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GROWTH RATES FOR THE HÖLDER COEFFICIENTS OF THE LINEAR STOCHASTIC FRACTIONAL HEAT EQUATION WITH ROUGH DEPENDENCE IN SPACE

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Abstract We study the linear stochastic fractional heat equation

$$\frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\frac{\alpha}{2}}u(t,x) + \dot{W}(t,x), \quad t>0,\ x\in\mathbb{R},$$

where $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian with $\alpha \in (1,2)$. The driving noise $\dot{W}$ is a centered Gaussian field that is white in time and has the covariance of a fractional Brownian motion with Hurst parameter $H \in \left(\frac{2-\alpha}{2}, \frac{1}{2}\right)$. We establish exact asymptotics for the solution as both temporal and spatial variables tend to infinity, and derive sharp growth rates for the Hölder coefficients. The proofs are based on Talagrand's majorizing measure theorem and Sudakov's minoration theorem.

Keywords Stochastic fractional heat equation; rough noise; global Hölder continuity; majorizing measure theorem

MSC2020 60H15; 60G17; 60G22

1 Introduction and main results

Consider the following linear stochastic fractional heat equation (SFHE, for short):

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\frac{\alpha}{2}}u(t,x) + \dot{W}(t,x), \quad t>0,\ x\in\mathbb{R}, \\ u(0,x) \equiv 0. \end{cases} \tag{1.1}$$

Here, $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian with $\alpha \in (1,2)$, $W(t,x)$ is a centered Gaussian field with the covariance given by

$$\mathbb{E}[W(t,x)W(s,y)] = \frac{1}{2}(s\wedge t)\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right), \tag{1.2}$$

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for some $H \in \left(0, \frac{1}{2}\right)$. Equivalently, the covariance of the noise $\dot{W}(t,x) = \frac{\partial^2 W}{\partial t \partial x}$ is

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta_0(t-s)\Lambda\left(x-y\right),$$

where Λ is a distribution, whose Fourier transform is the measure $\mu(d\xi) = c_H|\xi|^{1-2H}d\xi$ with

$$c_H = \frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H). \tag{1.3}$$

When $\alpha = 2$ and $H = \frac{1}{2}$, Equation (1.1) reduces to the classical stochastic heat equation (SHE, for short) driven by space-time white noise. In this case, it is well-known that the stochastic heat equation admits a unique mild solution [11, 27]. The study of stochastic partial differential equations driven by fractional Brownian motion (fBm, for short) or other general Gaussian noises has developed rapidly in recent years. For an overview, see, e.g., [4, 6, 12, 15, 17, 20–22, 26] and the references. Among these works, Balan and Tudor [6] gave necessary and sufficient conditions on the Hurst index for the existence of solutions to the stochastic heat equation driven by the following Gaussian noises:

- fractional noise in time with Hurst index $H \in (1/2, 1)$;
- spatially correlated noise with covariance given by Riesz, Bessel, or Poisson kernel functions.

Herrell, Song, Wu, and Xiao [13] investigated the sample path regularity of solutions to the stochastic heat equation driven by fractional-colored Gaussian noise.

The global spatial behavior of solutions to the SHE driven by space-time white noise and colored noise has been studied by Conus, Joseph, and Khoshnevisan [9, Theorem 1.2] and by Conus, Joseph, Khoshnevisan, and Shiu [10, Theorem 2.3]. Recently, Hu and Wang [16] extended these results to spatial rough noise with covariance structure given by (1.2). Specifically, for the SHE driven by Gaussian noise that is white in time and spatially fractional with Hurst index $H \in (\frac{1}{4}, \frac{1}{2})$, they established the precise asymptotic behavior of solutions as both temporal and spatial variables tend to infinity. Furthermore, Hu and Wang [16] established sharp estimates for the domain-size dependence of the Hölder coefficients. Very recently, Liu, Hu, and Wang [22] investigated analogous properties for solutions to the stochastic wave equation in arbitrary spatial dimensions, considering additive Gaussian noise with both temporal and spatial fractional structure.

In the present work, we extend the research program initiated in [16] to the fractional Laplacian framework for Equation (1.1). Specifically, our investigation focuses on two aspects:

- the asymptotic behavior of solutions to Equation (1.1);
- the sharp growth rates of the Hölder regularity coefficients.

We introduce the following key function that characterizes the space-time scaling in our analysis:

$$\Psi(t,L) := 1 + \sqrt{\log_2\left(\frac{L}{t^{1/\alpha}} \vee 1\right)}, \quad t > 0,\ L > 0. \tag{1.4}$$

This function plays a critical role throughout our work, particularly in establishing the precise relationship between spatial and temporal variables.

First, we estimate sharp bounds for $\mathbb{E}\left[\sup_{0\leqslant t\leqslant T, -L\leqslant x\leqslant L} u(t,x)\right]$ and $\mathbb{E}\left[\sup_{-L\leqslant x\leqslant L} u(t,x)\right]$ as T and L tend to infinity.

Theorem 1.1 (a) There exist some positive constants $c_{1,1}$ and $c_{1,2}$ such that for any T and $L > 0$,

$$c_{1,1}T^{\frac{2H+\alpha-2}{2\alpha}}\Psi(T,L) \leqslant \mathbb{E}\left[\sup_{\substack{0\leqslant t\leqslant T,\\ -L\leqslant x\leqslant L}} u(t,x)\right] \leqslant c_{1,2}T^{\frac{2H+\alpha-2}{2\alpha}}\Psi(T,L). \tag{1.5}$$

(b) There exist some positive constants $c_{1,3}$ and $c_{1,4}$ such that for any $L \geqslant t^{\frac{1}{\alpha}} > 0$,

$$c_{1,3}t^{\frac{2H+\alpha-2}{2\alpha}}\Psi(t,L) \leqslant \mathbb{E}\left[\sup_{-L\leqslant x\leqslant L} u(t,x)\right] \leqslant c_{1,4}t^{\frac{2H+\alpha-2}{2\alpha}}\Psi(t,L). \tag{1.6}$$

Next, we investigate the asymptotic behavior of the solution under the following three scaling regimes:

(1) temporal and spatial domain size $T, L \to \infty$;

(2) spatial domain size $L \to \infty$;

(3) spatial variable $|x| \to \infty$.

For any $T > 0$ and $\delta > 0$, let

$$\Upsilon(T,\delta) := [0,T] \times \left[-T^{\frac{1+\delta}{\alpha}}, T^{\frac{1+\delta}{\alpha}}\right]. \tag{1.7}$$

We have the following asymptotics results.

Proposition 1.2 (a) For any $\delta > 0$,

$$\begin{aligned} c_{1,1}\sqrt{\frac{\delta}{\alpha}} &\leqslant \liminf_{T\to\infty} \frac{\sup_{(t,x)\in\Upsilon(T,\delta)} u(t,x)}{T^{\frac{2H+\alpha-2}{2\alpha}}\sqrt{\log_2 T}} \\ &\leqslant \limsup_{T\to\infty} \frac{\sup_{(t,x)\in\Upsilon(T,\delta)} u(t,x)}{T^{\frac{2H+\alpha-2}{2\alpha}}\sqrt{\log_2 T}} \leqslant c_{1,2}\sqrt{\frac{\delta}{\alpha}}, \quad \text{a.s.} \end{aligned} \tag{1.8}$$

(b) For any $t > 0$,

$$\begin{aligned} c_{1,3}t^{\frac{2H+\alpha-2}{2\alpha}} &\leqslant \liminf_{L\to\infty} \frac{\sup_{x\in[-L,L]} u(t,x)}{\sqrt{\log_2 L}} \\ &\leqslant \limsup_{L\to\infty} \frac{\sup_{x\in[-L,L]} u(t,x)}{\sqrt{\log_2 L}} \leqslant c_{1,4}t^{\frac{2H+\alpha-2}{2\alpha}}, \quad \text{a.s.} \end{aligned} \tag{1.9}$$

(c) For any $t > 0$,

$$\limsup_{|x|\to\infty} \frac{u(t,x)}{\sqrt{\log|x|}} = \sqrt{2c_{\alpha,H}}\,t^{\frac{2H+\alpha-2}{2\alpha}}, \quad \text{a.s.}, \tag{1.10}$$

where

$$c_{\alpha,H} := \frac{c_H}{2H+\alpha-2}2^{\frac{2H+\alpha-2}{\alpha}}\Gamma\left(\frac{2-2H}{\alpha}\right). \tag{1.11}$$

Here, the constants $c_{1,i}, i = 1,\cdots,4$ are the constants given in Theorem 1.1.

Remark 1.3 Proposition 1.2(b) extends Theorem 1.2 of [9], which concerned the SHE with space-time white noise, to the more general case of the SFHE driven by spatially rough noise. Proposition 1.2(c) further unifies Theorem 2.3 of [10] and Equation (6.3) of [19] within this same rough-noise framework. For example, when $\alpha = 2$ and $H = 1/2$, (1.10) recovers the following result of Khoshnevisan, Kim, and Xiao [19, (6.3)]:

$$\limsup_{|x|\to\infty} \frac{u(t,x)}{\sqrt{\log|x|}} = \sqrt{2}\left(\frac{t}{\pi}\right)^{\frac{1}{4}}, \quad \text{a.s.}$$

Next, we investigate the precise domain-size dependence of the spatial Hölder regularity coefficient. Denote

$$\Delta_h u(t,x) := u(t,x+h) - u(t,x). \tag{1.12}$$

Theorem 1.4 For any $0 \leqslant \theta \leqslant \frac{2H+\alpha-2}{2}$, there exist some positive constants $c_{1,5}$ and $c_{1,6}$ such that

$$c_{1,5}|h|^{\frac{2H+\alpha-2}{2}}\Psi(t,L) \leqslant \mathbb{E}\left[\sup_{-L\leqslant x\leqslant L}\Delta_h u(t,x)\right] \leqslant c_{1,6}t^{\frac{2H+\alpha-2-2\theta}{2\alpha}}|h|^{\theta}\Psi(t,L), \tag{1.13}$$

for all $L \geqslant t^{\frac{1}{\alpha}}$ and $0 < |h| \leqslant \frac{3}{64}t^{\frac{1}{\alpha}}$.

Proposition 1.5 (a) For any $0 \leqslant \theta \leqslant \frac{2H+\alpha-2}{2}$ and $0 < |h| \leqslant \frac{3}{64}t^{\frac{1}{\alpha}}$,

$$\begin{aligned} c_{1,5}|h|^{\frac{2H+\alpha-2}{2}} &\leqslant \liminf_{L\to\infty}\frac{\sup_{-L\leqslant x\leqslant L}\Delta_h u(t,x)}{\sqrt{\log_2 L}} \\ &\leqslant \limsup_{L\to\infty}\frac{\sup_{-L\leqslant x\leqslant L}\Delta_h u(t,x)}{\sqrt{\log_2 L}} \leqslant c_{1,6}t^{\frac{2H+\alpha-2-2\theta}{2\alpha}}|h|^{\theta}, \quad \text{a.s.,} \end{aligned} \tag{1.14}$$

where $c_{1,5}$ and $c_{1,6}$ denote the constants given in Theorem 1.4.

(b) For any $t > 0$ and $0 < |h| \leqslant \frac{3}{64}t^{\frac{1}{\alpha}}$,

$$\limsup_{|x|\to\infty}\frac{\Delta_h u(t,x)}{\sqrt{\log|x|}} = \sqrt{2}\|\Delta_h u(t,0)\|_{L^2(\Omega)}, \quad \text{a.s.} \tag{1.15}$$

Remark 1.6 By Lemma 2.3 below, there exist some positive constants $c_{1,7}$ and $c_{1,8}$ such that for any $t > 0$ and $0 < |h| \leqslant \frac{3}{64}t^{\frac{1}{\alpha}}$,

$$c_{1,7}|h|^{\frac{2H+\alpha-2}{2}} \leqslant \limsup_{|x|\to\infty}\frac{\Delta_h u(t,x)}{\sqrt{\log|x|}} \leqslant c_{1,8}|h|^{\frac{2H+\alpha-2}{2}}, \quad \text{a.s.}$$

Finally, we establish sharp bounds for the domain-size dependence of the temporal Hölder regularity coefficient. Denote

$$\mathcal{D}_\tau u(t,x) := u(t+\tau,x) - u(t,x). \tag{1.16}$$

Theorem 1.7 For any $0 \leqslant \theta \leqslant \frac{2H+\alpha-2}{\alpha}$, there exist some positive constants $c_{1,9}$ and $c_{1,10}$ such that

$$c_{1,9}\tau^{\frac{2H+\alpha-2}{2\alpha}}\Psi(t,L) \leqslant \mathbb{E}\left[\sup_{-L\leqslant x\leqslant L}\mathcal{D}_\tau u(t,x)\right] \leqslant c_{1,10}\tau^{\frac{\theta}{2}}t^{\frac{2H+\alpha-2-\alpha\theta}{2\alpha}}\Psi(t,L), \tag{1.17}$$

for all $L \geqslant t^{\frac{1}{\alpha}}$ and $0 < \tau \leqslant \left(\frac{3}{32}\right)^{\alpha} t$.

Proposition 1.8 (a) For any $0 \leqslant \theta \leqslant \frac{2H+\alpha-2}{\alpha}$ and $0 < \tau \leqslant \left(\frac{3}{32}\right)^{\alpha} t$,

$$\begin{aligned} c_{1,9}\tau^{\frac{2H+\alpha-2}{2\alpha}} &\leqslant \liminf_{L\to\infty}\frac{\sup_{-L\leqslant x\leqslant L}\mathcal{D}_\tau u(t,x)}{\sqrt{\log_2 L}} \\ &\leqslant \limsup_{L\to\infty}\frac{\sup_{-L\leqslant x\leqslant L}\mathcal{D}_\tau u(t,x)}{\sqrt{\log_2 L}} \leqslant c_{1,10}\tau^{\frac{\theta}{2}}t^{\frac{2H+\alpha-2-\alpha\theta}{2\alpha}}, \quad \text{a.s.,} \end{aligned} \tag{1.18}$$

where $c_{1,9}$ and $c_{1,10}$ denote the constants given in Theorem 1.7.

(b) For any $t > 0$ and $0 < \tau \leqslant \left(\frac{3}{32}\right)^{\alpha} t$,

$$\limsup_{|x|\to\infty} \frac{\mathcal{D}_\tau u(t,x)}{\sqrt{\log|x|}} = \sqrt{2}\|\mathcal{D}_\tau u(t,0)\|_{L^2(\Omega)}, \quad \text{a.s.} \tag{1.19}$$

Remark 1.9 By Lemma 2.3 below, there exist some positive constants $c_{1,11}$ and $c_{1,12}$ such that for any $t > 0$ and $0 < \tau \leqslant \left(\frac{3}{32}\right)^{\alpha} t$,

$$c_{1,11}\tau^{\frac{2H+\alpha-2}{2\alpha}} \leqslant \limsup_{|x|\to\infty} \frac{\mathcal{D}_\tau u(t,x)}{\sqrt{\log|x|}} \leqslant c_{1,12}\tau^{\frac{2H+\alpha-2}{2\alpha}}, \quad \text{a.s.}$$

The precise asymptotics and sharp growth rates for Hölder coefficients established in our main theorems are theoretically significant. Moreover, these results provide valuable tools for constructing power-decay functions, which have direct applications to the well-posedness analysis of nonlinear stochastic heat equation driven by rough noise, as demonstrated in [16]. Specifically, building upon the foundational work of Hu and Wang [16], Qian and Wang [23] established the well-posedness of solutions to a nonlinear fractional stochastic heat equation in one spatial dimension:

$$\frac{\partial}{\partial t} u(t,x) = -(-\Delta)^{\alpha/2} u(t,x) + \sigma(t,x,u(t,x))\dot{W}(t,x), \tag{1.20}$$

where $\dot{W}$ denotes a space-time noise that is white in time and fractional in space with Hurst index $\frac{3-\alpha}{4} < H < \frac{1}{2}$ and $1 < \alpha < 2$. Compared with the earlier work of Liu and Mao [20], the technical assumption $\sigma(0) = 0$ is removed in [23].

To construct a suitable decay weight function, it is necessary to derive lower bounds for the following semi-norm, which quantifies the local variability of $u(t,x)$,

$$\mathcal{N}_{\frac{1}{2}-H} u(t,x) := \left(\int_{\mathbb{R}} |u(t,x+h) - u(t,x)|^2 |h|^{2H-2} dh\right)^{\frac{1}{2}}. \tag{1.21}$$

Proposition 1.10 For any fixed $t > 0$, there exists a positive constant c_t depending on t such that for any $L \geqslant t^{\frac{1}{\alpha}}$,

$$\mathbb{E}\left[\sup_{-L\leqslant x\leqslant L} \mathcal{N}^2_{\frac{1}{2}-H} u(t,x)\right] \geqslant c_t \log_2(L). \tag{1.22}$$

Note that when $\alpha = 2$, the above estimates reduce to those obtained by Hu and Wang [16] for the stochastic heat equation with rough noise.

The remainder of the paper is organized as follows. In Section 2, we review foundational properties of solutions and establish sharp moment bounds. In Section 3, we present the proofs of the main results, while technical lemmas are provided in the appendix.

2 Moment estimates for the temporal and spatial increments

2.1 Stochastic integral and well-posedness of the solution

Let us recall some notations from [14] and [16]. Let $\mathcal{D}(\mathbb{R})$ be the space of real-valued infinitely differentiable functions with compact support on $\mathbb{R}$. The Fourier transform of a function $f \in \mathcal{D}(\mathbb{R})$ is defined as

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}} e^{-i\xi x} f(x) dx.$$

The Green kernel associated with the operator $-(-\Delta)^{\frac{\alpha}{2}}$, where $\alpha \in (1,2]$, is denoted by G_α and defined via its Fourier transform

$$(\mathcal{F}G_\alpha(t,\cdot))(\xi) = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}, \quad t > 0. \tag{2.1}$$

See, e.g., [2, 7, 8].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ denote the space of real-valued, infinitely differentiable functions with compact support in $\mathbb{R}_+ \times \mathbb{R}$. The noise $\dot{W}$ is a zero-mean Gaussian family $\{W(\phi), \phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$, whose covariance structure is given by

$$\langle \phi, \psi \rangle_{\mathcal{H}} := \mathbb{E}\big[W(\phi)W(\psi)\big] = c_H \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\phi(s,\xi)\overline{\mathcal{F}\psi(s,\xi)}|\xi|^{1-2H} d\xi ds, \tag{2.2}$$

where $H \in (0,1)$, c_H is given in (1.3), and $\mathcal{F}\phi(s,\xi)$ is the Fourier transform of $\phi(s,x)$ with respect to the spatial variable x.

Let $\mathcal{H}$ be the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The mapping $\varphi \mapsto W(\varphi)$ can be extended to all $\varphi \in \mathcal{H}$, which is called the Wiener integral.

Definition 2.1 ([3, Definition 4.1]) The random field $\{u(t,x); (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) W(ds, dy) \tag{2.3}$$

is called a solution of Equation (1.1), provided that the stochastic integral on the right-hand side of (2.3) is well-defined. That is, for every $t > 0$ and $x \in \mathbb{R}$, the function

$$g_{t,x} := G_\alpha(t-\cdot, x-\cdot)\mathbf{1}_{[0,t]}$$

belongs to the space $\mathcal{H}$.

According to Balan [3, Remark 5.4], the existence of a solution to Equation (1.1) can be stated as follows.

Proposition 2.2 Equation (1.1) admits a unique solution $\{u(t,x); (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ of the form (2.3) if and only if

$$H \in \left(1 - \frac{\alpha}{2}, 1\right).$$

In this case, for any $t > 0$ and $x \in \mathbb{R}$,

$$\|u(t,x)\|^2_{L^2(\Omega)} = c_{\alpha,H} t^{\frac{2H+\alpha-2}{\alpha}}, \tag{2.4}$$

where the constant $c_{\alpha,H}$ is given by (1.11).

Proof Using (2.1), for any $\xi, x \in \mathbb{R}$ and $t > 0$, we have

$$(\mathcal{F}G_\alpha(t, x-\cdot))(\xi) = e^{-t|\xi|^\alpha - i\xi x}. \tag{2.5}$$

By (2.2), (2.5), and the change of variables $\eta = 2(t-s)\xi^\alpha$, we obtain that for all $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left[|u(t,x)|^2\right] &= 2c_H \int_0^t \int_0^\infty e^{-2(t-s)|\xi|^\alpha} |\xi|^{1-2H} d\xi ds \\
&= \frac{c_H}{\alpha} 2^{\frac{2H+\alpha-2}{\alpha}} \int_0^\infty e^{-\eta} \eta^{\frac{2-2H}{\alpha}-1} d\eta \int_0^t (t-s)^{\frac{2H-2}{\alpha}} ds \\
&= \frac{c_H}{\alpha} 2^{\frac{2H+\alpha-2}{\alpha}} \Gamma\left(\frac{2-2H}{\alpha}\right) \cdot \int_0^t (t-s)^{\frac{2H-2}{\alpha}} ds.
\end{aligned} \tag{2.6}$$

Note that the last integral converges if and only if $H > 1 - \frac{\alpha}{2}$. In this case, (2.4) holds. The proof is complete. □

2.2 Hölder continuity of the solution

In this part, we give a sharp moment estimate for the increment

$$d_1((t,x),(s,y)) := \left(\mathbb{E}\left[|u(t,x)-u(s,y)|^2\right]\right)^{\frac{1}{2}}. \tag{2.7}$$

Lemma 2.3 There exist some positive constants $c_{2,1}$ and $c_{2,2}$ such that for any (t,x), $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$,

$$c_{2,1}\tilde{d}_1((t,x),(s,y)) \leqslant d_1((t,x),(s,y)) \leqslant c_{2,2}\tilde{d}_1((t,x),(s,y)), \tag{2.8}$$

where

$$\tilde{d}_1((t,x),(s,y)) := |x-y|^{\frac{2H+\alpha-2}{2}} \wedge (t\wedge s)^{\frac{2H+\alpha-2}{2\alpha}} + |t-s|^{\frac{2H+\alpha-2}{2\alpha}}.$$

Proof Without loss of generality, we assume $t > s > 0$. By (2.3), we have

$$\begin{aligned} u(t,x)-u(s,y) &= \int_0^s\int_{\mathbb{R}} \left[G_\alpha(t-r,x-z)-G_\alpha(s-r,y-z)\right]W(dr,dz) \\ &\quad + \int_s^t\int_{\mathbb{R}} G_\alpha(t-r,x-z)W(dr,dz) \\ &=: I_1 + I_2. \end{aligned}$$

The independence of the stochastic integrals over the time intervals $[0,s]$ and $[s,t]$ gives

$$\mathbb{E}\left[|u(t,x)-u(s,y)|^2\right] = \mathbb{E}\left[I_1^2\right] + \mathbb{E}\left[I_2^2\right]. \tag{2.9}$$

As in the proof of (2.6), we obtain

$$\mathbb{E}\left[I_2^2\right] = c_{\alpha,H}(t-s)^{\frac{2H+\alpha-2}{\alpha}}. \tag{2.10}$$

By (2.2) and (2.5), we have

$$\begin{aligned} \mathbb{E}\left[I_1^2\right] &= c_H\int_0^s\int_{\mathbb{R}} \left|\mathcal{F}G_\alpha(t-r,x-\cdot)(\xi)-\mathcal{F}G_\alpha(s-r,y-\cdot)(\xi)\right|^2|\xi|^{1-2H}d\xi dr \\ &= c_H\int_0^s\int_{\mathbb{R}} e^{-2(s-r)|\xi|^\alpha}\left[1+e^{-2(t-s)|\xi|^\alpha}-2e^{-(t-s)|\xi|^\alpha}\cos(\xi(x-y))\right]|\xi|^{1-2H}d\xi dr \\ &= c_H\int_0^\infty \left(1-e^{-2s\xi^\alpha}\right)\left[1+e^{-2(t-s)\xi^\alpha}-2e^{-(t-s)\xi^\alpha}\cos(\xi(x-y))\right]\xi^{1-2H-\alpha}d\xi. \end{aligned} \tag{2.11}$$

Next, we derive matching upper and lower bounds for the above integral.

Upper bound. By (2.11), we have

$$\begin{aligned} \mathbb{E}\left[|u(s,x)-u(s,y)|^2\right] &= 2c_H\int_0^\infty \left(1-e^{-2s\xi^\alpha}\right)\left[1-\cos(\xi(x-y))\right]\xi^{1-2H-\alpha}d\xi \\ &\leqslant 2c_H\int_0^\infty \left[1-\cos(\xi(x-y))\right]\xi^{1-2H-\alpha}d\xi \\ &= 2c_H\frac{\Gamma(3-2H-\alpha)}{2H+\alpha-2}\cos\left(\left(H+\frac{\alpha}{2}-1\right)\pi\right)|x-y|^{2H+\alpha-2}. \end{aligned} \tag{2.12}$$

Here, the following identity is used in the last step: for any $\gamma \in (0,1)$ and $\xi \in \mathbb{R}$,

$$\int_0^\infty \frac{1-\cos(\xi z)}{z^{1+\gamma}}dz = \frac{\Gamma(1-\gamma)}{\gamma}\cos\left(\frac{\pi\gamma}{2}\right)|\xi|^\gamma. \tag{2.13}$$

See, e.g., [5, Lemma D.1].

By (2.9), (2.10), (2.11), and the change of variables $\eta=(t-s)\xi^\alpha$, we have

$$\begin{aligned}
&\mathbb{E}\left[|u(t,x)-u(s,x)|^2\right]\\
&=c_H\int_0^\infty\left(1-e^{-2s\xi^\alpha}\right)\left(1-e^{-(t-s)\xi^\alpha}\right)^2\xi^{1-2H-\alpha}d\xi+c_{\alpha,H}(t-s)^{\frac{2H+\alpha-2}{\alpha}}\\
&\leqslant c_H\int_0^\infty\left(1-e^{-(t-s)\xi^\alpha}\right)\xi^{1-2H-\alpha}d\xi+c_{\alpha,H}(t-s)^{\frac{2H+\alpha-2}{\alpha}}\\
&=\frac{c_H}{\alpha}(t-s)^{\frac{2H+\alpha-2}{\alpha}}\int_0^\infty\left(1-e^{-\eta}\right)\eta^{\frac{2-2H-2\alpha}{\alpha}}d\eta+c_{\alpha,H}(t-s)^{\frac{2H+\alpha-2}{\alpha}}\\
&=\left(\frac{c_H}{2H+\alpha-2}\Gamma\left(\frac{2-2H}{\alpha}\right)+c_{\alpha,H}\right)(t-s)^{\frac{2H+\alpha-2}{\alpha}}.
\end{aligned}\tag{2.14}$$

Using the Minkowski inequality, (2.4), (2.12), and (2.14), there exists a constant $c_{2,3}>0$ such that

$$\begin{aligned}
d_1((t,x),(s,y))&\leqslant d_1((s,x),(s,y))+d_1((t,x),(s,x))\\
&\leqslant c_{2,3}\left(|x-y|^{\frac{2H+\alpha-2}{2}}\wedge s^{\frac{2H+\alpha-2}{2\alpha}}+(t-s)^{\frac{2H+\alpha-2}{2\alpha}}\right),
\end{aligned}$$

which is the upper bound in (2.8).

Lower bound. By (2.11) and the change of variables $\tilde{\xi}:=|x-y|\xi$, we have

$$\begin{aligned}
\mathbb{E}\left[I_1^2\right]&\geqslant c_H\int_0^\infty\left(1-e^{-2s\xi^\alpha}\right)\left[1-e^{-(t-s)\xi^\alpha}\cos(\xi(x-y))\right]^2\xi^{1-2H-\alpha}d\xi\\
&=c_H|x-y|^{2H+\alpha-2}\cdot\int_0^\infty\left[1-\exp\left(-\frac{2s\tilde{\xi}^\alpha}{|x-y|^\alpha}\right)\right]\\
&\qquad\cdot\left[1-\exp\left(-\frac{(t-s)\tilde{\xi}^\alpha}{|x-y|^\alpha}\right)\cos\left(\tilde{\xi}\right)\right]^2\tilde{\xi}^{1-2H-\alpha}d\tilde{\xi}\\
&\geqslant c_H|x-y|^{2H+\alpha-2}\cdot\sum_{n=1}^\infty\int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}}\left[1-\exp\left(-\frac{2s\xi^\alpha}{|x-y|^\alpha}\right)\right]\xi^{1-2H-\alpha}d\xi,
\end{aligned}$$

where the last inequality holds because $\cos(\xi)\leqslant0$ on the interval $\left[2n\pi+\frac{\pi}{2},2n\pi+\frac{3\pi}{2}\right]$.

Next, we derive a lower bound of the above series. For convenience, we consider the following two cases:

$$|x-y|\leqslant s^{\frac{1}{\alpha}}\qquad\text{and}\qquad|x-y|>s^{\frac{1}{\alpha}}.$$

Case 1. When $|x-y|\leqslant s^{\frac{1}{\alpha}}$, we have

$$\mathbb{E}\left[I_1^2\right]\geqslant c_H|x-y|^{2H+\alpha-2}\cdot\sum_{n=1}^\infty\int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}}\left(1-e^{-2\xi^\alpha}\right)\xi^{1-2H-\alpha}d\xi.\tag{2.15}$$

Here, the series is convergent because $\alpha+2H>2$.

Case 2. The case $|x-y|>s^{\frac{1}{\alpha}}$ is a little complicated. Denote

$$n_0:=\inf\left\{n\in\mathbb{N}:2n\pi+\frac{\pi}{2}\geqslant\frac{\pi}{2}\frac{|x-y|}{s^{1/\alpha}}\right\}.\tag{2.16}$$

Then,

$$\frac{\pi}{2}\frac{|x-y|}{s^{1/\alpha}}\leqslant2n_0\pi+\frac{\pi}{2}\leqslant\frac{\pi}{2}\frac{|x-y|}{s^{1/\alpha}}+2\pi.\tag{2.17}$$

Since for all $\xi \geqslant 2n_0\pi + \frac{\pi}{2}$,

$$1 - \exp\left(-\frac{2s\xi^\alpha}{|x-y|^\alpha}\right) \geqslant 1 - \exp(-2^{1-\alpha}\pi^\alpha),$$

we have

$$\begin{aligned}\mathbb{E}\left[I_1^2\right] &\geqslant c_H|x-y|^{2H+\alpha-2}\sum_{n=n_0}^{\infty}\int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}}\left[1-\exp\left(-\frac{2s\xi^\alpha}{|x-y|^\alpha}\right)\right]\xi^{1-2H-\alpha}d\xi \\ &\geqslant c_H\left(1-\exp(-2^{1-\alpha}\pi^\alpha)\right)|x-y|^{2H+\alpha-2}\sum_{n=n_0}^{\infty}\int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}}\xi^{1-2H-\alpha}d\xi \\ &\geqslant c_{2,4}|x-y|^{2H+\alpha-2}\left(2n_0\pi+\frac{\pi}{2}\right)^{2-2H-\alpha},\end{aligned} \tag{2.18}$$

where the last inequality follows from the monotonicity of $\xi^{1-2H-\alpha}$ on $(0,\infty)$, and

$$c_{2,4} = \frac{c_H(1-\exp(-2^{1-\alpha}\pi^\alpha))}{2(2H+\alpha-2)}.$$

When $|x-y| > s^{\frac{1}{\alpha}}$, by (2.17), we have

$$\begin{aligned}\mathbb{E}\left[I_1^2\right] &\geqslant c_{2,4}|x-y|^{2H+\alpha-2}\left(2n_0\pi+\frac{\pi}{2}\right)^{2-2H-\alpha} \\ &\geqslant c_{2,4}|x-y|^{2H+\alpha-2}\left(\frac{\pi}{2}\cdot\frac{|x-y|}{s^{1/\alpha}}+2\pi\right)^{2-2H-\alpha} \\ &= c_{2,4}\left(\frac{\pi}{2}\cdot\frac{1}{s^{1/\alpha}}+2\pi\cdot\frac{1}{|x-y|}\right)^{2-2H-\alpha} \\ &\geqslant c_{2,4}\left(\frac{5\pi}{2}\right)^{2-2H-\alpha} s^{\frac{2H+\alpha-2}{\alpha}}.\end{aligned} \tag{2.19}$$

Putting (2.9), (2.10), (2.15), and (2.19) together, we have

$$d_1((t,x),(s,y)) \geqslant c_{2,5}\left(|x-y|^{\frac{2H+\alpha-2}{2}} \wedge s^{\frac{2H+\alpha-2}{2\alpha}} + (t-s)^{\frac{2H+\alpha-2}{2\alpha}}\right),$$

which is the lower bound in (2.8). The proof is complete. □

2.3 Moment bounds for the spatial increment

Recall the operator Δ_h defined by (1.12) for the spatial increment. We now give the sharp bounds for its natural metric:

$$d_{2,t,h}(x,y) := \left(\mathbb{E}\left[|\Delta_h u(t,x) - \Delta_h u(t,y)|^2\right]\right)^{\frac{1}{2}}. \tag{2.20}$$

Lemma 2.4 (a) (Upper bound) For any $0 \leqslant \theta \leqslant \frac{2H+\alpha-2}{2}$, there exists a positive constant $c_{2,6}$ such that for all $t, h \geqslant 0$ and $x, y \in \mathbb{R}$,

$$d_{2,t,h}(x,y) \leqslant c_{2,6}h^\theta\left(|x-y|^{\frac{2H+\alpha-2-2\theta}{2}} \wedge t^{\frac{2H+\alpha-2-2\theta}{2\alpha}}\right). \tag{2.21}$$

(b) (Lower bound) There exists a positive constant $c_{2,7}$ such that for all $|x-y| \geqslant t^{\frac{1}{\alpha}}$ and $h \leqslant \min\left\{\frac{3}{64}|x-y|, \frac{\pi}{4}\left(\frac{2}{\log 2}\right)^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}}\right\}$,

$$d_{2,t,h}(x,y) \geqslant c_{2,7}h^{\frac{2H+\alpha-2}{2}}. \tag{2.22}$$

Proof The upper bound in (2.21) follows directly from (1.12), Lemma 2.3, and the Minkowski inequality. Let us prove the lower bound in (2.22).

Assume $|x-y| \geqslant t^{\frac{1}{\alpha}}$ and $h \leqslant \frac{3}{64}|x-y|$. By (2.2) and (2.5), we have

$$\begin{aligned} d^2_{2,t,h}(x,y) &= c_H \int_0^t \int_{\mathbb{R}} |\mathcal{F}G_\alpha(t-s,x+h-\cdot)(\xi) - \mathcal{F}G_\alpha(t-s,x-\cdot)(\xi) \\ &\qquad -\mathcal{F}G_\alpha(t-s,y+h-\cdot)(\xi) + \mathcal{F}G_\alpha(t-s,y-\cdot)(\xi)|^2 \, |\xi|^{1-2H} d\xi ds \\ &= c_H \int_0^t \int_{\mathbb{R}} e^{-2(t-s)|\xi|^\alpha} \left|1-e^{-ih\xi}\right|^2 \left|1-e^{-i(x-y)\xi}\right|^2 |\xi|^{1-2H} d\xi ds \\ &= 4c_H \int_{\mathbb{R}_+} \left(1-e^{-2t\xi^\alpha}\right) [1-\cos(h\xi)] \, [1-\cos(|x-y|\xi)] \, \xi^{1-2H-\alpha} d\xi. \end{aligned} \tag{2.23}$$

Note that for any $\xi \geqslant \frac{\pi|x-y|}{4h}$ and $h \leqslant \frac{\pi}{4}\left(\frac{2}{\log 2}\right)^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}}$,

$$1-\exp\left(-\frac{2t\xi^\alpha}{|x-y|^\alpha}\right) \geqslant \frac{1}{2}.$$

The elementary inequality

$$1-\cos(x) \geqslant \frac{x^2}{4} \quad \text{for } |x| \leqslant \frac{\pi}{2}, \tag{2.24}$$

implies that for any $0<\xi \leqslant \frac{|x-y|\pi}{2h}$,

$$1-\cos\left(\frac{h\xi}{|x-y|}\right) \geqslant \frac{h^2\xi^2}{4|x-y|^2}.$$

Therefore, by (2.23) and the change of variables $\tilde{\xi} := |x-y|\xi$, we have

$$\begin{aligned} d^2_{2,t,h}(x,y) &= 4c_H|x-y|^{2H+\alpha-2} \int_{\mathbb{R}_+} \left[1-\exp\left(-\frac{2t\tilde{\xi}^\alpha}{|x-y|^\alpha}\right)\right] \left[1-\cos\left(\frac{h\tilde{\xi}}{|x-y|}\right)\right] \\ &\qquad \cdot \left[1-\cos\left(\tilde{\xi}\right)\right] \tilde{\xi}^{1-2H-\alpha} d\tilde{\xi} \\ &\geqslant 2c_H|x-y|^{2H+\alpha-2} \int_{\frac{|x-y|\pi}{4h}}^{\infty} \left[1-\cos\left(\frac{h\xi}{|x-y|}\right)\right] [1-\cos(\xi)]\xi^{1-2H-\alpha} d\xi \\ &\geqslant \frac{c_H}{2} h^2 |x-y|^{2H+\alpha-4} \int_{\frac{|x-y|\pi}{4h}}^{\frac{|x-y|\pi}{2h}} [1-\cos(\xi)]\xi^{3-2H-\alpha} d\xi. \end{aligned} \tag{2.25}$$

Define

$$k_0 := \inf\left\{k \in \mathbb{Z}_+ : \frac{(6k+1)\pi}{3} \geqslant \frac{|x-y|\pi}{4h}\right\},$$
$$k_1 := \sup\left\{k \in \mathbb{Z}_+ : \frac{(6k+5)\pi}{3} \leqslant \frac{|x-y|\pi}{2h}\right\}.$$

Equivalently,

$$k_0 = \left\lfloor \frac{\frac{3|x-y|}{4h}-1}{6} \right\rfloor + 1, \quad k_1 = \left\lfloor \frac{\frac{3|x-y|}{2h}-5}{6} \right\rfloor.$$

When $h \leqslant \frac{3}{64}|x-y|$, we have $k_0 < k_1$. Define

$$I_k := \left(\frac{(6k+1)\pi}{3}, \frac{(6k+5)\pi}{3}\right].$$

Then,

$$\bigcup_{k=k_0}^{k_1} I_k \subseteq \left[\frac{|x-y|\pi}{4h}, \frac{|x-y|\pi}{2h}\right].$$

Consequently, we have

$$\begin{aligned}
\int_{\frac{|x-y|\pi}{4h}}^{\frac{|x-y|\pi}{2h}} (1-\cos\xi)\xi^{3-2H-\alpha}\,d\xi &\geqslant \sum_{k=k_0}^{k_1}\int_{I_k}(1-\cos\xi)\xi^{3-2H-\alpha}\,d\xi \\
&\geqslant \frac{1}{2}\sum_{k=k_0}^{k_1}\int_{I_k}\xi^{3-2H-\alpha}\,d\xi \\
&\geqslant \frac{1}{4}\int_{\frac{(6k_0+1)\pi}{3}}^{\frac{(6k_1+5)\pi}{3}}\xi^{3-2H-\alpha}\,d\xi.
\end{aligned} \tag{2.26}$$

Here, the fact that $1-\cos\xi \geqslant \frac{1}{2}$ for $\xi \in I_k$ is used in the second inequality, and the monotonicity of $\xi^{3-2H-\alpha}$ is used in the last step.

From the definitions of k_0 and k_1, when $h \leqslant \frac{3}{64}|x-y|$,

$$\begin{aligned}
\frac{(6k_0+1)}{3} &\leqslant \frac{|x-y|}{4h}+2 \leqslant \frac{11|x-y|}{32h}, \\
\frac{(6k_1+5)}{3} &> \frac{|x-y|}{2h}-2 \geqslant \frac{13|x-y|}{32h}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_{\frac{(6k_0+1)\pi}{3}}^{\frac{(6k_1+5)\pi}{3}}\xi^{3-2H-\alpha}d\xi &= \frac{1}{4-2H-\alpha}\left[\left(\frac{(6k_1+5)\pi}{3}\right)^{4-2H-\alpha}-\left(\frac{(6k_0+1)\pi}{3}\right)^{4-2H-\alpha}\right] \\
&\geqslant \frac{\pi^{4-2H-\alpha}}{4-2H-\alpha}\left[\left(\frac{13}{32}\right)^{4-2H-\alpha}-\left(\frac{11}{32}\right)^{4-2H-\alpha}\right]\left(\frac{|x-y|}{h}\right)^{4-2H-\alpha}.
\end{aligned} \tag{2.27}$$

From (2.25), (2.26), and (2.27), we obtain (2.22). The proof is complete. □

2.4 Moment bounds for the temporal increment

Recall the temporal difference operator $\mathcal{D}_\tau$ defined in (1.16). In this part, we establish sharp bounds for its canonical metric defined by

$$d_{3,t,\tau}(x,y) := \left(\mathbb{E}\left[|\mathcal{D}_\tau u(t,x)-\mathcal{D}_\tau u(t,y)|^2\right]\right)^{1/2}, \tag{2.28}$$

where $x,y \in \mathbb{R}$ represent spatial coordinates and $t,\tau > 0$ denote time parameters.

Lemma 2.5 (a) (Upper bound) There exists a positive constant $c_{2,8}$ such that for all $0 \leqslant \theta \leqslant \frac{2H+\alpha-2}{\alpha}$, $x,y \in \mathbb{R}$ and $0 < \tau \leqslant t$,

$$d_{3,t,\tau}(x,y) \leqslant c_{2,8}\tau^{\frac{\theta}{2}}\left(|x-y|^{\frac{2H+\alpha-2-\alpha\theta}{2}} \wedge t^{\frac{2H+\alpha-2-\alpha\theta}{2\alpha}}\right). \tag{2.29}$$

(b) (Lower bound) There exists a positive constant $c_{2,9}$ such that for all $|x-y| \geqslant t^{\frac{1}{\alpha}}$ and $0 < \tau \leqslant \left(\frac{3}{32}\right)^{\alpha}|x-y|^{\alpha}$,

$$d_{3,t,\tau}(x,y) \geqslant c_{2,9}\tau^{\frac{2H+\alpha-2}{2\alpha}}. \tag{2.30}$$

Proof The upper bound in (2.29) follows immediately from Lemma 2.3 and the Minkowski inequality. Now, we prove the lower bound (2.30).

Using the Fourier representation (2.2) and (2.5), along with the independence of stochastic integrals over $[0,t]$ and $[t,t+\tau]$, we have

$$\begin{aligned}
d^2_{3,t,\tau}(x,y) \geqslant& \mathbb{E}\left[\left|\int_t^{t+\tau}\int_{\mathbb{R}} [G_\alpha(t+\tau-s,x-z)-G_\alpha(t+\tau-s,y-z)]\,W(ds,dz)\right|^2\right] \\
=& c_H\int_t^{t+\tau}\int_{\mathbb{R}}\left|1-e^{i(x-y)\xi}\right|^2 e^{-2(t+\tau-s)|\xi|^\alpha}|\xi|^{1-2H}d\xi ds \\
=& 2c_H\int_{\mathbb{R}_+}\left[1-e^{-2\tau\xi^\alpha}\right][1-\cos(\xi|x-y|)]\,\xi^{1-2H-\alpha}d\xi \\
=& 2c_H|x-y|^{2H+\alpha-2}\int_{\mathbb{R}_+}\left[1-\exp\left(-\frac{2\tau\tilde{\xi}^\alpha}{|x-y|^\alpha}\right)\right]\left[1-\cos(\tilde{\xi})\right]\tilde{\xi}^{1-2H-\alpha}d\tilde{\xi} \\
\geqslant& 2c_H|x-y|^{2H+\alpha-2}\int_{\frac{|x-y|\pi}{2\tau^{1/\alpha}}}^{\frac{|x-y|\pi}{\tau^{1/\alpha}}}\left[1-\exp\left(-\frac{2\tau\xi^\alpha}{|x-y|^\alpha}\right)\right][1-\cos(\xi)]\,\xi^{1-2H-\alpha}d\xi \\
\geqslant& 2c_H\left(1-\exp\left(-2^{1-\alpha}\pi^\alpha\right)\right)|x-y|^{2H+\alpha-2}\int_{\frac{|x-y|\pi}{2\tau^{1/\alpha}}}^{\frac{|x-y|\pi}{\tau^{1/\alpha}}}[1-\cos(\xi)]\,\xi^{1-2H-\alpha}d\xi,
\end{aligned} \tag{2.31}$$

where the change of variables $\xi\mapsto\tilde{\xi}/|x-y|$ is used in the third-last step, and the elementary inequality

$$1-\exp\left(-\frac{2\tau\xi^\alpha}{|x-y|^\alpha}\right)\geqslant 1-\exp\left(-2^{1-\alpha}\pi^\alpha\right),\quad \text{if}\;\; \xi\geqslant\frac{|x-y|\pi}{2\tau^{1/\alpha}},$$

is used in the last step.

For any $\xi\in\left[\frac{|x-y|\pi}{2\tau^{1/\alpha}},\frac{|x-y|\pi}{\tau^{1/\alpha}}\right]$, by (2.24), we have

$$1\geqslant 1-\cos\left(\frac{\tau^{1/\alpha}\xi}{2|x-y|}\right)\geqslant\frac{\tau^{2/\alpha}\xi^2}{16|x-y|^2}.$$

Consequently, when $\tau\leqslant\left(\frac{3}{32}\right)^\alpha|x-y|^\alpha$, it holds that

$$\begin{aligned}
\int_{\frac{|x-y|\pi}{2\tau^{1/\alpha}}}^{\frac{|x-y|\pi}{\tau^{1/\alpha}}}(1-\cos(\xi))\,\xi^{1-2H-\alpha}d\xi \geqslant& \frac{1}{16}\tau^{\frac{2}{\alpha}}|x-y|^{-2}\int_{\frac{|x-y|\pi}{2\tau^{1/\alpha}}}^{\frac{|x-y|\pi}{\tau^{1/\alpha}}}(1-\cos(\xi))\,\xi^{3-2H-\alpha}d\xi \\
\geqslant& c_{2,10}\tau^{\frac{2H+\alpha-2}{\alpha}}|x-y|^{2-2H-\alpha},
\end{aligned} \tag{2.32}$$

where $c_{2,10}>0$, and the last inequality follows from (2.26) and (2.27).

Combining (2.31) and (2.32), we obtain (2.30). The proof is complete. □

3 Proof of main results

3.1 Proof of Theorem 1.1

Proof [Proof of Theorem 1.1] The proof follows the strategy of [16, Theorem 1.1]. It hinges on two key probabilistic tools: Talagrand's majorizing measure theorem (Theorem A.1) for the upper bound, and Sudakov's minoration theorem (Theorem A.2) for the lower bound.

For brevity, let

$$\mathbb{T}:=[0,T]\quad\text{and}\quad\mathbb{L}:=[-L,L].$$

Upper bound. We begin by constructing a dyadic partition of the parameter space $\mathbb{T}\times\mathbb{L}$. For each integer $n\geqslant 1$, we define

- Temporal partition:

$$[0,T] = \bigcup_{j=0}^{2^{2^{n-1}}-1} \Big[j \cdot 2^{-2^{n-1}} T,\ (j+1) \cdot 2^{-2^{n-1}} T \Big). \tag{3.1}$$

- Spatial partition:

$$[-L,L] = \bigcup_{k=-2^{2^{n-2}}}^{2^{2^{n-2}}-1} \Big[k \cdot 2^{-2^{n-2}} L, (k+1) \cdot 2^{-2^{n-2}} L \Big). \tag{3.2}$$

For any (t,x) such that

$$j 2^{-2^{n-1}} T \leqslant t < (j+1) 2^{-2^{n-1}} T,\ \ k 2^{-2^{n-2}} L \leqslant x < (k+1) 2^{-2^{n-2}} L,$$

we define

$$A_n(t,x) := \Big[j 2^{-2^{n-1}} T,\ (j+1) 2^{-2^{n-1}} T \Big) \times \Big[k 2^{-2^{n-2}} L,\ (k+1) 2^{-2^{n-2}} L \Big).$$

By Theorem A.1, we have

$$\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u(t,x) \right] \leqslant c_{3,1} \sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} \sum_{n=0}^{\infty} 2^{n/2} \mathrm{diam}(A_n(t,x)), \tag{3.3}$$

where $\mathrm{diam}(A_n(t,x))$ denotes the diameter with respect to the metric d_1 defined in (2.7).

By Lemma 2.3, we obtain the following uniform bound for all $t \in [0,T]$ and $x \in \mathbb{R}$:

$$\mathrm{diam}(A_n(t,x)) \leqslant c_{2,2} \left[\left(2^{-\left(\frac{2H+\alpha-2}{2}\right) 2^{n-2}} L^{\frac{2H+\alpha-2}{2}} \right) \wedge T^{\frac{2H+\alpha-2}{2\alpha}} + 2^{-\left(\frac{2H+\alpha-2}{2\alpha}\right) 2^{n-1}} T^{\frac{2H+\alpha-2}{2\alpha}} \right]. \tag{3.4}$$

We proceed by considering two distinct cases, depending on the relationship between the spatial and temporal scales:

$$L < T^{1/\alpha} \qquad \text{and} \qquad L \geqslant T^{1/\alpha}.$$

Case 1. When $L < T^{\frac{1}{\alpha}}$, by (3.3) and (3.4), we have

$$\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u(t,x) \right] \leqslant c_{3,2} T^{\frac{2H+\alpha-2}{2\alpha}}. \tag{3.5}$$

Case 2. When $L \geqslant T^{\frac{1}{\alpha}}$, let

$$\begin{aligned} N_0 &:= \inf\left\{ n \geqslant 2 : 2^{-2^{n-2}} \leqslant \frac{T^{\frac{1}{\alpha}}}{L} \wedge \frac{1}{2} \right\} \\ &= \inf\left\{ n \geqslant 2 : 2^{n-2} \geqslant \log_2 \left(\frac{L}{T^{\frac{1}{\alpha}}} \vee 2 \right) \right\}. \end{aligned} \tag{3.6}$$

By (3.4) and the definition of N_0, we have

$$\begin{aligned} &\sum_{n=N_0+1}^{\infty} 2^{\frac{n}{2}} \mathrm{diam}\left(A_n(t,x)\right) \\ &\leqslant c_{2,2} \left(\sum_{n=N_0+1}^{\infty} 2^{\frac{n}{2}} \cdot 2^{-\left(\frac{2H+\alpha-2}{2}\right) \cdot 2^{n-2}} L^{\frac{2H+\alpha-2}{2}} + \sum_{n=N_0+1}^{\infty} 2^{\frac{n}{2}} \cdot 2^{-\left(\frac{2H+\alpha-2}{2\alpha}\right) 2^{n-1}} T^{\frac{2H+\alpha-2}{2\alpha}} \right) \\ &\leqslant c_{2,2} T^{\frac{2H+\alpha-2}{2\alpha}} \left[\sum_{n=N_0+1}^{\infty} 2^{\frac{n}{2}} \left(\frac{2^{2^{N_0-2}}}{2^{2^{n-2}}} \right)^{\frac{2H+\alpha-2}{2}} + \sum_{n=N_0+1}^{\infty} 2^{\frac{n}{2} - \left(\frac{2H+\alpha-2}{2\alpha}\right) 2^{n-1}} \right] \\ &\leqslant c_{3,3} T^{\frac{2H+\alpha-2}{2\alpha}}. \end{aligned} \tag{3.7}$$

By the definitions of $\Psi(T,L)$ in (1.4) and N_0 in (3.6), we have

$$2^{\frac{N_0}{2}} \leqslant 2^{\frac{3}{2}}\Psi(T,L). \tag{3.8}$$

This, together with (3.3) and (3.7), implies that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u(t,x)\right]\\
&\leqslant c_{3,1}\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}}\sum_{n=0}^{N_0} 2^{\frac{n}{2}}\operatorname{diam}\left(A_n(t,x)\right) + c_{3,1}\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}}\sum_{n=N_0+1}^{\infty} 2^{\frac{n}{2}}\operatorname{diam}\left(A_n(t,x)\right)\\
&\leqslant 2\sqrt{2}\left(\sqrt{2}+1\right)c_{3,1}c_{2,2}T^{\frac{2H+\alpha-2}{2\alpha}}2^{\frac{N_0}{2}} + c_{3,1}c_{3,3}T^{\frac{2H+\alpha-2}{2\alpha}}\\
&\leqslant \left(2^{\frac{5}{2}}\sqrt{2}\left(\sqrt{2}+1\right)c_{3,1}c_{2,2}+c_{3,1}c_{3,3}\right)T^{\frac{2H+\alpha-2}{2\alpha}}\Psi(T,L).
\end{aligned} \tag{3.9}$$

From (3.5) and (3.9), we obtain the upper bound in (1.5).

Lower bound. When $L \geqslant T^{\frac{1}{\alpha}}$, we consider the sequence $\left\{u(T,x_i), i=0,1,\cdots,\pm\left\lfloor L/T^{\frac{1}{\alpha}}\right\rfloor\right\}$, where

$$x_j := jT^{\frac{1}{\alpha}}, \quad \text{for} \quad j=0,\pm1,\cdots,\pm\left\lfloor L/T^{\frac{1}{\alpha}}\right\rfloor.$$

By (2.8), we have that for any $i\neq j$,

$$d_1((T,x_i),(T,x_j)) \geqslant c_{2,1}T^{\frac{2H+\alpha-2}{2\alpha}}.$$

Applying Theorem A.2 yields

$$\begin{aligned}
\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u(t,x)\right] &\geqslant \mathbb{E}\left[\sup_{-\left\lfloor L/T^{\frac{1}{\alpha}}\right\rfloor\leqslant i\leqslant\left\lfloor L/T^{\frac{1}{\alpha}}\right\rfloor} u(T,x_i)\right]\\
&\geqslant \frac{1}{2}c_{A,1}c_{2,1}T^{\frac{2H+\alpha-2}{2\alpha}}\Psi(T,L).
\end{aligned} \tag{3.10}$$

When $L < T^{\frac{1}{\alpha}}$, by (2.8) and Theorem A.2, we have

$$\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u(t,x)\right] \geqslant \mathbb{E}\left[u\left(\frac{T}{2},0\right)\vee u(T,0)\right] \geqslant c_{A,1}c_{2,1}T^{\frac{2H+\alpha-2}{2\alpha}}. \tag{3.11}$$

From (3.10) and (3.11), we obtain the lower bound in (1.5).

The proof of (1.6) is identical to that of (1.5) with the necessary modifications, and is omitted. This completes the proof. □

3.2 Proof of Proposition 1.2

Proof [Proof of Proposition 1.2] **(a)** Recall $\Upsilon(T,\delta)$ given by (1.7) for any $T,\delta>0$. Define

$$\begin{aligned}
f(T) &:= \sup_{(t,x)\in\Upsilon(T,\delta)} u(t,x),\\
g(T) &:= T^{\frac{2H+\alpha-2}{2\alpha}}\left(1+\sqrt{\log_2\left(T^{\frac{\delta}{\alpha}}\right)}\right).
\end{aligned}$$

Given $\varepsilon>0$, we choose a subsequence $T(n)=n^{\beta}$ for some $\beta > \frac{2\alpha c_{\alpha,H}\log 2}{\delta c_{1,1}^2\varepsilon^2}$. By (1.5) and (2.4), we have

$$\lambda_{H,\alpha}(n) := \mathbb{E}\left[f\left(n^{\beta}\right)\right] \geqslant c_{1,1}g\left(n^{\beta}\right),$$

and

$$\sigma_{H,\alpha}^2(n) := \sup_{(t,x)\in\Upsilon(n^\beta,\delta)} \mathbb{E}\left[|u(t,x)|^2\right] = c_{\alpha,H} n^{\beta\left(\frac{2H+\alpha-2}{\alpha}\right)}.$$

By Borell's inequality (Theorem A.3), we obtain that for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\left(\left|f\left(n^\beta\right) - \mathbb{E}\left[f\left(n^\beta\right)\right]\right| \geqslant \varepsilon\mathbb{E}\left[f\left(n^\beta\right)\right]\right) &\leqslant 2\exp\left(-\frac{\left(\varepsilon\lambda_{H,\alpha}(n)\right)^2}{2\sigma_{H,\alpha}^2(n)}\right)\\ &\leqslant 2n^{-\frac{\beta\delta}{\alpha}\cdot\frac{c_{1,1}^2\varepsilon^2}{2c_{\alpha,H}\log 2}}.\end{aligned}$$

By Borel-Cantelli's lemma, we have almost surely

$$\limsup_{n\to\infty} \frac{\left|f\left(n^\beta\right) - \mathbb{E}\left[f\left(n^\beta\right)\right]\right|}{\mathbb{E}\left[f\left(n^\beta\right)\right]} \leqslant \varepsilon. \tag{3.12}$$

The monotonicity of the functions f and g allows us to derive (1.8) via standard interpolation techniques, using the bounds provided in (1.5) and (3.12).

(b) For any $\varepsilon > 0$, we consider a subsequence $L^\beta(n) = n^\beta$ for some $\beta > \frac{2c_{\alpha,H}\log 2}{c_{1,3}^2\varepsilon^2}$ and denote

$$\mathbb{L}^\beta(n) := \left[-n^\beta, n^\beta\right].$$

By (1.6), we have

$$\mathbb{E}\left[\sup_{x\in\mathbb{L}^\beta(n)} u(t,x)\right] \geqslant c_{1,3} t^{\frac{2H+\alpha-2}{2\alpha}} \left(1 + \sqrt{\log_2\left(n^\beta/t^{\frac{1}{\alpha}}\right)}\right).$$

The result (1.9) follows by adapting the proof of (1.8), with minor modifications. Details are omitted for brevity.

(c) We prove part (c) by verifying the conditions in Theorem A.4.

Let $\rho_1(x,y)$ denote the correlation function between $u(t,x)$ and $u(t,y)$, that is

$$\rho_1(x,y) := \frac{\mathbb{E}[u(t,x)u(t,y)]}{\sqrt{\mathbb{E}[u(t,x)^2]\cdot\mathbb{E}[u(t,y)^2]}}.$$

By the spatial stationarity of $\{u(t,x)\}_{x\in\mathbb{R}}$, $\rho_1(x,y)$ has the following representation in terms of L^2-increments:

$$\rho_1(x,y) = 1 - \frac{\|u(t,x)-u(t,y)\|_{L^2(\Omega)}^2}{2\mathbb{E}[u(t,x)^2]}. \tag{3.13}$$

Combining Lemma 2.3 with (3.13) yields, for any $0 < |x-y| < t^{1/\alpha}$,

$$1 - \frac{c_{2,2}^2}{2\|u(t,x)\|_{L^2(\Omega)}^2}|x-y|^{2H+\alpha-2} \leqslant \rho_1(x,y) \leqslant 1 - \frac{c_{2,1}^2}{2\|u(t,x)\|_{L^2(\Omega)}^2}|x-y|^{2H+\alpha-2}. \tag{3.14}$$

This verifies Condition (1) in Theorem A.4.

By (2.2) and (2.5), we obtain the following representation for the correlation function:

$$\begin{aligned}\rho_1(x,y) &= \frac{c_H}{2\|u(t,x)\|_{L^2(\Omega)}^2}\int_{\mathbb{R}}\left(1-e^{-2t|\xi|^\alpha}\right)e^{-i(x-y)\xi}|\xi|^{1-2H-\alpha}d\xi\\ &= \frac{c_H}{\|u(t,x)\|_{L^2(\Omega)}^2}\int_0^\infty\left(1-e^{-2t\xi^\alpha}\right)\xi^{1-2H-\alpha}\cos\left(\xi|x-y|\right)d\xi.\end{aligned} \tag{3.15}$$

Let

$$h(\xi) := \left(1-e^{-2t\xi^\alpha}\right)\xi^{1-2H-\alpha} \quad \text{for } \xi > 0.$$

We claim that

$$\int_0^\infty h(\xi)\cos(\xi|x-y|)\,d\xi = O\left(|x-y|^{\frac{H-1}{2}}\right), \quad \text{as} \quad |x-y|\to\infty. \tag{3.16}$$

This asymptotic estimate verifies Condition (2) in Theorem A.4. The proof of (3.16) will be presented later; we first address the estimate in (1.10).

For a fixed parameter $\lambda > 0$, consider the function

$$\phi_\lambda(x) := \sqrt{\lambda \log x} \quad \text{for } x > 0.$$

Let $I(\phi_\lambda)$ denote the integral defined in (A.4). We can prove the following dichotomy:

- If $\lambda > 2$, then $I(\phi_\lambda) < \infty$;
- If $\lambda \leqslant 2$, then $I(\phi_\lambda) = \infty$.

An application of Theorem A.4 consequently yields (1.10).

It remains to verify the claim (3.16). To do this, we divide the integral into two parts:

$$\begin{aligned}
&\int_0^\infty h(\xi)\cos(\xi|x-y|)\,d\xi\\
=&\int_0^{|x-y|^{-\frac14}} h(\xi)\cos(\xi|x-y|)\,d\xi + \int_{|x-y|^{-\frac14}}^\infty h(\xi)\cos(\xi|x-y|)\,d\xi\\
=:&\, J_1 + J_2.
\end{aligned} \tag{3.17}$$

Using the elementary inequality

$$1 - e^{-x} \leqslant x \quad \text{for any } x \geqslant 0, \tag{3.18}$$

we have

$$J_1 \leqslant 2t \int_0^{|x-y|^{-\frac14}} \xi^{1-2H} d\xi = \frac{t}{1-H}|x-y|^{\frac{H-1}{2}}. \tag{3.19}$$

Using the integration by parts formula and (3.18), we have

$$\begin{aligned}
J_2 &= |x-y|^{-1}\int_{|x-y|^{-\frac14}}^\infty h(\xi)d(\sin(\xi|x-y|))\\
&= |x-y|^{-1}\left[h(\xi)\sin(\xi|x-y|)\Big|_{|x-y|^{-\frac14}}^\infty - \int_{|x-y|^{-\frac14}}^\infty \sin(\xi|x-y|)d(h(\xi))\right]\\
&\leqslant |x-y|^{-1}\left[h\left(|x-y|^{-\frac14}\right) + \int_{|x-y|^{-\frac14}}^\infty |h'(\xi)|d\xi\right]\\
&\leqslant 2t|x-y|^{\frac{2H-5}{4}} + |x-y|^{-1}\int_{|x-y|^{-\frac14}}^\infty |h'(\xi)|d\xi.
\end{aligned} \tag{3.20}$$

Using the change of variables $\eta := \sqrt{2t}\xi^{\frac{\alpha}{2}}$, we have

$$\begin{aligned}
&\int_{|x-y|^{-\frac{1}{4}}}^{\infty} |h'(\xi)| d\xi \\
&\leqslant 2t\alpha \int_{|x-y|^{\frac{1}{4}}}^{\infty} e^{-2t\xi^{\alpha}} \xi^{-2H} d\xi + (2H+\alpha-1) \int_{|x-y|^{-\frac{1}{4}}}^{\infty} \left(1-e^{-2t\xi^{\alpha}}\right) \xi^{-2H-\alpha} d\xi \\
&\leqslant 2^{\frac{2H+2\alpha-1}{\alpha}} t^{\frac{2H+\alpha-1}{\alpha}} \int_0^{\infty} e^{-\eta^2} \eta^{\frac{2-4H-\alpha}{\alpha}} d\eta + (2H+\alpha-1) \int_{|x-y|^{-\frac{1}{4}}}^{\infty} \xi^{-2H-\alpha} d\xi \\
&= (2t)^{\frac{2H+\alpha-1}{\alpha}} \Gamma\left(\frac{1-2H}{\alpha}\right) + |x-y|^{\frac{2H+\alpha-1}{4}}.
\end{aligned} \tag{3.21}$$

By (3.17), (3.19), (3.20) and (3.21), we have

$$\begin{aligned}
&\int_0^{\infty} h(\xi) \cos\left(\xi|x-y|\right) d\xi \\
&\leqslant |x-y|^{\frac{H-1}{2}} \left(\frac{t}{1-H} + 2t|x-y|^{-\frac{3}{4}} + (2t)^{\frac{2H+\alpha-1}{\alpha}} \Gamma\left(\frac{1-2H}{\alpha}\right) |x-y|^{-\frac{1+H}{2}} + |x-y|^{-\frac{3-\alpha}{4}} \right),
\end{aligned}$$

which implies (3.16). The proof is complete. □

3.3 Proof of Theorem 1.4

Proof [Proof of Theorem 1.4] The proof of Theorem 1.4 follows the approach of Theorem 1.2 in [16] and Theorem 1.1. It relies on Talagrand's majorizing measure theorem and Sudakov's minoration theorem. The main steps are outlined below.

For a fixed $L \geqslant t^{1/\alpha}$, consider the partition of $[-L, L]$ introduced in (3.2). Using (2.21), the diameter of $A_n(x)$ under the metric $d_{2,t,h}(x,y)$ defined in (2.20) satisfies

$$\operatorname{diam}\left(A_n(t,x)\right) \leqslant c_{2,6} h^{\theta} \left((2^{-2^{n-2}} L)^{\frac{2H+\alpha-2-2\theta}{2}} \wedge t^{\frac{2H+\alpha-2-2\theta}{2\alpha}} \right).$$

Let

$$N_1 := \inf\left\{ n \geqslant 2 : 2^{-2^{n-2}} \leqslant \frac{t^{\frac{1}{\alpha}}}{L} \wedge \frac{1}{2} \right\}. \tag{3.22}$$

Applying Theorem A.1 in the same way as in the proofs of Theorem 1.1 and (3.8), we obtain that for all $\theta \in \left[0, \frac{2H+\alpha-2}{2}\right]$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{x\in\mathbb{L}} \Delta_h u(t,x)\right] &\leqslant c_{3,1} \sup_{x\in\mathbb{L}} \sum_{n=0}^{N_1} 2^{\frac{n}{2}} \operatorname{diam}\left(A_n(t,x)\right) + c_{3,1} \sup_{x\in\mathbb{L}} \sum_{n=N_1+1}^{\infty} 2^{\frac{n}{2}} \operatorname{diam}\left(A_n(t,x)\right) \\
&\leqslant c_{3,1} c_{2,6} h^{\theta} t^{\frac{2H+\alpha-2-2\theta}{2\alpha}} \left[\sum_{n=0}^{N_1} 2^{\frac{n}{2}} + \sum_{n=N_1+1}^{\infty} 2^{\frac{n}{2}} \left(\frac{2^{2^{N_1-2}}}{2^{2^{n-2}}} \right)^{\frac{2H+\alpha-2-2\theta}{2}} \right] \\
&\leqslant c_{3,4} h^{\theta} t^{\frac{2H+\alpha-2-2\theta}{2\alpha}} \cdot 2^{\frac{N_1}{2}} \\
&\leqslant 2^{\frac{3}{2}} c_{3,4} h^{\theta} t^{\frac{2H+\alpha-2-2\theta}{2\alpha}} \Psi(t,L).
\end{aligned} \tag{3.23}$$

Consider the interval $[-L, L]$ with $L \geqslant t^{1/\alpha}$. We select the grid points

$$x_j = j t^{1/\alpha} \quad \text{for} \quad j = 0, \pm 1, \ldots, \pm \left\lfloor L/t^{1/\alpha} \right\rfloor.$$

Applying Theorem A.2 as in (3.10), we obtain the lower bound for all $0 < h \leqslant \frac{3}{64} t^{1/\alpha}$,

$$\mathbb{E}\left[\sup_{-L\leqslant x\leqslant L} \Delta_h u(t,x)\right] \geqslant \frac{1}{2} c_{A,1} c_{2,7} h^{\frac{2H+\alpha-2}{2}} \Psi(t,L). \tag{3.24}$$

Combining the upper bound (3.23) with the lower bound (3.24) yields the desired result (1.13). The proof is complete. □

3.4 Proof of Proposition 1.5

Proof [Proof of Proposition 1.5] The result follows the argument outlined in the proof of Proposition 1.2; the key steps are summarized below.

Step 1. Fix an arbitrary $\varepsilon > 0$ and suppose that $\beta > \frac{2c_{2,2}^2 \log 2}{c_{1,5}^2 \varepsilon^2}$. Denote $\mathbb{L}^\beta(n) := \left[-n^\beta, n^\beta\right]$. By (1.13) and (2.8), we have that for any $0 < h \leqslant \frac{3}{64} t^{\frac{1}{\alpha}}$,

$$\mathbb{E}\left[\sup_{x \in \mathbb{L}^\beta(n)} \Delta_h u(t,x)\right] \geqslant c_{1,5} h^{\frac{2H+\alpha-2}{2}} \left(1 + \sqrt{\log_2\left(\frac{n^\beta}{t^{1/\alpha}}\right)}\right),$$

and

$$\sup_{x \in \mathbb{L}^\beta(n)} \mathbb{E}\left[|\Delta_h u(t,x)|^2\right] \leqslant c_{2,2}^2 h^{2H+\alpha-2}.$$

Consequently, combining Borell's inequality (A.3) with the Borel-Cantelli lemma and applying the estimate (1.13), we establish (1.14).

Step 2. Let $\rho_2(x,y)$ be the correlation function between $\Delta_h u(t,x)$ and $\Delta_h u(t,y)$, that is,

$$\rho_2(x,y) := \frac{\mathbb{E}[\Delta_h u(t,x)\Delta_h u(t,y)]}{\sqrt{\mathbb{E}[\Delta_h u(t,x)^2] \cdot \mathbb{E}[\Delta_h u(t,y)^2]}}.$$

Proceeding similarly to the proof of (3.14) and applying Lemma 2.4, we can find some positive constants $c_{3,5}, c_{3,6}$ and $c_{3,7}$ such that for all $0 < |x-y| < c_{3,5}h$,

$$1 - \frac{c_{3,6}}{\|\Delta_h u(t,x)\|_{L^2(\Omega)}^2}|x-y|^{2H+\alpha-2} \leqslant \rho_2(x,y) \leqslant 1 - \frac{c_{3,7}}{\|\Delta_h u(t,x)\|_{L^2(\Omega)}^2}|x-y|^{2H+\alpha-2}.$$

This verifies Condition (1) in Theorem A.4.

By (2.2), (2.5), and (3.16), we have

$$\begin{aligned}\rho_2(x,y) &= \frac{c_H}{\|\Delta_h u(t,x)\|_{L^2(\Omega)}^2} \int_{\mathbb{R}} \left(1 - e^{-2t|\xi|^\alpha}\right)(1-\cos(h\xi))\,|\xi|^{1-2H-\alpha}\cos(\xi|x-y|)d\xi \\ &\leqslant \frac{4c_H}{\|\Delta_h u(t,x)\|_{L^2(\Omega)}^2} \int_0^\infty \left(1 - e^{-2t\xi^\alpha}\right)\xi^{1-2H-\alpha}\cos(\xi|x-y|)d\xi \\ &= O\left(|x-y|^{\frac{H-1}{2}}\right), \quad \text{as} \quad |x-y| \to \infty.\end{aligned}$$

This verifies Condition (2) in Theorem A.4. The same argument as in Proposition 1.2(c) gives (1.15). The proof is complete. □

3.5 Proof of Theorem 1.7

Proof [Proof of Theorem 1.7] The proof of Theorem 1.7 follows the strategy of Theorem 1.1 and [16, Theorem 1.3], combining Talagrand's majorizing measure theorem with Sudakov's minoration theorem. The main steps are outlined below.

For a fixed $L > 0$, consider the partition of $[-L, L]$ introduced in (3.2). Under the metric $d_{3,t,\tau}(x,y)$ defined in (2.28), the diameter of $A_n(x)$ is bounded via Lemma 2.5 as follows:

$$\operatorname{diam}\left(A_n(t,x)\right) \leqslant c_{2,8}\tau^{\frac{\theta}{2}}\left((2^{-2^{n-2}}L)^{\frac{2H+\alpha-2-\alpha\theta}{2}} \wedge t^{\frac{2H+\alpha-2-\alpha\theta}{2\alpha}}\right).$$

For $0 < \tau \leqslant t$, we apply Theorem A.1 in the same way as in Theorem 1.4 together with the bound from (3.8) to obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{x\in\mathbb{L}} \mathcal{D}_\tau u(t,x)\right] &\leqslant c_{3,1} \sup_{x\in\mathbb{L}} \sum_{n=0}^{N_1} 2^{\frac{n}{2}} \operatorname{diam}\left(A_n(t,x)\right) + c_{3,1} \sup_{x\in\mathbb{L}} \sum_{n=N_1+1}^{\infty} 2^{\frac{n}{2}} \operatorname{diam}\left(A_n(t,x)\right) \\
&\leqslant c_{3,1} c_{2,8} \tau^{\frac{\theta}{2}} t^{\frac{2H+\alpha-2-\alpha\theta}{2\alpha}} \left[\sum_{n=0}^{N_1} 2^{\frac{n}{2}} + \sum_{n=N_1}^{\infty} 2^{\frac{n}{2}} \left(\frac{2^{2^{N_1-2}}}{2^{2^{n-2}}}\right)^{\frac{2H+\alpha-2-\alpha\theta}{2}}\right] \qquad (3.25) \\
&\leqslant c_{3,8} \tau^{\frac{\theta}{2}} t^{\frac{2H+\alpha-2-\alpha\theta}{2\alpha}} \cdot 2^{\frac{N_1}{2}} \\
&\leqslant 2^{\frac{3}{2}} c_{3,8} \tau^{\frac{\theta}{2}} t^{\frac{2H+\alpha-2-\alpha\theta}{2\alpha}} \Psi(t,L),
\end{aligned}$$

where N_1 is given by (3.22).

Following the argument in the proof of (3.10) and applying Theorem A.2, we obtain that for all $0 < \tau \leqslant \left(\frac{3}{32}\right)^{\alpha} t$,

$$\mathbb{E}\left[\sup_{-L\leqslant x\leqslant L} \mathcal{D}_\tau u(t,x)\right] \geqslant \frac{1}{2} c_{A,1} c_{2,7} \tau^{\frac{2H+\alpha-2}{2\alpha}} \Psi(t,L). \qquad (3.26)$$

Combining (3.25) and (3.26), we have (1.17). The proof is complete. □

3.6 Proof of Proposition 1.8

Proof [Proof of Proposition 1.8] The proof follows the structure of Proposition 1.2; only the key steps are presented here.

Step 1. For any $\varepsilon > 0$, assume $\beta > \frac{2c_{2,2}^2 \log 2}{c_{1,9}^2 \varepsilon^2}$. Denote $L^\beta(n) = n^\beta$. By (1.17) and (2.8), we have that for any $0 < \tau \leqslant \left(\frac{3}{32}\right)^{\alpha} t$,

$$\mathbb{E}\left[\sup_{x\in\mathbb{L}^\beta(n)} \mathcal{D}_\tau u(t,x)\right] \geqslant c_{1,9} \tau^{\frac{2H+\alpha-2}{2\alpha}} \left(1 + \sqrt{\log_2\left(\frac{n^\beta}{t^{1/\alpha}}\right)}\right),$$

and

$$\sup_{x\in\mathbb{L}^\beta(n)} \mathbb{E}\left[|\mathcal{D}_\tau u(t,x)|^2\right] \leqslant c_{2,2}^2 \tau^{\frac{2H+\alpha-2}{\alpha}}.$$

Using Borell's inequality (A.3) and the Borel-Cantelli lemma with (1.17), we obtain (1.18).

Step 2. To prove (1.19), we verify the conditions of Theorem A.4. The correlation function between $\mathcal{D}_\tau u(t,x)$ and $\mathcal{D}_\tau u(t,y)$ is given by

$$\rho_3(x,y) := \frac{\mathbb{E}[\mathcal{D}_\tau u(t,x)\mathcal{D}_\tau u(t,y)]}{\sqrt{\mathbb{E}[\mathcal{D}_\tau u(t,x)^2] \cdot \mathbb{E}[\mathcal{D}_\tau u(t,y)^2]}}.$$

Applying the approach of (3.14) through Lemma 2.5, there exist some positive constants $c_{3,9}$, $c_{3,10}$, and $c_{3,11}$ such that for all x, y with $0 < |x-y| < c_{3,9} h$,

$$1 - \frac{c_{3,10}}{\|\mathcal{D}_\tau u(t,x)\|_{L^2(\Omega)}^2} |x-y|^{2H+\alpha-2} \leqslant \rho_3(x,y) \leqslant 1 - \frac{c_{3,11}}{\|\mathcal{D}_\tau u(t,x)\|_{L^2(\Omega)}^2} |x-y|^{2H+\alpha-2},$$

where $\|\mathcal{D}_\tau u(t,x)\|_{L^2(\Omega)}^2$ does not depend on x; see (2.14).

By (2.2) and (2.5), we have

$$\begin{aligned}\rho_3(x,y) &= \frac{c_H}{2\|\mathcal{D}_\tau u(t,x)\|^2_{L^2(\Omega)}}\int_{\mathbb{R}}\left[\left(1-e^{-2t|\xi|^\alpha}\right)\left(1-e^{-\tau|\xi|^\alpha}\right)^2+\left(1-e^{-2\tau|\xi|^\alpha}\right)\right]\\ &\qquad\qquad\qquad\cdot|\xi|^{1-2H-\alpha}\cos(\xi|x-y|)d\xi\\ &\leqslant \frac{c_H}{\|\mathcal{D}_\tau u(t,x)\|^2_{L^2(\Omega)}}\left[\int_0^\infty\left(1-e^{-2t\xi^\alpha}\right)\xi^{1-2H-\alpha}\cos(\xi|x-y|)d\xi\right.\\ &\qquad\qquad\left.+\int_0^\infty\left(1-e^{-2\tau\xi^\alpha}\right)\xi^{1-2H-\alpha}\cos(\xi|x-y|)d\xi\right]\\ &= O\left(|x-y|^{\frac{H-1}{2}}\right),\quad \text{as}\quad |x-y|\to\infty,\end{aligned}$$

where (3.16) is used in the last step.

Having verified the conditions of Theorem A.4, the estimate (1.19) follows in parallel to the proof of Proposition 1.2(c). This completes the proof. □

3.7 Proof of Proposition 1.10

Proof [Proof of Proposition 1.10] To prove (1.22), we adapt the approach of [16, Proposition 3.11] and apply the Sudakov minoration theorem.

Consider the probability density function defined as

$$\varrho(h) := c_{3,12}\left(|h|^{-\frac{H}{2}-\frac{\alpha}{4}}\mathbf{1}_{\{|h|\leqslant 1\}}+|h|^{2H-2}\mathbf{1}_{\{|h|\geqslant 1\}}\right),$$

where $c_{3,12}=2^{-1}\left(\frac{4}{4-2H-\alpha}+\frac{1}{1-2H}\right)^{-1}$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}}\Delta_h u(t,x)\varrho(h)dh\right| &\leqslant \left(\int_{\mathbb{R}}|\Delta_h u(t,x)|^2\cdot|h|^{2H-2}dh\right)^{\frac12}\cdot\left(\int_{\mathbb{R}}\varrho^2(h)|h|^{2-2H}dh\right)^{\frac12}\\ &= c_{3,13}^{\frac12}\mathcal{N}_{\frac12-H}u(t,x),\end{aligned}\tag{3.27}$$

where $c_{3,13}=2c_{3,12}^2\left(\frac{2}{6-6H-\alpha}+\frac{1}{1-2H}\right)$. This, together with Jensen's inequality, implies that

$$\begin{aligned}\mathbb{E}\left[\sup_{-L\leqslant x\leqslant L}\mathcal{N}^2_{\frac12-H}u(t,x)\right] &\geqslant c_{3,13}^{-1}\mathbb{E}\left[\left(\sup_{-L\leqslant x\leqslant L}\int_{\mathbb{R}}\Delta_h u(t,x)\varrho(h)dh\right)^2\right]\\ &\geqslant c_{3,13}^{-1}\left(\mathbb{E}\left[\sup_{-L\leqslant x\leqslant L}\int_{\mathbb{R}}\Delta_h u(t,x)\varrho(h)dh\right]\right)^2.\end{aligned}\tag{3.28}$$

Denote

$$u_\varrho(t,x) := \int_{\mathbb{R}}\Delta_h u(t,x)\varrho(h)dh.$$

By stochastic Fubini's theorem (see, e.g., [18, Corollary 2.9]), we have

$$u_\varrho(t,x) = \int_0^t\int_{\mathbb{R}}\left(\int_{\mathbb{R}}[G_\alpha(t-s,x+h-z)-G_\alpha(t-s,x-z)]\varrho(h)dh\right)W(ds,dz).$$

By (2.2) and (2.5), we have

$$\begin{aligned}\mathbb{E}\left[u_\varrho^2(t,x)\right] &= c_H \int_0^t \int_{\mathbb{R}} \left|\mathcal{F}\left(\int_{\mathbb{R}} [G_\alpha(t-s,x+h-\cdot) - G_\alpha(t-s,x-\cdot)]\varrho(h)dh\right)(\xi)\right|^2 |\xi|^{1-2H} d\xi ds\\ &= c_H \int_0^t \int_{\mathbb{R}} \left|\int_{\mathbb{R}} \left[e^{-(t-s)|\xi|^\alpha - i(x+h)\xi} - e^{-(t-s)|\xi|^\alpha - ix\xi}\right] \varrho(h)dh\right|^2 |\xi|^{1-2H} d\xi ds\\ &= c_H \int_0^\infty \left(1-e^{-2t\xi^\alpha}\right)\left(\int_{\mathbb{R}} (1-\cos(h\xi))\varrho(h)dh\right)^2 \xi^{1-2H-\alpha} d\xi\\ &\leqslant 4c_H \int_0^\infty \left(1-e^{-2t\xi^\alpha}\right)\xi^{1-2H-\alpha} d\xi.\end{aligned}$$

The last integral is convergent since $\alpha + 2H > 2$, which guarantees that $u_\varrho(t,x)$ is a well-defined Gaussian random field.

Next, we establish the lower bound for the canonical metric $d_{t,\varrho}$, which is defined by

$$d_{t,\varrho} := \left(\mathbb{E}\left[|u_\varrho(t,x) - u_\varrho(t,y)|^2\right]\right)^{\frac{1}{2}} \quad \text{for } |x-y| \geqslant t^{\frac{1}{\alpha}}. \tag{3.29}$$

The argument follows the approach used in (3.10).

Using (2.2) and (2.5), we obtain

$$\begin{aligned}d_{t,\varrho}^2 &= c_H \int_0^t \int_{\mathbb{R}} \Big|\mathcal{F}\Big(\int_{\mathbb{R}} [G_\alpha(t-s,x+h-\cdot) - G_\alpha(t-s,x-\cdot)\\ &\qquad - G_\alpha(t-s,y+h-\cdot) + G_\alpha(t-s,y-\cdot)]\varrho(h)dh\Big)(\xi)\Big|^2 |\xi|^{1-2H} d\xi ds\\ &= c_H \int_0^t \int_{\mathbb{R}} \left|\int_{\mathbb{R}} e^{-(t-s)|\xi|^\alpha} e^{-ix\xi}\left(1-e^{-i(y-x)\xi}\right)\left(e^{-ih\xi}-1\right)\varrho(h)dh\right|^2 |\xi|^{1-2H} d\xi ds\\ &= 4c_H \int_0^\infty \left(1-e^{-2t\xi^\alpha}\right)(1-\cos(|x-y|\xi))\left(\int_0^\infty (1-\cos(h\xi))\varrho(h)dh\right)^2 \xi^{1-2H-\alpha} d\xi.\end{aligned} \tag{3.30}$$

When $\xi \geqslant 1$, by (2.24), we have

$$\begin{aligned}\int_0^\infty (1-\cos(h\xi))\varrho(h)dh &\geqslant c_{3,12} \int_0^1 (1-\cos(h\xi)) h^{-\frac{H}{2}-\frac{\alpha}{4}} dh\\ &= c_{3,12}\xi^{\frac{H}{2}+\frac{\alpha}{4}-1} \int_0^\xi (1-\cos(h)) h^{-\frac{H}{2}-\frac{\alpha}{4}} dh\\ &\geqslant c_{3,12}\xi^{\frac{H}{2}+\frac{\alpha}{4}-1} \int_0^1 (1-\cos(h)) h^{-\frac{H}{2}-\frac{\alpha}{4}} dh\\ &\geqslant \frac{c_{3,12}}{4}\xi^{\frac{H}{2}+\frac{\alpha}{4}-1} \int_0^1 h^{-\frac{H}{2}-\frac{\alpha}{4}+2} dh\\ &= \frac{c_{3,12}}{12-2H-\alpha}\xi^{\frac{H}{2}+\frac{\alpha}{4}-1}.\end{aligned} \tag{3.31}$$

This, together with (3.30), implies that

$$\begin{aligned}d_{t,\varrho}^2 &\geqslant \frac{4c_H c_{3,12}^2}{(12-2H-\alpha)^2}\left(1-e^{-2t}\right)\int_1^\infty (1-\cos(|x-y|\xi))\xi^{-H-\frac{\alpha}{2}-1} d\xi\\ &= \frac{4c_H c_{3,12}^2}{(12-2H-\alpha)^2}\left(1-e^{-2t}\right)|x-y|^{H+\frac{\alpha}{2}} \int_{|x-y|}^\infty (1-\cos(\xi))\xi^{-H-\frac{\alpha}{2}-1} d\xi.\end{aligned} \tag{3.32}$$

Define

$$k_3 := \left\lfloor \frac{|x-y| - \frac{\pi}{2}}{2\pi} \right\rfloor + 1. \tag{3.33}$$

Then, there exists a positive constant $c_{3,13}$ such that

$$\begin{aligned}
\int_{|x-y|}^{\infty} (1-\cos(\xi))\xi^{-H-\frac{\alpha}{2}-1} d\xi &\geqslant \sum_{k=k_3}^{\infty} \int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} (1-\cos(\xi))\xi^{-H-\frac{\alpha}{2}-1} d\xi \\
&\geqslant \pi \sum_{k=k_3}^{\infty} \left(2k\pi + \frac{3\pi}{2}\right)^{-H-\frac{\alpha}{2}-1} \\
&\geqslant c_{3,13} \left(2k_3\pi + \frac{3\pi}{2}\right)^{-H-\frac{\alpha}{2}} \\
&\geqslant c_{3,13} \left(|x-y| + 3\pi\right)^{-H-\frac{\alpha}{2}},
\end{aligned} \tag{3.34}$$

where the second inequality holds because $\cos(\xi)$ is negative on the intervals $\left[2k\pi + \frac{\pi}{2}, 2k\pi + \frac{3\pi}{2}\right]$ and $\xi^{-H-\frac{\alpha}{2}-1}$ is a decreasing function on $(0,\infty)$.

Combining (3.34) with (3.32), we have that for any $|x-y| \geqslant t^{\frac{1}{\alpha}}$,

$$\begin{aligned}
d_{t,\varrho}^2 &\geqslant \frac{4c_H c_{3,13} c_{3,12}^2}{(12-2H-\alpha)^2} \left(1-e^{-2t}\right) \left(\frac{1}{1+\frac{3\pi}{|x-y|}}\right)^{H+\frac{\alpha}{2}} \\
&\geqslant \frac{4c_H c_{3,13} c_{3,12}^2}{(12-2H-\alpha)^2} \left(1-e^{-2t}\right) \left(\frac{t^{\frac{1}{\alpha}}}{t^{\frac{1}{\alpha}}+3\pi}\right)^{H+\frac{\alpha}{2}}.
\end{aligned} \tag{3.35}$$

For the interval $[-L, L]$, where $L \geqslant t^{1/\alpha}$, we define a set of grid points

$$x_j = jt^{1/\alpha} \quad \text{for} \quad j = 0, \pm 1, \ldots, \pm \left\lfloor L/t^{1/\alpha} \right\rfloor.$$

Using (3.35) and Theorem A.2 as in (3.10), there exists a positive constant c_t depending on t such that

$$\mathbb{E}\left[\sup_{-L\leqslant x\leqslant L} \int_{\mathbb{R}} \Delta_h u(t,x)\varrho(h)dh\right] \geqslant c_t\sqrt{\log_2 L}. \tag{3.36}$$

Combining with (3.28) and (3.36), we obtain (1.10). The proof is complete. □

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Appendix

Let $\mathbb{T}$ be an index set and $\{X_t\}_{t\in\mathbb{T}}$ a centered Gaussian process. Define the canonical metric d by

$$d(t,s)=\sqrt{\mathbb{E}\left[|X_t-X_s|^2\right]}.$$

Theorem A.1 (Talagrand's majorizing measure theorem, [25, Theorem 2.4.1]) There exists a constant $c>1$ such that

$$c^{-1}\inf_{\mathcal{A}}\sup_{t\in\mathbb{T}}\sum_{n\geqslant 0}2^{\frac{n}{2}}\operatorname{diam}\left(A_n(t)\right)\leqslant\mathbb{E}\Big[\sup_{t\in\mathbb{T}}X_t\Big]\leqslant c\inf_{\mathcal{A}}\sup_{t\in\mathbb{T}}\sum_{n\geqslant 0}2^{\frac{n}{2}}\operatorname{diam}\left(A_n(t)\right),\tag{A.1}$$

where the infimum is taken over all increasing sequence $\mathcal{A}:=\{\mathcal{A}_n,n=1,2,\cdots\}$ of partitions of $\mathbb{T}$ such that $\#\mathcal{A}_n\leqslant 2^{2^n}$ ($\#A$ denotes the number of elements in the set A), $A_n(t)$ denotes the unique element of $\mathcal{A}_n$ that contains t, and $\operatorname{diam}(A_n(t))$ is the diameter of $A_n(t)$ with respect to the metric d.

Theorem A.2 (Sudakov's minoration theorem, [25, Lemma 2.4.2]) Let $\{X_{t_i}, i=1,\cdots,n\}$ be a centered Gaussian family with natural distance d and assume that

$$\forall p,q \leqslant n,\ p \neq q \Rightarrow d(t_p,t_q) \geqslant \delta.$$

Then, we have

$$\mathbb{E}\left[\sup_{1\leqslant i\leqslant n} X_{t_i}\right] \geqslant c_{A,1}\delta\sqrt{\log_2(n)}, \tag{A.2}$$

where $c_{A,1}$ is a universal constant.

Theorem A.3 (Borell's inequality, [1, Theorem 2.1]) Let $\{X_t\}_{t\in\mathbb{T}}$ be a centered separable Gaussian process on some topological index set $\mathbb{T}$ with almost surely bounded sample paths. Then $\mathbb{E}\left[\sup_{t\in\mathbb{T}} X_t\right] < \infty$, and for all $\lambda > 0$

$$\mathbb{P}\left(\left|\sup_{t\in\mathbb{T}} X_t - \mathbb{E}\left[\sup_{t\in\mathbb{T}} X_t\right]\right| > \lambda\right) \leqslant 2\exp\left(-\frac{\lambda^2}{2\sigma^2}\right), \tag{A.3}$$

where $\sigma^2 := \sup_{t\in\mathbb{T}} \mathbb{E}\left[X_t^2\right]$.

Theorem A.4 ([24, Theorems 1.1, 2.1 and 2.2]) Let $\{X(t)\}_{t\in\mathbb{R}}$ be a real-valued separable stationary Gaussian process with $\mathbb{E}[X(t)] \equiv 0$ and $\mathrm{Var}[X(t)] \equiv 1$. Assume that its correlation function $\rho(s,t) = \mathbb{E}[X(s)X(t)]$ is continuous and satisfies the following conditions:

(1) There are some positive constants $\delta, c_{A,2}, c_{A,3}, T$, and $\alpha \in (0,2]$ such that

$$1 - c_{A,2}h^{\alpha} \leqslant \rho(t,t+h) \leqslant 1 - c_{A,3}h^{\alpha}, \quad \text{for all} \ \ 0 \leqslant h \leqslant \delta, \quad t \geqslant T;$$

(2) There exists a constant $\gamma > 0$ such that

$$\rho(t,t+h) = O(h^{-\gamma}), \quad \text{as} \ \ h \to \infty.$$

Then, for every positive non-decreasing function $\phi(t) : [a,\infty) \to \mathbb{R}_+$,

$$\mathbb{P}\Big(\exists t_0(\omega) > a : X(t) < \phi(t) \ \ \text{for all} \ \ t \geqslant t_0(\omega)\Big) = 1 \ \ \text{or} \ \ 0,$$

according as the integral

$$I(\phi) := \int_a^{\infty} \phi(t)^{2/\alpha - 1} \exp\left(-\phi(t)^2/2\right) dt \tag{A.4}$$

converges or diverges.