}]$, which is a strictly increasing function in c_0, and decreasing function in d_i, w_i, h_j, and c_j with all other arguments fixed.

Proof: To prove Claim 1). It is easy to check that $P(0) = 0 < 1$ and $P(1) = \prod_{i=1}^{q_1} (d_i + 2)^{w_i} \prod_{j=1}^{q_2} (h_j + 1)^{c_j} > 1$. Since $P(x)$ is a continuous function, (3) has at least one positive real root in (0,1). On the other hand, $P(x)$ is a strictly increasing of x over $[0, +\infty)$, implying that there is a unique positive real root $x = x_1 \in (0, 1)$ for (3).

To prove Claim 2). Suppose that $z = a + b\mathrm{i}$ is another root of (3), where $a, b \in \mathbb{R}$, and i is the imaginary unit. When $b = 0$, since (3) has only one positive real root x_1, it is clear that $a \leq 0 < x_1$. For $b \neq 0$, we have

$$\begin{aligned} &|P(z)| = 1 \\ &= |(a+b\mathrm{i})^{c_0}| \left| \prod_{i=1}^{q_1} (a + b\mathrm{i} + d_i + 1)^{w_i} \right| \left| \prod_{j=1}^{q_2} (a + b\mathrm{i} + h_j)^{c_j} \right|. \end{aligned}$$

Hence, if $a \leq 0$, then clearly $a \leq 0 < x_1$. Otherwise, one has

$$\begin{aligned} 1 &= |(a+b\mathrm{i})^{c_0}| \left| \prod_{i=1}^{q_1} (a + b\mathrm{i} + d_i + 1)^{w_i} \right| \left| \prod_{j=1}^{q_2} (a + b\mathrm{i} + h_j)^{c_j} \right| \\ &> |a^{c_0}| \left| \prod_{i=1}^{q_1} (a + d_i + 1)^{w_i} \right| \left| \prod_{j=1}^{q_2} (a + h_j)^{c_j} \right| = P(a). \end{aligned}$$

Since $P(x_1) = 1$ and $P(x)$ is strictly monotonically increasing of x, one has $a < x_1$.

To prove Claim 3). Define

$$\begin{aligned} &s = x_1(c_0, D, W, H, C), \ r_1 = x_1(c_0 + l, D, W, H, C) \\ &r_2 = x_1(c_0, D, w_1, \ldots, w_* + l, \ldots, w_{q_1}, H, C) \\ &r_3 = x_1(c_0, D, W, H, c_1, \ldots, c_* + l, \ldots, c_{q_2}) \\ &r_4 = x_1(c_0, d_1, \ldots, d_* + l, \ldots, d_{q_1}, W, H, C) \\ &r_5 = x_1(c_0, D, W, h_1, \ldots, h_* + l, \ldots, h_{q_2}, C). \end{aligned}$$

One needs to check that $r_1 > s$, $r_2 < s$, $r_3 < s$, $r_4 < s$, and $r_5 < s$ for any integer $l > 0$. Thus, we have

$$s^{c_0} \prod_{i=1}^{q_1} (s + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (s + h_j)^{c_j} = 1$$

$$r_1^{c_0 + l} \prod_{i=1}^{q_1} (r_1 + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (r_1 + h_j)^{c_j} = 1$$

$$\Downarrow$$

$$r_1^{c_0} \prod_{i=1}^{q_1} (r_1 + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (r_1 + h_j)^{c_j} = \frac{1}{r_1^l} > 1$$

$$r_2^{c_0} (r_2 + d_* + 1)^{w_* + l} \prod_{i=1, i \neq *}^{q_1} (r_2 + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (r_2 + h_j)^{c_j} = 1$$

$$\Downarrow$$

$$r_2^{c_0} \prod_{i=1}^{q_1} (r_2 + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (r_2 + h_j)^{c_j} = \frac{1}{(r_2 + d_* + 1)^l} < 1$$

$$r_3^{c_0} (r_3 + h_*)^{c_* + l} \prod_{i=1}^{q_1} (r_3 + d_i + 1)^{w_i} \prod_{j=1, j \neq *}^{q_2} (r_3 + h_j)^{c_j} = 1$$

$$\Downarrow$$

$$r_3^{c_0} \prod_{i=1}^{q_1} (r_3 + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (r_3 + h_j)^{c_j} = \frac{1}{(r_3 + h_*)^l} < 1$$

$$r_4^{c_0} (r_4 + d_* + l + 1)^{w_*} \prod_{i=1, i \neq *}^{q_1} (r_4 + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (r_4 + h_j)^{c_j} = 1$$

$$\Downarrow$$

$$r_4^{c_0} \prod_{i=1}^{q_1} (r_4 + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (r_4 + h_j)^{c_j} = \left(\frac{r_4 + d_* + 1}{r_4 + d_* + l + 1} \right)^{w_*} < 1$$

$$r_5^{c_0} (r_5 + h_* + l)^{c_*} \prod_{i=1}^{q_1} (r_5 + d_i + 1)^{w_i} \prod_{j=1, j \neq *}^{q_2} (r_5 + h_j)^{c_j} = 1$$

$$\Downarrow$$

$$r_5^{c_0} \prod_{i=1}^{q_1} (r_5 + d_i + 1)^{w_i} \prod_{j=1}^{q_2} (r_5 + h_j)^{c_j} = \left(\frac{r_5 + h_*}{r_5 + h_* + l} \right)^{c_*} < 1$$

where $*$ represents the index of a certain parameter. Since $P(x)$ is strictly monotonically increasing with respect to x over $[0, +\infty)$, it is evident that $r_1 > s$, and $r_2 < s$, $r_3 < s$, $r_4 < s$, and $r_5 < s$; that is, x_1 increases in c_0 and decreases in d_i, w_i, h_j, and c_j, respectively.

For a directed acyclic network with n nodes, suppose that adding m directed edges causes the generation of a directed cycle with $p\,(p \geq m)$ nodes, where the nodes on the new network can be divided into three types as follows:

Type I.A m nodes on the cycle as tail nodes of m directed edges. Define w_i nodes with in-degree of $d_i + 1$, satisfying $\sum_{i=1}^{q_1} w_i = m$, and $1 \leq d_1 \leq \cdots \leq d_m$ and $1 \leq q_1 \leq m$. See purple nodes in Fig. 2.

Type I.B $p - m$ nodes on the cycle, which are not the tail nodes of m directed edges. Define c_j nodes with in-degree of h_j, satisfying $\sum_{j=1}^{q_2} c_j = p - m$, and $1 \leq h_1 \leq \cdots \leq h_{p-m}$ and $0 \leq q_2 \leq p - m$. Here, $q_2 = 0$ means that there are no Type I.B nodes. See green nodes in Fig. 2.

Type II. The remaining $n - p$ nodes which are not on the directed cycle. Define e_k nodes with in-degree of g_k, satisfying $\sum_{k=1}^{q_3} e_k = n - p$, and $0 = g_1 < g_2 \leq \cdots \leq g_{n-p}$, $1 \leq q_3 \leq n - p$. See blue nodes in Fig. 2.

Note that q_1, q_2, and q_3, respectively, represent the number of in-degree types for Type I.A, Type I.B, and Type II.

Then, we define $h_0 = \min\{h_1, d_1 + 1\}$ and $h_{\max} = \max\{h_{p-m}, d_m + 1\}$ as the minimum and maximum node in-degree of the cycle, and c_0 and $c_{\max}$ as the number of nodes with in-degree h_0 and $h_{\max}$. Notice that

$$c_0 = \begin{cases} w_1, & h_0 = d_1 + 1 \\ c_1, & h_0 = h_1 \end{cases}, c_{\max} = \begin{cases} c_{q_1}, & h_{\max} = d_m + 1 \\ c_{q_2}, & h_{\max} = h_{p-m}. \end{cases}$$

Furthermore, we obtain the effect of this directed cycle on the convergence rate of the network in the following theorem.

Theorem 1: For the n-sized directed acyclic network with an added directed cycle having m Type I.A nodes, $p - m$ Type I.B nodes, and $n - p$ Type II nodes, one has

1) Only p eigenvalues of the Laplacian are changed and the other $n - p$ eigenvalues remain unchanged;
2) if $p = m$ and $d_1 = d_2 = \cdots = d_{q_1} = d$ hold, or $d_1 + 1 = \cdots = d_{q_1} + 1 = h_1 = \cdots = h_{q_2} = d + 1$ holds, then the convergence rate remains unchanged, i.e., $\lambda_2 = \min\{d, g_2\}$;
3) the p changed eigenvalues is determined as follows:

$$\prod_{i=1}^{q_1} (d_i + 1 - \lambda)^{w_i} \prod_{j=1}^{q_2} (h_j - \lambda)^{c_j} = 1 \quad (4)$$

that always admits a simple real root $\lambda_{\min} \in (h_0 - 1, h_0)$, which depends on c_0, d_i, w_i, and h_j, c_j and can be explicitly denoted as $\lambda_{\min}(c_0, D, W, H, C)$;
4) $\lambda_{\min}(c_0, D, W, H, C)$ is a strictly decreasing function of c_0 and increasing function of d_i, w_i, h_j, and c_j, respectively.
5) $\lambda_2 = \min\{\lambda_{\min}, g_2\}$ is the convergence rate of the new network. In particular, $\lambda_2 = \lambda_{\min}$ is the convergence rate when $h_0 = h_1 = 1$.

Proof: By Lemma 1, the characteristic equation of Laplacian L in original directed acyclic network can be obtained as follows:

$$\prod_{i=1}^{q_1} (\lambda - d_i)^{w_i} \prod_{j=1}^{q_2} (\lambda - h_j)^{c_j} \prod_{k=1}^{q_3} (\lambda - g_k)^{e_k} = 0 \quad (5)$$

and $\lambda_2(L) = \min\{d_1, h_1, g_2\}$. After adding a directed cycle, the characteristic equation of new Laplacian L_1 is given as follows:

$$\prod_{i=1}^{q_1} (\lambda - d_i - 1)^{w_i} \prod_{j=1}^{q_2} (\lambda - h_j)^{c_j} \prod_{k=1}^{q_3} (\lambda - g_k)^{e_k} + (-1)^{2n-p+1} \prod_{k=1}^{q_3} (\lambda - g_k)^{e_k} 1^p = 0$$

namely,

$$\prod_{k=1}^{q_3} (\lambda - g_k)^{e_k} \left[\prod_{i=1}^{q_1} (d_i + 1 - \lambda)^{w_i} \prod_{j=1}^{q_2} (h_j - \lambda)^{c_j} - 1 \right] = 0. \quad (6)$$

For Claim 1), comparing (6) with (5), we find that (6) has e_k multiple eigenvalues at g_k for $\sum_{k=1}^{q_3} e_k = n - p$, which are the same as (5), and the remaining p eigenvalues are determined as follows:

$$\prod_{i=1}^{q_1} (d_i + 1 - \lambda)^{w_i} \prod_{j=1}^{q_2} (h_j - \lambda)^{c_j} = 1. \quad (7)$$

For Claim 2), if $p = m$ and $d_1 = d_2 = \cdots = d_{q_1} = d$ hold, or $d_1 + 1 = \cdots = d_{q_1} + 1 = h_1 = \cdots = h_{q_2} = d + 1$ holds, then (7) is rewritten as follows:

$$(d + 1 - \lambda)^p = 1. \quad (8)$$

All roots of (8) on the complex plane can be represented as $\lambda(k) = d + 1 - \cos \frac{2k\pi}{m} - \mathrm{i} \sin \frac{2k\pi}{m}$ for $k = 0, 1, \ldots, m - 1$. Hence, the root with the smallest nonzero real part is $\lambda(0) = d$, implying that under this case, the convergence rate is $\lambda_2(L_1) = \min\{d, g_2\}$, which is same as that of the original directed acyclic network.

For Claims 3)–5), define

$$x = h_0 - \lambda, \tilde{d}_i = d_i - h_0, \tilde{h}_j = h_j - h_0.$$

Equation (7) is equivalent to

$$x^{c_0} \prod_{i=1}^{q_1} (\tilde{d}_i + 1 + x)^{w_i} \prod_{j=2}^{q_2} (\tilde{h}_j + x)^{c_j} = 1$$

when $h_0 = h_1$, and

$$x^{c_0} \prod_{i=2}^{q_1} (\tilde{d}_i + 1 + x)^{w_i} \prod_{j=1}^{q_2} (\tilde{h}_j + x)^{c_j} = 1$$

when $h_0 = d_1 + 1$, both which satisfy Claims 1)–3) of Lemma 4. Therefore, (7) has a simple real root $\lambda_{\min} \in (h_0 - 1, h_0)$, which is a strictly decreasing function of c_0 and increasing function of d_i, w_i, c_j, and h_j. Moreover, $\lambda_2 = \min\{\lambda_{\min}, g_2\}$ is the convergence rate of the new network. When $h_0 = 1$ and $\lambda_2 = \lambda_{\min} \in (0, 1)$ is the smallest nonzero real root with algebraic multiplicity one; that is, the convergence rate.

Remark 1: For uniformity, it is specified that $m = 0$, $p = 0$, and $h_0 = 0$ for directed acyclic networks. Theorem 1 indicates that a directed cycle may maintain or decrease the convergence rate of a directed acyclic network when the new network still has a root node. This directed cycle does not change the convergence rate of the directed acyclic network if and only if all its nodes have the same in-degree or the minimum in-degree of nodes outside the cycle is less than that of nodes on the cycle.

By Theorem 1, the conditions for the directed cycle to decrease the convergence rate are given by the following corollaries.

Corollary 1: A sufficient condition for a directed cycle to decrease the convergence rate of a directed acyclic network is that there exists one node with in-degree of 1 on the cycle.

Corollary 2: A necessary and sufficient condition for a directed cycle to decrease the convergence rate of a directed acyclic network is $g_2 \geq h_0$ (or $n = p + 1$) and $h_1 \leq d_1$.

Remark 2: The more nodes with in-degree of 1 on the directed cycle, the smaller the convergence rate, while the more nodes with in-degree greater than 1 on the cycle, the greater the convergence rate, but the variation in convergence rate induced by this directed cycle is less than 1. Thus, to enhance consensus performance in directed networks, it is necessary to avoid forming directed cycles, especially those with a large number of in-degree 1.

Example 1: For a directed chain with node numbers 1 to n, if adding one reverse edge from node j to node i $(1 < i < j \leq n)$, then λ_2 is the simple root of smallest real part for the following:

$$(1-\lambda)^{j-i}(2-\lambda) = 1$$

and only depends on $j - i$. Moreover, λ_2 is strictly decreasing in $j - i$.

Case II: *Multiple disjoint directed cycle*. Now, we present the effect of ℓ disjoint directed cycles on the convergence rate of directed networks.

Define the number of nodes on the κth directed cycle as p_κ, and the number of added edges as m_κ for $\kappa = 1, \ldots, \ell$. For the κth cycle, one has

Type I.A m_κ nodes on the κth cycle as tail nodes of m_κ added directed edges. Define $w_{\kappa i}$ nodes with in-degree of $d_{\kappa i} + 1$, satisfying $\sum_{i=1}^{q_{\kappa 1}} w_{\kappa i} = m_\kappa$, and $1 \leq d_{\kappa 1} \leq \cdots \leq d_{\kappa m_\kappa}$ and $1 \leq q_{\kappa 1} \leq m_\kappa$.

Type I.B $p_\kappa - m_\kappa$ nodes on the κth cycle, which are not the tail nodes of m_κ directed edges. Define $c_{\kappa j}$ nodes with in-degree of $h_{\kappa j}$, satisfying $\sum_{j=1}^{q_{\kappa 2}} c_{\kappa j} = p_\kappa - m_\kappa$, and $1 \leq h_{\kappa 1} \leq \cdots \leq h_{\kappa(p_\kappa - m_\kappa)}$ and $0 \leq q_{\kappa 2} \leq p_\kappa - m_\kappa$. Here, $q_{\kappa 2} = 0$ means that there are no Type I.B nodes for the κth cycle.

Type II. The remaining $n - \sum_{\kappa=1}^{\ell} p_\kappa$ nodes which are not on any directed cycle. Define e_k nodes with in-degree of g_k, satisfying $\sum_{k=1}^{q_3} e_k = n - \sum_{\kappa=1}^{\ell} p_\kappa \triangleq n - \rho$, and $0 = g_1 < g_2 \leq \cdots \leq g_{n-\rho}$ and $1 \leq q_3 \leq n - \rho$.

We obtain the influence of ℓ disjoint directed cycles on the convergence rate in Theorem 2.

Theorem 2: Suppose that adding edges forms ℓ disjoint directed cycles, then we have the following conclusions:

1) Only $\sum_{\kappa=1}^{\ell} p_\kappa$ eigenvalues of the Laplacian are changed and the other $n - \sum_{\kappa=1}^{\ell} p_\kappa$ eigenvalues remain unchanged;
2) the convergence rate is only determined by the directed cycle with the minimum node in-degree $h_{\min}$, and is independent of other directed cycles when $g_2 \geq h_{\min}$.

Proof: To prove Claim 1). The characteristic equation of the new Laplacian is as follows:

$$\prod_{k=1}^{q_3}(\lambda - g_k)^{e_k} \prod_{\kappa=1}^{\ell}\left(\prod_{i=1}^{q_{\kappa 1}}(d_{\kappa i}+1-\lambda)^{w_{\kappa i}} \prod_{j=1}^{q_{\kappa 2}}(h_{\kappa j}-\lambda)^{c_{\kappa j}} - 1\right) = 0. \tag{9}$$

For the derivation details of (9), the reader is referred to the Part I of Supplementary Materials (https://qinruidai.github.io/files/Supplementary_Materials.pdf). Comparing (9) with (5), only $\sum_{\kappa=1}^{\ell} p_\kappa$ eigenvalues of the Laplacian are changed, which is the same as the number of nodes on these ℓ disjoint directed cycles.

To prove Claim 2). When $g_2 \geq h_{\min}$, the convergence rate of the new network is determined by the minimal real root of the following:

$$\prod_{\kappa=1}^{\ell}\left(\prod_{i=1}^{q_{\kappa 1}}(d_{\kappa i}+1-\lambda)^{w_{\kappa i}} \prod_{j=1}^{q_{\kappa 2}}(h_{\kappa j}-\lambda)^{c_{\kappa j}} - 1\right) = 0. \tag{10}$$

By Theorem 1, the convergence rate is $\lambda_2 = \lambda_{\min} \in (h_{\min} - 1, h_{\min})$, in which $h_{\min} = \min_{\kappa=1,\ldots,\ell}\{h_{\kappa 0}\}$, and $h_{\kappa 0} = \min_{\kappa=1,\ldots,\ell}\{d_{\kappa 1}+1, h_{\kappa 1}\}$ is the minimum node in-degree of the κth cycle. Thus, the convergence rate is only determined by the directed cycle that has the minimum node in-degree $h_{\min}$. ■

Remark 3: When more than one directed cycle has the minimum node in-degree, define R as the ratio of the number of nodes with minimum in-degree to the number of nodes with the maximum in-degree on a directed cycle, i.e., $R = \frac{c_0}{c_{\max}}$, then the convergence rate is generally determined by the directed cycle with larger R.

Example 2: For a directed chain with node numbers 1 to n, if adding two nonintersected reverse edges from node j to node i and node k to node l $(1 < i < j < l < k \leq n)$, then the dominant equation is as follows:

$$[(1-\lambda)^{j-i}(2-\lambda) - 1][(1-\lambda)^{k-l}(2-\lambda) - 1] = 0.$$

Further, λ_2 is the simple root of smallest real part for the following:

$$(1-\lambda)^{S}(2-\lambda) = 1$$

with $S = \min\{j-i, k-l\}$.

Case III: Multiple intersected directed cycle. The simplest type of intersected directed cycles is full intersection. For nonfull intersection, there may be some disjoint directed cycles, making it difficult to describe the characteristic equation. To investigate the effect of ℓ_1 fully intersected directed cycles on the convergence rate of directed networks, Lemma 5 is given below.

Lemma 5: For the polynomial equation

$$P(x) \triangleq x^{c_0}\prod_{i=1}^{q_1}(x+d_i+1)^{w_i}\prod_{j=1}^{q_2}(x+h_j)^{c_j} = \ell_1 \tag{11}$$

with integers $c_0 \geq 1$, $q_1, q_2 \geq 1$, $w_i, c_j \geq 1$, $d_i, h_j \geq 1$, $\ell_1 \geq 2$, $i = 1, \ldots, q_1$, and $j = 1, \ldots, q_2$, one has

1) Equation (11) has a unique positive real root $x = x_1 > 0$. Specially, $x = x_1 \in (0,1)$ when $\prod_{i=1}^{q_1}(2+d_i)^{w_i}\prod_{j=1}^{q_2}(1+h_j)^{c_j} > \ell_1$ and $x = x_1 \geq 1$ when $\prod_{i=1}^{q_1}(2+d_i)^{w_i}\prod_{j=1}^{q_2}(1+h_j)^{c_j} \leq \ell_1$;
2) the real part of the other roots of (11) is smaller than the positive real root x_1;
3) $x_1(\ell_1)$ is a strictly increasing function in ℓ_1 with all other arguments fixed.

Proof: To prove Claim 1). $P(x)$ is a strictly increasing continuous function of x over $[0, +\infty)$ with $P(0) = 0 < \ell_1$, $P(1) = \prod_{i=1}^{q_1}(2+d_i)^{w_i}\prod_{j=1}^{q_2}(1+h_j)^{c_j}$, and $\lim_{x\to+\infty} P(x) = +\infty$. When $P(1) > \ell_1$ is true, we have $x = x_1 \in (0,1)$. When $P(1) \leq \ell_1$ holds, it indicates that $x = x_1 \geq 1$.

To prove Claim 2). Suppose that $z = a + bi$ is another root of (11), where $a, b \in \mathbb{R}$. If $b = 0$, since (11) has only one positive real root x_1, it is clear that $a \leq 0 < x_1$. For $b \neq 0$, one has

$$|P(z)| = \ell_1$$
$$= |(a+bi)^{c_0}|\left|\prod_{i=1}^{q_1}(a+bi+d_i+1)^{w_i}\right|\left|\prod_{j=1}^{q_2}(a+bi+h_j)^{c_j}\right|.$$

Hence, if $a \leq 0$, then clearly $a \leq 0 < x_1$. Otherwise, we have

$$\ell_1 = |(a+bi)^{c_0}|\left|\prod_{i=1}^{q_1}(a+bi+d_i+1)^{w_i}\right|\left|\prod_{j=1}^{q_2}(a+bi+h_j)^{c_j}\right|$$
$$> |a^{c_0}|\left|\prod_{i=1}^{q_1}(a+d_i+1)^{w_i}\right|\left|\prod_{j=1}^{q_2}(a+h_j)^{c_j}\right| = P(a).$$

Since $P(x_1) = \ell_1$ and $P(x)$ is strictly monotonically increasing of x, one has $a < x_1$.

To prove Claim 3). Define $s = x_1(\ell_1)$ and $r_1 = x_1(\ell_1 + l)$. One needs to check that $r_1 > s$ for any integer $l > 0$. Thus, we have

$$s^{c_0}\prod_{i=1}^{q_1}(s+d_i+1)^{w_i}\prod_{j=1}^{q_2}(s+h_j)^{c_j} = \ell_1$$
$$r_1^{c_0+l}\prod_{i=1}^{q_1}(r_1+d_i+1)^{w_i}\prod_{j=1}^{q_2}(r_1+h_j)^{c_j} = \ell_1 + l > \ell_1.$$

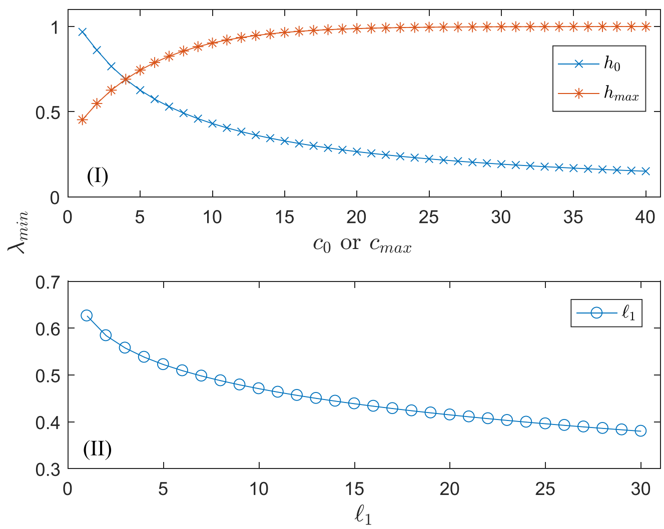

Fig. 3. (I) $\lambda_{\min}$ gradually decreases as the increase in c_0 (for fixed 3 nodes with in-degree 4, and 4 nodes with in-degree 2 on the cycle), while it increases as the increase in $c_{\max}$ (for fixed 5 nodes with in-degree 1, and 4 nodes with in-degree 2 on the cycle) when there is a single directed cycle. (II) $\lambda_{\min}$ gradually decreases as the increase in the number of intersected cycle (for fixed 5 nodes with in-degree 1, 4 nodes with in-degree 2, and 3 nodes with in-degree 4 on each cycle).

Since $P(x)$ is strictly monotonically increasing with respect to x over $[0, +\infty)$, it is clear that $r_1 > s$. Hence, $x_1(\ell_1)$ is a strictly increasing function in ℓ_1 when all other arguments are fixed. ■

Then we obtain the influence of ℓ_1 intersected directed cycles on the convergence rate in Theorem 3.

Theorem 3: Suppose that adding edges forms ℓ_1 fully intersected directed cycles, and each directed cycle contains p_1 nodes and the corresponding nodes have the same in-degree, then the following conclusions are true:

1) Only p_1 eigenvalues of Laplacian are changed and the other $n - p_1$ eigenvalues remain unchanged;
2) The more intersected cycles there are, i.e., the larger ℓ_1, the smaller the convergence rate with all other arguments fixed.

Proof: To prove Claim 1). The characteristic equation of Laplacian is as follows:

$$\prod_{k=1}^{q_3}(\lambda - g_k)^{e_k}\left[\prod_{i=1}^{q_1}(d_i + 1 - \lambda)^{w_i}\prod_{j=1}^{q_2}(h_j - \lambda)^{c_j} - \ell_1 Q(\lambda)\right] = 0 \tag{12}$$

where

$$Q(\lambda) = \frac{\prod_{i=1}^{q_1}(d_i + 1 - \lambda)^{w_i}\prod_{j=1}^{q_2}(h_j - \lambda)^{c_j}}{\prod_{i=1}^{q_1{}'}(d_i{}' + 1 - \lambda)^{w_i{}'}\prod_{j=1}^{q_2{}'}(h_j{}' - \lambda)^{c_j{}'}}$$

in which w_i and c_j represent the number of nodes for all cycles with in-degrees $d_i + 1$ and h_j, respectively; $w_i{}'$ and $c_j{}'$ represent the number of nodes on a cycle with in-degrees $d_i{}' + 1$ and $h_j{}'$, respectively, satisfying $\sum_{i=1}^{q_1{}'} w_i' + \sum_{j=1}^{q_2{}'} c_j' = p_1$; e_k is the number of nodes, which are not on any directed cycle, having in-degrees g_k. For the derivation details of (12), the reader is referred to the Part II of Supplementary Materials.

Comparing (12) with (5), it shows that p_1 eigenvalues of Laplacian are changed and the other $n - p_1$ eigenvalues remain unchanged.

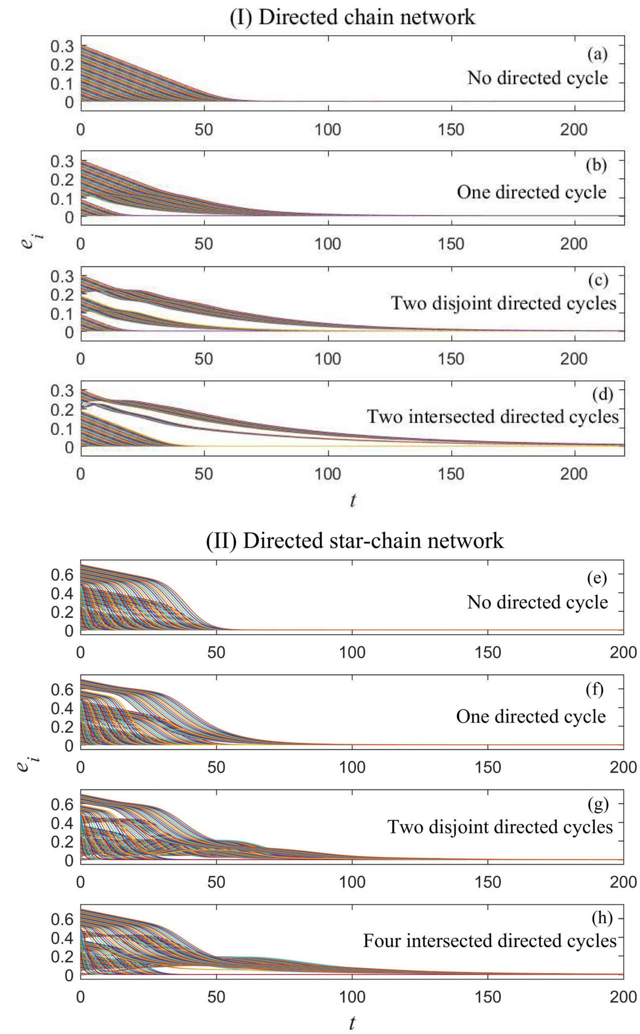

Fig. 4. (I) Consensus of a directed chain with 60 nodes versus time t, where $e_i = x_i - x_1$, and x_i is the state of node i $(1 \le i \le 60)$: (a) Directed chain without directed cycle; (b) directed chain with a directed cycle of 15 nodes; (c) directed chain with two disjoint directed cycles of 15 and 22 nodes; (d) directed chain with two intersected directed cycles of 15 and 22 nodes. (II) Consensus of a directed star-chain with three chains and 40 nodes on each chain versus time t, where $e_i = x_i - x_1$, and x_i is the state of node i $(1 \le i \le 121)$: (e) Directed star-chain without directed cycle; (f) directed star-chain with a directed cycle of 9 nodes located on one of the chains; (g) directed star-chain with two disjoint directed cycles of 48 and 9 nodes, where the nodes of the large cycle are distributed on three chains, with 16 nodes on each chain, and the small cycle is located on a certain chain; (h) directed star-chain with four intersected directed cycles of 48, 9, 9, and 9 nodes, where the nodes of the large cycle are distributed on three chains, with 16 nodes on each chain, and three small cycles on the three different chains.

Furthermore, p_1 changed eigenvalues are determined as follows:

$$\prod_{i=1}^{q_1'}(d_i{}' + 1 - \lambda)^{w_i{}'}\prod_{j=1}^{q_2'}(h_j{}' - \lambda)^{c_j{}'} = \ell_1. \tag{13}$$

To prove Claim 2). Define

$$x = h_0 - \lambda, \tilde{d}_i = d_i' - h_0, \tilde{h}_j = h_j' - h_0 \tag{14}$$

where $h_0 = \min\{h'_1, d'_1 + 1\}$ is the minimum node in-degree of these ℓ_1 cycles, the corresponding number of nodes c'_0.

Then, (13) becomes

$$x^{c'_0} \prod_{i=1}^{q'_1} (\tilde{d}_i + 1 + x)^{w'_i} \prod_{j=2}^{q'_2} (\tilde{h}_j + x)^{c'_j} = \ell_1$$

when $h_0 = h'_1$, and

$$x^{c'_0} \prod_{i=2}^{q'_1} (\tilde{d}_i + 1 + x)^{w'_i} \prod_{j=1}^{q'_2} (\tilde{h}_j + x)^{c'_j} = \ell_1$$

when $h_0 = d'_1 + 1$.

By Lemma 5, the minimum real root of (13) is $\lambda_{\min} \in (h_0 - 1, h_0)$ when $\prod_{i=1}^{q'_1} (2 + \tilde{d}_i)^{w'_i} \prod_{j=1}^{q'_2} (\tilde{h}_j + 1)^{c'_j} > \ell_1$, and $\lambda_{\min} \le h_0 - 1$ when $\prod_{i=1}^{q'_1} (2 + \tilde{d}_i)^{w'_i} \prod_{j=1}^{q'_2} (\tilde{h}_j + 1)^{c'_j} \le \ell_1$. Moreover, $\lambda_2 = \lambda_{\min}$ is the convergence rate when $g_2 > h_0$ holds. Thus, the more intersected cycles there are, the smaller the convergence rate. ■

Remark 4: For a network with other types of intersected directed cycles, its characteristic polynomials are more complex. For example, when the number of nodes for each directed cycle in Theorem 3 is different, the characteristic equation of Laplacian becomes

$$\prod_{k=1}^{q_3} (\lambda - g_k)^{e_k} \left[\prod_{i=1}^{q_1} (d_i + 1 - \lambda)^{w_i} \prod_{j=1}^{q_2} (h_j - \lambda)^{c_j} - \sum_{\kappa=1}^{\ell_1} Q_\kappa(\lambda) \right] = 0$$

where $Q_\kappa(\lambda)$ is similar to Q in the proof of Theorem 3, representing the remaining part of polynomial $\prod_{i=1}^{q_1} (d_i + 1 - \lambda)^{w_i} \prod_{j=1}^{q_2} (h_j - \lambda)^{c_j}$ after removing the polynomial associated with κth directed cycle.

Example 3: For a directed chain with node numbers 1 to n, if adding two intersected reverse edges from node j to node i and node k to node l $(1 < i < l < j < k \le n)$, then the dominant equation is as follows:

$$(2 - \lambda)^2 (1 - \lambda)^{k-i-1} - (2 - \lambda)(1 - \lambda)^{l-i-1} - (1 - \lambda)^{k-j} = 1.$$

Remark 5: In an undirected connected network, cycles increase the convergence rate of the network, and the more cycles there are, the greater the convergence rate [15], which is contrary to the conclusions for a directed network. Furthermore, adding more long-range edges improves synchronizability or propagation in the undirected network, while adding long-range reverse edge into directed network inhibits the network consensus.

Remark 6: For a weighted directed network with directed cycles, the form of its characteristic equation is the same as that of the unweighted network, except for differences in certain coefficients. Therefore, the methodology of this article can be extended to analyze consensus in weighted directed networks.

IV. Numerical Simulation

In this section, we take several directed networks as examples to verify the validity of the theoretical results.

For a directed acyclic star-chain, if adding m edges forms a directed cycle of p nodes, then when $p > m$, the convergence rate $\lambda_2(p, m)$ is the simple root of the smallest real part for equation $(1 - \lambda)^{p-m} (2 - \lambda)^m = 1$, which only depends on p and m. Moreover, $\lambda_2(p, m)$ is strictly decreasing in p and increasing in m, respectively. Fig. 2 shows one example of the star-chain networks, where the directed cycle decreases the convergence rate, i.e., $\lambda_2 = 0.38196$ ($p = 6$ and $m = 3$). The variation in the convergence rate of directed star-chains is only determined by the number of added edges and nodes on the directed cycle, independent of the position of added edges and the number of nodes in the star-chain.

Fig. 3(I) shows that fixing other arguments for a directed cycle, $\lambda_{\min}$ gradually decreases as the increase in the number of nodes with minimum in-degree h_0 on the cycle, and increases as the increase in the number of nodes with maximum in-degree $h_{\max}$. In addition, $\lambda_{\min}$ decreases as the number of intersected directed cycles increases [see Fig. 3(II)]. Fig. 4 shows the consensus profile for directed chain and directed star-chain. It is seen that directed cycles weaken the synchronizability of these directed networks and cause the generation of cluster during the synchronization process. The inhibition caused by two disjoint directed cycles is determined by the large cycle, while multiple intersected directed cycles enhance the inhibitory effect on the consensus.

V. Conclusion

This article has investigated the influence of directed cycles on the consensus of directed networks. The main conclusions can be summarized as follows. First, it has been found that the convergence rate of the network is only determined by the in-degree of nodes on directed cycles, particularly those with in-degree of 1. Second, unlike in undirected networks, the directed cycle exerts an inhibitory effect on the convergence rate of any directed network. The more nodes on the cycle with in-degree of 1, the stronger the inhibition exerted by the cycle on the convergence rate. Conversely, a higher number of nodes with in-degrees exceeding 1 weakens this inhibitory effect. We have established a necessary and sufficient condition under which a directed cycle reduces the convergence rate. In addition, the impact of multiple directed cycles is analyzed. The influence of disjoint cycles on the convergence rate of directed networks is only determined by the cycle with the minimum node in-degree, and is independent of other cycles. However, the greater the number of intersected directed cycles, the slower the convergence rate becomes. Finally, some numerical examples are presented to verify the validity of the theoretical results.

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