Further on Synchronization of Dynamical Networks via Adaptive Intermittent Control

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***Abstract*—In recent years, the synchronization problem of dynamical networks under adaptive intermittent control has been extensively studied, but it has not been well solved up to now. In this article, we further investigate this problem. The updating laws adopted here is simpler than that in much of the literature. First, boundedness of the adaptive control gain is proved to ensure practicality of adaptive control. Then, synchronization criteria for dynamical networks under adaptive intermittent control are derived based on the gain boundedness analysis. Finally, examples are provided to demonstrate the theoretical results.**

***Index Terms*—Adaptive intermittent control (AIC), dynamical network, linear intermittent control (LIC), synchronization.**

I. Introduction

Complex networks are widespread in the real world, such as neural networks, citation networks, and power grids. As an important collective behavior in nature, synchronization of dynamical networks has been extensively studied for several decades [1], [2], [3], [4], [5].

Quite often, a network is not able to realize synchronization by itself, and thus external feedback control is indispensable in general. In the literature, many control schemes have been applied to realize network synchronization, including adaptive control [6], intermittent control [7], pinning control [8], and impulsive control [9]. In reality, state feedback information could become inaccessible due to sensor failure or external interruption, which results in inapplicability of continuous control. In contrast, intermittent control is more effective as it does not have to receive feedback information all the time.

In recent years, intermittent control has been widely adopted for network synchronization. In [10], [11], [12], [13], [14], and [15], linear intermittent control (LIC) is adopted to realize asymptotic synchronization or exponential synchronization. It is noteworthy that the design of the control gain of LIC heavily relies on a prior knowledge of network parameters. To deal with such problem, the adaptive intermittent control (AIC) scheme can be adopted. In [16], [17], [18], [19], [20], [21], and [22], updating laws of the adaptive control gain $\xi_i(t)$ is designed in form of

$$\dot{\xi}_i(t) = \eta_i\|e_i(t)\|^2\exp(at) \tag{1}$$

where e_i means the synchronization error and η_i, a can be any positive numbers. The term $\exp(at)$ is critical in guaranteeing convergence of the Lyapunov functionals constructed in these studies. However, it is hard to determine whether $\dot{\xi}_i$ converges to zero even if convergence of e_i has been proved. If not, ξ_i might increase quickly to infinity, which is unrealistic. Hence, analysis on boundedness of ξ_i is fundamental but neglected in [16], [17], [18], [19], [20], [21], and [22]. In [23], the updating laws is designed as

$$\dot{\xi}_i(t) = \eta_i\|e_i(t)\|^2. \tag{2}$$

Unfortunately, the critical proof after inequality (31) therein seems to be incorrect. In [24] and [25], the updating laws takes the following form:

$$\dot{\xi}(t) = \|e(t)\|^2 \tag{3}$$

where e is the synchronization error of the entire network. It should be highlighted that updating laws (3) needs global state information of the network, which is generally impractical for large-scale networks. To sum up, updating laws (2) is simpler and more practical, but AIC with updating laws (2) has not yet been successfully used for network synchronization. The main difficulty lies in that the technique widely used in [16], [17], [18], [19], [20], [21], and [22] is inapplicable when removing the term $\exp(at)$ from $\dot{\xi}_i$.

Motivated by the above discussions, this note further investigates synchronization of dynamical networks under AIC. The main contributions are as follows.

1) The updating laws of AIC is designed in form of (2), which is successfully addressed in the synchronization problem for the first time.
2) In order to ensure practicality of adaptive control, boundedness of the adaptive control gain is theoretically proved, which is missing in [16], [17], [18], [19], [20], [21], [22], [23], [24], and [25].
3) Based on boundedness of the control gain, synchronization criteria are established for networks under AIC.

Notations: $\mathbb{R}_+, \mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ denote the set of nonnegative real numbers, real vectors of dimension n, and real matrices of dimension $n\times m$, respectively; $\|\cdot\|$ is the two-norm of a matrix or a vector; $\otimes$ is the Kronecker product; the superscript $\top$ stands for the transpose of a vector or a matrix; I_n denotes the identity matrix of dimension n. Given vectors $x_1, x_2, \ldots, x_N$, denote $[x_r^\top, x_{r+1}^\top, \ldots, x_s^\top]^\top$ as $\mathrm{vec}_{r\le i\le s}\{x_i\}$, where r and s are integers such that $1\le r < s\le N$. In particular, denote $\mathrm{vec}_{1\le i\le s}\{x_i\}$ as $\mathrm{vec}_s\{x_i\}$.

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II. Preliminaries

A. Problem Formulation

Consider a dynamical network consisting of N nonlinear systems

$$\dot{x}_i(t) = g(x_i(t), t) + c\sum_{j=1}^{N} a_{ij}\Phi x_j(t) + u_i,\ i \in \Im \tag{4}$$

where the set $\Im = \{1, 2, \ldots, N\}$; $x_i \in \mathbb{R}^n$ represents the state vector of node i; $g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a nonlinear function; $c > 0$ represents the coupling strength; $A = (a_{ij})_{N\times N}$ describes the weights of connections among the nodes and satisfies $\sum_{j=1}^{N} a_{ij} = 0$; $\Phi \in \mathbb{R}^{n\times n}$ stands for the inner coupling matrix; u_i is the adaptive control input to be designed.

The main objective is to design appropriate AIC to synchronize network (4) to a given state $w(t)$ governed by

$$\dot{w}(t) = g(w(t), t). \tag{5}$$

The synchronization error is denoted as $e(t) = \mathrm{vec}_N\{e_i(t)\}$, where $e_i(t) = x_i(t) - w(t)$. Control u_i is designed as follows:

$$u_i(t) = \begin{cases} -\xi_i(t)e_i(t), & t \in [t_k, t_k + \omega_k] \\ 0, & t \in (t_k + \omega_k, t_{k+1}) \end{cases} \tag{6}$$

where $0 = t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to+\infty} t_k = +\infty$, $\omega_k > 0$ for $k \geq 1$, and $\xi_i(t)$ represents the adaptive control gain, which satisfies $\xi_i(0) \geq 0$ and $\xi_i(t_{k+1}) = \xi_i(t_k + \omega_k)$. The control gain is updated according to

$$\dot{\xi}_i(t) = \eta_i\|e_i(t)\|^2,\ t \in [t_k, t_k + \omega_k] \tag{7}$$

where $k \geq 1$, $i \in \Im$, and $\eta_i > 0$ is a constant.

Remark 1: Control (6) is referred to as aperiodical AIC. In particular, it reduces to periodical AIC if $t_k = (k-1)T^*$ and $\omega_k = \omega^*$ for $k \geq 1$.

B. Mathematical Preliminaries

Assumption 1: For any $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^n$, function g satisfies

$$\|g(x,t) - g(y,t)\| \leq \vartheta\|x - y\| \tag{8}$$

where ϑ is a nonnegative constant.

Assumption 2: There exist positive constants ω^* and T^* such that

$$\begin{array}{l} \inf_{k\geq 1}\{\omega_k\} = \omega^* > 0 \\ \sup_{k\geq 1}\{t_{k+1} - t_k\} = T^* < +\infty \end{array} \tag{9}$$

where t_k and ω_k are defined in (6).

Lemma 1: Let $\{\alpha_k\}_{k\geq 1}$, $\{\beta_k\}_{k\geq 1}$, and $\{\mu_k\}_{k\geq 1}$ be nonnegative number sequences satisfying

$$\mu_{k+1} \leq \alpha_k\mu_k + \beta_k,\ k \geq k_0 \tag{10}$$

where $k_0 \geq 1$ is an integer. If $\sup_{k\geq 1}\{\alpha_k\} < 1$ and $\sum_{k=1}^{+\infty}\beta_k$ converges, then $\sum_{k=1}^{+\infty}\mu_k$ converges.

Proof: Denote $\alpha = \sup_{k\geq 1}\{\alpha_k\}$, then $\alpha \in [0, 1)$. If $\alpha = 0$, one has $0 \leq \mu_{k+1} \leq \beta_k$ for $k \geq k_0$, from which it can be derived that $\sum_{k=k_0+1}^{+\infty}\mu_k \leq \sum_{k=k_0}^{+\infty}\beta_k$. By the convergence of $\sum_{k=1}^{+\infty}\beta_k$, it is clear that $\sum_{k=1}^{+\infty}\mu_k$ converges.

It is next to deal with the case of $\alpha \neq 0$. Applying (10), one has $\mu_{k+1} \leq \alpha\mu_k + \beta_k$ for $k \geq k_0$. Hence

$$\begin{aligned} \frac{\mu_{k+1}}{\alpha^{k+1}} &\leq \frac{\mu_k}{\alpha^k} + \frac{\beta_k}{\alpha^{k+1}} \\ &\leq \frac{\mu_{k-1}}{\alpha^{k-1}} + \frac{\beta_{k-1}}{\alpha^k} + \frac{\beta_k}{\alpha^{k+1}} \\ &\ \ \vdots \\ &\leq \frac{\mu_{k_0}}{\alpha^{k_0}} + \frac{\beta_{k_0}}{\alpha^{k_0+1}} + \frac{\beta_{k_0+1}}{\alpha^{k_0+2}} + \cdots + \frac{\beta_k}{\alpha^{k+1}} \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \mu_{k+1} \leq{}& \mu_{k_0}\alpha^{k-k_0+1} + \beta_{k_0}\alpha^{k-k_0} + \beta_{k_0+1}\alpha^{k-k_0-1} \\ &+ \cdots + \beta_{k-1}\alpha + \beta_k,\ k \geq k_0. \end{aligned} \tag{11}$$

Summing (11) from $k = k_0$ to $k = m$ results in

$$\begin{aligned} \sum_{k=k_0}^{m}\mu_{k+1} \leq{}& \mu_{k_0}\sum_{k=k_0}^{m}\alpha^{k-k_0+1} + \beta_{k_0}\sum_{k=k_0}^{m}\alpha^{k-k_0} \\ &+ \beta_{k_0+1}\sum_{k=k_0+1}^{m}\alpha^{k-k_0-1} + \cdots \\ &+ \beta_{m-1}\sum_{k=m-1}^{m}\alpha^{k-m+1} + \beta_m \\ ={}& \mu_{k_0}\frac{\alpha(1-\alpha^{m-k_0+1})}{1-\alpha} + \beta_{k_0}\frac{1-\alpha^{m-k_0+1}}{1-\alpha} \\ &+ \beta_{k_0+1}\frac{1-\alpha^{m-k_0}}{1-\alpha} + \cdots + \beta_{m-1}\frac{1-\alpha^2}{1-\alpha} + \beta_m \\ \leq{}& \frac{\alpha}{1-\alpha}\mu_{k_0} + \frac{1}{1-\alpha}(\beta_{k_0} + \beta_{k_0+1} + \cdots + \beta_m). \end{aligned}$$

Therefore, convergence of $\sum_{k=1}^{+\infty}\beta_k$ implies convergence of $\sum_{k=1}^{+\infty}\mu_k$. □

Lemma 2: Let σ be a positive constant and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying

$$\phi(\hat{t}_0 + t) \leq \phi(\hat{t}_0)\exp(\sigma t)\ \forall t, \hat{t}_0 \in \mathbb{R}_+.$$

If $\alpha, \beta,$ and γ are constants such that $\gamma > \beta > \alpha \geq 0$, then

$$\int_\beta^\gamma \phi(s)\mathrm{d}s \leq \left(\frac{\varrho\exp[\sigma(\gamma-\alpha)]}{\varrho - 1} - 1\right)\int_\alpha^\beta \phi(s)\mathrm{d}s \tag{12}$$

where $\varrho = \exp[\sigma(\beta - \alpha)]$.

Proof: Denote $\frac{\gamma-\beta}{\beta-\alpha} = p + \varepsilon$, where p is a nonnegative integer and $\varepsilon \in [0, 1)$. Then

$$\begin{aligned} \int_\beta^\gamma \phi(s)\mathrm{d}s &\leq \sum_{\ell=1}^{p+1}\int_{\alpha+\ell(\beta-\alpha)}^{\alpha+(\ell+1)(\beta-\alpha)}\phi(s)\mathrm{d}s \\ &= \sum_{\ell=1}^{p+1}\int_\alpha^\beta \phi[s + \ell(\beta-\alpha)]\mathrm{d}s \\ &\leq \sum_{\ell=1}^{p+1}\varrho^\ell\int_\alpha^\beta\phi(s)\mathrm{d}s = \frac{\varrho(\varrho^{p+1}-1)}{\varrho-1}\int_\alpha^\beta\phi(s)\mathrm{d}s. \end{aligned}$$

Considering $p \leq \frac{\gamma-\beta}{\beta-\alpha}$, one has $\varrho^{p+1} \leq \exp[\sigma(\gamma-\alpha)]$, then inequality (12) can be derived. □

III. Main Results

In this section, it is first to analyze boundedness of the adaptive control gain $\xi_i(t)$, and then to establish synchronization criteria for network (4).

A. Control Gain Analysis

According to updating laws (7), it is easy to find that the adaptive control gain $\xi_i(t)$ keeps increasing as long as $e_i(t) \neq 0$. Since controllers cannot provide infinitely large control gain, it is necessary to analyze the boundedness of $\xi_i(t)$.

Theorem 1: Consider network (4), and let the intermittent control u_i for $i \in \Im$ be designed as (6) with updating laws (7). If Assumptions 1 and 2 hold, then the adaptive control gain $\xi_i(t)$ is bounded for $i \in \Im$.

Proof: From (9), one has $\omega_k \geq \omega^*$ and $t_{k+1} - t_k \leq T^*$ for $k \geq 1$. Denote

$$\beta = 2\vartheta + 2c\|A\|\|\Phi\| + c^2\|A\|^2\|\Phi\|^2$$

$$\xi_0 = \max\left\{\frac{\alpha+\beta}{2},\ \xi_1(0),\ \cdots,\ \xi_N(0)\right\}$$

$$\Lambda = \{i \in \Im | \xi_i(t) < \xi_0\ \forall t \geq 0\}$$

where ϑ is defined in Assumption 1 and α is a large enough parameter to be determined.

Suppose that there are r elements in Λ, then $0 \leq r \leq N$. If $r = N$, then $\xi_1(t), \ldots, \xi_N(t)$ are all bounded. Otherwise, it suffices to prove that $\xi_i(t)$ is bounded for $i \notin \Lambda$.

Next consider the case of $0 \leq r < N$. Without loss of generality, let $\Lambda = \{1, 2, \ldots, r\}$. Define $\kappa(t) = \mathrm{vec}_r\{e_i(t)\}$, $\zeta(t) = \mathrm{vec}_{r+1\leq i\leq N}\{e_i(t)\}$, and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} \in \mathbb{R}^{r\times r}$ and $A_{22} \in \mathbb{R}^{(N-r)\times(N-r)}$. In particular, one has $\kappa(t) = 0$ and $\zeta(t) = e(t)$ if $r = 0$.

Consider the Lyapunov candidate $V_1(t) = \zeta^\top(t)\zeta(t)$. By (4) and (5), it is clear that

$$\begin{aligned}
\dot{V}_1(t) &= 2\sum_{i=r+1}^{N} e_i^\top(t)[g(x_i(t),t) - g(w(t),t)] \\
&\quad + 2c\sum_{i=r+1}^{N}\sum_{j=1}^{N} a_{ij} e_i^\top(t)\Phi e_j(t) - 2\sum_{i=r+1}^{N}\xi_i(t)\|e_i(t)\|^2 \\
&\leq 2\vartheta\|\zeta(t)\|^2 + 2c\zeta^\top(t)(A_{22}\otimes\Phi)\zeta(t) \\
&\quad + 2c\zeta^\top(t)(A_{21}\otimes\Phi)\kappa(t) \\
&\quad - 2\sum_{i=r+1}^{N}\xi_i(t)\|e_i(t)\|^2\ \forall t \in [t_k, t_k+\omega_k].
\end{aligned} \tag{13}$$

Let μ be a vector satisfying $\|\mu\| = 1$ and $\|A_{22}\| = \|A_{22}\mu\|$, and denote $\bar{\mu} = [0, \mu^\top]^\top \in \mathbb{R}^N$. Then

$$\|A\| \geq \|A\bar{\mu}\| = \sqrt{\|A_{12}\mu\|^2 + \|A_{22}\mu\|^2} \geq \|A_{22}\|$$

which leads to

$$\zeta^\top(t)(A_{22}\otimes\Phi)\zeta(t) \leq \|A_{22}\|\|\Phi\|\|\zeta(t)\|^2 \leq \|A\|\|\Phi\|\|\zeta(t)\|^2.$$

Similarly, one has $\|A\| \geq \|A_{21}\|$ and

$$\begin{aligned}
2c\zeta^\top(t)(A_{21}\otimes\Phi)\kappa(t) &\leq 2c\|A\|\|\Phi\|\|\zeta(t)\|\|\kappa(t)\| \\
&\leq c^2\|A\|^2\|\Phi\|^2\|\zeta(t)\|^2 + \|\kappa(t)\|^2.
\end{aligned}$$

Since $\xi_i(t)$ is nondecreasing, it follows from the definition of r that there exists an integer $k_0 \geq 1$ satisfying

$$\xi_i(t) \geq \xi_0\ \forall t \in T_1$$

where $r < i \leq N$ and $T_1 = \bigcup_{k=k_0}^{+\infty}[t_k, t_k+\omega_k]$. Then, it can be deduced from (13) that

$$\begin{aligned}
\dot{V}_1(t) &\leq 2\vartheta\|\zeta(t)\|^2 + 2c\|A\|\|\Phi\|\|\zeta(t)\|^2 + \\
&\qquad + c^2\|A\|^2\|\Phi\|^2\|\zeta(t)\|^2 + \|\kappa(t)\|^2 - 2\xi_0\|\zeta(t)\|^2 \\
&\leq \beta\|\zeta(t)\|^2 + \psi(t) - 2\times\frac{\alpha+\beta}{2}\|\zeta(t)\|^2 \\
&= -\alpha V_1(t) + \psi(t)\ \forall t \in T_1
\end{aligned} \tag{14}$$

where $\psi(t) = \|\kappa(t)\|^2$. Similarly, one gets

$$\dot{V}_1(t) \leq \beta V_1(t) + \psi(t) \quad \forall t \in T_2 \tag{15}$$

where $T_2 = \bigcup_{k=k_0}^{+\infty}(t_k+\omega_k, t_{k+1})$.

Rewrite inequality (14) as

$$\frac{\mathrm{d}}{\mathrm{d}t}[V_1(t)\exp(\alpha t)] \leq \psi(t)\exp(\alpha t) \quad \forall t \in T_1. \tag{16}$$

Let k be any integer no smaller than k_0. For $t \in [t_k, t_k+\omega_k]$, one obtains

$$V_1(t) \leq V_1(t_k)\exp[\alpha(t_k - t)] + \int_{t_k}^{t}\psi(s)\exp[\alpha(s-t)]\mathrm{d}s. \tag{17}$$

Since $\int_{t_k}^{t_k+\omega_k}\exp[\alpha(t_k-t)]\mathrm{d}t = \frac{1-\exp(-\alpha\omega_k)}{\alpha} \leq \frac{1}{\alpha}$ and

$$\begin{aligned}
&\int_{t_k}^{t_k+\omega_k}\mathrm{d}t\int_{t_k}^{t}\psi(s)\exp[\alpha(s-t)]\mathrm{d}s \\
&= \int_{t_k}^{t_k+\omega_k}\psi(s)\exp(\alpha s)\mathrm{d}s\int_{s}^{t_k+\omega_k}\exp(-\alpha t)\mathrm{d}t \\
&= \int_{t_k}^{t_k+\omega_k}\psi(s)\frac{1-\exp[-\alpha(t_k+\omega_k-s)]}{\alpha}\mathrm{d}s \\
&\leq \frac{1}{\alpha}\int_{t_k}^{t_k+\omega_k}\psi(s)\mathrm{d}s
\end{aligned}$$

integrating (17) from $t = t_k$ to $t = t_k+\omega_k$ yields

$$\int_{t_k}^{t_k+\omega_k}V_1(t)\mathrm{d}t \leq \frac{1}{\alpha}V_1(t_k) + \frac{1}{\alpha}\int_{t_k}^{t_k+\omega_k}\psi(s)\mathrm{d}s. \tag{18}$$

Taking $t = t_k+\omega_k$, it can also be derived from (17) that

$$V_1(t_k+\omega_k) \leq V_1(t_k)\exp(-\alpha\omega_k) + \int_{t_k}^{t_k+\omega_k}\psi(s)\mathrm{d}s.$$

Solving (15) in a similar way, one has

$$\begin{aligned}
V_1(t_{k+1}) &\leq V_1(t_k+\omega_k)\exp[\beta(t_{k+1}-t_k-\omega_k)] \\
&\quad + \int_{t_k+\omega_k}^{t_{k+1}}\psi(s)\exp[\beta(t_{k+1}-s)]\mathrm{d}s \\
&\leq \left[V_1(t_k+\omega_k) + \int_{t_k+\omega_k}^{t_{k+1}}\psi(s)\mathrm{d}s\right] \\
&\quad \times\exp[\beta(t_{k+1}-t_k-\omega_k)] \\
&\leq \left[V_1(t_k)\exp(-\alpha\omega_k) + \int_{t_k}^{t_{k+1}}\psi(s)\mathrm{d}s\right] \\
&\quad \times\exp[\beta(t_{k+1}-t_k-\omega_k)].
\end{aligned} \tag{19}$$

Consider another Lyapunov candidate $V_2(t) = e^\top(t)e(t)$. Similar to (13), one obtains

$$\dot{V}_2(t) \leq 2\vartheta V_2(t) + 2ce^\top(t)(A \otimes \Phi)e(t) \leq \gamma V_2(t) \quad (20)$$

where $\gamma = 2(\vartheta + c\|A\|\|\Phi\|)$. Then, for any $\hat{t}_0 \geq 0$

$$V_2(\hat{t}_0 + t) \leq V_2(\hat{t}_0)\exp(\gamma t)\ \forall t \geq 0. \quad (21)$$

Applying Lemma 2 gives

$$\begin{aligned}\int_{t_k}^{t_{k+1}} V_2(s)\mathrm{d}s &= \int_{t_k}^{t_k+\omega_k} V_2(s)\mathrm{d}s + \int_{t_k+\omega_k}^{t_{k+1}} V_2(s)\mathrm{d}s \\ &\leq \frac{\varrho_k \exp[\gamma(t_{k+1}-t_k)]}{\varrho_k - 1}\int_{t_k}^{t_k+\omega_k} V_2(s)\mathrm{d}s\end{aligned}$$

where $\varrho_k = \exp(\gamma\omega_k)$. Combining inequality (18), $V_2(t) = \psi(t) + V_1(t)$, and

$$\frac{\varrho_k \exp[\gamma(t_{k+1}-t_k)]}{\varrho_k - 1} \leq \frac{\exp(\gamma\omega^*)\exp(\gamma T^*)}{\exp(\gamma\omega^*) - 1} \triangleq \varrho$$

it can be obtained that

$$\begin{aligned}\int_{t_k}^{t_{k+1}} \psi(s)\mathrm{d}s \leq& \int_{t_k}^{t_{k+1}} V_2(s)\mathrm{d}s \\ &\leq \varrho\int_{t_k}^{t_k+\omega_k} V_2(s)\mathrm{d}s \\ &= \varrho\int_{t_k}^{t_k+\omega_k}\psi(s)\mathrm{d}s + \varrho\int_{t_k}^{t_k+\omega_k} V_1(s)\mathrm{d}s \\ &\leq \frac{\varrho}{\alpha}V_1(t_k) + \frac{(\alpha+1)\varrho}{\alpha}\int_{t_k}^{t_k+\omega_k}\psi(s)\mathrm{d}s. \quad (22)\end{aligned}$$

Denoting $\mu_k = V_1(t_k)$, $\bar{\alpha}_k = \exp[\beta(t_{k+1} - t_k - \omega_k) - \alpha\omega_k]$, and $\bar{\beta}_k = \exp[\beta(t_{k+1} - t_k - \omega_k)]$, it is deduced from (19) and (22) that

$$\begin{aligned}\mu_{k+1} &\leq \bar{\alpha}_k\mu_k + \bar{\beta}_k\int_{t_k}^{t_{k+1}}\psi(s)\mathrm{d}s \\ &\leq \left(\bar{\alpha}_k + \frac{\varrho\bar{\beta}_k}{\alpha}\right)\mu_k + \frac{(\alpha+1)\varrho\bar{\beta}_k}{\alpha}\int_{t_k}^{t_k+\omega_k}\psi(s)\mathrm{d}s \\ &\triangleq \alpha_k\mu_k + \beta_k. \quad (23)\end{aligned}$$

Let $h(\alpha) = \exp[\beta(T^* - \omega^*) - \alpha\omega^*] + \frac{\varrho\exp[\beta(T^*-\omega^*)]}{\alpha}$. Clearly, $\lim_{\alpha\to 0^+} h(\alpha) = +\infty$ and $\lim_{\alpha\to+\infty} h(\alpha) = 0$. Hence, there exists a $\alpha^* > 0$ such that $h(\alpha^*) = \frac{1}{2}$. Take $\alpha = \alpha^*$, then $\sup_{k\geq 1}\{\alpha_k\} \leq h(\alpha) < 1$.

In addition, the updating laws (7) indicates

$$\begin{aligned}\sum_{k=k_0}^{+\infty}\int_{t_k}^{t_k+\omega_k}\eta_i\|e_i(s)\|^2\mathrm{d}s &= \sum_{k=k_0}^{+\infty}[\xi_i(t_k+\omega_k) - \xi_i(t_k)] \\ &= \lim_{k\to+\infty}\xi_i(t_k+\omega_k) - \xi_i(t_{k_0}) \quad (24)\end{aligned}$$

where $i \in \mathfrak{I}$. Since $\xi_i(t)$ is nondecreasing and $\xi_i(t) \leq \xi_0$ for $1 \leq i \leq r$, the limit $\lim_{k\to+\infty}\xi_i(t_k+\omega_k)$ exists and

$$\lim_{k\to+\infty}\xi_i(t_k+\omega_k) \leq \xi_0,\ 1 \leq i \leq r.$$

Denote $\eta^* = \min_{1\leq i\leq r}\{\eta_i\}$, then

$$\begin{aligned}\sum_{k=k_0}^{+\infty}\int_{t_k}^{t_k+\omega_k}\psi(s)\mathrm{d}s &= \sum_{i=1}^{r}\sum_{k=k_0}^{+\infty}\int_{t_k}^{t_k+\omega_k}\|e_i(s)\|^2\mathrm{d}s \\ &\leq \sum_{i=1}^{r}\frac{\xi_0 - \xi_i(t_{k_0})}{\eta^*} \quad (25)\end{aligned}$$

which indicates that

$$\sum_{k=k_0}^{+\infty}\beta_k \leq \beta^*\sum_{k=k_0}^{+\infty}\int_{t_k}^{t_k+\omega_k}\psi(s)\mathrm{d}s \leq \beta^*\sum_{i=1}^{r}\frac{\xi_0 - \xi_i(t_{k_0})}{\eta^*}$$

where

$$\beta^* = \frac{(\alpha+1)\varrho\bar{\beta}_k}{\alpha} \leq \frac{(\alpha+1)\varrho\exp[\beta(T^*-\omega^*)]}{\alpha}.$$

Hence, $\sum_{k=1}^{+\infty}\beta_k$ converges.

By Lemma 1, it is concluded that $\sum_{k=1}^{+\infty}\mu_k$ converges. By (18) and (25), the convergence of $\sum_{k=k_0}^{+\infty}\int_{t_k}^{t_k+\omega_k}V_1(t)\mathrm{d}t$ can be verified. Since $\xi_i(t)$ is nondecreasing, it follows from (24) that

$$\begin{aligned}\lim_{t\to+\infty}\xi_i(t) =& \lim_{k\to+\infty}\xi_i(t_k+\omega_k) \\ &= \xi_i(t_{k_0}) + \sum_{k=k_0}^{+\infty}\int_{t_k}^{t_k+\omega_k}\eta_i\|e_i(s)\|^2\mathrm{d}s \\ &\leq \xi_i(t_{k_0}) + \eta_i\sum_{k=k_0}^{+\infty}\int_{t_k}^{t_k+\omega_k}V_1(s)\mathrm{d}s,\ r < i \leq N\end{aligned}$$

which indicates that $\xi_i(t)$ is also bounded for $r < i \leq N$. The proof is complete. □

Remark 2: In Theorem 1, boundedness of the adaptive control gains $\xi_1(t), \ldots, \xi_N(t)$ is theoretically proved. In the literature [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], AIC has been widely adopted to realize network synchronization, but boundedness analysis of the adaptive control gain is missing.

B. Synchronization Criteria

Based on boundedness analysis on $\xi_i(t)$, it is now ready to derive synchronization criteria for network (4).

Theorem 2: If Assumptions 1 and 2 hold, then network (4) can realize synchronization under AIC (6) with updating laws (7).

Proof: By Theorem 1, there exist positive constants ϱ_i such that $\xi_i(t) \leq \varrho_i$ for $i \in \mathfrak{I}$. Then, it follows from the updating laws (7) that

$$\sum_{k=1}^{+\infty}\int_{t_k}^{t_k+\omega_k}\eta_i\|e_i(s)\|^2\mathrm{d}s \leq \varrho_i,\ i \in \mathfrak{I}. \quad (26)$$

Letting $V(t) = e^\top(t)e(t)$, (26) implies that

$$\sum_{k=1}^{+\infty}\int_{t_k}^{t_k+\omega_k}V(s)\mathrm{d}s \leq \sum_{i=1}^{N}\frac{\rho_i}{\eta_i}. \quad (27)$$

It is first to prove that

$$\lim_{t\in T_c, t\to+\infty} e(t) = 0 \quad (28)$$

where $T_c = \bigcup_{k=1}^{+\infty}[t_k, t_k+\omega_k]$. If (28) does not hold, then there exists a number sequence $\{s_k\}_{k\geq 1}$ in strictly increasing order and a positive constant ε such that $s_k \in T_c$, $\lim_{k\to+\infty}s_k = +\infty$, and $\|e(s_k)\| \geq \varepsilon$. For any $k \geq 1$, there exists a unique n_k such that $s_k \in [t_{n_k}, t_{n_k} + \omega_{n_k}]$. It is clear that $\lim_{k\to+\infty}n_k = +\infty$. It then follows from (27) that

$$\lim_{k\to+\infty}\int_{t_{n_k}}^{t_{n_k}+\omega_{n_k}}V(s)\mathrm{d}s = 0. \quad (29)$$

If $s_k - t_{n_k} \geq \frac{\omega^*}{2}$, inequality (20) indicates

$$V(t) \geq V(s_k)\exp[\gamma_1(t-s_k)] \quad \forall t \leq s_k \tag{30}$$

where ω^* is defined in (9) and $\gamma_1 = 2(\vartheta + c\|A\|\|\Phi\|)$. Then

$$\int_{t_{n_k}}^{t_{n_k}+\omega_{n_k}} V(s)\mathrm{d}s \geq \int_{s_k-\frac{\omega^*}{2}}^{s_k} V(s_k)\exp[\gamma_1(s-s_k)]\mathrm{d}s \geq \frac{\varepsilon^2}{\gamma_1}[1-\exp(-\gamma_1\omega^*/2)] \triangleq \epsilon_1.$$

If $s_k - t_{n_k} < \frac{\omega^*}{2}$, similar to (13), one has

$$\dot{V}(t) \geq -2\vartheta V(t) + 2ce^\top(t)(A\otimes\Phi)e(t) - 2\sum_{i=1}^N \xi_i(t)\|e_i(t)\|^2 \geq -2(\vartheta + c\|A\|\|\Phi\| + \max_{i\in\mathfrak{I}}\{\varrho_i\})V(t) \triangleq -\gamma_2 V(t)$$

which further yields

$$V(t) \geq V(s_k)\exp[-\gamma_2(t-s_k)] \quad \forall t \geq s_k. \tag{31}$$

Since $t_{n_k} + \omega_{n_k} - s_k > \omega_{n_k} - \frac{\omega^*}{2} \geq \frac{\omega^*}{2}$, one gets

$$\int_{t_{n_k}}^{t_{n_k}+\omega_{n_k}} V(s)\mathrm{d}s \geq \int_{s_k}^{s_k+\frac{\omega^*}{2}} V(s_k)\exp[-\gamma_2(s-s_k)]\mathrm{d}s \geq \frac{\varepsilon^2}{\gamma_2}[1-\exp(-\gamma_2\omega^*/2)] \triangleq \epsilon_2.$$

Hence, $\int_{t_{n_k}}^{t_{n_k}+\omega_{n_k}} V(s)\mathrm{d}s \geq \min\{\epsilon_1, \epsilon_2\} > 0$, which contradicts (29). Therefore, (28) is verified.

It is next to prove that $\lim_{t\to+\infty} e(t) = 0$. Let $\{\hat{t}_k\}_{k=1}^{+\infty}$ be any sequence in increasing order such that $\lim_{k\to+\infty}\hat{t}_k = +\infty$. It suffices to prove that $\lim_{k\to+\infty} V(\hat{t}_k) = 0$.

There exists a sequence $\{\hat{s}_k\}_{k=1}^{+\infty}$ such that $\hat{s}_k \in T_c$ and $0 \leq \hat{t}_k - \hat{s}_k \leq T^*$. It can be derived from (20) and (21) that

$$|V(\hat{t}_k) - V(\hat{s}_k)| = |(\hat{t}_k - \hat{s}_k)\dot{V}(\alpha_k)| \leq \gamma T^* V(\alpha_k) \leq \gamma T^* \exp(\gamma T^*) V(\hat{s}_k)$$

where $\alpha_k \in [\hat{s}_k, \hat{t}_k]$. By (28), one has $\lim_{k\to+\infty} V(\hat{s}_k) = 0$, which further leads to $\lim_{k\to+\infty} V(\hat{t}_k) = 0$. This completes the proof. □

Remark 3: In Theorem 2, synchronization criteria of network (4) under AIC are established. Although bounds ω^* and T^* for the control width ω_k and the control interval $t_{k+1} - t_k$ are required, the synchronization criteria here are independent of the exact values of these bounds. In contrast, if the LIC scheme is adopted, the synchronization criteria and the design of the control gain would depend on ω^* and T^*.

Remark 4: In [16], [17], [18], [19], [20], [21], and [22], updating laws during the control interval $[t_k, t_k + \omega_k]$ is designed in form of $\dot{\xi}_i(t) = \eta_i\|e_i(t)\|^2\exp(at)$, and the Lyapunov functional is constructed as (or slightly different)

$$W(t) = \sum_{i=1}^N e_i^\top(t)e_i(t) + \sum_{i=1}^N \frac{(\xi_i(t) - \bar{\xi})^2}{\eta_i}\exp(-at) \triangleq W_1(t) + W_2(t)$$

where $\bar{\xi}$ is a constant and the term $W_2(t)$ is used to address AIC. The term $\exp(at)$ in $\dot{\xi}_i(t)$ ensures exponential convergence of $W(t)$ on $[t_k, t_k + \omega_k]$, which is critical in deriving that $\lim_{t\to+\infty} W(t) = 0$. If removing the term $\exp(at)$ from $\dot{\xi}_i(t)$, however, exponential convergence of $W(t)$ on $[t_k, t_k + \omega_k]$ might be destroyed, and thus the above proof procedure would not work. Differing from the technique used in [16], [17], [18], [19], [20], [21], and [22], convergence of $e(t)$ is proved here based on boundedness of $\xi_i(t)$.

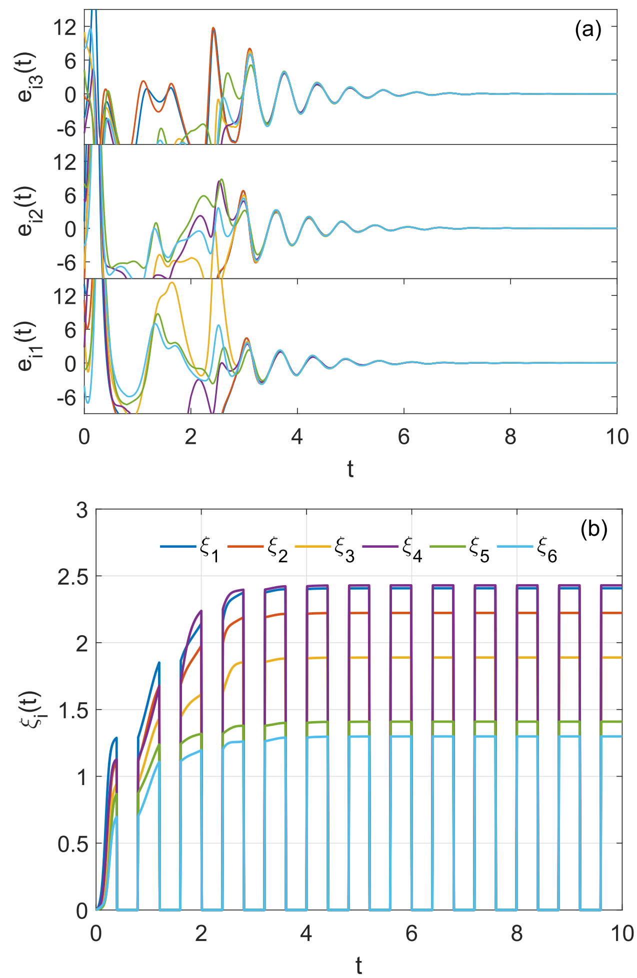

Fig. 1. (a) Synchronization error $e_{ij}(t)$ and (b) adaptive control gain $\xi_i(t)$ of network (33) under periodical AIC, where $1 \leq i \leq 6$ and $1 \leq j \leq 3$.

IV. Numerical Examples

In this section, two numerical examples are provided, where the node system is taken as the Lorenz system [26]

$$\begin{bmatrix}\dot{w}_1(t)\\ \dot{w}_2(t)\\ \dot{w}_3(t)\end{bmatrix} = \begin{bmatrix} c_1[w_2(t) - w_1(t)] \\ c_3 w_1(t) - w_2(t) - w_1(t)w_3(t) \\ -c_2 w_3(t) + w_1(t)w_2(t)\end{bmatrix} \tag{32}$$

where $c_1 = 10$, $c_2 = 8/3$, and $c_3 = 28$. Assumption 1 can be verified due to boundedness of the Lorenz system (see [27] for details).

A. Example 1: Synchronization Via Periodical AIC

Consider a directed network of $N = 6$ Lorenz systems

$$\dot{x}_i(t) = g(x_i(t), t) + c\sum_{j=1}^6 a_{ij}\Phi x_j(t) + u_i(t) \tag{33}$$

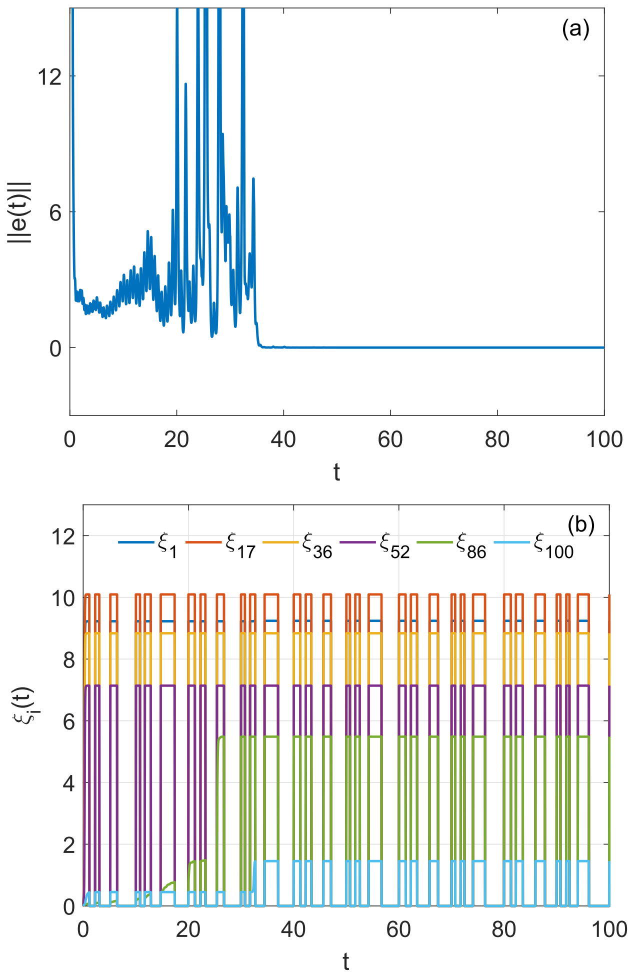

Fig. 2. (a) Norm of synchronization error $e(t)$ and (b) adaptive control gains $\xi_1(t)$, $\xi_{17}(t)$, $\xi_{36}(t)$, $\xi_{52}(t)$, $\xi_{86}(t)$, and $\xi_{100}(t)$ of network (35) under aperiodical AIC.

where $c = 1$, $\Phi = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, and

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 2 & -5 & 0 & 0 & 0 \\ 0 & 0 & 4 & -7 & 0 & 3 \\ 0 & 0 & 0 & 3 & -3 & 0 \\ 2 & 0 & 1 & 1 & 0 & -4 \end{bmatrix}.$$

The periodical AIC scheme is adopted to achieve synchronization of network (33), where the time sequence $\{t_k\}_{k\geq 1}$ and the control width ω_k is chosen as

$$t_k = 0.8(k-1),\ \omega_k = 0.4. \tag{34}$$

Since Assumption 1 has been verified and Assumption 2 clearly holds, it follows from Theorems 1 and 2 that the adaptive control gains $\xi_1(t), \ldots, \xi_6(t)$ are bounded and network (33) can realize synchronization.

Choose $\eta_i = 0.002$, $\xi_i(0) = 0$, $z(0) = [-5, 1, -2]^\top$, and

$$x_i(0) = 10\left[\cos\left(\frac{i\pi}{7}\right), \cos\left(\frac{3i\pi}{7}\right), \cos\left(\frac{5i\pi}{7}\right)\right]^\top$$

where $1 \leq i \leq 6$. The trajectories of $e_{ij}(t)$ and $\xi_i(t)$ are as illustrated in Fig. 1. It is found that synchronization of network (33) is realized and $\xi_i(t)$ is bounded. Thus, effectiveness of the theoretical results is demonstrated.

B. Example 2: Synchronization Via Aperiodical AIC

Consider an undirected ring network consisting of $N = 100$ Lorenz systems

$$\dot{x}_i(t) = g(x_i(t), t) + c\sum_{j=1}^{100} a_{ij}\Phi x_j(t) + u_i(t) \tag{35}$$

where $c = 5$, $\Phi = I_3$, and

$$a_{ij} = \begin{cases} 1, & \text{if } |i-j| = \pm 1 \pmod{100} \\ -2, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

The aperiodical AIC scheme with

$$t_k = \begin{cases} 10m - 10, & \text{if } k = 3m-2 \\ 10m - 8 + 0.3\sin k & \text{if } k = 3m-1 \\ 10m - 5 + \sin k, & \text{if } k = 3m \end{cases}$$

$$\omega_k = \begin{cases} 1 + 0.3\cos k, & \text{if } k = 3m-2 \\ 1 + 0.5\cos k, & \text{if } k = 3m-1 \\ 2 + 0.7\cos k, & \text{if } k = 3m \end{cases}$$

is adopted for synchronization of network (35), where $m \geq 1$. By Theorems 1 and 2, $\xi_i(t)$ is bounded for $1 \leq i \leq 100$ and network (35) can realize synchronization, as shown in Fig. 2, where only $\|e(t)\|$ and a part of the control gains are presented for clarity.

Some initial values and parameters are taken as $\eta_i = 0.1$, $\xi_i(0) = 0$, $z(0) = [-2, 1, -1]^\top$, and $x_i(0) = [-5, -1, -2]^\top + 0.02i[1, 2, -1]^\top$, where $1 \leq i \leq 100$.

V. Conclusion

In this note, AIC has been adopted for network synchronization. Boundedness of the control gains has been proved, which is fundamental for guaranteeing practicality of adaptive control. Based on the boundedness analysis, synchronization criteria for networks under AIC have been established. Since existing updating laws needs the state information all the time, the communication cost is high. Hence, we will further adopt AIC with sampled-data updating laws to realize synchronization in future work.

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