Introducing a New Edge Centrality Measure: The Connectivity Rank Index

Jin Zhou, Yanqi Zhang, Jun-an Lu, and Guanrong Chen, *Life Fellow, IEEE*

***Abstract*—A new edge centrality measure, connectivity rank index (CRI), is proposed based on the effect of an edge on the network algebraic connectivity. Compared with the existing indices, the CRI can determine the importance of a present edge as well as an absent edge. For large-scale networks, the algorithm based on original CRI definition has high-time complexity. Therefore, an approximation algorithm is designed using the eigenvector elements corresponding to the second smallest Laplacian eigenvalue. This algorithm can identify the most influential edges and the least influential ones easily, which reduces the time complexity from the exhaustive searching scheme with $O(N^5)$ to $O(N^3)$ in a network of size N. Some examples are shown to verify the effectiveness of the algorithm and the theoretical results.**

***Index Terms*—Algebra connectivity, complex network, edge centrality, eigenvalue, eigenvector.**

I. Introduction

COMPLEX networks provide an efficient framework for describing complex systems. A network structure is described by nodes and edges. Finding which nodes or edges are the most influential, referred to as node centrality and edge centrality respectively, is an important task for studying disease spreading [1], network synchronization [2], [3], [4], attack prevention [5] and failure diagnosis [6]. Various node centrality indicators have been proposed, such as degree centrality [7], closeness centrality [8], betweenness centrality [9], PageRank [10], k-shell [11], eigenvector centrality [12], ControlRank [13], [14], etc.

Node centrality emphasizes on the influence of a node by itself, while edge centrality considers a node and its neighbors. As the most popular index, betweenness centrality of an edge is proportional to the number of shortest paths passing it [15]. In [16], the edge betweenness centrality is generalized to k-path edge centrality, which is proportional to the number of l ($l \leq k$) paths passing it at random. Relative effectiveness of an edge is measured by the decrease of the maximum Laplacian eigenvalue due to edge removal [17]. All these edge centrality indicators rank the importance of the present edges in a network. It should be noted that the edge centrality in a network includes two aspects, namely, the importance of the edges (abbreviation for the present edges) and that of the absent edges. For examples, to defend against a network attack, it is necessary to identify the most essential edges in the network and protect them; while to accelerate the diffusion through a network by adding edges, one should identify the most influential absent edges and add them into the network. This article investigates the edge centrality in both situations by introducing a new measure, called connectivity rank index (CRI).

Algebraic connectivity, characterized by the second smallest Laplacian eigenvalue [18], bridges the topology and the dynamics of a network. It has been verified that many dynamical properties of a network are directly related to its second smallest Laplacian eigenvalue, such as synchronizability [19], [20], [21], diffusion rate [22], stability and robustness [23], distributed consensus [24], [25], [26], and the dynamical behaviors of many distributed algorithms [24], [27].

The proposed CRI applies the increase or decrease of the algebraic connectivity caused by edge addition or edge removal in a network. Further, for large-scale networks, the time complexity of calculating the variation of the second smallest eigenvalue is high. Therefore, an approximation algorithm is designed to simplify the calculation of the CRI, with rigorous theoretical analysis. The edge centrality computed by the approximation algorithm is measured by the eigenvector elements corresponding to the second smallest Laplacian eigenvalue. By using this algorithm, the time complexity of calculating all the edges in a network with N nodes is reduced from the exhaustive searching scheme with $O(N^5)$ to $O(N^3)$. Some examples are shown to verify the effectiveness of the algorithm and the theoretical results.

The remaining parts of this article are organized as follows. Section II-A presents some mathematical preliminaries and Section II-B introduces the CRI. Section III-A analyzes the variation range of the second smallest Laplacian eigenvalue caused by deleting or adding an edge in the network; Section III-B designs an approximation algorithm for calculating the CRI of large-scale networks. Section IV shows some examples. Finally, Section V concludes this article.

Manuscript received 18 October 2023; accepted 21 December 2023. Date of publication 23 January 2024; date of current version 17 April 2024. This work was supported in part by the National Natural Science Foundation of China under Grant 62173254, Grant 61773294, and Grant 62176099; and in part by the Hong Kong Research Grants Council under the GRF Grant CityU 11206320. This article was recommended by Associate Editor W. Chen. *(Corresponding authors: Yanqi Zhang; Jun-an Lu.)*

Jin Zhou, Yanqi Zhang, and Jun-an Lu are with the School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China (e-mail: zhangyanqi@whu.edu.cn; jalu@whu.edu.cn).

Guanrong Chen is with the Department of Electrical Engineering, City University of Hong Kong, Hong Kong, China.

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TSMC.2024.3349407.

Digital Object Identifier 10.1109/TSMC.2024.3349407

2168-2216 © 2024 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission.
See https://www.ieee.org/publications/rights/index.html for more information.

Authorized licensed use limited to: Wuhan University. Downloaded on May 31,2024 at 12:56:25 UTC from IEEE Xplore. Restrictions apply.

II. Connectivity Rank Index

A diffusively connected, unweighted and undirected network with N nodes is considered throughout this manuscript. Let the original network, the original Laplacian matrix, the network and the Laplacian matrix after deleting (adding) edge ε_{ij} be G, L, G^- (G^+) and L^- (L^+), respectively. For simplicity, G^- is assumed to be connected. Notationally, $G^- = G - \varepsilon_{ij}$, $G^+ = G + \varepsilon_{ij}$, $\lambda_2(L)\boldsymbol{x_2(L)} = L\boldsymbol{x_2(L)}$, where $\lambda_2(\cdot)$ and $\boldsymbol{x_2(\cdot)}$ are the second smallest eigenvalue and the corresponding eigenvector of a matrix, respectively.

A. Mathematical Preliminaries

To establish the main results, some lemmas are first prepared.

Lemma 1 [18]: For an undirected, unweighted and connected graph G, the second smallest Laplacian eigenvalue satisfies

$$\lambda_2(L) = \min_{z\perp\mathbf{1}, \|z\|=1} z^T L z = \min_{z\perp\mathbf{1}, \|z\|=1} \sum_{\varepsilon_{mn}\in G} (z_m - z_n)^2 \qquad (1)$$

where $\mathbf{1}$ is a column vector with all the components being 1, and z_m is the mth component of vector $\boldsymbol{z}$.

Therefore, the calculation of $\lambda_2(L)$ can be regarded as an optimization problem of minimizing the sum of some quadratic terms by distributing each component of $\boldsymbol{z}$.

Lemma 2 [28]: If a graph $G_1 \subseteq G_2$, then $\lambda_2(L_1) \le \lambda_2(L_2)$, where L_1 and L_2 are the Laplacian matrices of graphs G_1 and G_2 respectively.

Lemma 3 [18]: If L is the Laplacian matrix of graph G, then $\lambda_2(L) \ge 0$, with equality if and only if G is disconnected.

It is seen from Lemma 3 that if deleting an edge render the network G^- disconnected, then this edge must be the most influential in the network, due to $\lambda_2(L^-) = 0$.

B. Connectivity Rank Index

Define the variation of the second smallest Laplacian eigenvalues as

$$\triangle\lambda_{ij}^- = \lambda_2\left(L^-\right) - \lambda_2(L) \qquad (2)$$

and

$$\triangle\lambda_{ij}^+ = \lambda_2\left(L^+\right) - \lambda_2(L) \qquad (3)$$

respectively, where $\triangle\lambda_{ij}^-$ ($\triangle\lambda_{ij}^+$) represents the decrease (increase) of the second smallest Laplacian eigenvalue caused by deleting (adding) edge ε_{ij} in the network.

The second smallest Laplacian eigenvalue is referred to as the algebraic connectivity of a network, which connects the topology and the dynamical properties [19], [20], [22]. Based on the algebraic connectivity, a new measure called CRI is defined to characterize the edge centrality in a network.

Definition 1 (CRI of an Edge): For any $\varepsilon_{ij} \in G$, the CRI of edge ε_{ij} is

$$I_{ij}^- = \frac{\triangle\lambda_{ij}^-}{\left|\min\limits_{\varepsilon_{mn}\in G} \triangle\lambda_{mn}^-\right|} \qquad (4)$$

where $I_{ij}^- = 0$ if $\min_{\varepsilon_{mn}\in G} \triangle\lambda_{mn}^- = 0$.

Definition 2 (CRI of an Absent Edge): For any $\varepsilon_{ij} \notin G$, the CRI of absent edge ε_{ij} is

$$I_{ij}^+ = \frac{\triangle\lambda_{ij}^+}{\left|\max\limits_{\varepsilon_{mn}\notin G} \triangle\lambda_{mn}^+\right|} \qquad (5)$$

where $I_{ij}^+ = 0$ if $\max_{\varepsilon_{mn}\notin G} \triangle\lambda_{mn}^+ = 0$.

According to Lemma 2, if an edge is deleted from (added into) a graph, $\lambda_2(L)$ will not increase (decrease). As a result, $\triangle\lambda_{ij}^-$ ($\triangle\lambda_{ij}^+$) and thus I_{ij}^- (I_{ij}^+) are nonpositive (nonnegative) real numbers. Therefore, when the denominator of (4) is 0, it means that $\triangle\lambda_{ij}^-$ caused by every edge $\varepsilon_{ij} \in G$ is 0. As mentioned in Definition 1, just let every I_{ij}^- be 0.

The index CRI is normalized, so its value is between -1 and 1. The closer the absolute value of the CRI of an edge (absent edge) is to 1, the more important it is.

III. Approximation of Connectivity Rank Index

This section is divided into two subsections. Subsections A presents the range of $\triangle\lambda_{ij}^-$ ($\triangle\lambda_{ij}^+$) caused by edge ε_{ij}, respectively. In Section III-B, for large network size N, $\triangle\lambda_{ij}^-$ ($\triangle\lambda_{ij}^+$) is approximated by its lower (upper) bound given in Section III-A; and an approximation algorithm is designed for computing the CRI.

A. Variation Range of the Second Smallest Laplacian Eigenvalue

In this section, the lower (upper) bound of the decrease (increase) in the second smallest eigenvalue caused by the edge removal (edge addition) is derived. First consider the edge addition case.

Theorem 1: Let L (L^+) be the Laplacian matrix of a network (the network by adding edge ε_{ij}). Then

$$\triangle\lambda_{ij}^+ = \lambda_2\left(L^+\right) - \lambda_2(L) \le \left(x_{2i}(L) - x_{2j}(L)\right)^2$$

where $x_{2i}(L)$ and $x_{2j}(L)$ are the ith and jth components of the unit eigenvector $\boldsymbol{x_2(L)}$ corresponding to the second smallest Laplacian eigenvalue $\lambda_2(L)$ respectively.

Proof: Denote $\boldsymbol{x^*}$ as the unit vector orthogonal to $\mathbf{1}$, which belongs to the set

$$\Theta = \left\{\bar{\boldsymbol{x}} \middle| \sum_{\varepsilon_{mn}\in G} (\bar{x}_m - \bar{x}_n)^2 = \min_{z\perp\mathbf{1}, \|z\|=1} \sum_{\varepsilon_{mn}\in G} (z_m - z_n)^2 \right\}.$$

Obviously, the existence of $\lambda_2(L)$ guarantees the existence of $\boldsymbol{x^*}$. Based on Lemma 1 and the definition of $\boldsymbol{x^*}$, one gets

$$\lambda_2(L) = \boldsymbol{x^{*T}} \boldsymbol{L} \boldsymbol{x^*} = \sum_{\varepsilon_{mn}\in G} \left(x_m^* - x_n^*\right)^2 \qquad (6)$$

and

$$\lambda_2\left(L^+\right) = \min_{z\perp\mathbf{1}, \|z\|=1} \sum_{\varepsilon_{mn}\in G^+} (z_m - z_n)^2 \le \sum_{\varepsilon_{mn}\in G^+} \left(x_m^* - x_n^*\right)^2. \qquad (7)$$

Authorized licensed use limited to: Wuhan University. Downloaded on May 31,2024 at 12:56:25 UTC from IEEE Xplore. Restrictions apply.

On the one hand, from $G^+ = G + \varepsilon_{ij}$ and (6), one has

$$\begin{aligned}\lambda_2(L^+) &\leq \sum_{\varepsilon_{mn}\in G^+} \left(x_m^* - x_n^*\right)^2 \\ &= \sum_{\varepsilon_{mn}\in G} \left(x_m^* - x_n^*\right)^2 + \left(x_i^* - x_j^*\right)^2 \\ &= \lambda_2(L) + \left(x_i^* - x_j^*\right)^2. \end{aligned} \tag{8}$$

On the other hand, left multiplying (6) by $\boldsymbol{x^*}$ shows that $\boldsymbol{x^*}$ is the unit eigenvector of L corresponding to $\lambda_2(L)$.

Replacing x_i^* and x_j^* in (8) with $x_{2i}(L)$ and $x_{2j}(L)$ respectively, one can rewrite (8) as

$$\lambda_2(L^+) \leq \lambda_2(L) + \left(x_{2i}(L) - x_{2j}(L)\right)^2.$$

This completes the proof. ■

For the case of edge removal, the following result can be similarly obtained.

Theorem 2: Let L (L^-) be the Laplacian matrix of a network (the network by deleting edge ε_{ij}). Then

$$|\triangle\lambda_{ij}^-| = \lambda_2(L) - \lambda_2(L^-) \geq \left(x_{2i}(L) - x_{2j}(L)\right)^2$$

where $x_{2i}(L)$ and $x_{2j}(L)$ are the ith and jth components of the unit eigenvector $\boldsymbol{x_2(L)}$ corresponding to the second smallest Laplacian eigenvalue $\lambda_2(L)$ respectively.

Proof: Let $\boldsymbol{x^*}$ be the same vector as that in the proof of Theorem 1. Note that $G^- = G - \varepsilon_{ij}$. Similarly to the derivation of Theorem 1, one gets

$$\begin{aligned}\lambda_2(L^-) &\leq \sum_{\varepsilon_{mn}\in G^-} \left(x_m^* - x_n^*\right)^2 \\ &= \sum_{\varepsilon_{mn}\in G} \left(x_m^* - x_n^*\right)^2 - \left(x_i^* - x_j^*\right)^2 \\ &= \lambda_2(L) - \left(x_i^* - x_j^*\right)^2. \end{aligned} \tag{9}$$

Thus, one obtains

$$\lambda_2(L) - \lambda_2(L^-) \geq \left(x_{2i}(L) - x_{2j}(L)\right)^2.$$

This completes the proof. ■

From the perspective of algebraic connectivity, the following corollary gives a criterion to judge the least important absent edge.

Corollary 1: Let L (L^+) be the Laplacian of a network (the network by adding edge ε_{ij}). If the ith and jth components of the unit eigenvector $\boldsymbol{x_2(L)}$ are equal, then $\lambda_2(L^+) = \lambda_2(L)$.

Corollary 1 is derived easily from Theorem 1 and Lemma 2. It implies that if two or more components of $\boldsymbol{x_2(L)}$ are equal, then adding the corresponding edges makes no difference to the second smallest Laplacian eigenvalue. The conclusion is consistent with the results of [29, Th. 3.1].

Now that the least variation of $\lambda_2(L)$ is 0 by adding an edge into a network, it is interesting to find the largest variation of $\lambda_2(L)$ by adding or deleting an edge. Consider adding edge ε_{ij} into a network. On the one hand, it is clear that $\triangle\lambda_{ij}^+ \leq (x_{2i}(L) - x_{2j}(L))^2$ based on Theorem 1. On the other hand, graph G can be regarded as $G = G^+ - \varepsilon_{ij}$ and so $\triangle\lambda_{ij}^+ \geq (x_{2i}(L^+) - x_{2j}(L^+))^2$ according to Theorem 2. Similar result can be obtained for edge removal.

Corollary 2: For a given network, the variation range of $\lambda_2(L)$ by adding or deleting an edge ε_{ij} is

$$\left(x_{2i}(L^+) - x_{2j}(L^+)\right)^2 \leq \triangle\lambda_{ij}^+ \leq \left(x_{2i}(L) - x_{2j}(L)\right)^2$$
$$\left(x_{2i}(L) - x_{2j}(L)\right)^2 \leq |\triangle\lambda_{ij}^-| \leq \left(x_{2i}(L^-) - x_{2j}(L^-)\right)^2.$$

Corollary 2 reveals that the components of the unit eigenvector corresponding to $\lambda_2(L)$ contribute to its variation range.

B. Approximation Algorithm for Large-Scale Networks

Definitions 1 and 2 present a CRI-based method to rank edges by calculating the variation of the second smallest Laplacian eigenvalues. The exhaustive searching time complexity of calculating the eigenvalue variation induced by deleting (adding) each edge (absent edge) is $O(N^5)$. Thus, for a large-scale network, it is necessary to improve the computational efficiency of CRI-based method. The approximate calculation method of second smallest eigenvalue changes is proposed below, so a more efficient calculation algorithm of approximate CRI is obtained.

Theorem 3: Given a network with sufficiently large size N. Let L (L^+) be the Laplacian matrix of the network (the network by adding edge ε_{ij}). Then

$$\triangle\lambda_{ij}^+ = \lambda_2(L^+) - \lambda_2(L) = \left(x_{2i}(L) - x_{2j}(L)\right)^2$$

where $x_{2i}(L)$ and $x_{2j}(L)$ are the ith and jth components of the unit eigenvector $\boldsymbol{x_2(L)}$ corresponding to the second smallest Laplacian eigenvalue $\lambda_2(L)$ respectively.

Proof: For $\varepsilon_{ij} \notin G$, ignoring the same elements in $L \triangleq (d_{ij})_{N\times N}$ and L^+, one has

$$L = \begin{pmatrix} d_{ii} & 0 \\ 0 & d_{jj} \end{pmatrix}, L^+ = \begin{pmatrix} d_{ii}+1 & -1 \\ -1 & d_{jj}+1 \end{pmatrix}.$$

Introduce a positive semi-definite matrix that has only 4 nonzero elements

$$P = \begin{pmatrix} N & -N \\ -N & N \end{pmatrix}.$$

It is obvious that $L^+ = L + (1/N)P$. Assume that

$$\begin{cases} \lambda_2(L^+) = \lambda_2(L) + \frac{1}{N}\lambda_2' \\ \boldsymbol{x_2(L^+)} = \boldsymbol{x_2(L)} + \frac{1}{N}\boldsymbol{x_2'}. \end{cases} \tag{10}$$

Substituting (10) into $\lambda_2(L^+)\boldsymbol{x_2(L^+)} = L^+\boldsymbol{x_2(L^+)}$, one obtains $(\lambda_2(L)+\frac{1}{N}\lambda_2')(\boldsymbol{x_2(L)}+\frac{1}{N}\boldsymbol{x_2'}) = (L+\frac{1}{N}P)(\boldsymbol{x_2(L)}+\frac{1}{N}\boldsymbol{x_2'})$. Expand the equation and left multiply it by $\boldsymbol{x_2^T(L)}$, then

$$\begin{aligned}&\frac{1}{N}\boldsymbol{x_2^T(L)}\lambda_2'\boldsymbol{x_2(L)} + \frac{1}{N^2}\boldsymbol{x_2^T(L)}\lambda_2'\boldsymbol{x_2'} \\ &= \frac{1}{N}\boldsymbol{x_2^T(L)}P\boldsymbol{x_2(L)} + \frac{1}{N^2}\boldsymbol{x_2^T(L)}P\boldsymbol{x_2'}. \end{aligned} \tag{11}$$

Authorized licensed use limited to: Wuhan University. Downloaded on May 31,2024 at 12:56:25 UTC from IEEE Xplore. Restrictions apply.

Note that

$$\begin{aligned}&\boldsymbol{x_2^T(L)}\lambda_2'\boldsymbol{x_2'}-\boldsymbol{x_2^T(L)}P\boldsymbol{x_2'}\\&=N\boldsymbol{x_2^T(L)}\big(\lambda_2(L^+)-\lambda_2(L)\big)\boldsymbol{x_2'}-N\boldsymbol{x_2^T(L)}\big(L^+-L\big)\boldsymbol{x_2'}\\&=N\boldsymbol{x_2^T(L)}\lambda_2(L^+)\boldsymbol{x_2'}-N\boldsymbol{x_2^T(L)}L^+\boldsymbol{x_2'}\\&=N\bigg(\boldsymbol{x_2^T(L^+)}-\frac{1}{N}\boldsymbol{x_2'^T}\bigg)\lambda_2(L^+)\boldsymbol{x_2'}-N\bigg(\boldsymbol{x_2^T(L^+)}-\frac{1}{N}\boldsymbol{x_2'^T}\bigg)L^+\boldsymbol{x_2'}\\&=-\boldsymbol{x_2'^T}\lambda_2(L^+)\boldsymbol{x_2'}+\boldsymbol{x_2'^T}L^+\boldsymbol{x_2'}.\end{aligned}\tag{12}$$

Ignoring $O(1/N^2)$ terms in (11), one gets

$$\lambda_2'=\frac{\boldsymbol{x_2^T(L)}P\boldsymbol{x_2(L)}}{\boldsymbol{x_2^T(L)}\boldsymbol{x_2(L)}}.$$

Since $\boldsymbol{x_2(L)}$ is a unit vector, one has

$$\lambda_2'=\boldsymbol{x_2^T(L)}P\boldsymbol{x_2(L)}=N\big(x_{2i}(L)-x_{2j}(L)\big)^2$$

and so $\lambda_2(L^+)=\lambda_2(L)+(x_{2i}(L)-x_{2j}(L))^2$. This completes the proof. ■

The conclusion is consistent with the numerical algorithm in [30], which is illustrated in a supergradient way.

Remark 1: Theorem 3 can be understood from the perspective of optimization. Let $\boldsymbol{x_2(L)}$ be the unit vector orthogonal to $\mathbf{1}$, which minimizes $\sum_{\varepsilon_{mn}\in G}(z_m-z_n)^2$. Then $\boldsymbol{x_2(L)}$ is potentially the vector that minimizes $\sum_{\varepsilon_{mn}\in G+\varepsilon_{ij}}(z_m-z_n)^2=\sum_{\varepsilon_{mn}\in G}(z_m-z_n)^2+(z_i-z_j)^2$ for sufficiently large N, because $\sum_{\varepsilon_{mn}\in G}(z_m-z_n)^2$ is the dominance. Based on Lemma 1, one obtains $\lambda_2(L^+)=\lambda_2(L)+(x_{2i}(L)-x_{2j}(L))^2$.

Theorem 1 gives an upper bound $(x_{2i}(L)-x_{2j}(L))^2$ of $\triangle\lambda_{ij}^+$, while Theorem 3 shows that $\triangle\lambda_{ij}^+$ is close to the bound when the network size is sufficiently large. Analogously, the following Theorem reveals that $-\triangle\lambda_{ij}^-$ is close to the lower bound presented in Theorem 2 for a large-scale network.

Theorem 4: Given a network with sufficiently large size N. Let L (L^-) be the Laplacian matrix of the network (the network by deleting edge ε_{ij}). Then

$$\triangle\lambda_{ij}^-=\lambda_2\big(L^-\big)-\lambda_2(L)=-\big(x_{2i}(L)-x_{2j}(L)\big)^2$$

where $x_{2i}(L)$ and $x_{2j}(L)$ are the ith and jth components of the unit eigenvector $\boldsymbol{x_2(L)}$ corresponding to the second smallest Laplacian eigenvalue $\lambda_2(L)$ respectively.

Proof: Similarly to the proof of Theorem 3, $L^-=L-(1/N)P$. Letting

$$\begin{cases}\lambda_2\big(L^-\big)=\lambda_2(L)-\frac{1}{N}\lambda_2'\\ \boldsymbol{x_2\big(L^-\big)}=\boldsymbol{x_2(L)}-\frac{1}{N}\boldsymbol{x_2'}\end{cases}$$

and ignoring $O(1/N^2)$ terms, one obtains

$$\lambda_2(L)\boldsymbol{x_2'}+\lambda_2'\boldsymbol{x_2(L)}=L\boldsymbol{x_2'}+P\boldsymbol{x_2(L)}$$

and further obtains

$$\lambda_2'=\boldsymbol{x_2^T(L)}P\boldsymbol{x_2(L)}=N\big(x_{2i}(L)-x_{2j}(L)\big)^2.$$

Therefore, $\lambda_2(L^-)=\lambda_2(L)-(x_{2i}(L)-x_{2j}(L))^2$. This completes the proof. ■

Based on the theorems, an approximation to the CRI in a large-scale network is proposed as follows.

Algorithm 1 CRI Algorithm

Step 1: Given the Laplacian matrix L of graph G with N nodes and an all-zero matrix P with the same dimension as L.
Step 2: Calculate the eigenvector $\boldsymbol{x_2(L)}$ corresponding to second smallest eigenvalue of L.
Step 3: For any $i\in[1,N-1], j\in[i+1,N]$, if the element in row i and column j of L is not equal to 0, then let the element in row i and column j of P be $-(x_{2i}(L)-x_{2j}(L))^2$. Else let the element in row i and column j of P be $(x_{2i}(L)-x_{2j}(L))^2$.
Step 4: Find the maximum $x_{\max}$ and minimum $x_{\min}$ values in P.
Step 5: Divide all numbers greater than 0 in P by $x_{\max}$ and all numbers smaller than 0 in P by $x_{\min}$.

Definition 3 (CRI of an Edge): For $\varepsilon_{ij}\in G$, an approximation to the CRI of edge ε_{ij} is

$$I_{ij}^-=\frac{-\big(x_{2i}(L)-x_{2j}(L)\big)^2}{\max\limits_{\varepsilon_{mn}\in G}(x_{2m}(L)-x_{2n}(L))^2}\tag{13}$$

where $I_{ij}^-=0$ if $\max_{\varepsilon_{mn}\in G}(x_{2m}(L)-x_{2n}(L))^2=0$.

Definition 4 (CRI of an Absent Edge): For $\varepsilon_{ij}\notin G$, an approximation to the CRI of absent edge ε_{ij} is

$$I_{ij}^+=\frac{\big(x_{2i}(L)-x_{2j}(L)\big)^2}{\max\limits_{\varepsilon_{mn}\notin G}(x_{2m}(L)-x_{2n}(L))^2}\tag{14}$$

where $I_{ij}^+=0$ if $\max_{\varepsilon_{mn}\notin G}(x_{2m}(L)-x_{2n}(L))^2=0$.

Remark 2: In a large-scale network, one can easily find the most influential edges by considering $\{\varepsilon_{i^*j^*}||x_{2i^*}-x_{2j^*}|=\max_{ij}|x_{2i}-x_{2j}|\}$, and identify the least influential edges by $\{\varepsilon_{i^*j^*}||x_{2i^*}-x_{2j^*}|=\min_{ij}|x_{2i}-x_{2j}|\}$.

Remark 3: Under certain circumstance, the unit eigenvector(s) corresponding to the second smallest Laplacian eigenvalue may not be unique. Without loss of generality, let $\boldsymbol{x_2^1(L)},\ldots,\boldsymbol{x_2^M(L)}(M\in[1,N-1])$ be the unit eigenvectors corresponding to $\lambda_2(L)$. In this case, $-(x_{2i}(L)-x_{2j}(L))^2$ in Definition 3 is not unique. From Theorem 2, $\min_{k=1,\ldots,M}\{-(x_{2i}^k(L)-x_{2j}^k(L))^2\}$ is the closest value of $\triangle\lambda_{ij}^-$ among the M values. Analogously, the closest value to $\triangle\lambda_{ij}^+$ is $\min_{k=1,\ldots,M}\{(x_{2i}^k(L)-x_{2j}^k(L))^2\}$. Therefore, an approximation to the CRI in a large-scale network is redefined as

$$I_{ij}^-=\frac{\min\limits_{k=1,\ldots,M}\Big\{-\Big(x_{2i}^k(L)-x_{2j}^k(L)\Big)^2\Big\}}{\max\limits_{\varepsilon_{mn}\in G}\max\limits_{k=1,\ldots,M}\Big\{\Big(x_{2m}^k(L)-x_{2n}^k(L)\Big)^2\Big\}}$$

$$I_{ij}^+=\frac{\min\limits_{k=1,\ldots,M}\Big\{\Big(x_{2i}^k(L)-x_{2j}^k(L)\Big)^2\Big\}}{\max\limits_{\varepsilon_{mn}\notin G}\min\limits_{k=1,\ldots,M}\Big\{\Big(x_{2m}^k(L)-x_{2n}^k(L)\Big)^2\Big\}}$$

where I_{ij}^- and I_{ij}^+ are both equal to 0 if the denominator equals 0.

Accordingly, the approximation algorithm based on Definitions 3 and 4 is proposed.

Obviously, for a network with N nodes, this algorithm reduces the time complexity of CRI-based method from $O(N^5)$ to $O(N^3)$. At the same time, according to Theorems 3, 4 and

Authorized licensed use limited to: Wuhan University. Downloaded on May 31,2024 at 12:56:25 UTC from IEEE Xplore. Restrictions apply.

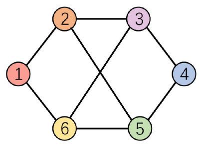

Fig. 1. (a) Network with 6 nodes and 8 edges. (b) Network after deleting edge ε_{12} from (a). (c) Network after adding edge ε_{13} into (a).

the explanation at Section IV-B, there is no excessive loss of CRI accuracy. As a result, the approximation to the CRI of all the edges (including the present and the absent edges) in a large-scale network can be calculated by using the elements of $\boldsymbol{x_2(L)}$ or those of the auxiliary eigen-matrix simultaneously. The CRIs allow us to rank the edges and the absent edges from the perspective of algebraic connectivity.

IV. Examples

In this section, some examples are shown to verify the effectiveness of the theoretical results.

A. Calculation of CRI

Take the network in Fig. 1(a) as an example. The network after deleting edge ε_{12} is depicted in Fig. 1(b). The second smallest Laplacian eigenvalue of the network in Fig. 1(a) and that in Fig. 1(b) are 1.4384 and 0.7639 respectively. So, $\triangle\lambda_{12}^- = 0.7639 - 1.4384 = -0.6745$. Repeating the above operation for the remaining seven edges, one obtains $|\min_{\varepsilon_{mn}\in G} \triangle\lambda_{mn}^-| = 0.6745$. By Definition 1, the CRI of edge ε_{12} is -1.

Consider the network after adding edge ε_{13} as shown in Fig. 1(c). Its second smallest Laplacian eigenvalue is 1.7857. It is clear that $\triangle\lambda_{13}^+ = 1.7857 - 1.4384 = 0.3473$. Repeating the same operation for the other absent edges, one obtains $|\max_{\varepsilon_{mn}\notin G} \triangle\lambda_{mn}^+| = 1.5616$, and the CRI of the absent edge ε_{13} is 0.2224.

Other CRIs of the edges and absent edges shown in Fig. 1(a) can be obtained similarly. All the CRI values are shown in Table I. From the table, the most important edges in Fig. 1(a) are $\varepsilon_{12}, \varepsilon_{16}, \varepsilon_{34}$, and ε_{45}, because their CRIs are -1. Whereas the most important absent edge is ε_{14}, and the least important edges are ε_{26} and ε_{35}. It is clear that the CRI values of the edges are negative and those of the absent edges are positive. The influence of an edge or an absent edge is proportional to the absolute value of CRI.

B. Comparison of the Approximations With the Precision of CRI

1) Network With 13 Edges: To compare the approximation with the exact CRI intuitively, a strongly symmetrical network with 12 nodes and 13 edges is considered, as shown in Fig. 2. According to Definition 1 and Definition 3, the exact and the approximate CRIs of all the edges are compared in Table II (as seen in the second and fourth columns respectively). Based on the CRIs, the exact and approximate ranks of edge centrality are shown in the third and fifth columns of Table II, respectively. The absolute values of CRI reveal the connectivity rank of edges. The results of edge centrality in Table II are visualized by Fig. 2, where the deeper color the edge is represented, the more influential of the edge is.

TABLE I
CRI of All the Edges in Fig. 1(a)

i \ j	1	2	3	4	5	6
1	0.0000	**-1.0000**	0.2224	**1.0000**	0.2224	**-1.0000**
2		0.0000	-0.6500	0.2224	-0.6500	**0.0000**
3			0.0000	**-1.0000**	**0.0000**	-0.6500
4				0.0000	**-1.0000**	0.2224
5					0.0000	-0.6500
6						0.0000

[1] The i-j pair element represents the CRI of edge ε_{ij} $(i \in [1, N], j > i)$.
[2] Negative values indicate the CRIs of the present edges, whereas positive values indicates the CRIs of the absent edges.

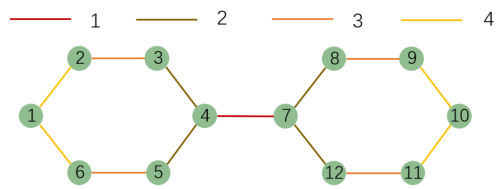

Fig. 2. Strongly symmetrical network with 12 nodes and 13 edges. Edge centrality in the network with exact or approximate CRIs.

TABLE II
Exact and Approximate CRIs of the Edges in Fig. 2

(i,j)	I_{ij}^-	Rank	Approximate I_{ij}^-	Approximate Rank
$(1,2)$	-0.0562	4	-0.0118	4
$(2,3)$	-0.2666	3	-0.0954	3
$(3,4)$	-0.4251	2	-0.2120	2
$(4,5)$	-0.4251	2	-0.2120	2
$(5,6)$	-0.2666	3	-0.0954	3
$(1,6)$	-0.0562	4	-0.0118	4
$(4,7)$	**-1.0000**	1	**-1.0000**	1
$(7,8)$	-0.4251	2	-0.2120	2
$(8,9)$	-0.2666	3	-0.0954	3
$(9,10)$	-0.0562	4	-0.0118	4
$(10,11)$	-0.0562	4	-0.0118	4
$(11,12)$	-0.2666	3	-0.0954	3
$(7,12)$	-0.4251	2	-0.2120	2

[1] The second and fourth columns are the exact and the approximate CRIs, according to Definition 1 and Definition 3 respectively. The third and fifth columns are the exact and the approximate rank of the edges, according to the values in the second and fourth columns respectively. The greater the absolute value in the second or fourth column, the more important the edge.

Remark 4: It can be observed that the approximate I_{ij}^- and the exact one have a distinct difference due to the small size of the network. Interestingly, even so, the approximate rank of the edges is consistent with that of the exact one. This shows that it is reasonable to evaluate the edge centrality by using the elements of $\boldsymbol{x_2(L)}$.

When it comes to the importance of the absent edges, similar results are obtained.

Results of more than 100 stochastic simulations suggest that although the edge rank based on the approximate CRIs is not always precisely consistent with the exact edge rank as shown

Authorized licensed use limited to: Wuhan University. Downloaded on May 31,2024 at 12:56:25 UTC from IEEE Xplore. Restrictions apply.

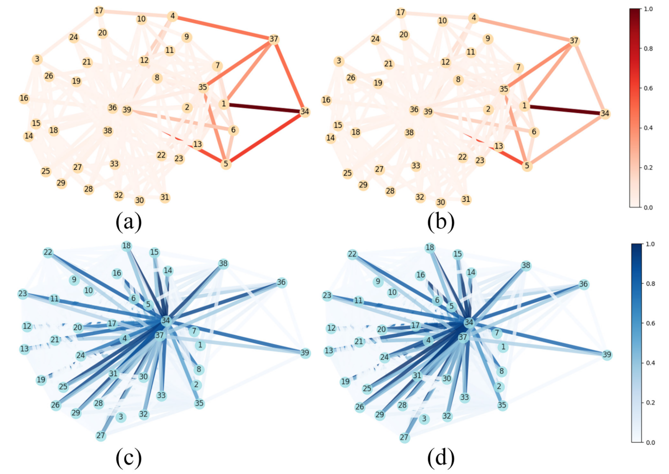

Fig. 3. Edge centrality based on the exact and the approximate CRIs of all the edges in (a). Network in (c) and that in (a) are complementary. The deeper color the edge (absent edge) is represented, the more influential of the edge (absent edge) is. (a) Original network with 39 nodes and 170 edges. Importance of the edges by the exact CRIs. (b) Importance of the edges by the approximate CRIs. (c) Importance of the absent edges by the exact CRIs. (d) Importance of the absent edges by the approximate CRIs.

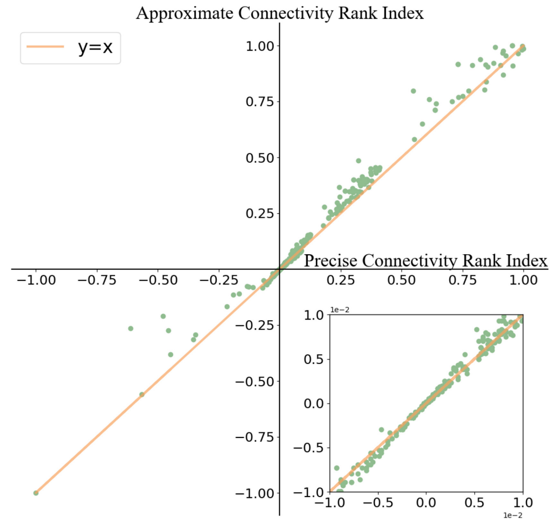

Fig. 4. Comparison of the approximate CRIs with the exact CRIs of all the edges in Fig. 3(a).

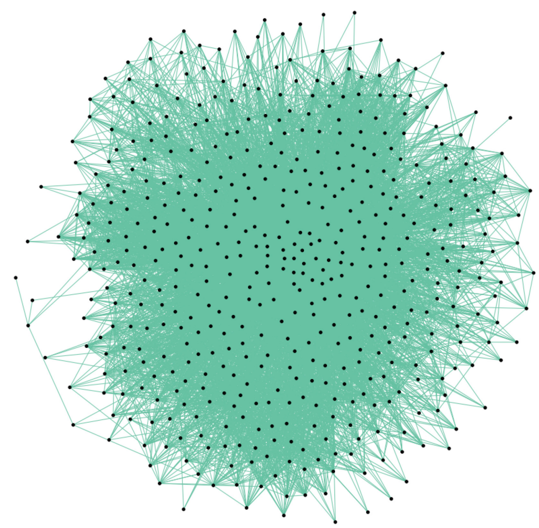

Fig. 5. Actual social network with 500 nodes and 10437 edges. The data is obtained from the website [31].

by this example, these ranks have a very high correlation, which will be illustrated in the coming subsection.

2) Network With 170 *Edges:* A network with 39 nodes and 170 edges is shown in Fig. 3(a). The result of edge centrality according to the exact and the approximate CRIs are shown in Fig. 3. The network in Fig. 3(c) and that in Fig. 3(a) are complementary. The edges in one is the absent edges in the other, and vice versa. The deeper color the edge (absent edge) is represented, the more influential the edge (absent edge) is. Fig. 3(a) and (b) display the importance of the edges by the exact and the approximate CRIs, respectively; Fig. 3(c) and (d) show the importance of the absent edges by the exact and the approximate CRIs, respectively. It is clear that the edge centralities by the two kinds of CRIs are highly positively correlated. It is worth mentioning that edge centrality here is based on the algebraic connectivity, accordingly the most essential edge removal (edge addition) has the most influence on the network connectivity.

The exact and the approximate CRIs of all the edges in the network are demonstrated in Fig. 4. Whereas the dots in the first quadrant (or, the third quadrant) show the exact and the approximate CRIs of the absent edges (or, edges). Overall the approximate results calculated by Definitions 3 and 4 are very close to the exact results. By performing Spearman rank correlation analysis and hypothesis testing on the data in the first and third quadrant, respectively, it was found that the Spearman rank correlation coefficients of the exact and the approximate CRIs are 0.9980 and 0.9960, respectively. This accounts for the highly positive linear correlation between the exact and the approximate CRIs. Therefore, for a large-scale network, it is appropriate to rank the edge centrality based on algebraic connectivity by applying the approximate CRIs, which greatly reduces the time complexity of the algorithm.

C. Application of CRI Indicator and Further Discussion

An actual social network with 500 nodes and 10437 edges is shown in Fig. 5 [31]. The network was generated using email data from a large European research institution. There is an edge (u, v) in the network if person u sent person v at least one email. The e-mails only represent communication among institution members (the core), and the dataset does not contain incoming messages from or outgoing messages to the rest of the world. By performing Spearman rank correlation analysis [32] and hypothesis testing on the exact and the approximate CRIs, it was found that the Spearman rank correlation coefficients of them are 0.9947. It accounts for the highly positive linear correlation between the exact and the approximate CRIs.

Diffusion on social networks refers to the process where opinions are spread via the connected nodes. The independent cascade model (IC model) is a widely adopted diffusion model

Authorized licensed use limited to: Wuhan University. Downloaded on May 31,2024 at 12:56:25 UTC from IEEE Xplore. Restrictions apply.

Algorithm 2 Independent Cascade Algorithm

Step 1: Given the graph G, probability p and collection of activated nodes $v_i (i \in [1, \dots, m])$.
Step 2: Traverse all inactive nodes. For each inactive node, traverse its neighboring nodes. If the neighboring node is active and the generated random number between 0 and 1 is less than p, then the current node is activated and the current traversal ends.
Step 3: Repeat the second step until no new active nodes are generated.

TABLE III
NUMBER OF ACTIVE NODES AFTER CUTTING OFF THE EDGE BETWEEN NODES i AND j OF NETWORK IN FIG. 5

(i, j)	Approximate CRI	Edge betweenness centrality	Number of active nodes
$(414, 449)$	-1.0000	0.0040	346.8
$(370, 414)$	-0.1952	0.0021	394.7
$(177, 479)$	-0.0190	0.0020	400.3
$(275, 479)$	-0.0183	0.0013	419.8
$(414, 415)$	-0.0119	0.0008	436.9
$(370, 427)$	-0.0002	0.0002	493.6
$(76, 443)$	-0.0001	0.0001	494.1
$(0, 1)$	-0.0000	0.0001	494.9

[1] The second column is the approximate CRIs, according to Definition 3 respectively.
[2] The third column is edge betweenness centrality of edge between nodes i and j of network in Fig. 5.
[3] Assumes that the network G is the network in Figure 5, p is equal to 0.1 and the activated node is node 0. The number of active nodes obtained based on independent cascading algorithm after cutting off the corresponding edge is recorded in the fourth column. This result is the average of 100 random trials.

in social networks where a node is assumed to be activated independently by any one of its neighbors [33]. Take the email network as an example. Assume that an opinion appears in the email network and the active node spreads the opinion. The opinion can spread from the active node to the inactive node with a certain probability. Therefore, when the diffusion process reaches a steady state, the number of active nodes reflects the degree of connectivity of the network generally. It is interesting to observe the impact of cutting off different important edges estimated by CRI on network connectivity.

As shown in the second column of Table III, the importance of edge $\varepsilon_{414-449}, \varepsilon_{370-414}, \varepsilon_{414-479}, \varepsilon_{370-427}$ and ε_{0-1} increase progressively one by one. And accordingly, the number of active nodes by cutting off edges between node pair 414–449, 370–414, 414–479, 370–427 and 0–1 are shown, respectively, in the fourth column of Table III. It is clear that the number of active nodes by cutting off the corresponding edge is in agreement with the sequence of edge importance of CRI.

In the third column of Table III, the edge betweenness centrality of edges is shown. It is seen that the approximated CRIs are consistent with the edge betweenness centralities. In addition, the running time of different indicators is compared in Table IV. To ensure consistency, we uniformly calculated various indicators of existing edges on the network in Fig. 5. It is seen that CRI is accurate but has a high-time complexity, 1535.12 s. The running time of edge betweenness centrality

TABLE IV
RUNNING TIME OF DIFFERENT INDICATORS BASED ON THE NETWORK IN FIG. 5

Edge Betweenness Centrality	CRI	Approximate CRI
2.18s	1535.45s	0.53s

[1] This result is the average of 100 trials.

and the approximate CRI are 2.18 s and 0.53 s, respectively. Compared to traditional indicators for measuring the importance of edges, the approximate CRI has a neat form and a better understandability from the aspect of dynamics. At the same time, it can be used not only to measure the importance of existing edges in the network, but also to calculate the importance of absent edges in the network.

V. CONCLUSION

This article has proposed a new index to measure edge centrality in a network: CRI. The CRI of an edge is normalized by its effect on the second smallest Laplacian eigenvalue. Compared with the other existing indices, CRI determines the importance of a present edge as well as an absent edge. An approximation algorithm has been designed, which can reduce the time complexity in calculating the CRIs of all the edges in a network of size N, from $O(N^5)$ to $O(N^3)$. Based on the eigenvector(s) corresponding to the second smallest Laplacian eigenvalue of the network, the approximation algorithm performs well. Particularly, in a large-scale network, one can easily find the most and the least influential edges by maximizing and minimizing $|x_{2i} - x_{2j}|$, respectively. Some examples have verified the highly positive correlation between the exact and the approximate CRIs. The proposed edge centrality measure can be applied to network synchronization, pinning control, information diffusion, epidemic spreading, route planning, attack survivability, and so on.

REFERENCES

[1] I. Mishkovski, M. Mirchev, S. Scepanovic, and L. Kocarev, "Interplay between spreading and random walk processes in multiplex networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 64, no. 10, pp. 2761–2771, Oct. 2017.
[2] S. Zhu, J. Zhou, X. Yu, and J.-A. Lu, "Synchronization of complex networks with nondifferentiable time-varying delay," *IEEE Trans. Cybern.*, vol. 52, no. 5, pp. 3342–3348, May 2022.
[3] X. Zhao, J. Zhou, and J.-A. Lu, "Pinning synchronization of multiplex delayed networks with stochastic perturbations," *IEEE Trans. Cybern.*, vol. 49, no. 12, pp. 4262–4270, Dec. 2019.
[4] J. Zhou, J. Chen, J. Lu, and J. L, "On applicability of auxiliary system approach to detect generalized synchronization in complex network," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3468–3473, Jul. 2017.
[5] L. Duan, A. W. Min, J. Huang, and K. G. Shin, "Attack prevention for collaborative spectrum sensing in cognitive radio networks," *IEEE J. Sel. Areas Commun.*, vol. 30, no. 9, pp. 1658–1665, Oct. 2012.
[6] A. R. G. Castro, and V. Miranda, "Knowledge discovery in neural networks with application to transformer failure diagnosis," *IEEE Trans. Power Syst.*, vol. 20, no. 2, pp. 717–724, May 2005.
[7] P. Bonacich, "Factoring and weighting approaches to status scores and clique identification," *J. Math Sociol.*, vol. 2, no. 1, pp. 113–120, Jan. 1972.
[8] G. Sabidussi, "The centrality index of a graph," *Psychometrika*, vol. 31, no. 4, pp. 581–603, Dec. 1966.
[9] L. C. Freeman, "A set of measures of centrality based on betweenness," *Sociometry*, vol. 40, no. 1, pp. 35–41, Mar. 1977.

Authorized licensed use limited to: Wuhan University. Downloaded on May 31,2024 at 12:56:25 UTC from IEEE Xplore. Restrictions apply.

[10] S. Brin and L. Page, "The anatomy of a large-scale hypertextual Web search engine," *Comput. Netw.*, vol. 30, nos. 1–7, pp. 107–117, Apr. 1998.
[11] M. Kitsak et al., "Identification of influential spreaders in complex networks," *Nat. Phys.*, vol. 6, no. 11, pp. 888–893, Nov. 2010.
[12] X. Qi et al., "Terrorist networks, network energy and node removal: A new measure of centrality based on Laplacian energy," *Soc. Netw.*, vol. 2, no. 1, pp. 19–31, Jan. 2013.
[13] J. Zhou, X. Yu, and J.-A. Lu, "Node importance in controlled complex networks," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 66, no. 3, pp. 437–441, Mar. 2019.
[14] H. Liu, X. Xu, J.-A. Lu, G. Chen, and Z. Zeng, "Optimizing pinning control of complex dynamical networks based on spectral properties of grounded Laplacian matrices," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 2, pp. 786–796, Feb. 2021.
[15] M. Girvan and M. E. J. Newman, "Community structure in social and biological networks," *Proc. Natl. Acad. Sci.*, vol. 99, no. 12, pp. 7821–7826, Jun. 2002.
[16] P. De Meo, E. Ferrara G. Fiumara, and A. Ricciardello, "A novel measure of edge centrality in social networks," *Knowl. Based. Syst.*, vol. 30, pp. 136–150, Jun. 2012.
[17] N. Chung, L. Y. Chew, J. Zhou, and C. H. Lai, "Impact of edge removal on the centrality betweenness of the best spreaders," *Europhys. Lett.*, vol. 98, no. 5, Jun. 2012, Art. no. 58004.
[18] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, New York, NY, USA: Springer, 2012, pp. 12–13.
[19] L. M. Pecora, and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Int. J. Bifurcat. Chaos*, vol. 9, no. 12, pp. 2315–2320, Dec. 1999.
[20] X. F. Wang and G. R. Chen, "Synchronization in small-world dynamical networks," *Int J. Bifurcat. Chaos*, vol. 12, no. 1, pp. 187–192, Jan. 2002.
[21] J. Lü, and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 841–846, Jun. 2005.
[22] S. Gomez, A. Diaz-Guilera, J. Gomez-Gardenes, C. J. Perez-Vicente, Y. Moreno, and A. Arenas, "Diffusion dynamics on multiplex networks," *Phys. Rev. Lett.*, vol. 110, no. 2, Jan. 2013, Art. no. 028701.
[23] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.
[24] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.
[25] Z. Li, G. Wen, Z. Duan, and W. Ren, "Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs," *IEEE Trans. Autom. Control*, vol. 60, no. 4, pp. 1152–1157, Apr. 2015.
[26] T. Li, M. Fu, L. Xie, and J.-F. Zhang, "Distributed consensus with limited communication data rate," *IEEE Trans. Autom. Control*, vol. 56, no. 2, pp. 279–292, Feb. 2011.
[27] I. Necoara, "Random coordinate descent algorithms for multi-agent convex optimization over networks," *IEEE Trans. Autom. Control*, vol. 58, no. 8, pp. 2001–2012, Aug. 2013.
[28] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslov. Math. J.*, vol. 23, no. 2, pp. 298–305, Jan. 1973.
[29] R. Merris, "Laplacian graph eigenvectors," *Linear Algebra Appl.*, vol. 278, nos. 1–3, pp. 221–236, Jul. 1998.
[30] A. Ghosh, and S. Boyd, "Growing well-connected graphs," in *Proc. 45th IEEE Conf. Decis. Control*, San Diego, CA, USA, 2006, pp. 6605–6611.
[31] *SNAP: Network Datasets: Email-Eu-Core Network*. Stanford, CA, USA: Stanford Univ., 2007. [Online]. Available: http://snap.stanford.edu/data/email-Eu-core.html
[32] C. Xiao, J. Ye, R. M. Esteves, and C. Rong, "Using spearman's correlation coefficients for exploratory data analysis on big dataset," *Concurr. Comput.-Prac. Exp.*, vol. 28, no. 14, pp. 3866–3878, Sep. 2016.
[33] Q. Bao, W. K. Cheung, Y. Zhang, and J. Liu, "A component-based diffusion model with structural diversity for social networks," *IEEE Trans. Cybern.*, vol. 47, no. 4, pp. 1078–1089, Apr. 2017.
[34] S. Jiang, J. Zhou, M. Small, J. A. Lu, and Y. Zhang, "Searching for key cycles in a complex network," *Phys. Rev. Lett.*, vol. 130, no. 18, May 2023, Art. no. 18740.

Jin Zhou received the B.Sc. degree in mathematics from Central China Normal University, Wuhan, China, in 2000, and the M.Sc. and Ph.D. degrees in complex networks from the School of Mathematics and Statistics, Wuhan University, Wuhan, in 2006 and 2009, respectively.

She is currently a Professor with the School of Mathematics and Statistics, Wuhan University. Her current research interests include multilayer networks, complex networks, nonlinear systems, and pinning control.

Prof. Zhou was a recipient of the Second Prize of the National Natural Science Award from the State Council of China in 2016, the First Prize of the Natural Science Award from Hubei Province, China, in 2013, and the First Prize of the Natural Science Award from the Ministry of Education of China in 2007.

Yanqi Zhang received the B.Sc. degree in mathematics from Wuhan University, Wuhan, China, in 2023.

Her research interests include multilayer networks, complex networks, and superdiffusion.

Jun-an Lu received the B.Sc. degree in geophysics from Peking University, Beijing, China, in 1968, and the M.Sc. degree in applied mathematics from Wuhan University, Wuhan, China, in 1982.

He is currently a Professor with the School of Mathematics and Statistics, Wuhan University. He has published over 200 journal papers in the aforementioned fields. His current research interests include complex networks, nonlinear systems, chaos control, as well as scientific and engineering computing.

Prof. Lu was a recipient of the Second Prize of the National Natural Science Award from the State Council of China in 2016, the First Prize of the Natural Science Award from Hubei Province, China, in 2013, the Second Prize of the National Natural Science Award from the State Council of China in 2008, the First Prize of the Natural Science Award from the Ministry of Education of China in 2007, and the Second Prize of the Natural Science Award from the Hubei Province, China, in 2006.

Guanrong Chen (Life Fellow, IEEE) received the M.Sc. degree in computer science from Sun Yat-sen University, Guangzhou, China, in 1981, and the Ph.D. degree in applied mathematics from Texas A&M University at College Station, College Station, TX, USA, in 1987.

Since 2000, he has been a Chair Professor and the Founding Director of the Centre for Chaos and Complex Networks, City University of Hong Kong, Hong Kong. His research interests are in the fields of complex networks, nonlinear dynamics, and control systems.

Prof. Chen is a Highly Cited Researcher in Engineering in the past ten years according to Thomson Reuters. He was awarded the 2011 Euler Gold Medal from Russia, and conferred the Honorary Doctor degrees by the Saint Petersburg State University, Russia, in 2011 and by the University of Normandy, France, in 2014. He has been a member of the Academy of Europe since 2014 and a Fellow of The World Academy of Sciences since 2015.

Authorized licensed use limited to: Wuhan University. Downloaded on May 31,2024 at 12:56:25 UTC from IEEE Xplore. Restrictions apply.