Adaptive Synchronization of Complex Dynamical Networks: Dealing With Uncertain Impulses

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***Abstract*—The synchronization problem of complex dynamical networks with impulsive effects has been extensively addressed. However, great challenges arise when applying the existing synchronization criteria to networks with uncertain impulses. In this article, we investigate the adaptive synchronization problem of complex networks with uncertain impulses. First, the adaptive control gain is proved to be bounded for both synchronizing and desynchronizing impulses. Then, adaptive synchronization criteria for impulsive networks are derived from the boundedness of the control gain. Finally, a numerical example is provided to validate the proposed criteria.**

***Index Terms*—Adaptive feedback control, complex network, impulse, linear feedback control, synchronization.**

I. Introduction

Complex networks are usually used to describe large-scale interconnected systems in the real world, including protein networks, biological neural networks, dynamical networks, and citation networks. Over the past decades, there has been a surge of interest in the study of complex networks, in regard to network structure [1], [2], statistical mechanics [3], synchronization [4], region of attraction [5], and controllability [6].

As one of the most essential problems in the field of complex dynamical networks, synchronization has been extensively investigated [7], [8], [9], [10]. Some networks can realize synchronization via the interactions among the nodes, while some can only realize synchronization by means of external control. *Linear feedback control* [11], [12], [13] is one of the most widely-used control schemes for synchronization as it is easy to implement. However, the design of the linear control gain heavily depends on network parameters. In other words, the linear control scheme is generally inapplicable if some network parameters are uncertain. To address the synchronization problem of networks with uncertain parameters, *adaptive feedback control* [14], [15], [16] can be adopted, because its control gain can be automatically adapted without knowing any parameter.

In reality, the states of a network may be subject to a sequence of abrupt instantaneous changes, also known as *impulsive effects* [17], [18], [19]. According to whether or not it is beneficial for synchronization, an impulse is classified as *synchronizing impulse* or *desynchronizing impulse*. In the literature, there have been a great deal of studies devoted to the synchronization problem of impulsive networks [18], [19], [20], [21], [22], [23], [24]. It is worth noting that these studies adopt the linear control approach, and the designed control gain depends on the impulsive strength and the impulsive interval. That is, the two parameters of each impulse should be known in advance, despite the fact that, in practice, it may not be possible to determine the exact parameters.

In the synchronization literature, adaptive control is often used to address uncertain network topology, while little attention has been paid to uncertain impulses. In [25], [26], [27], and [28], the synchronization problem of impulsive networks under adaptive control was considered. In [25], only synchronizing impulses are considered, while desynchronizing impulses that are more troublesome are discounted. Essentially, the impulsive strength is assumed to be known. Desynchronizing impulses are addressed in [26], [27], and [28], but the initial value of the adaptive control gain depends on the impulse parameters. In consequence, the adaptive control scheme in [26], [27], and [28] is not practical to apply as the impulse parameters may be uncertain. To the best of authors' knowledge, the adaptive synchronization problem of networks with uncertain impulses has not yet been successfully solved. The main challenge lies with the fact that desynchronizing impulses could prevent the Lie derivative of the Lyapunov candidate functions studied in much of the literature from being negative semidefinite. To overcome such difficulty, we will derive synchronization criteria by analyzing the boundedness of the control gain.

Motivated by the above discussions, this article investigates the adaptive synchronization problem of complex dynamical networks with uncertain impulses. The main contributions are as follows.

1) The boundedness of the adaptive control gain is theoretically guaranteed, no matter that the impulses are synchronizing or desynchronizing.
2) In contrast to [26], [27], and [28], the initial value of the control gain is independent of the impulse parameters and thereby the adaptive control proposed here is more practical.
3) The adaptive synchronization criteria for impulsive networks are derived by proving the boundedness of the control gain. It should be highlighted that uncertain impulses are addressed for the first time via adaptive control in the synchronization problem.

The rest of this article is organized as follows. Section II presents some preliminaries. Section III derives the adaptive synchronization criteria for complex dynamical networks with uncertain impulses. Section IV provides a numerical example to verify the effectiveness of the proposed criteria. Finally, Section V concludes this article.

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II. Preliminaries

A. Notation and Problem Formulation

$\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all the $(n \times m)$-dimensional real matrices, respectively; $\mathbb{R}_+$ is the set of nonnegative real numbers; the superscript $\top$ represents the transpose of a vector or a matrix; $\|\cdot\|$ denotes the two-norm of a vector or a matrix; $\otimes$ represents the Kronecker product; I_n is the identity matrix of dimension n. The Dini derivative of a function ψ is defined as $D^+\psi(t) := \limsup_{h\to 0^+}[(\psi(t+h)-\psi(t))/h]$.

Consider a controlled complex dynamical network consisting of N equations of dimension n

$$\dot{x}_i(t) = f(t, x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t) + u_i \tag{1}$$

where $1 \le i \le N$; $x_i \in \mathbb{R}^n$ represents the state of the ith node; $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear function; $c > 0$ is the coupling strength; $A = (a_{ij})_{N\times N}$ denotes the weighted outer coupling matrix; $\Gamma \in \mathbb{R}^{n\times n}$ denotes the inner coupling matrix; u_i is the control input to the ith node. For $i \ne j$, if there exists a connection from node j to node i, then $a_{ij} > 0$; otherwise, $a_{ij} = 0$; the diagonal entries of A are defined by $a_{ii} = -\sum_{j=1, j\ne i}^{N} a_{ij}$.

The target is to synchronize all the nodes to a desired state trajectory $z(t)$ such that

$$\dot{z}(t) = f(t, z(t)). \tag{2}$$

The synchronization errors of the ith node and network (1) are, respectively, denoted as $e_i(t) = x_i(t) - z(t)$ and

$$e(t) = [e_1^\top(t), e_2^\top(t), \ldots, e_N^\top(t)]^\top. \tag{3}$$

During signal transmission, the states x_i may be subject to sudden changes at discrete time instants, which can be well described by differential equations with impulses [23]. To be more practical, the following impulsive dynamical network is considered:

$$\begin{cases} \dot{x}_i(t) = f(t, x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t) + u_i, & t \ne t_k \\ \Delta x_i(t_k) = B_{ik} e_i(t_k^-), & t = t_k \end{cases} \tag{4}$$

where $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ represents the impulse for the ith node at t_k; $B_{ik} \in \mathbb{R}^{n\times n}$ is a constant matrix; $\{t_k\}_{k\ge 1}$ is an impulsive sequence such that $0 < t_k < t_{k+1}$ for all $k \ge 1$ and $\lim_{k\to+\infty} t_k = +\infty$. Assume that $x_i(t)$ is right-hand continuous at $t = t_k$, that is, $x_i(t_k) = x_i(t_k^+)$ for all k. Then, one obtains the following error system from (2) and (4):

$$\begin{cases} \dot{e}_i(t) = f(t, x_i(t)) - f(t, z(t)) \\ \qquad\qquad + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + u_i, & t \ne t_k \\ e_i(t_k) = (I_n + B_{ik}) e_i(t_k^-), & t = t_k. \end{cases} \tag{5}$$

The impulses can be classified into two types according to the value of $\|I_n + B_{ik}\|$. The impulse for the ith node at t_k is *synchronizing* if $\|I_n + B_{ik}\| \le 1$ (as $\|e_i(t_k)\| \le \|e_i(t_k^-)\|$) or *desynchronizing* if $\|I_n + B_{ik}\| > 1$.

Remark 1: The network model (4) considered here is quite standard and general, where the coupling matrices A and Γ can be asymmetric. Although the connections among the nodes are beneficial for achieving network synchronization, appropriate adaptive control u_i for each and every node is still necessary in the presence of uncertain impulses (which might be strong desynchronizing impulses).

Remark 2: In most if not all of the existing studies [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], the values of B_{ik} and t_k are assumed to be known in advance. Assuming the advance knowledge of the impulse times and values over an infinite time horizon is often impractical. To overcome the uncertainty introduced into the problem in the absence of such an assumption, it is necessary to replace the strategy of linear feedback control used by most existing studies with an adaptive feedback control scheme.

B. Assumptions and Lemmas

Assumption 1 (A1): Suppose that there exists a nonnegative constant ρ satisfying

$$\|f(t, x) - f(t, y)\| \le \rho\|x - y\| \tag{6}$$

for any $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$.

Assumption 2 (A2): Suppose there are constants $\mu > 0$ and $\tau_{\min} > 0$, such that the impulses in (4) satisfy

$$\|I_n + B_{ik}\| \le \mu \tag{7}$$

$$t_{k+1} - t_k \ge \tau_{\min} \tag{8}$$

for all $1 \le i \le N$ and $k \ge 1$.

Remark 3: Condition (7) imposes a bound on the desynchronizing effect of the impulses, whereas (8) limits the frequency of the impulses. It is noteworthy that advance knowledge of μ and $\tau_{\min}$ is not required here, only the existence of the bounds is needed.

Lemma 1 ([29], Th. 16 and 19]): Let $\xi_1, \xi_2, \cdots, \xi_m \ge 0$. Then, the following inequalities hold:

$$\left(\sum_{i=1}^{m} \xi_i\right)^p \le m^{p-1}\sum_{i=1}^{m} \xi_i^p, \left(\sum_{i=1}^{m} \xi_i\right)^q \le \sum_{i=1}^{m} \xi_i^q$$

where $p \ge 1$ and $0 < q \le 1$.

Lemma 2: Let $E \subset \mathbb{R}_+$ be an interval in the form of $[\sigma_1, \sigma_2)$ or (σ_1, σ_2), where σ_1 may be $-\infty$ and σ_2 may be $+\infty$. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a right-hand continuous function. If function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable and satisfies

$$\dot{\varphi}(t) \le \alpha\varphi(t) + \sqrt{\varphi(t)}g(t), \quad t \in E \tag{9}$$

then

$$D^+\omega(t) \le \frac{\alpha}{2}\omega(t) + \frac{1}{2}g(t), \quad t \in E \tag{10}$$

where $\alpha \in \mathbb{R}$ and $\omega(t) = \sqrt{\varphi(t)}$.

For conciseness, the proofs of Lemma 2 and the following lemmas are presented in the Appendix.

Remark 4: When dealing with network synchronization, an inequality in the form of (9) may be established to estimate the convergence speed of the Lyapunov function. Since inequality (9) is generally not easy to solve, it would be better to simplify (9) into the form of (10). If $\varphi > 0$, one can easily derive (10), which even holds for the true derivative of ω (as ω is differentiable at any t with $\varphi(t) > 0$). The proof of Lemma 2 addresses the additional difficulty of bounding the derivative of ω at points $\hat{t}_0$ where $\varphi(\hat{t}_0) = 0$.

Remark 5: It should be highlighted that $g \ge 0$ is a necessary condition in this lemma. If the condition $g \ge 0$ does not hold, then there exists a $\hat{t}_0$ such that $g(\hat{t}_0) < 0$. Taking $\varphi \equiv 0$ gives

$$D^+\omega(\hat{t}_0) = 0 > \frac{1}{2}g(\hat{t}_0) = \frac{\alpha}{2}\omega(\hat{t}_0) + \frac{1}{2}g(\hat{t}_0)$$

a contradiction with (10).

Lemma 3: Suppose that the integral $\int_{t_0}^{+\infty} g^p(t)\mathrm{d}t$ converges, where g is a nonnegative function, $t_0 \in \mathbb{R}_+$, and $p \geq 1$. If $W : \mathbb{R}_+ \to \mathbb{R}_+$ is a piecewise continuous function satisfying

$$W(t) \leq c_1 \exp(-\alpha(t-t_0)) + c_2 \int_{t_0}^{t} g(\theta)\exp(-\alpha(t-\theta))\mathrm{d}\theta,\ t \geq t_0 \tag{11}$$

with $c_1, c_2,$ and α being positive constants, then $\int_{t_0}^{+\infty} W^p(t)\mathrm{d}t$ converges.

Lemma 4: Let $\{\hat{t}_k\}_{k\geq 0}$ be a sequence satisfying $\hat{t}_{k+1} - \hat{t}_k \geq \tau_0$, where $\tau_0 > 0$ and $\hat{t}_0 = 0$. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a right-hand continuous function such that

$$\begin{cases} D^+\varphi(t) \leq \alpha\varphi(t), & t \neq \hat{t}_k \\ \varphi(\hat{t}_k) \leq \mu\varphi(\hat{t}_k^-), & t = \hat{t}_k \end{cases} \tag{12}$$

then $\varphi(t) \leq \varphi(0)\exp(c_0 t)$, where $\alpha \in \mathbb{R}$, $k \geq 1$, $\mu > 0$, and $c_0 = \max\{\alpha, \alpha + \frac{\ln\mu}{\tau_0}\}$.

Lemma 5: Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} and A_{22} are square matrices. Then, $\|A_{11}\| \leq \|A\|$ and $\|A_{22}\| \leq \|A\|$.

III. Main Results

The purpose of this section is to establish synchronization criteria for the impulsive network (4) under the adaptive controller

$$u_i(t) = -d_i(t)e_i(t),\ \ 1 \leq i \leq N \tag{13}$$

with adaptation laws

$$\dot{d}_i(t) = \zeta_i\|e_i(t)\|^p \tag{14}$$

where $\zeta_i > 0$, $p \geq 1$, and $d_i(0) \geq 0$ are constants.

A. Boundedness of the Control Gain

From the adaptation laws (14), we find that the adaptive control gain $d_i(t)$ keeps increasing if $e_i(t) \neq 0$. Since controllers cannot provide infinitely large control gain in reality, $d_i(t)$ must be designed so as to remain bounded.

Theorem 1: Let the assumptions (A1) and (A2) hold. If the control u_i of impulsive network (4) is designed as (13) with adaptation laws (14), then the control gain $d_i(t)$ is necessarily bounded, where $1 \leq i \leq N$.

Proof: The proof is divided into two steps.

Step 1: We first prove that $d_i(t)$ is bounded on the interval $[0, T]$ for any $T > 0$. Consider the Lyapunov function $V_1(t) = e^\top(t)e(t) = \sum_{i=1}^N e_i^\top(t)e_i(t)$. Differentiating $V_1(t)$ along the solution of (5) gives

$$\begin{aligned}\dot{V}_1(t) &= 2\sum_{i=1}^N e_i^\top(t)[f(t,x_i(t)) - f(t,z(t))] \\ &\quad + 2c\sum_{i=1}^N\sum_{j=1}^N a_{ij}e_i^\top(t)\Gamma e_j(t) \\ &\quad - 2\sum_{i=1}^N d_i(t)\|e_i(t)\|^2,\ \text{if}\ t \neq t_k.\end{aligned}$$

Recalling assumption (A1), one has

$$\begin{aligned}\dot{V}_1(t) &\leq 2\rho\sum_{i=1}^N \|e_i(t)\|^2 + 2c\sum_{i=1}^N\sum_{j=1}^N a_{ij}e_i^\top(t)\Gamma e_j(t) \\ &\quad - 2\sum_{i=1}^N d_i(t)\|e_i(t)\|^2,\ \text{if}\ t \neq t_k.\end{aligned} \tag{15}$$

Since $d_i(t) \geq d_i(0) \geq 0$, it follows that:

$$\begin{aligned}\dot{V}_1(t) &\leq 2\rho\|e(t)\|^2 + 2ce^\top(t)(A\otimes\Gamma)e(t) \\ &\leq 2(\rho + c\|A\|\|\Gamma\|)V_1(t),\ \text{if}\ t \neq t_k.\end{aligned} \tag{16}$$

When $t = t_k$, it follows from (5) and assumption (A2) that:

$$V_1(t_k) \leq \mu^2 V_1(t_k^-).$$

According to Lemma 4, there exists a $c_0 > 0$ such that $V_1(t) \leq V_1(0)\exp(2c_0 t)$, leading to

$$\|e_i(t)\| \leq \|e(t)\| = \sqrt{V_1(t)} \leq \|e(0)\|\exp(c_0 t).$$

Then, for $t \in [0, T]$, one derives from (14) that

$$\begin{aligned}d_i(t) &= d_i(0) + \zeta_i\int_0^t \|e_i(\theta)\|^p\mathrm{d}\theta \\ &\leq d_i(0) + \zeta_i T\|e(0)\|^p\exp(c_0 pT),\ 1 < i \leq N.\end{aligned}$$

Therefore, $d_i(t)$ is bounded on $[0, T]$ for $1 < i \leq N$.

Step 2: We next prove that $d_i(t)$ is bounded on $\mathbb{R}$.

Let α be a positive constant, and denote

$$\begin{aligned}\mu_0 &= \max\{\mu, 1\} \\ d_0 &= \rho + c\|A\|\|\Gamma\| + \frac{\ln\mu_0}{\tau_{\min}} + \alpha \\ \Lambda &= \{i \in \{1,\ldots,N\}|\ d_i(t) < d_0\ \forall t \geq 0\}\end{aligned}$$

where μ is as defined in assumption (A2).

Up to a permutation of the indices, $\Lambda = \{1,\ldots,\ell\}$. Clearly, if $\ell = N$ then all the $d_i(t)$ are bounded. Next consider $0 \leq \ell < N$, and denote

$$\begin{aligned}\hat{e}(t) &= [e_1^\top(t),\ldots,e_\ell^\top(t)]^\top \\ \tilde{e}(t) &= [e_{\ell+1}^\top(t),\ldots,e_N^\top(t)]^\top \\ A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\end{aligned}$$

where $A_{11} \in \mathbb{R}^{\ell\times\ell}$ and $A_{22} \in \mathbb{R}^{(N-\ell)\times(N-\ell)}$. In particular, one has $\hat{e} = 0$, $\tilde{e} = e$, and $A_{22} = A$ if $\ell = 0$.

Consider another Lyapunov candidate

$$V_2(t) = \tilde{e}^\top(t)\tilde{e}(t) = \sum_{i=\ell+1}^N e_i^\top(t)e_i(t).$$

Similar to (15), one has

$$\begin{aligned}\dot{V}_2(t) &\leq 2\rho\sum_{i=\ell+1}^N \|e_i(t)\|^2 + 2c\sum_{i=\ell+1}^N\sum_{j=1}^N a_{ij}e_i^\top(t)\Gamma e_j(t) \\ &\quad - 2\sum_{i=\ell+1}^N d_i(t)\|e_i(t)\|^2 \\ &= 2\rho\sum_{i=\ell+1}^N \|e_i(t)\|^2 + 2c\sum_{i=\ell+1}^N\sum_{j=\ell+1}^N a_{ij}e_i^\top(t)\Gamma e_j(t) \\ &\quad + 2c\sum_{i=\ell+1}^N\sum_{j=1}^{\ell} a_{ij}e_i^\top(t)\Gamma e_j(t) - 2\sum_{i=\ell+1}^N d_i(t)\|e_i(t)\|^2 \\ &= 2\rho\|\tilde{e}(t)\|^2 + 2c\tilde{e}^\top(t)(A_{22}\otimes\Gamma)\tilde{e}(t) \\ &\quad + 2c\tilde{e}^\top(t)(A_{21}\otimes\Gamma)\hat{e}(t) - 2\sum_{i=\ell+1}^N d_i(t)\|e_i(t)\|^2\end{aligned}$$

$$\leq 2(\rho + c\|A_{22}\|\|\Gamma\|)V_2(t) + 2c\|A_{21}\|\|\Gamma\|\sqrt{V_2(t)}\|\hat{e}(t)\|$$
$$-2\sum_{i=\ell+1}^{N} d_i(t)\|e_i(t)\|^2, \text{ if } t \neq t_k. \tag{17}$$

From (14) and the definition of ℓ, it follows that there exists a $k_0 \geq 1$ such that:

$$d_i(t) \geq d_0 \; \forall t \geq t_{k_0}$$

for $\ell < i \leq N$. By Lemma 5, one has $\|A_{22}\| \leq \|A\|$. Then

$$\begin{aligned}\dot{V}_2(t) \leq & -2(d_0 - \rho - c\|A\|\|\Gamma\|)V_2(t) \\ & + 2c\|A_{21}\|\|\Gamma\|\sqrt{V_2(t)}\|\hat{e}(t)\| \\ = & -2\alpha_0 V_2(t) + 2\sqrt{V_2(t)}g(t), \text{ if } t \neq t_k, t \geq t_{k_0}.\end{aligned}$$

where $\alpha_0 = \frac{\ln \mu_0}{\tau_{\min}} + \alpha$ and $g(t) = c\|A_{21}\|\|\Gamma\|\|\hat{e}(t)\|$. Denoting

$$W(t) = \sqrt{V_2(t)} = \|\tilde{e}(t)\|$$

it follows from Lemma 2 that:

$$D^+W(t) \leq -\alpha_0 W(t) + g(t), \text{ if } t \neq t_k, t \geq t_{k_0}.$$

When $t = t_k$, it follows from (5) and assumption (A2) that

$$W(t_k) \leq \mu W(t_k^-). \tag{18}$$

For any $k \geq k_0$, consider the following comparison system:

$$\begin{cases}\dot{U}(t) = -\alpha_0 U(t) + g(t) + \varepsilon, & t \in (t_k, t_{k+1}) \\ U(t) = W(t), & t = t_k\end{cases}$$

where $\varepsilon > 0$. According to the comparison principle, one has

$$\begin{aligned}W(t) < U(t) = & \exp(-\alpha_0(t - t_k))U(t_k) \\ & + \int_{t_k}^{t} [g(\theta) + \varepsilon] \exp(-\alpha_0(t - \theta))\mathrm{d}\theta\end{aligned}$$

for $t \in (t_k, t_{k+1})$. Letting $\varepsilon \to 0^+$ gives

$$W(t) \leq \exp(-\alpha_0(t - t_k))W(t_k) + \hat{g}_k(t, \alpha_0) \tag{19}$$

where

$$\hat{g}_k(t, s) = \int_{t_k}^{t} g(\theta)\exp(-s(t - \theta))\mathrm{d}\theta$$

with $t \in (t_k, t_{k+1})$ and $s > 0$. From (18), one has

$$W(t_{k+1}) \leq \mu \exp(-\alpha_0(t_{k+1} - t_k))W(t_k) + \mu\hat{g}_k(t_{k+1}, \alpha_0).$$

Since $t_{k+1} - t_k \geq \tau_{\min}$ and $\mu_0 \geq \mu$, it is derived that

$$\begin{aligned}\alpha_0(t_{k+1} - t_k) = & \frac{\ln \mu_0}{\tau_{\min}}(t_{k+1} - t_k) + \alpha(t_{k+1} - t_k) \\ \geq & \ln \mu + \alpha(t_{k+1} - t_k).\end{aligned} \tag{20}$$

Combining $\alpha_0 \geq \alpha$ and inequality (20) yields

$$W(t_{k+1}) \leq \exp(-\alpha(t_{k+1} - t_k))W(t_k) + \mu\hat{g}_k(t_{k+1}, \alpha). \tag{21}$$

Denoting $\hat{W}_k = W(t_k)\exp(\alpha t_k)$, (21) can be rewritten in the following compact form:

$$\hat{W}_{k+1} \leq \hat{W}_k + \mu\int_{t_k}^{t_{k+1}} g(\theta)\exp(\alpha\theta)\mathrm{d}\theta$$

which further gives

$$\hat{W}_k \leq \hat{W}_{k_0} + \mu\int_{t_{k_0}}^{t_k} g(\theta)\exp(\alpha\theta)\mathrm{d}\theta, \; k \geq k_0. \tag{22}$$

Since $\alpha_0 \geq \alpha$, it follows from (19) that:

$$W(t) \leq \exp(-\alpha(t - t_k))W(t_k) + \hat{g}_k(t, \alpha), \; t \in [t_k, t_{k+1}).$$

For any $t \geq t_{k_0}$, there exists a $k \geq k_0$ such that $t \in [t_k, t_{k+1})$. Then, considering $\mu_0 \geq 1$ and $\mu_0 \geq \mu$, one has

$$\begin{aligned}W(t)\exp(\alpha t) \leq & \hat{W}_k + \int_{t_k}^{t} g(\theta)\exp(\alpha\theta)\mathrm{d}\theta \\ \leq & \hat{W}_{k_0} + \mu_0\int_{t_{k_0}}^{t} g(\theta)\exp(\alpha\theta)\mathrm{d}\theta.\end{aligned} \tag{23}$$

We now verify the convergence of $\int_0^{+\infty} g^p(t)\mathrm{d}t$. If $\ell = 0$, one has $g = 0$, so $\int_0^{+\infty} g^p(t)\mathrm{d}t$ converges. Otherwise, one has $\ell > 0$ and

$$d_i(+\infty) < d_0, \; 1 \leq i \leq \ell.$$

Then, it follows from (14) that $\int_0^{+\infty}\|e_i(t)\|^p\mathrm{d}t$ converges for $1 \leq i \leq \ell$. Applying Lemma 1 gives

$$g^p = c_0^p\left(\sum_{i=1}^{\ell}\|e_i\|^2\right)^{\frac{p}{2}} \leq c_0^p \max\{1, \ell^{\frac{p}{2}-1}\}\sum_{i=1}^{\ell}\|e_i\|^p \tag{24}$$

where $c_0 = c\|A_{21}\|\|\Gamma\|$. Hence, $\int_0^{+\infty} g^p(t)\mathrm{d}t$ also converges for $\ell > 0$.

Combining the convergence of $\int_0^{+\infty} g^p(t)\mathrm{d}t$ and inequality (23), it follows from Lemma 3 that $\int_{t_{k_0}}^{+\infty} W^p(t)\mathrm{d}t$ converges. It has been proved in *Step 1* that $d_i(t_{k_0})$ is finite. Then

$$\begin{aligned}d_i(t) = & d_i(t_{k_0}) + \zeta_i\int_{t_{k_0}}^{t}\|e_i(\theta)\|^p\mathrm{d}\theta \\ \leq & d_i(t_{k_0}) + \zeta_i\int_{t_{k_0}}^{t} W^p(\theta)\mathrm{d}\theta, \;\; t \geq t_{k_0}\end{aligned}$$

where $\ell < i \leq N$. This accounts for the boundedness of $d_i(t)$ for $\ell < i \leq N$. This completes the proof. □

Remark 6: According to Theorem 1, the boundedness of the control gain $d_i(t)$ can be theoretically guaranteed, no matter that the impulses are synchronizing or desynchronizing. In fact, desynchronizing impulses could enlarge the synchronization error and make $d_i(t)$ increase more rapidly. In this sense, desynchronizing impulses are more difficult to address than synchronizing impulses.

B. Adaptive Synchronization Criteria

After analyzing the boundedness of the control gain $d_i(t)$, we are now in position to establish the adaptive synchronization criteria for impulsive network (4).

Theorem 2: If the assumptions (A1) and (A2) hold, then impulsive network (4) under the adaptive control (13) with adaptation laws (14) synchronizes asymptotically to the desired signal $z(t)$.

Proof: The boundedness of d_i implies that $\int_0^{+\infty} \|e_i(t)\|^p \mathrm{d}t$ converges for $1 \le i \le N$. Applying Lemma 1 gives

$$\|e\|^p = \left(\sum_{i=1}^N \|e_i\|^2\right)^{\frac{p}{2}} \le \max\{1, N^{\frac{p}{2}-1}\} \sum_{i=1}^N \|e_i\|^p$$

which implies the convergence of $\int_0^{+\infty} \|e(t)\|^p \mathrm{d}t$.

It has been proved in (16) that $\dot{V}_1(t) \le 2\beta V_1(t)$, where $t \ne t_k$ and $\beta = \rho + c\|A\|\|\Gamma\|$. Denoting $W(t) = \sqrt{V_1(t)} = \|e(t)\|$, it follows from Lemma 2 that

$$D^+W(t) \le \beta W(t), \text{ if } t \ne t_k. \tag{25}$$

When $t = t_k$, one has $W(t_k) \le \mu W(t_k^-)$.

Suppose that the synchronization error $e(t)$ does not converge to zero. Then, there exists a positive constant ϵ and a sequence $\{\omega_m\}$ in strictly increasing order such that $\lim_{m\to+\infty} \omega_m = +\infty$ and $W(\omega_m) \ge \mu_0 \epsilon$ for $m \ge 1$.

The number of impulses occurring in the interval $(\omega_m - \tau_{\min}, \omega_m]$ does not exceed 1. If the interval $(\omega_m - \tau_{\min}, \omega_m]$ contains no impulses, then applying the comparison principle, one deduces from (25) that $W(t) \ge \mu_0 \epsilon \exp(-\beta(\omega_m - t))$ for $t \in (\omega_m - \tau_{\min}, \omega_m]$. Recalling $\mu_0 = \max\{\mu, 1\}$, one has

$$W(t) \ge \epsilon \exp(-\beta(\omega_m - t)) \triangleq h(t). \tag{26}$$

On the other hand, if the interval contains a single impulse, then there exists a $t_{k_1} \in (\omega_m - \tau_{\min}, \omega_m]$. Then

$$W(t) \ge \mu_0 h(t), \ t \in [t_{k_1}, \omega_m] \tag{27}$$

which yields $W(t_{k_1}^-) \ge \frac{1}{\mu} W(t_{k_1}) \ge h(t_{k_1})$. Applying the comparison principle again gives

$$\begin{aligned} W(t) &\ge W(t_{k_1}^-) \exp(-\beta(t_{k_1} - t)) \\ &\ge h(t), \ t \in (\omega_m - \tau_{\min}, t_{k_1}). \end{aligned} \tag{28}$$

Combining (26)–(28), one gets

$$\|e(t)\| = W(t) \ge h(t), t \in (\omega_m - \tau_{\min}, \omega_m] \tag{29}$$

which further gives

$$\begin{aligned} \int_{\omega_m-\tau_{\min}}^{\omega_m} \|e\|^p(t)\mathrm{d}t &\ge \int_{\omega_m-\tau_{\min}}^{\omega_m} h^p(t)\mathrm{d}t \\ &= \epsilon^p \exp(-p\beta\tau_{\min}) \frac{\exp(p\beta\tau_{\min}) - 1}{p\beta} \\ &\ge \tau_{\min}\epsilon^p \exp(-p\beta\tau_{\min}) \triangleq \kappa_0 \end{aligned}$$

where $\kappa_0 > 0$ is a constant that does not depend on m. This contradicts the convergence of $\int_0^{+\infty} \|e(t)\|^p \mathrm{d}t$. Hence, one has $\lim_{t\to+\infty} e(t) = 0$. □

Remark 7: At this point, it may be useful to compare the methods of the preceding proof with standard methods in the literature, where a widely used technique for proving asymptotic synchronization in the adaptive setting (see, e.g., [30], [31], and [32]) is to study a Lyapunov function candidate of the form

$$\widetilde{V} = \sum_{i=1}^N e_i^\top(t) e_i(t) + \sum_{i=1}^N \frac{(d_i(t) - d^*)^2}{\zeta_i}$$

where d^* is a constant. This technique works well for synchronizing impulses. In the presence of desynchronizing impulses, however, $\sum_{i=1}^N e_i^\top(t) e_i(t)$ could increase abruptly at the impulsive instant, causing the standard technique to fail. This difficulty motivated the more global approach in this article, considering explicitly the possible interactions between these abrupt jumps and the adaptive control gain.

TABLE I
T_{sync} AND d^* FOR DIFFERENT VALUES OF p

p	2	1.8	1.6	1.4	1.2	1	0.8	0.6	0.4	0.2
T_{sync}	1.39	1.47	1.49	1.80	2.10	2.40	3.27	5.46	5.17	7.17
d^*	33.01	29.92	26.53	24.36	21.99	16.60	12.46	10.14	11.85	11.35

IV. Numerical Simulations

A. Example

The well-known Lorenz system [33] is described by

$$\dot{z}(t) = f(z(t)) = \begin{bmatrix} c_1(z_2(t) - z_1(t)) \\ c_3 z_1(t) - z_2(t) - z_1(t) z_3(t) \\ -c_2 z_3(t) + z_1(t) z_2(t) \end{bmatrix} \tag{30}$$

where $c_1 = 10$, $c_2 = 8/3$, and $c_3 = 28$. Then, the assumption (A1) holds due to the boundedness of the Lorenz system [34].

Example 1: Consider impulsive network (4) consisting of $N = 5$ Lorenz systems, where the controllers are as given in Theorem 1. In this network, set $c = 1$, $B_{ik} = (\mu - 1)I_n$, $t_k = 0.1\,k$, $\Gamma = I_3$, $p = 2$, $\zeta_i = 1$, and

$$A = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -5 & 2 & 0 & 2 \\ 2 & 0 & -3 & 1 & 0 \\ 0 & 1 & 1 & -4 & 2 \\ 3 & 2 & 0 & 2 & -7 \end{bmatrix}$$

where $1 \le i \le 5$, $k \ge 1$, and $\mu > 0$ is a parameter to be discussed.

Let the initial values be $d_i(0) = 1$, $z(0) = [0, 2, -3]^\top$, and $x_i(0) = 3i + [-5, -10, -15]^\top$, where $1 \le i \le 5$. The synchronization error $e_{ij}(t)$ and the control gain $d(t)$ of this impulsive network are presented in Figs. 1 and 2 when the impulsive strength $\mu = 0.5$ and 2, respectively. It is found that the synchronization is realized no matter the impulses are synchronizing or desynchronizing, which verifies the theoretical results.

B. Further Discussion

Revisiting Example 1, we find that the adaptive control gain is large when $\mu = 2$. Next, we discuss how the parameter p affect the control gain $d_i(t)$ when $\mu = 2$. The other parameters (except p) and initial conditions keep the same.

For simplicity, the synchronization is considered to be realized numerically at

$$T_{\text{sync}} \triangleq \inf\{t_0 | \ \|e(t)\| \le \epsilon \ \forall t \ge t_0\}$$

where $\epsilon = 10^{-4}$. The ultimate control gain d^* of the entire network is defined as

$$d^* = \max_{1\le i\le 5} d_i(T_{\text{sync}}).$$

The values of T_{sync} and d^* for several different values of p are given in Table I. It is shown that the control gain d^* needed decreases when p decreases from 2 to 1. Therefore, we can choose a small p so that d^* does not exceed the maximum control gain that controllers can provide. On the other hand, the synchronization speed reduces when p becomes small.

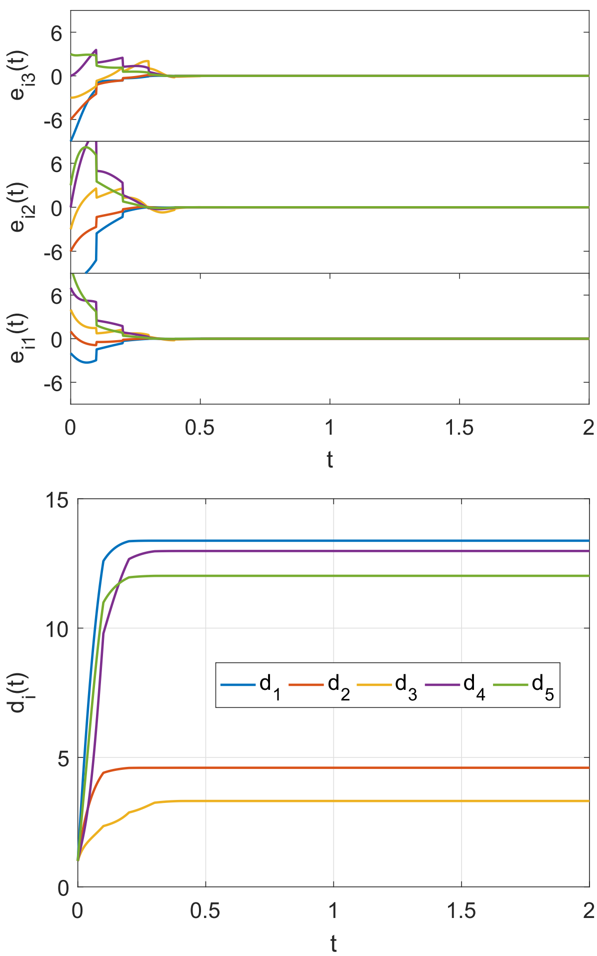

Fig. 1. Synchronization error $e_{ij}(t)$ and control gain $d_i(t)$ of the impulsive network in Example 1, where the impulsive strength $\mu = 0.5$, $1 \leq i \leq 5$, and $1 \leq j \leq 3$.

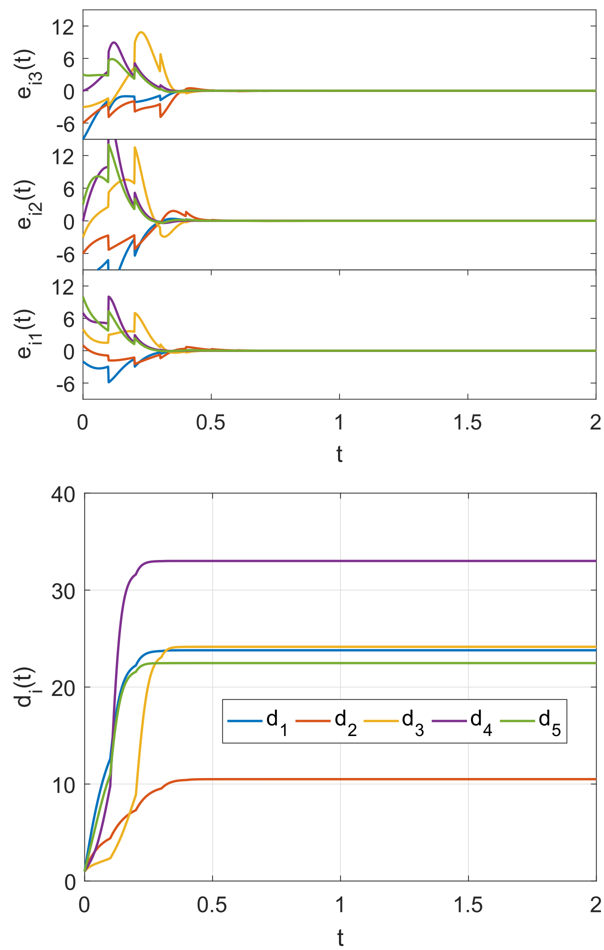

Fig. 2. Synchronization error $e_{ij}(t)$ and control gain $d_i(t)$ of the impulsive network in Example 1, where the impulsive strength $\mu = 2$, $1 \leq i \leq 5$, and $1 \leq j \leq 3$.

The synchronization can be realized when $p \geq 1$ according to Theorem 1, while simulation results in Table I give that the synchronization may be realized even when $p \in (0, 1)$. When reducing p from 1 to 0.6, the control gain needed continues to decrease; when further reducing p from 0.6 to 0.2, the change of d^* has no evident tendency. The reason may be that the minimum control gain needed is around 10.14 (corresponding to $p = 0.6$).

V. Conclusion

In this article, the adaptive synchronization problem of impulsive dynamical networks has been studied. Since conventional techniques for implementing adaptive control do not work well in the presence of desynchronizing impulses, boundedness analysis of the adaptive control gain has been conducted, based on which sufficient conditions for synchronization in networks with uncertain impulses have been established. In addition, the boundedness analysis here is also fundamental to guarantee the practicality of adaptive synchronization. Compared with the existing studies, the criteria derived here are independent of the impulse parameters and thereby are more practical.

Another notable benefit derived from the methods developed in this article is the flexibility provided by the parameter $p \geq 1$ for tuning the upper bound on the control gains. While prior studies have routinely taken $p = 2$, both our theory (Theorem 1) and the simulation results in Section IV indicate that p could be tuned to ensure that saturation bounds of actual controllers are not exceeded. Moreover, the simulations indicate that synchronization with $p \in (0, 1)$ is still possible, and a topic for further study.

Appendix

Proof of Lemma 2: Let $\hat{t}_0$ be any constant in E. If $\varphi(\hat{t}_0) \neq 0$, then ω is differentiable and

$$D^+\omega(\hat{t}_0) = \dot{\omega}(\hat{t}_0) = \frac{\dot{\varphi}(\hat{t}_0)}{2\sqrt{\varphi(\hat{t}_0)}} \leq \frac{\alpha}{2}\omega(\hat{t}_0) + \frac{1}{2}g(\hat{t}_0). \tag{31}$$

Hence, inequality (10) holds for $\varphi(\hat{t}_0) \neq 0$.

Next consider the case of $\varphi(\hat{t}_0) = 0$. Since $\omega(\hat{t}_0) = \sqrt{\varphi(\hat{t}_0)} = 0$, one obtains

$$D^+\omega(\hat{t}_0) = \overline{\lim}_{h \to 0^+} \frac{\omega(\hat{t}_0 + h)}{h}.$$

Let ε be any positive number. Considering that φ is continuous on E and g is right-hand continuous, there exists a positive number h_0 satisfying

$$|\alpha\omega(t)| < \varepsilon, \ |g(t) - g(\hat{t}_0)| < \varepsilon \tag{32}$$

for any $t \in [\hat{t}_0, \hat{t}_0 + h_0] \subset E$. For $h \in (0, h_0]$, define

$$\hat{t}_1 = \hat{t}_1(h) = \sup\{t \in [\hat{t}_0, \hat{t}_0 + h] \mid \varphi(t) = 0\}.$$

The continuity of φ gives $\varphi(\hat{t}_1)=0$, which further leads to $\omega(\hat{t}_1)=0$. If $\hat{t}_1=\hat{t}_0+h$, then

$$\frac{\omega(\hat{t}_0+h)}{h}=\frac{\omega(\hat{t}_1)}{h}=0\le\frac{1}{2}g(\hat{t}_0). \tag{33}$$

If $\hat{t}_1<\hat{t}_0+h$, then one has $\varphi(t)\neq 0$ for any $t\in(\hat{t}_1,\hat{t}_0+h]$. It follows that ω is differentiable on $(\hat{t}_1,\hat{t}_0+h]$. Then, there exists $\xi\in(\hat{t}_1,\hat{t}_0+h)$ such that

$$\frac{\omega(\hat{t}_0+h)}{h}=\frac{\omega(\hat{t}_0+h)-\omega(\hat{t}_1)}{h}=\dot{\omega}(\xi). \tag{34}$$

Since $\xi\in(\hat{t}_1,\hat{t}_0+h)\subset[\hat{t}_0,\hat{t}_0+h_0]$, inequality (32) yields $|\alpha\omega(\xi)|<\varepsilon$ and $g(\xi)<g(\hat{t}_0)+\varepsilon$. Recalling (31) and (34), it is derived that

$$\frac{\omega(\hat{t}_0+h)}{h}\le\frac{\alpha}{2}\omega(\xi)+\frac{1}{2}g(\xi)\le\frac{1}{2}g(\hat{t}_0)+\varepsilon. \tag{35}$$

Then, it follows from (33) and (35) that $D^+\omega(\hat{t}_0)\le\frac{1}{2}g(\hat{t}_0)+\varepsilon$. Letting $\varepsilon\to 0^+$, one has

$$D^+\omega(\hat{t}_0)\le\frac{1}{2}g(\hat{t}_0)=\frac{\alpha}{2}\omega(\hat{t}_0)+\frac{1}{2}g(\hat{t}_0).$$

This completes the proof. □

Proof of Lemma 3: Define $\tilde{g}(t)=\int_{t_0}^{t}g(\theta)\exp(-\alpha(t-\theta))\mathrm{d}\theta$. Then, applying Lemma 1 to inequality (11) gives

$$W^p(t)\le 2^{p-1}c_1^p\exp(-\alpha p(t-t_0))+2^{p-1}c_2^p\tilde{g}^p(t) \tag{36}$$

where $t\ge t_0$. It is easy to verify that $\int_{t_0}^{+\infty}\exp(-\alpha p(t-t_0))\mathrm{d}t$ converges. Since W is piecewise continuous, it follows from (36) that $\int_{t_0}^{+\infty}W^p(t)\mathrm{d}t$ converges if $\int_{t_0}^{+\infty}\tilde{g}^p(t)\mathrm{d}t$ converges.

It needs to be shown that $\int_{t_0}^{+\infty}\tilde{g}^p(t)\mathrm{d}t$ converges for $p=1$ and $p>1$, respectively.

Case I: If $p=1$, one has

$$\begin{aligned}\int_{t_0}^{T}\tilde{g}(t)\mathrm{d}t&=\int_{t_0}^{T}g(\theta)\exp(\alpha\theta)\mathrm{d}\theta\int_{\theta}^{T}\exp(-\alpha t)\mathrm{d}t\\&=\frac{1}{\alpha}\int_{t_0}^{T}g(\theta)[1-\exp(\alpha(\theta-T))]\mathrm{d}\theta\\&\le\frac{1}{\alpha}\int_{t_0}^{T}g(\theta)\mathrm{d}\theta.\end{aligned} \tag{37}$$

Since $\int_{t_0}^{+\infty}g(t)\mathrm{d}t$ converges, the convergence of $\int_{t_0}^{+\infty}\tilde{g}^p(t)\mathrm{d}t$ for $p=1$ is derived.

Case II: If $p>1$, there exists a unique constant $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Applying the Hölder inequality [29, Th. 189], one has

$$\begin{aligned}\tilde{g}(t)&=\int_{t_0}^{t}g(\theta)\exp\left(-\frac{\alpha(t-\theta)}{p}\right)\exp\left(-\frac{\alpha(t-\theta)}{q}\right)\mathrm{d}\theta\\&\le\left[\int_{t_0}^{t}g^p(\theta)\exp\left(-\alpha(t-\theta)\right)\mathrm{d}\theta\right]^{\frac{1}{p}}\\&\quad\times\left[\int_{t_0}^{t}\exp\left(-\alpha(t-\theta)\right)\mathrm{d}\theta\right]^{\frac{1}{q}}\\&\le\left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\left[\int_{t_0}^{t}g^p(\theta)\exp\left(-\alpha(t-\theta)\right)\mathrm{d}\theta\right]^{\frac{1}{p}}.\end{aligned}$$

It then follows that:

$$\int_{t_0}^{T}\tilde{g}^p(t)\mathrm{d}t\le\left(\frac{1}{\alpha}\right)^{\frac{p}{q}}\int_{t_0}^{T}\mathrm{d}t\int_{t_0}^{t}g^p(\theta)\exp\left(-\alpha(t-\theta)\right)\mathrm{d}\theta.$$

Similar to (37), it can be proved that

$$\int_{t_0}^{T}\mathrm{d}t\int_{t_0}^{t}g^p(\theta)\exp\left(-\alpha(t-\theta)\right)\mathrm{d}\theta\le\frac{1}{\alpha}\int_{t_0}^{T}g^p(\theta)\mathrm{d}\theta$$

which further leads to

$$\int_{t_0}^{T}\tilde{g}^p(t)\mathrm{d}t\le\left(\frac{1}{\alpha}\right)^{p}\int_{t_0}^{T}g^p(\theta)\mathrm{d}\theta.$$

Since $\int_{t_0}^{+\infty}g^p(t)\mathrm{d}t$ converges, the integral $\int_{t_0}^{+\infty}\tilde{g}^p(t)\mathrm{d}t$ also converges for $p>1$. □

Proof of Lemma 4: According to the comparison principle, for $k\ge 1$ and $t\in[\hat{t}_k,\hat{t}_{k+1})$, one has

$$\varphi(t)\le\varphi(\hat{t}_k)\exp(\alpha(t-\hat{t}_k))\le\mu\varphi(\hat{t}_k^-)\exp(\alpha(t-\hat{t}_k))$$

leading to $\varphi(\hat{t}_{k+1}^-)\le\mu\varphi(\hat{t}_k^-)\exp(\alpha(\hat{t}_{k+1}-\hat{t}_k))$. It then follows that:

$$\begin{aligned}\varphi(\hat{t}_k^-)&\le\mu^{k-1}\varphi(\hat{t}_1^-)\exp(\alpha(\hat{t}_k-\hat{t}_1))\\&\le\mu^{k-1}\times\varphi(\hat{t}_0)\exp(\alpha(\hat{t}_1-\hat{t}_0))\times\exp(\alpha(\hat{t}_k-\hat{t}_1))\\&=\mu^{k-1}\varphi(0)\exp(\alpha\hat{t}_k).\end{aligned}$$

Therefore, $\varphi(t)\le\mu^k\varphi(0)\exp(\alpha t)$ for $t\in[\hat{t}_k,\hat{t}_{k+1})$. Since $\hat{t}_{k+1}-\hat{t}_k\ge\tau_0$, one has $t\ge\hat{t}_k\ge k\tau_0$. If $\mu\ge 1$, then

$$\varphi(t)\le\mu^{t/\tau_0}\varphi(0)\exp(\alpha t)\le\varphi(0)\exp(c_0t).$$

Otherwise, $\varphi(t)\le\varphi(0)\exp(\alpha t)\le\varphi(0)\exp(c_0t)$. □

Proof of Lemma 5: Suppose $A_{11}\in\mathbb{R}^{k\times k}$, where k is a positive integer. Taking $u\in\mathbb{R}^k$, one has

$$\begin{aligned}\|A_{11}u\|&\le\sqrt{\|A_{11}u\|^2+\|A_{21}u\|^2}\\&=\left\|\begin{bmatrix}A_{11}u\\A_{21}u\end{bmatrix}\right\|=\left\|A\begin{bmatrix}u\\0\end{bmatrix}\right\|.\end{aligned}$$

It then follows that:

$$\max_{\|u\|=1}\|A_{11}u\|\le\max_{\|u\|=1}\left\|A\begin{bmatrix}u\\0\end{bmatrix}\right\|\le\max_{\|w\|=1}\|Aw\|$$

which indicates $\|A_{11}\|\le\|A\|$. Similarly, it can be concluded that $\|A_{22}\|\le\|A\|$. □

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